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Existence and Nonexistence of Nontrivial Solutions for Some Nonlinear Elliptic Systems

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ABSTRACT. In this paper we give some existence and nonexistence results of non trivial solutions of nonlinear elliptic systems involving the p -Laplacian.

0. INTRODUCTION

In this paper, we give some existence and nonexistence results concerning nonlinear elliptic systems. The case of one equation has been studied by many authors.

Let Ω be a bounded regular open set in \mathbb{R}^n and consider the problem

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$$(P_\lambda) \quad \begin{cases} \text{Find } u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \text{ such that} \\ -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f(u) \in C^{0,\alpha}(\mathbb{R})$, $0 < \alpha < 1$, is such that: $f(0) = 0$ and $|f(u)| \leq A + B|u|^m$.

Any solution u^* of (P_λ) satisfies the Pohožaev's identity [21]:

$$n \int_{\Omega} \lambda \left[\frac{n-2}{2n} u^* f(u^*) - \int_0^{u^*} f(s) ds \right] dx = -\frac{1}{2} \int_{\partial\Omega} |\nabla u^*|^2 (x \cdot \nu) d\sigma,$$

whence $u^* = 0$ if Ω is starshaped and

$$\lambda \left[\frac{n-2}{2n} u^* f(u^*) - \int_0^{u^*} f(s) ds \right] > 0.$$

On the other hand, if

$$0 < m+1 < \frac{2n}{n-2},$$

Pohožaev [21] has shown that (P_λ) admits an eigenfunction $u^* \neq 0$ corresponding to λ .

Always in the scalar case, Ôtani [19], [20] and de Thélin [25] generalize these results for the p -Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

For example, they give the following results concerning the equation

$$(E_\lambda) \quad -\Delta_p u = \lambda |u|^{m-1} u$$

- If Ω is a strictly starshaped open set and $(m+1)(n-p) \geq np$ the only solution $u^* \in W_0^{1,p}(\Omega)$ of (E_λ) is $u^* \equiv 0$.

- If $(m+1)(n-p) < np$ and $m+1 \neq p$, then for any $\lambda > 0$, (E_λ) admits a positive solution $u^* \in W_0^{1,p}(\Omega)$.

- If $m+1 = p$, we have an eigenvalue problem [3].

More recently, in [32], we have given some results concerning the existence and nonexistence of a nontrivial solution $(u^*, v^*) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of the following system

$$\begin{cases} -\Delta_p u = u |u|^{\alpha-1} |v|^{\beta+1} & \text{in } \Omega \\ -\Delta_q v = |u|^{\alpha+1} v |v|^{\beta-1}. \end{cases}$$

We prove

1) nonexistence results when

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} \geq 1$$

when Ω is a strictly starshaped open set;

2) existence results when

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} < 1$$

and when

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} \neq 1.$$

Now, in this paper, we extend the study of existence and nonexistence of positive solutions of the nonlinear elliptic problem

$$(P) \quad \begin{cases} -\Delta_p u = f(x; u, v) & \text{in } \Omega \\ -\Delta_q v = g(x; u, v) & \text{in } \Omega \\ u = 0, v = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that (P) is a potential system if there is a C^1 function H such that

$$f(x;s,t) = \frac{\partial H}{\partial s}(x;s,t), \quad g(x;s,t) = \frac{\partial H}{\partial t}(x;s,t).$$

In a first part, following Egnell [10] and Pucci-Serrin [22], we obtain a Pohožaev type identity for potential systems. In the case when Ω is a starshaped bounded open set, this identity gives nonexistence results.

In a second part, we give some existence results for non potential systems. Following Deuel and Hess [7], we construct appropriate sub-supersolutions for (P) and use a suitable comparison principle.

In a third part, we give some existence results for potential systems. Following Nirenberg [18], we apply Mountain-Pass Lemma to find a nontrivial solution; after that, we extend an iterative method previously used by Ôtani [20] for the equation (E_λ) to prove that the solution is bounded.

Concerning the systems, we can notice the existence results obtained in [4], [6], [11], [12], [28]. Independently, [13], [22] give nonexistence results.

1. NONEXISTENCE RESULT

In this first section, we propose to extend the non-existence study, made by de Thélin [26] and Egnell [10] in the scalar case, to the following problem (P)

$$(P) \quad \left\{ \begin{array}{l} \text{Find } (u,v) \in X \cap [L^\infty(\Omega)]^2 \text{ such that} \\ (1) \quad -\Delta_p u = \frac{\partial H}{\partial u}(x;u,v) \quad \text{in } \Omega \\ (2) \quad -\Delta_q v = \frac{\partial H}{\partial v}(x;u,v) \quad \text{in } \Omega \\ \quad \quad u > 0 \quad \quad \quad \text{in } \Omega \\ \quad \quad v > 0 \quad \quad \quad \text{in } \Omega \end{array} \right.$$

Hereafter, X denotes the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

1.1. Properties and Results.

Theorem 1.1. *Assume the following hypotheses*

$$i) H(x;0,0) = 0 \text{ and } \frac{\partial H}{\partial s}(x;0,0) = \frac{\partial H}{\partial t}(x;0,0) = 0$$

$$ii) \frac{\partial H}{\partial s}(x;s,t), \frac{\partial H}{\partial t}(x;s,t) \text{ are in } C(\Omega \times \mathbb{R} \times \mathbb{R}) \text{ and } \frac{\partial H}{\partial s}(x;s,t) \geq 0$$

$$\frac{\partial H}{\partial t}(x;s,t) \geq 0 \text{ for any } s, t \geq 0 \text{ and } x \in \Omega$$

$$iii) \forall (s,t) \in \mathbb{R}^2$$

$$H(x;s,t) \leq \frac{n-p}{np} \left\{ s \frac{\partial H}{\partial s}(x;s,t) \right\} + \frac{n-q}{nq} \left\{ t \frac{\partial H}{\partial t}(x;s,t) \right\} - \frac{x}{n} \cdot \nabla_x H(x;s,t)$$

$$iv) \Omega \text{ is a bounded strictly starshaped domain in } \mathbb{R}^n \text{ containing } 0.$$

Then, $(u^*, v^*) \equiv 0$ is the only solution of (P) in $X \cap [L^\infty(\Omega)]^2$.

Corollary 1.1. *Let Ω be a bounded strictly starshaped domain in \mathbb{R}^n and $H(x;s,t) = |s|^{\alpha+1} |t|^{\beta+1}$.*

If

$$(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} \geq 1,$$

(P) has only the trivial solution $(0,0)$ in $X \cap [L^\infty(\Omega)]^2$.

Proof of the Corollary 1.1. Since

$$(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} \geq 1,$$

we have

$$H(x; s, t) \leq \left[(\alpha + 1) \frac{n-p}{np} + (\beta + 1) \frac{n-q}{nq} \right] H(x; s, t)$$

(1.1)

$$\leq \frac{n-p}{np} \left\{ s \frac{\partial H}{\partial s}(x; s, t) \right\} + \frac{n-q}{nq} \left\{ t \frac{\partial H}{\partial t}(x; s, t) \right\}$$

and all the hypotheses of Theorem 1.1 are satisfied.

The proof of Theorem 1.1 needs the following lemma which extends Egnell's one [10].

Lemma 1.1.1. *Let (u^*, v^*) be a solution of the problem (P); then for all x on the boundary of Ω , we have: $|\nabla u^*(x)| \neq 0$ and $|\nabla v^*(x)| \neq 0$.*

Proof. Let $x_0 \in \partial\Omega$; there is a ball $B_{r_0} \subset \Omega$.

By translation we assume that $B_{r_0} = \{x \in \Omega; |x| < r_0\}$ and, proceeding as in [10], we introduce the function

$$g(x) = k(e^{-\alpha|x|^2} - e^{-\alpha r_0^2}).$$

For $p > 1$, a suitable choice of α gives g_p such that

$$(1.2) \quad -\operatorname{div}(|\nabla g_p|^{m-2} \nabla g_p) \leq a g_p^{m-1} \text{ in } B_{r_0} \setminus B_{r_0/2}$$

Multiplying (1) and (1.2)_p [resp. (2) and (1.2)_q] by the test function $\varphi_p = (g_p - u^*)_+$ [resp. $\varphi_q = (g_q - v^*)_+$] and integrating on the set $B_p^+ = \{x \in B_{r_0} \setminus B_{r_0/2}; \varphi_p > 0\}$ [resp. B_q^+] where u^* and v^* are regular, we obtain

$$0 \leq \int_{B_p^+} (|\nabla g_p|^{p-2} \nabla g_p - |\nabla u^*|^{p-2} \nabla u^*) \cdot \nabla \varphi_p dx \leq - \int_{B_p^+} \frac{\partial H}{\partial u}(x; u^*, v^*) \varphi_p dx$$

whence, $g_p \leq u^*$ in $B_{r_0} \setminus B_{r_0/2}$.

By construction $g_p(x_0) = u^*(x_0) = 0$, therefore

$$(1.3) \quad |\nabla u^*(x_0)| > 2k_p \alpha_p e^{-\alpha_r} > 0$$

Proof of Theorem 1.1. Let (u^*, v^*) be a nontrivial solution of (P). For $i = 1, \dots, n$; $l = 1, \dots, n$ let

$$P_i = \sum_{l=1}^n |\nabla u^*|^{p-2} \frac{\partial u^*}{\partial x_l} x_l \frac{\partial u^*}{\partial x_i} \quad \text{and} \quad Q_i = \sum_{l=1}^n |\nabla v^*|^{q-2} \frac{\partial v^*}{\partial x_l} x_l \frac{\partial v^*}{\partial x_i}$$

Let $K_p = \{x \in \Omega; |\nabla u^*(x)| = 0\}$, $K_q = \{x \in \Omega; |\nabla v^*(x)| = 0\}$.

Lemma 1.1. allows us to consider as in [10], the sets $\tilde{\Omega}_k$ and $\tilde{\Omega}_k^i$ such that $K_p \subset \tilde{\Omega}_k \subset\subset \Omega$, $K_q \subset \tilde{\Omega}_k^i \subset\subset \Omega$, with $dist(K_p; \partial\tilde{\Omega}_k) \rightarrow 0$, $dist(K_q; \partial\tilde{\Omega}_k^i) \rightarrow 0$, as $k \rightarrow +\infty$ and we define $\Omega_k = \Omega \setminus \tilde{\Omega}_k$, $\Omega_k^i = \Omega \setminus \tilde{\Omega}_k^i$.

$$\sum_{i=1}^n \int_{\Omega_i} \frac{\partial P_i}{\partial x_i} dx = \sum_{i=1}^n \int_{\Omega_i} \sum_{l=1}^n x_l \frac{\partial u^*}{\partial x_l} \frac{\partial}{\partial x_i} \left(|\nabla u^*|^{p-2} \frac{\partial u^*}{\partial x_i} \right) dx + \int_{\Omega_i} |\nabla u^*|^p dx$$

$$+ \sum_{i=1}^n \int_{\Omega_i} \sum_{l=1}^n \frac{\partial u^*}{\partial x_l} x_l |\nabla u^*|^{p-2} \frac{\partial}{\partial x_i} \left(\frac{\partial u^*}{\partial x_l} \right) dx$$

$$(1.4) \quad = - \int_{\Omega_i} \sum_{l=1}^n x_l \frac{\partial u^*}{\partial x_l} \frac{\partial H}{\partial u}(x; u^*, v^*) dx + \int_{\Omega_i} |\nabla u^*|^p dx$$

$$+ \int_{\Omega_i} \sum_{l=1}^n \frac{\partial}{\partial x_l} \left(x_l \frac{1}{p} |\nabla u^*|^p \right) dx - \frac{n}{p} \int_{\Omega_i} |\nabla u^*|^p dx$$

∇u^* do not vanish in Ω_k and therefore u^* is of class C^2 in Ω_k , so we can use the Gauss's formula to obtain

$$(1.5) \quad \int_{\Omega_i} \sum_{i=1}^n \frac{\partial P}{\partial x_i} dx = \int_{\partial\Omega_i} \sum_{i=1}^n P_i v_i d\sigma = \int_{\partial\Omega_i} |\nabla u^*|^{p-2} (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma$$

and

$$(1.6) \quad \int_{\Omega_i} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(x_i \frac{1}{p} |\nabla u^*|^p \right) = \int_{\partial\Omega_i} \frac{1}{p} |\nabla u^*|^p (x \cdot v) d\sigma$$

Whence, by (1.4), (1.5) and (1.6)

$$(1.7) \quad \int_{\partial\Omega_i} |\nabla u^*|^{p-2} (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma - \frac{1}{p} \int_{\partial\Omega_i} |\nabla u^*|^p (x \cdot v) d\sigma \\ = - \int_{\Omega_i} \sum_{i=1}^n x_i \frac{\partial u^*}{\partial x_i} \frac{\partial H}{\partial u} (x; u^*, v^*) dx + \frac{p-n}{p} \int_{\Omega_i} u^* \frac{\partial H}{\partial u} (x; u^*, v^*) dx$$

In the same way, an analogous relation is also obtained relatively to v^* . Summing up these relations, we have

$$\int_{\partial\Omega_i} |\nabla u^*|^{p-2} (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma + \int_{\partial\Omega_i} |\nabla v^*|^{q-2} (x \cdot \nabla v^*) (v \cdot \nabla v^*) d\sigma \\ - \frac{1}{p} \int_{\partial\Omega_i} |\nabla u^*|^p (x \cdot v) d\sigma - \frac{1}{q} \int_{\partial\Omega_i} |\nabla v^*|^q (x \cdot v) d\sigma$$

(1.8)

$$\begin{aligned}
&= \frac{p-n}{p} \int_{\Omega_i} u^* \frac{\partial H}{\partial u}(x; u^*, v^*) dx + \frac{q-n}{q} \int_{\Omega_i} v^* \frac{\partial H}{\partial v}(x; u^*, v^*) dx \\
&- \int_{\Omega_i} \sum_{l=1}^n x_l \left\{ \frac{\partial u^*}{\partial x_l} \frac{\partial H}{\partial u}(x; u^*, v^*) \right\} dx - \int_{\Omega_i} \sum_{l=1}^n x_l \left\{ \frac{\partial v^*}{\partial x_l} \frac{\partial H}{\partial v}(x; u^*, v^*) \right\} dx.
\end{aligned}$$

Passing to the limit on k in this equality, as u^* and $v^* \equiv 0$ on $\partial\Omega$ and using the results of Egnell (2.1 [10, p. 64]).

$$\begin{aligned}
&\frac{p-1}{p} \int_{\partial\Omega} |\nabla u^*|^p(x \cdot \nu) d\sigma + \frac{q-1}{q} \int_{\partial\Omega} |\nabla v^*|^q(x \cdot \nu) d\sigma \\
&= -\frac{n-p}{p} \int_{\Omega} u^* \frac{\partial H}{\partial u}(x; u^*, v^*) dx - \frac{n-q}{q} \int_{\Omega} v^* \frac{\partial H}{\partial v}(x; u^*, v^*) dx
\end{aligned}$$

(1.9)

$$- \int_{\Omega} \sum_{l=1}^n x_l \left\{ \frac{\partial u^*}{\partial x_l} \frac{\partial H}{\partial u}(x; u^*, v^*) + \frac{\partial v^*}{\partial x_l} \frac{\partial H}{\partial v}(x; u^*, v^*) \right\} dx.$$

We have the following relation

$$\sum_{l=1}^n \frac{\partial}{\partial x_l} \{x_l H(x; s, t)\} = nH(x; s, t) + x \cdot \nabla_x H(x; s, t)$$

(1.10)

$$+ \sum_{i=1}^n x_i \left\{ \frac{\partial s}{\partial x_i} \frac{\partial H}{\partial s}(x; s, t) + \frac{\partial t}{\partial x_i} \frac{\partial H}{\partial t}(x; s, t) \right\}.$$

Moreover, since the application $x \rightarrow H(x; u^*(x), v^*(x))$ is of class $C^1(\bar{\Omega})$, using again the Gauss's formula then we have from hypothesis *i*) $\int_{\partial\Omega} H(x; u^*(x), v^*(x)) (x \cdot \nu) d\sigma = 0$. Hence, we obtain

$$\begin{aligned} & - \left(\frac{p-1}{p} \int_{\partial\Omega} |\nabla u^*|^p (x \cdot \nu) d\sigma + \frac{q-1}{q} \int_{\partial\Omega} |\nabla v^*|^q (x \cdot \nu) d\sigma \right) \\ & = \int_{\Omega} \left[-x \cdot \nabla_x H(x; u^*, v^*) - nH(x; u^*, v^*) + \frac{n-p}{p} \left\{ u^* \frac{\partial H}{\partial u}(x; u^*, v^*) \right\} \right. \\ & \quad \left. + \frac{n-q}{q} \left\{ v^* \frac{\partial H}{\partial v}(x; u^*, v^*) \right\} \right] dx \end{aligned}$$

According to the hypothesis *iii*) the integral on Ω is nonnegative, whence a contradiction.

2. EXISTENCE RESULTS VIA COMPARISON ARGUMENTS

Ω denotes a bounded regular open set in \mathbb{R}^n and $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Throughout this second section, we shall prove some existence results for the following problem.

$$(P) \quad \begin{cases} \text{Find } (u, v) \in X \text{ such that} \\ -\Delta_p u = f(x; u, v) & \text{on } \Omega \\ -\Delta_q v = g(x; u, v) & \text{on } \Omega. \end{cases}$$

We make the following assumptions

(H1) f and g belong to $C(\Omega \times \mathbb{R} \times \mathbb{R})$
 moreover, for any $s \geq 0$, $t \geq 0$; $f(x;s,t) \geq 0$ and $g(x;s,t) \geq 0$

(H2) There are nonnegative constants:

$\alpha > 0$, $\beta > 0$, p_i, q_i ($i = 1, 2$)
 a_j, b_j ($j = 1, \dots, 6$) where $a_1 > 0$, $a_3 > 0$, $b_1 > 0$, $b_3 > 0$ satisfying
 (H2)_a and (H2)_b:

$$(H2)_a: \begin{cases} \frac{\alpha+1}{p} + \frac{\beta+1}{p} < 1 \\ 1 < p_1 < p; \quad 0 < q_1 - 1 < \frac{q}{p^*} \\ 0 < p_2 - 1 < \frac{p}{q^*}; \quad 1 < q_2 < q \end{cases}$$

$$(H2)_b: \begin{cases} a_1 s |s|^{\alpha-1} |t|^{\beta+1} - a_2 |s|^{p-1} \leq f(x;s,t) \leq a_3 s |s|^{\alpha-1} |t|^{\beta+1} + a_4 |s|^{p-1} \\ \quad + a_5 |t|^{q-1} + a_6 \\ b_1 |s|^{\alpha+1} |t|^{\beta-1} - b_2 |t|^{q-1} \leq g(x;s,t) \leq b_3 s^{\alpha+1} |t|^{\beta-1} + b_4 |s|^{p-1} \\ \quad + b_5 |t|^{q-1} + b_6 \end{cases}$$

We have the following existence theorem:

Theorem 2.1. Under hypotheses (H1) and (H2), (P) has a nontrivial solution (u^*, v^*) in $X \cap [L^\infty(\Omega)]^2$.

Example: existence result for $f(x;s,t) = a(x)s |s|^{\alpha-1} |t|^{\beta+1}$ and $g(x;s,t) = b(x) |s|^{\alpha+1} |t|^{\beta-1}$.

Corollary 2.1. Let f and g be as above where a and b are

nonnegative continuous functions and assume that $\alpha > 0$ and $\beta > 0$ are such that

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} < 1; \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1.$$

Then, the corresponding problem (P) has a nontrivial solution in $X \cap [L^\infty(\Omega)]^2$.

The proof of Theorem 2.1 is in three steps.

1st step: Construction of sub-supersolutions of (P).

Definition 2.1. A pair $[(u_0, v_0), (u^0, v^0)]$ is said a weak sub-super solution for the Dirichlet problem (P) if the following conditions are satisfied:

$$(1): \begin{cases} (u_0, v_0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2 \\ (u^0, v^0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2 \end{cases}$$

$$(2.1) \quad \begin{cases} -\Delta_p u_0 - f(x; u_0, v_0) \leq 0 \leq -\Delta_p u^0 - f(x; u^0, v) & \text{in } \Omega \quad \forall v \in [v_0, v^0] \\ -\Delta_q v_0 - g(x; u, v_0) \leq 0 \leq -\Delta_q v^0 - g(x; u, v^0) & \text{in } \Omega \quad \forall u \in [u_0, u^0] \\ u_0 \leq u^0 & \text{in } \Omega \\ v_0 \leq v^0 & \text{in } \Omega \\ u_0 \leq 0 \leq u^0 & \text{on } \partial\Omega \\ v_0 \leq 0 \leq v^0 & \text{on } \partial\Omega \end{cases}$$

Similar definitions can be found in Díaz-Hernández [8], Díaz-Herrero [9], Hernández [16].

Proposition 2.1. Assume (H2) and

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1;$$

then, for any $M > 0$, the problem (P) admits a pair $[(u_0, v_0), (u^0, v^0)]$ of sub-super solution satisfying $u_0(x) \leq M \leq u^0(x)$, $v_0(x) \leq M \leq v^0(x)$ in Ω .

Proof. a) Construction of (u^0, v^0)

Consider $R > 0$ such that $\Omega \subset B(0; R)$. We seek for u^0, v^0 in the following forms:

$$(2.2) \quad \begin{aligned} u^0(x) &= \varphi^0(r) = ar^{p^*} + b \\ v^0(x) &= \psi^0(r) = cr^{q^*} + d \end{aligned}$$

$$\begin{aligned} &a < 0 \text{ and } c < 0 \\ \text{with: } &b > 0 \text{ and } d > 0 \\ &\|x\| = r. \end{aligned}$$

We fix a real $M > 0$ and choose

$$(2.3) \quad a = -\frac{b-M}{R^{p^*}} \text{ and } c = -\frac{d-M}{R^{q^*}},$$

we have, for b and d greater than M

$$(2.4) \quad M \leq u^0(x); M \leq v^0(x) \quad \forall x \in \Omega.$$

and for each point x in Ω , we have:

$$(2.5) \quad \Delta_p u^0(x) = (p-1)|\varphi'(r)|^{p-2} \varphi''(r) + \frac{n-1}{r} |\varphi'(r)|^{p-2} \varphi'(r) = -np^* |a|^{p-1} = np^* \left(\frac{b-M}{R^{p^*}} \right)^{p-1}$$

For $u \leq u^0$, $v \leq v^0$ and $a < 0$; $c < 0$ we have

$$(2.6) \quad \begin{cases} \Delta_p u^0 + f(x; u^0, v) \leq -np * \left(\frac{b-M}{R^{p^*}} \right)^{(p-1)} + a_3 b^\alpha d^{\beta+1} \\ \qquad \qquad \qquad + a_4 b^{p_1-1} + a_5 d^{q_1-1} + a_6, \quad \forall v_0 \leq v \leq v^0 \\ \Delta_q v^0 + g(x; u, v^0) \leq -nq * \left(\frac{d-M}{R^{q^*}} \right)^{(q-1)} + b_3 b^{\alpha+1} d^\beta \\ \qquad \qquad \qquad + b_4 b^{p_2-1} + b_5 d^{q_2-1} + b_6, \quad \forall u_0 \leq u \leq u^0. \end{cases}$$

Let $k > 0$, $b = k^{1/p}$ and $d = k^{1/q}$. Comparing, the growth of the different terms in (2.6) for large k , we obtain

$$(2.7) \quad \begin{cases} \Delta_p u^0 + f(x; u^0, v) \leq 0 \quad \forall v^0 \leq v \leq v^0 \\ \Delta_q v^0 + g(x; u, v^0) \leq 0 \quad \forall u_0 \leq u \leq u^0. \end{cases}$$

b) Construction of (u_0, v_0) . Consider $x_0 \in \Omega$, and $R > 0$ such that $B(x_0; R) \subset \Omega$; we can assume $0 \in \Omega$.

As in [11], [26], we seek (u_0, v_0) in the following form

$$(2.8) \quad u_0(x) = \varphi_0(r) = \begin{cases} Ar^{p^*} + B & \text{for } 0 \leq r \leq \frac{nR}{n+1}, \\ C(R-r)^{p^*} & \text{for } \frac{nR}{n+1} \leq r \leq R, \\ 0 & \text{for } R < r, \end{cases}$$

$$(2.9) \quad v_0(x) = \psi_0(r) = \begin{cases} \bar{A}r^{q^*} + \bar{B} & \text{for } 0 \leq r \leq \frac{nR}{n+1}, \\ \tilde{C}(R-r)^{q^*} & \text{for } \frac{nR}{n+1} \leq r \leq R, \\ 0 & \text{for } R < r \end{cases}$$

Take

$$A = -B \left(\frac{n+1}{n} \right)^{p^*-1} \frac{1}{R^{p^*}}, \quad \tilde{A} = -\tilde{B} \left(\frac{n+1}{n} \right)^{q^*-1} \frac{1}{R^{q^*}}$$

$$(2.10) \quad C = -An^{p^*-1}, \quad \tilde{C} = -\tilde{A}n^{q^*-1}$$

$$B > 0, \quad \tilde{B} > 0.$$

By (2.10) u_0 and v_0 are in $C^1(\bar{\Omega})$ and moreover they vanish on $\partial\Omega$.

First consider x such that

$$\frac{nR}{n+1} \leq r = \|x\| \leq R;$$

we have

$$(2.11) \quad \begin{cases} 0 \leq u_0(x) \leq C \left(R - \frac{nR}{n+1} \right)^{p^*} \\ 0 \leq v_0(x) \leq \tilde{C} \left(R - \frac{nR}{n+1} \right)^{q^*} \end{cases}$$

Consequently

$$(2.12) \quad \begin{aligned} \Delta_p u_0(x) &= p^{*p-1} C^{p-1} \left\{ 1 - (n-1) \frac{R-r}{r} \right\} \\ &\geq \frac{p^{*p-1} C^{p-1}}{n} \end{aligned}$$

Whence for any $(u, v) \in [u_0, u^0] \times [v_0, v^0]$ and for sufficiently small R :

$$(2.13) \quad \begin{cases} \Delta_p u_0 + f(x; u_0, v_0) \geq C^{p-1} \left\{ \frac{p^{*p-1}}{n} - a_2 \left(\frac{R}{n+1} \right)^p \right\} \geq 0 \\ \Delta_q v_0 + g(x; u_0, v_0) \geq \tilde{C}^{q-1} \left\{ \frac{q^{*q-1}}{n} - b_2 \left(\frac{R}{n+1} \right)^q \right\} \geq 0 \end{cases}$$

Now consider $x \in \Omega$ such that:

$$0 \leq \|x\| \leq \frac{nR}{n+1}$$

We have in this case

$$(2.14) \quad 0 \leq u_0(x) \leq B \text{ and } 0 \leq v_0(x) \leq \tilde{B}.$$

Moreover

$$(2.15) \quad \Delta_p u_0(x) = -B^{(p-1)} \frac{n+1}{R^p} p^{*(p-1)}$$

Using the hypothesis (H2), for any $(u, v) \in [u_0, v_0] \times [v_0, v^0]$, we obtain

$$(2.16) \quad \begin{cases} -B^{p-1} \frac{n+1}{R^{p^*}} (p^*)^{p-1} + a_1 B^{\alpha} \tilde{B}^{\beta+1} \frac{1}{(n+1)^{\alpha+\beta+1}} - a_2 B^{p-1} \leq \Delta_p u_0 + f(x; u_0, v_0) \\ -\tilde{B}^{q-1} \frac{n+1}{R^{q^*}} (q^*)^{q-1} + b_1 B^{\alpha+1} \tilde{B}^{\beta} \frac{1}{(n+1)^{\alpha\beta+1}} - b_2 \tilde{B}^{q-1} \leq \Delta_q v_0 + g(x; u_0, v_0) \end{cases}$$

Hence the conclusion follows for $B = D^{1/p}$, $\tilde{B} = D^{1/q}$, $D > 0$ sufficiently small.

2nd Step: The troncated problem (\tilde{P}) associated to (P).

Following [7], we define a troncated problem (\tilde{P}), associated to (P).

$$(\tilde{P}) \quad \begin{cases} \text{Find } (u,v) \in X \text{ such that} \\ (\tilde{1}) \quad -\Delta_p u = \tilde{f}(x;u,v) - \gamma_1(x,u) & \text{in } \Omega \\ (\tilde{2}) \quad -\Delta_q v = \tilde{g}(x;u,v) - \gamma_2(x,v) & \text{in } \Omega \end{cases}$$

Where

$$(2.17) \quad \begin{aligned} \gamma_1(x,u(x)) &= -(u_0(x) - u(x))_+^{p-1} + (u(x) - u^0(x))_+^{p-1} \\ \gamma_2(x,v(x)) &= -(v_0(x) - v(x))_+^{q-1} + (v(x) - v^0(x))_+^{q-1} \\ \tilde{f}(x;u(x),v(x)) &= f(x;U(x),V(x)) \\ \tilde{g}(x;u(x),v(x)) &= g(x;U(x),V(x)) \end{aligned}$$

With

$$(2.18) \quad \begin{aligned} U(x) &= u(x) + (u_0(x) - u(x))_+ - (u(x) - u^0(x))_+ \\ V(x) &= v(x) + (v_0(x) - v(x))_+ - (v(x) - v^0(x))_+ \end{aligned}$$

For any $(u,v) \in X, (\hat{u},\hat{v}) \in X$, we define:

$$(2.19) \quad \begin{aligned} A(u,v) &= - \begin{pmatrix} \Delta_p & 0 \\ 0 & \Delta_q \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1(x;u) - \tilde{f}(x;u,v) \\ \gamma_2(x;v) - \tilde{g}(x;u,v) \end{pmatrix} \\ &= - \begin{pmatrix} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) \\ \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|\nabla v|^{q-2} \frac{\partial v}{\partial x_i} \right) \end{pmatrix} + \begin{pmatrix} \gamma_1(\cdot;u) - \tilde{f}(x;u,v) \\ \gamma_2(\cdot;v) - \tilde{g}(x;u,v) \end{pmatrix} \end{aligned}$$

$$a[(u,v);(\hat{u},\hat{v})] = \int_{\Omega} A(u,v) \cdot W dx$$

$$\text{with } W = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}$$

We have

$$(2.20) \quad \begin{aligned} a[(u,v);(\hat{u},\hat{v})] &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \hat{u} dx + \int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \hat{v} dx \\ &\quad - \int_{\Omega} \tilde{f}(x;u,v) \hat{u} dx - \int_{\Omega} \tilde{g}(x;u,v) \hat{v} dx \\ &\quad + \int_{\Omega} \gamma_1(x,u) \hat{u} dx + \int_{\Omega} \gamma_2(x,v) \hat{v} dx. \end{aligned}$$

Lemma 2.1. *A is a bounded operator from X to X^* .*

Proof [31].

Definition 2.2 (C.f [17]). *An operator $A : X \rightarrow X^*$ is called a calculus of variations operator, if it is bounded and if it can be represented in the form*

$$(1) \quad A(u,v) = \mathcal{A}[(u,v);(u,v)]$$

where $((u,v),(\hat{u},\hat{v})) \rightarrow \mathcal{A}[(u,v);(\hat{u},\hat{v})]$ is an operator $X \times X \rightarrow X^*$ which satisfies

$$\left\{ \begin{array}{l} \forall (u,v) \in X; (\hat{u}, \hat{v}) \rightarrow \mathcal{A}[(u,v); (\hat{u}, \hat{v})] \text{ is a hemicontinuous bounded} \\ \text{operator } X \rightarrow X^* \text{ and} \\ \langle \mathcal{A}[(u,v); (u,v)] - \mathcal{A}[(u,v); (\hat{u}, \hat{v})], (u,v) - (\hat{u}, \hat{v}) \rangle \geq 0; \forall (u,v), (\hat{u}, \hat{v}) \in X \end{array} \right. \quad (2)$$

$$\begin{array}{l} \text{For any } (\hat{u}, \hat{v}) \in X, (u,v) \rightarrow \mathcal{A}[(u,v); (\hat{u}, \hat{v})] \\ \text{is a bounded hemicontinuous operator } X \rightarrow X^*. \end{array} \quad (3)$$

If $(u_\mu, v_\mu) \rightarrow (u,v)$ weakly in X
and

$$\begin{array}{l} \text{if } \langle \mathcal{A}[(u_\mu, v_\mu), (u_\mu, v_\mu)] - \mathcal{A}[(u_\mu, v_\mu), (u,v)], (u_\mu - u, v_\mu - v) \rangle \rightarrow 0 \\ \text{then, for any } (\hat{u}, \hat{v}) \text{ in } X \\ \text{the sequence } \mathcal{A}[(u_\mu, v_\mu), (\hat{u}, \hat{v})] \text{ converges weakly to } \mathcal{A}[(u,v), (\hat{u}, \hat{v})] \\ \text{in } X^*. \end{array} \quad (4)$$

$$\begin{array}{l} \text{If } (u_\mu, v_\mu) \rightarrow (u,v) \text{ in } X \\ \text{and if } \mathcal{A}[(u_\mu, v_\mu), (\hat{u}, \hat{v})] \rightarrow (\phi, \psi) \text{ weakly in } X^* \end{array} \quad (5)$$

then

$$\langle \mathcal{A}[(u_\mu, v_\mu), (\hat{u}, \hat{v})]; (u_\mu, v_\mu) \rangle_{X^*, X} \rightarrow \langle (\phi, \psi), (u,v) \rangle_{X^*, X}.$$

In our problem, we define \mathcal{A} by the following relation; for any $(u_1, v_1), (u_2, v_2), (\hat{u}, \hat{v})$:

$$\begin{aligned} \langle \mathcal{A} [(u_1, v_1), (u_2, v_2)]; (\hat{u}, \hat{v}) \rangle &= \int_{\Omega} |\nabla u_2|^{p-2} \nabla u_2 \nabla \hat{u} dx + \int_{\Omega} |\nabla v_2|^{q-2} \nabla v_2 \nabla \hat{v} dx \\ (2.21) \quad &- \int_{\Omega} \tilde{f}(x; u_1, v_1) \hat{u} dx - \int_{\Omega} \tilde{g}(x; u_1, v_1) \hat{v} dx \\ &+ \int_{\Omega} \gamma_1(x, u_1) \hat{u} dx + \int_{\Omega} \gamma_2(x, v_1) \hat{v} dx \end{aligned}$$

Lemma 2.2. \mathcal{A} is a calculus of variations operator.

Proof. (c.f [31])

Lemma 2.3. Let V be a Banach space and let A be a coercive calculus of variations operator.

Then, for any f in V^* , the equation $A(u) = f$ has a solution u in V .

Proof (c.f [17], proposition 2.6, theorem 2.7, p. 180-181).

Lemma 2.4. *If the application \tilde{f} , \tilde{g} , γ_1 and γ_2 are defined as above, then the problem (\tilde{P}) has a solution (\bar{u}, \bar{v}) in X .*

3st Step: Existence of a non-trivial solution for (P) .

Now, we prove that $u_0 \leq \bar{u} \leq u^0$ $v_0 \leq \bar{v} \leq v^0$, in Ω .

We show for example $\bar{u} \leq u^0$.

Consider $\hat{u} = (\bar{u} - u^0)_+$ and $\hat{v} = (\bar{v} - v^0)_+$.

Multiplying $(\tilde{1})$ by \hat{u} and $(\tilde{2})$ by \hat{v} , we have

$$(2.22) \quad \int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \hat{u} dx - \int_{\Omega} \tilde{f}(x; \bar{u}, \bar{v}) \hat{u} dx + \|(\bar{u} - u^0)_+\|_{L^p(\Omega)}^p = 0$$

but, according to the definition of u^0 , $\forall v \in [v_0, v^0]$, we have

$$(2.23) \quad \int_{\Omega} |\nabla u^0|^{p-2} \nabla u^0 \nabla \hat{u} dx - \int_{\Omega} f(x; u^0, v) \hat{u} dx \geq 0$$

Thus, combining (2.22) and (2.23), we obtain

$$(2.24) \quad 0 \geq \int_{\Omega} (|\nabla \bar{u}|^{p-2} \nabla \bar{u} - |\nabla u^0|^{p-2} \nabla u^0) \nabla (\bar{u} - u^0)_+ dx + \int_{\Omega} (f(x; u^0, v) - \tilde{f}(x; \bar{u}, \bar{v})) (\bar{u} - u^0)_+ dx + \|(\bar{u} - u^0)_+\|_{L^p(\Omega)}^p$$

Take $v = \bar{V}$ where \bar{V} is associated to \bar{v} as in (2.18). On the set $\{x \in \Omega; \bar{u}(x) - u^0(x) > 0\}$, we have $\bar{U}(x) = u^0(x)$,

(2.27)

$$\int_{\Omega} (f(x; u^0, \bar{V}) - f(x; \bar{u}, \bar{v})) (\bar{u} - u^0)_+(x) dx = \int_{\Omega} (f(x; u^0, \bar{V}) - f(x; \bar{U}, \bar{V})) (\bar{u} - u^0)_+(x) dx = 0$$

By monotonicity of $-\Delta_p$, we get that $0 \geq \|(\bar{u} - u^0)_+\|_{L^p(\Omega)}^p \geq 0$.

Thus $\bar{u} \leq u^0$ on Ω and similarly $\bar{v} \leq v^0$ on Ω .

3. EXISTENCE RESULTS VIA VARIATIONAL METHODS

3.0. Introduction. We present in this final section an existence result for the following problem (P)

$$\begin{cases} \text{Find } (u, v) \in X \text{ such that} \\ (1^*) \quad -\Delta_p u = \frac{\partial H}{\partial u}(x; u, v) & \text{in } \Omega \\ (2^*) \quad -\Delta_q v = \frac{\partial H}{\partial v}(x; u, v) & \text{in } \Omega \end{cases}$$

This result extends to a potential system those obtained by L. Nirenberg [18] and F. de Thélin [26], in the scalar case. Our existence result follows from an appropriate adaptation of the variational method given by Ambrosetti-Rabinowitz [2].

Recal that $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

In the next section, we shall prove that in fact $(u, v) \in X \cap [L^\infty(\Omega)]^2$.

We make the following assumptions

(H1) $H \in C^1(\Omega \times \mathbb{R} \times \mathbb{R})$

(H2) There exist two positive real numbers δ, A , with $\delta < A$ such that, for a partition of \mathbb{R}^2 in D_1, D_2, D_3 respectively defined by

$$D_1 = \{(s,t) \in \mathbb{R}^2; |s| \geq A \text{ or } |t| \geq A\}$$

$$D_2 = \{(s,t) \in \mathbb{R}^2 \setminus D_1; |s| > \delta \text{ and } |t| > \delta\}$$

$$D_3 = \mathbb{R}^2 \setminus (D_1 \cup D_2)$$

We have:

(H2)_a there exists a nonnegative constant C and

$$p' \in \left] p, \frac{np}{n-p} \right[, q' \in \left] q, \frac{nq}{n-q} \right[,$$

such that $0 \leq H(x,s,t) \leq C(|s|^{p'} + |t|^{q'})$, for any $x \in \Omega$ and for any pair $(s,t) \in D_3$.

(H2)_b There exists a positive function $a \in L^\infty(\Omega)$ such that $H(x,s,t) = a(x)|s|^{\alpha+1}|t|^{\beta+1}$ for any $x \in \Omega$ and $(s,t) \in D_1$.

Remark. We are interested by the nonnegative solutions for the problem (P), so we can add the following hypothesis

(H3) For any $x \in \Omega$, $s \leq 0$ or $t \leq 0$;

$$\frac{\partial H}{\partial s}(x,s,t)=0 \text{ and } \frac{\partial H}{\partial t}(x,s,t)=0.$$

For any (u,v) in X , we define:

$$(3.0) \quad J(u,v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q dx - \int_{\Omega} H(x,u,v) dx$$

We shall use the Mountain-Pass Lemma to obtain an existence theorem for (P). The nontrivial solution is obtained as a critical point of J .

Theorem 3.1. We suppose that the hypotheses (H1) and (H2) are satisfied and that the real numbers α and β in $(H2)_b$ are such that

$$\begin{cases} 1) & (\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} < 1 \\ 2) & \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1, \end{cases}$$

then, the problem (P) possesses a nontrivial solution (u^*, v^*) in $X \cap [L^\infty(\Omega)]^2$.

Corollary 3.1. All the hypotheses of Theorem 3.1. are satisfied for $H(x; s, t) = a(x) |s|^{\alpha+1} |t|^{\beta+1}$.

If

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} < 1, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$$

then, the corresponding problem possesses a nontrivial solution (u^*, v^*) in $X \cap [L^\infty(\Omega)]^2$.

Proof of Corollary 3.1. Consider a truncature \tilde{H} of the application H

$$\tilde{H}(x; s, t) = \begin{cases} 0 & \text{if } s \leq 0 \text{ or } t \leq 0 \\ H(x; s, t) & \text{otherwise} \end{cases}$$

\tilde{H} satisfies the hypotheses (H1), (H2). For proving $(H2)_a$, we write for any real s and t

$$(*) \quad |s|^{\alpha+1} |t|^{\beta+1} \leq C(|s|^{\lambda p} + |t|^{\mu q})$$

Where λ and μ are such that

$$\frac{\alpha+1}{\lambda p} + \frac{\beta+1}{\mu q} = 1, \quad 1 < \lambda < \frac{n}{n-p} \quad \text{and} \quad 1 < \mu < \frac{n}{n-q}.$$

3.1. Existence of a solution in X .

Lemma 3.1.1. *If*

$$(\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} < 1,$$

there exist γ_1 and γ_2 such that

$$\begin{cases} \frac{\alpha+1}{\gamma_1} + \frac{\beta+1}{\gamma_2} = 1 \\ \gamma_1 \in \left[1, \frac{np}{n-p}\right], \gamma_2 \in \left[1, \frac{nq}{n-q}\right] \end{cases}$$

Moreover, if (u_k, v_k) is bounded in X , the applications

$$x \rightarrow u_k(x) |u_k(x)|^{\alpha-1} |v_k(x)|^{\beta+1} \quad \text{and} \quad x \rightarrow v_k(x) |v_k(x)|^{\beta-1} |u_k(x)|^{\alpha+1}$$

are bounded in $L^{\gamma_1}(\Omega)$ and L^{γ_2} respectively.

Lemma 3.1.2. *If*

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1,$$

J satisfies the Palais-Smale (P.S) condition.

Proof. Let $\{(u_k, v_k); k \in \mathcal{N}\}$ be a sequence in X such that

$$\text{there exist } M > 0, \quad |J(u_k, v_k)| \leq M \quad (P.S)_1$$

$J'(u_k, v_k) \rightarrow 0$ strongly in X^* as k goes to $+\infty$ $(P.S)_2$.

We claim that this sequence is bounded in X .

By contradiction, suppose that we can extract from (u_k, v_k) a subsequence denoted again by (u_k, v_k) such that $\|(u_k, v_k)\|_X \rightarrow +\infty$.

Hereafter, we set

$$e_k = \frac{1}{p} \int_{\Omega} |\nabla u_k|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v_k|^q dx.$$

The $(P.S)_1$ condition implies that

$$(3.1.1) \quad -\frac{M}{e_k} \leq 1 - \frac{1}{e_k} \int_{\Omega} H(x; u_k, v_k) dx \leq \frac{M}{e_k}.$$

Let $\Omega_{i,k} = \{x \in \Omega: (u_k(x), v_k(x)) \in D_i\}$, for $i = 1, 2, 3$; we obtain

$$(3.1.2) \quad -\frac{M}{e_k} \leq 1 - \frac{1}{e_k} \left\{ \int_{\Omega_{1,k}} a(x) u_k^{\alpha+1} v_k^{(\beta+1)} dx + \int_{\Omega_{2,k}} H(x; u_k, v_k) dx \right\} \leq \frac{M}{e_k}.$$

On the other hand, by $(PS)_2$ we have:

$$-\varepsilon \|(u_k, v_k)\|_X \leq J'(u_k, v_k) \left(\frac{u_k}{p}, \frac{v_k}{q} \right) \leq \varepsilon \|(u_k, v_k)\|_X.$$

That means

$$-\varepsilon \|(u_k, v_k)\|_X \leq e_k - \frac{1}{p} \int_{\Omega_{1,k}} u_k \frac{\partial H}{\partial u}(x; u_k, v_k) dx - \frac{1}{q} \int_{\Omega_{2,k}} v_k \frac{\partial H}{\partial v}(x; u_k, v_k) dx$$

$$(3.1.3) \quad -\frac{1}{p} \int_{\Omega} u_k \frac{\partial H}{\partial u}(x; u_k, v_k) dx - \frac{1}{q} \int_{\Omega} v_k \frac{\partial H}{\partial v}(x; u_k, v_k) dx$$

$$\leq \varepsilon \|(u_k, v_k)\|_X$$

Then, taking the limit with respect to k in the inequalities (3.1.2) and (3.1.3), we obtain respectively

$$(3.1.4) \quad \lim_{k \rightarrow +\infty} \frac{1}{e_k} \int_{\Omega_{1,1}} a(x) u_k^{\alpha+1} v_k^{\beta+1} dx = 1$$

$$\lim_{k \rightarrow +\infty} \frac{1}{e_k} \int_{\Omega_{1,1}} a(x) u_k^{\alpha+1} v_k^{\beta+1} dx = \frac{1}{\frac{\alpha+1}{p} + \frac{\beta+1}{q}}$$

But, this contradicts the hypothesis

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1.$$

Thus, there exist positive constants C_1 et C_2 such that: $\|u_k\|_{1,p} \leq C_1$ and $\|v_k\|_{1,q} \leq C_2$.

Denoting again by $\{u_k; k \in \mathcal{N}\}$ and $\{v_k; k \in \mathcal{N}\}$ the extracted subsequences, they converge strongly in the spaces $L^p(\Omega)$ and $L^q(\Omega)$ respectively; we claim that the subsequence $\{(u_k, v_k); k \geq 0\}$ converges strongly in X .

In fact, for any integer pair (m, l)

$$(3.1.5) \quad \int_{\Omega} (F_p(\nabla u_m) - F_p(\nabla u_l)) \nabla(u_m - u_l) dx = A_{m,l}$$

where

$$A_{m,l} = \langle J'_{p,q}(u_m, v_m) - J'_{p,q}(u_l, v_l); (u_m - u_l, 0) \rangle_{X,X^*} + \int_{\Omega} \left\{ \frac{\partial H}{\partial u}(x; u_m, v_m) - \frac{\partial H}{\partial u}(x; u_l, v_l) \right\} (u_m - u_l) dx$$

and

$$(3.1.6) \quad \int_{\Omega} (F_q(\nabla v_m) - F_q(\nabla v_l)) \nabla(v_m - v_l) dx = B_{m,l}$$

where

$$B_{m,l} = \langle J'(u_m, v_m) - J'(u_l, v_l); (0, v_m - v_l) \rangle_{X,X^*} + \int_{\Omega} \left\{ \frac{\partial H}{\partial v}(x; u_m, v_m) - \frac{\partial H}{\partial v}(x; u_l, v_l) \right\} (v_m - v_l) dx$$

By $(P.S)_2$ it is easy to remark that $\langle J'_{p,q}(u_m, v_m) - J'_{p,q}(u_l, v_l); (u_m - u_l, 0) \rangle_{X,X^*}$ converges to 0 as m and l tend to $+\infty$.

From the hypotheses $(H1)$ and $(H2)$, there exist two constants A_1 and A_2 such that for any (s, t) in \mathbb{R}^2 and x in Ω

$$(3.1.7) \quad \left| \frac{\partial H}{\partial s}(x; s, t) \right| \leq A_1 + A_2 |s|^\alpha |t|^{\beta+1}.$$

By use of Lemma 3.1.,

$$\int_{\Omega} \left\{ \frac{\partial H}{\partial v}(x; u_m, v_m) - \frac{\partial H}{\partial v}(x; u_l, v_l) \right\} (v_m - v_l) dx$$

converges to 0 and therefore $A_{m,l}$ converges to 0.

We have the following algebraic relation [24]:

$$|\nabla u_m - \nabla u_l|^p \leq C([F_p(\nabla u_m) - F_p(\nabla u_l)](\nabla v_m - \nabla v_l))^{s/2} (|\nabla u_m|^p + |\nabla u_l|^p)^{(1-s/2)}$$

$$(3.1.8) \quad \text{with } s = \begin{cases} p & \text{for } 1 < p \leq 2 \\ 2 & \text{for } 2 < p \end{cases}$$

Integrating (3.1.8) on Ω and using Hölder's inequality in the right hand side, we obtain

$$(3.1.9) \quad \|u_m - u_l\|_{1,p}^p \leq C |A_{m,l}|^{s/2} (\|u_m\|_{1,p}^p + \|u_l\|_{1,p}^p)^{(1-s/2)}$$

and

$$(3.1.10) \quad \|v_m - v_l\|_{1,q}^q \leq C' |B_{m,l}|^{t/2} (\|v_m\|_{1,q}^q + \|v_l\|_{1,q}^q)^{(1-t/2)}$$

From the convergence results related above, these inequalities give strong convergence of $\{(u_k, v_k); k \in \mathcal{N}\}$.

Lemma 3.1.3. *Under the hypotheses of Theorem 3.1.*

1) *There exist two positive real numbers ρ, ν_1 and a neighborhood V_ρ of the origin of X such that for any element (u, v) on the boundary of V_ρ : $J(u, v) \geq \nu_1 > 0$.*

2) *There exist (ϕ, ψ) in X such that $J(\phi, \psi) < 0$.*

Proof. 1) By (H1) and (H2)

$$(3.1.11) \quad \int_{\Omega} H(x; u, v) dx \leq C \int_{\Omega_1} (|u|^{p'} + |v|^{q'}) dx + \int_{\Omega_2} B dx + \int_{\Omega_3} a(x) |u|^{\alpha+1} |v|^{\beta+1} dx$$

$$\leq C(\|u\|_{1,p}^{p'} + \|v\|_{1,q}^{q'}) + b_\delta \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} dx + \int_{\Omega} a(x) |u|^{\alpha+1} |v|^{\beta+1} dx$$

By lemma 3.1.1., we obtain

$$(3.1.12) \quad \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} dx \leq \|u\|_{L^{\alpha+1}(\Omega)}^{\alpha+1} \cdot \|v\|_{L^{\beta+1}(\Omega)}^{\beta+1} \leq M \|u\|_{1,p}^{\alpha+1} \cdot \|v\|_{1,q}^{\beta+1}$$

Therefore, we get

$$(3.1.13) \quad \int_{\Omega} H(x;u,v) dx \leq C(\|u\|_{1,p}^{p'} + \|v\|_{1,q}^{q'} + (b_{\delta} + \|a\|_{\infty}) \{ \|u\|_{1,p}^{r(\alpha+1)} + \|v\|_{1,q}^{r*(\beta+1)} \})$$

where b_{δ} is a positive constant $B = b_{\delta} \delta^{\alpha+\beta+2}$, δ fixed,

$$r = 1 + \frac{p}{q} \frac{\beta+1}{\alpha+1} \quad \text{and} \quad r* = 1 + \frac{q}{p} \frac{\alpha+1}{\beta+1}.$$

Denoting by θ and η respectively $\|u\|_{1,p}$ and $\|v\|_{1,q}$, we therefore obtain the following minoration of J for any $(u,v) \in X$,

(3.1.14)

$$J(u,v) \geq \theta^p \left[1 - C\theta^{p'-p} - (b_{\delta} + \|a\|_{\infty})\theta^{(r(\alpha+1)-p)} \right] + \eta^q \left[1 - C\eta^{q'-q} - (b_{\delta} + \|a\|_{\infty})\eta^{(r*(\beta+1)-q)} \right]$$

Whence,

$$(3.1.15) \quad J(u,v) \geq v_1 > 0$$

2) Let $\phi \in W_0^{1,p}(\Omega)$ and $\psi \in W_0^{1,q}(\Omega)$ be positive in Ω , for any $\sigma > 0$, we have

$$J(\sigma^{\frac{1}{p}}\phi; \sigma^{\frac{1}{q}}\psi) = \sigma \|\phi\|_{1,p}^p + \sigma \|\psi\|_{1,q}^q - \int_{\Omega} H(x; \sigma^{\frac{1}{p}}\phi, \sigma^{\frac{1}{q}}\psi) dx$$

(3.1.16)

$$= \sigma \|\phi\|_{1,p}^p + \sigma \|\psi\|_{1,q}^q - \int_{\Omega} H(x; \sigma^{\frac{1}{p}}\phi, \sigma^{\frac{1}{q}}\psi) dx - \sigma^{\frac{\alpha+1}{p} + \frac{\beta+1}{q}} \int_{\Omega} |\phi|^{\alpha+1} |\psi|^{\beta+1} dx$$

Taking σ sufficiently large to have $|\Omega_1| > 0$, we obtain

$$\lim_{\sigma \rightarrow +\infty} J(\sigma^{\frac{1}{p}}\tilde{\phi}; \sigma^{\frac{1}{q}}\tilde{\psi}) = -\infty, \text{ since } \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1.$$

By the continuity for $J(\cdot, \cdot)$ on X , we find a pair (ϕ, ψ) in $X \setminus B_\rho(0)$ such that $J(\phi, \psi) < 0$.

Proof of the theorem 3.1. (1st part). By Mountain-Pass Lemma [2], there exist a pair (u^*, v^*) in X which is a critical point of J . This means that for any $(w_1, w_2) \in X$, $J'(u^*, v^*) \cdot (w_1, w_2) = 0$, i.e

$$\begin{cases} -\Delta_p u^* = \frac{\partial H}{\partial u}(x; u^*, v^*) & \text{in } \Omega \\ -\Delta_q v^* = \frac{\partial H}{\partial v}(x; u^*, v^*) & \text{in } \Omega. \end{cases}$$

So, we have proved that (P) possesses a nontrivial solution in X . The second part is devoted to prove that the solutions are bounded in Ω .

Moreover, [26] (c.f the definition for H) ensure $u^* \geq 0$ and $v^* \geq 0$ in Ω .

3.2. L^∞ -Estimate of the solution

3.2.0. Introduction. In this part, we use an iterative method to estimate the solution (u^*, v^*) obtained in section 3.1. We prove here that in fact $(u^*, v^*) \in [L^\infty(\Omega)]^2$.

In this matter, the crucial point is the construction of two strictly increasing unbounded sequences $\{\lambda_k; k \geq 0\}$ and $\{\mu_k; k \geq 0\}$ such that u^* and v^* verify:

$$\text{If } \begin{cases} u^* \in L^{\lambda_k}(\Omega) \\ v^* \in L^{\mu_k}(\Omega) \end{cases} \quad \text{then } \begin{cases} u^* \in L^{\lambda_{k+1}}(\Omega) \\ v^* \in L^{\mu_{k+1}}(\Omega) \end{cases}$$

We shall present some properties deriving to the fact that u^* and v^* belong to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ respectively. In a second step, we shall proceed to the appropriate construction for these sequences.

It is very important to note that this iterative schema use some regularity properties of u^* and v^* , for example (u^*, v^*) belong to $[C^2(\Omega) \cap C^1(\bar{\Omega})]^2$. The study of regularized equations (cf. [20], [26]) allows us to suppose u^* and v^* smooth throughout all this part. Though we do not make extensive development about our iterative method, more detailed proofs are given in [31].

Proposition 3.2. *Suppose that all the hypotheses of Theorem 3.1. are satisfied. Then, there exist sequences $\{\lambda_k; k \geq 0\}$ and $\{\mu_k; k \geq 0\}$ such that*

- 1) For each k , u^* and v^* belong respectively to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$.
- 2) There exist two real constants A_p and A_q be such that

$$\|u^*\|_{\infty} \leq \overline{\lim}_{k \rightarrow +\infty} \|u^*\|_{L^{\lambda_k}(\Omega)} \leq A_p$$

$$\|v^*\|_{\infty} \leq \overline{\lim}_{k \rightarrow +\infty} \|v^*\|_{L^{\mu_k}(\Omega)} \leq A_q$$

Lemma 3.2.1. *Let π_p (resp. π_q) be such that*

$$1 < \pi_p < \frac{np}{n-p} \quad (\text{resp. } 1 < \pi_q < \frac{nq}{n-q}),$$

and for any $k \geq 0$

$$a_k = \lambda_k \left(1 - \frac{\alpha}{\lambda_k} - \frac{\beta+1}{\mu_k} \right) - 1 \quad (1)_k$$

$$b_k = \mu_k \left(1 - \frac{\alpha+1}{\lambda_k} - \frac{\beta}{\mu_k} \right) - 1 \quad (2)_k$$

Then there are some constants c and c' such that for any $u^* \in L^\lambda(\Omega)$ and $v^* \in L^\mu(\Omega)$ we have

$$\int_{\Omega} |u^*|^{(1+\frac{a_k}{p})\pi_k} dx \leq c \left(1 + \frac{a_k}{p} \right)^{\pi_k} \theta_k^{(\pi_k/p)}, \quad \int_{\Omega} |v^*|^{(1+\frac{b_k}{q})\pi_k} dx \leq c' \left(1 + \frac{b_k}{q} \right)^{\pi_k} \Phi_k^{(\pi_k/p)}$$

where θ_k and Φ_k are defined as

$$\theta_k = \int_{\Omega} \frac{\partial H}{\partial u}(x; u^*, v^*) u^* |u^*|^{a_k} dx, \quad \Phi_k = \int_{\Omega} \frac{\partial H}{\partial v}(x; u^*, v^*) v^* |v^*|^{b_k} dx.$$

Proof of the Lemma 3.2.1. Multiplying (1*) by $u^* |u^*|^{a_k}$ and integrating on Ω , we obtain

$$(3.2.1) \quad \int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \nabla [u^* |u^*|^{a_k}] = \int_{\Omega} \frac{\partial H}{\partial u}(x; u^*, v^*) u^* |u^*|^{a_k} dx$$

On the other hand, we have,

$$(3.2.2) \quad \int_{\Omega} \left| \nabla (u^*)^{1+\frac{a_k}{p}} \right|^p = \left(1 + \frac{a_k}{p} \right)^p \int_{\Omega} |u^*|^{a_k} |\nabla u^*|^p dx$$

Since, u^* is in $C^1(\bar{\Omega})$, so is $\{u^*\}^{1+a_k/p}$ and consequently $\{u^*\}^{1+a_k/p}$ belongs to $W_0^{1,p}(\Omega)$. The continuous imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^\lambda(\Omega)$ implies the existence of a constant $c > 0$ such that

$$(3.2.3) \quad \left(\int_{\Omega} |u^*|^{(1+\frac{a_k}{p})\pi_p} dx \right)^{1/\pi_p} \leq c \left(\int_{\Omega} |\nabla(u^*)^{1+\frac{a_k}{p}}|^p dx \right)^{1/p}$$

Since a_k is nonnegative, (3.2.1), (3.2.2), (3.2.3) give,

$$(3.2.4) \quad \int_{\Omega} |u^*|^{(1+\frac{a_k}{p})\pi_p} \leq c \left(1 + \frac{a_k}{p} \right)^{\pi_p} \left[\int_{\Omega} |\nabla u^*|^p |u^*|^{a_k} dx \right]^{\pi_p/p}$$

$$\leq c \left(1 + \frac{a_k}{p} \right)^{\pi_p} \theta_k^{\pi_p/p}$$

Lemma 3.2.2. Assume that

$$\lambda_{k+1} \leq \left(1 + \frac{a_k}{p} \right) \pi_p \quad (3)_k, \quad \mu_{k+1} \leq \left(1 + \frac{b_k}{q} \right) \pi_q \quad (4)_k.$$

Then, if $u^* \in L^{\lambda_k}(\Omega)$ and $v^* \in L^{\mu_k}(\Omega)$, we have

$$(3.2.5) \quad \|u^*\|_{L^{\lambda_{k+1}}(\Omega)}^{\lambda_{k+1}} \leq K_p^{\lambda_{k+1}} \left\{ c^{\frac{1}{\pi_p}} \left(1 + \frac{a_k}{p} \right) \left\{ A_1 \|u^*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k} + A_2 \left(\|u^*\|_{L^{\lambda_k}(\Omega)}^{\mu_k} \right)^{\frac{\alpha+a_k+1}{\mu_k}} \left(\|v^*\|_{L^{\mu_k}(\Omega)}^{\lambda_k} \right)^{\frac{\beta+1}{\lambda_k}} + A_3 \right\}^{\frac{1}{p}} \right\}^{\frac{\lambda_{k+1}}{1+\frac{\alpha_k}{p}}}$$

where $A_i (i=1;2;3)$ are positive constants.

Proof. We first call (c.f (3.1.7)) that the hypotheses on H imply the existence of positive constants A_i ($i=1;2$) such that for any real numbers s and t ,

$$\frac{\partial H}{\partial s}(x;s,t) \leq A_1 + A_2 |s|^\alpha |t|^{\beta+1}$$

Thus, by Hölder's inequality we obtain

$$\begin{aligned} \int_{\Omega} \frac{\partial H}{\partial u}(x;u^*,v^*) u^* |u^*|^{a_i} dx &\leq A_1 \int_{\Omega} |u^*|^{a_i+1} dx + A_2 \int_{\Omega} |u^*|^{\alpha+a_i+1} |v^*|^{\beta+1} dx \\ (3.2.6) \quad &\leq A_1 \int_{\Omega} |u^*|^{\lambda_i} dx + A_2 \int_{\Omega} |u^*|^{\alpha+a_i+1} |v^*|^{(\beta+1)} dx + A_3 \\ &\leq A_1 \int_{\Omega} |u^*|^{\lambda_i} dx + A_2 \left(\int_{\Omega} |u^*|^{\lambda_i} dx \right)^{\frac{\alpha+a_i+1}{\lambda_i}} \left(\int_{\Omega} |v^*|^{\mu_i} dx \right)^{\frac{(\beta+1)}{\mu_i}} + A_3 \end{aligned}$$

That implies with (3.2.4),

(3.2.7)

$$\begin{aligned} \int_{\Omega} |u^*|^{\left(1+\frac{a_i}{p}\right)^{\pi_p}} &\leq c \left(1+\frac{a_i}{p}\right)^{\pi_p} \left[\int_{\Omega} |\nabla u^*|^p |u^*|^{a_i} dx \right]^{\pi_p/p} \\ &\leq c \left(1+\frac{a_i}{p}\right)^{\pi_p} \left[A_1 \int_{\Omega} |u^*|^{\lambda_i} dx + A_2 \left(\int_{\Omega} |u^*|^{\lambda_i} dx \right)^{\frac{\alpha+a_i+1}{\lambda_i}} \left(\int_{\Omega} |v^*|^{\mu_i} dx \right)^{\frac{(\beta+1)}{\mu_i}} + A_3 \right]^{\pi_p/p} \end{aligned}$$

Now, by (3), $L^{(1+a_i/p)\pi_p}(\Omega)$ is continuously imbedded into $L^{\lambda_i}(\Omega)$, so there exists a constant K_p such that

$$\left(\int_{\Omega} |u^*|^{\lambda_{i+1}} dx\right)^{1/\lambda_{i+1}} \leq K_p \left(\int_{\Omega} |u^*|^{\left(1+\frac{a_i}{p}\right)\pi_p} dx\right)^{1/\left(1+\frac{a_i}{p}\right)\pi_p}.$$

Combined with (3.2.7), we have

(3.2.8)

$$\int_{\Omega} |u^*|^{\lambda_{i+1}} dx \leq K_p^{\lambda_{i+1}} \left\{ c^{\frac{1}{\pi_p}} \left(1 + \frac{a_k}{p}\right) \left[A_1 \int_{\Omega} |u^*|^{\lambda_i} dx + A_2 \left(\int_{\Omega} |u^*|^{\lambda_i} dx\right)^{\frac{\alpha+a_i+1}{\lambda_i}} \left(\int_{\Omega} |v^*|^{\mu_i} dx\right)^{\frac{(\beta+1)}{\mu_i}} + A_3 \right]^{\frac{1}{p}} \right\}^{\frac{\lambda_{i+1}}{1+\frac{a_i}{p}}}.$$

An analogous result is obtained for v^* .

3.2.1. Definition and construction of sequences $\{\lambda_k; k \in \mathcal{N}\}$ and $\{\mu_k; k \in \mathcal{N}\}$. Here, we construct the sequences $\{\lambda_k; k \in \mathcal{N}\}$ and $\{\mu_k; k \in \mathcal{N}\}$. This construction requires similar tools as in [20], [26] or [27] use for the study of first eigenvalue, but here the problem is different from [27], because α and β do not verify

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Here, the first terms of each sequence cannot be determined directly by using the Rellich-Kondrachov's continuous imbedding result. So, we first construct Lebesgue spaces of exponents $\hat{\lambda}_k$ and $\hat{\mu}_k$ containing respectively u^* and v^* . By an appropriate choice for $k_0 \in \mathcal{N}$, $\hat{\lambda}_{k_0}$ and $\hat{\mu}_{k_0}$ give the respective first terms of $\{\lambda_k; k \geq 0\}$ and $\{\mu_k; k \geq 0\}$. After that, we shall show that u^* and v^* are estimated independently to k by a same constant in every $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ spaces respectively. This is not always the case when we are limiting us only to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ spaces.

a) Construction of $(\hat{\lambda}_k; k > 0)$ and $(\hat{\mu}_k; k > 0)$. We consider here α and β satisfying the relations

$$(3.2.9) \quad \frac{\alpha+1}{p} \left(\frac{n-p}{n} \right) + \frac{\beta+1}{q} \left(\frac{n-q}{n} \right) < 1$$

$$\frac{\alpha+1}{p} (1) + \frac{\beta+1}{q} (1) > 1$$

So, we can find $\hat{C} > 1$ and (λ, μ) such that

$$(3.2.10) \quad \left\{ \begin{array}{l} 1 < \lambda < \frac{n}{(n-p)\hat{C}} \\ 1 < \mu < \frac{n}{(n-q)\hat{C}} \\ \frac{\alpha+1}{\lambda p} + \frac{\beta+1}{\mu q} = 1 \end{array} \right.$$

Now, we take $\hat{\lambda}_k = \lambda p \hat{C}^k$, $\hat{\mu}_k = \mu q \hat{C}^k$.

From $(1)_k$ and $(2)_k$, we get

$$(3.2.11) \quad \left\{ \begin{array}{l} \hat{a}_k = \hat{\lambda}_k - \lambda p \\ \hat{b}_k = \hat{\mu}_k - \mu q \end{array} \right.$$

Lemma 3.2.3. For each $k \in \mathcal{N}$, u^* and v^* belong respectively to $L^{\hat{\lambda}_k}(\Omega)$ and $L^{\hat{\mu}_k}(\Omega)$.

Proof. We give a proof by induction.

By Sobolev imbedding Theorem, we have $u^* \in L^{\lambda p}(\Omega)$; $v^* \in L^{\mu q}(\Omega)$.

Then the Lemma is proved for $k = 0$. Suppose that it is true for all integer k' such that $0 \leq k' \leq k \in \mathcal{N}$

Take $\pi_p = \lambda p \hat{C}$ and $\pi_q = \mu q \hat{C}$, and $u^* \in L^{\lambda}(\Omega)$. The relation:

$$\left(1 + \frac{a_k}{p}\right) \pi_p = \lambda^2 p \hat{C}^{k+1} + \lambda p \hat{C} - \lambda^2 p \hat{C} \geq \lambda p \hat{C}^{k+1} = \hat{\lambda}_{k+1},$$

and Lemma 3.2.1. give $u^* \in L^{\hat{\lambda}_{k+1}}(\Omega)$ and $v^* \in L^{\hat{\mu}_{k+1}}(\Omega)$.

b) Construction of sequences $\{\lambda_k; k \in N\}$ and $\{\mu_k; k \in N\}$. Let

$$C = \min\left(\frac{n}{n-p}, \frac{n}{n-q}\right), \quad \gamma = \frac{\alpha+1}{\lambda p} + \frac{\beta+1}{\mu q}, \quad \delta = (M - (\gamma-1))C,$$

with $M > \gamma-1$; we define the sequences $\{\lambda_k; k \in \mathcal{N}\}$ and $\{\mu_k; k \in \mathcal{N}\}$ by

$$\lambda_k = p f_k, \quad \mu_k = q f_k$$

where f_k denotes the sequence

$$(3.2.12) \quad f_k = \frac{C}{C-1} [\delta C^{k-1} + (\gamma-1)].$$

Remark the sequences $\{\lambda_k; k \in \mathcal{N}\}$ and $\{\mu_k; k \in \mathcal{N}\}$ are strictly increasing and tend to $+\infty$, furthermore, we have the iterative relation

$$f_{k+1} = C[f_k^{-(\gamma-1)}] \quad (5)_k.$$

Proof of Proposition 3.2. We proceed again by induction.

First, we use the fact that the sequences $\hat{\lambda}_k$ and $\hat{\mu}_k$ are strictly increasing to establish the existence of an integer k_0 such that $\lambda_0 \geq \hat{\lambda}_{k_0}$ and $\mu_0 \geq \hat{\mu}_{k_0}$; we obtain from Lemma 3.2.3. that $u^* \in L^{\lambda_0}(\Omega)$ and $v^* \in L^{\mu_0}(\Omega)$.

Suppose that the proposition is true for $0 \leq k' \leq k$. Let $\pi_p = Cp$ and $\pi_q = Cq$, (1)_k and (2)_k give: $a_k = p(f_k - \gamma)$ and $b_k = q(f_k - \gamma)$.

So,

$$1 + \frac{a_k}{p} = 1 + f_k - \gamma < \frac{C}{C-1} \left[\frac{\delta}{C} + (\gamma - 1) \right] C^k.$$

Moreover by (5)_k we obtain

$$\lambda_{k+1} = \left(1 + \frac{a_k}{p} \right) \pi_p,$$

and similarly

$$\mu_{k+1} = \left(1 + \frac{b_k}{q} \right) \pi_q.$$

Then, we conclude with Lemma 3.2.2. that $u^* \in L^{\lambda_{k+1}}(\Omega)$, according to (3.1.6) and taking

$$A = \frac{C}{C-1} \left[\frac{\delta}{C} + (\gamma - 1) \right],$$

$$\|u^*\|_{L^{\lambda_{k+1}}(\Omega)}^{\lambda_{k+1}} \leq C \left(1 + \frac{a_k}{p} \right)^{Cp} \left\{ A_1 \|u^*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k} + A_2 \left(\|u^*\|_{L^{\lambda_k}(\Omega)} \right)^{\frac{\alpha + a_k + 1}{\lambda_k}} \left(\|v^*\|_{L^{\mu_k}(\Omega)}^{\mu_k} \right)^{\frac{\beta + 1}{\mu_k}} \right\}^C$$

(3.2.13)

$$\leq A^C C^{kCp} \max \left(1; \|u^*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k}; \|v^*\|_{L^{\mu_k}(\Omega)}^{\mu_k} \right)^C.$$

Considering the equality

$$-\Delta_q v^* = \frac{\partial H}{\partial v}(x; u^*, v^*),$$

we obtain an analogous inequality

$$\|v^*\|_{L^{\mu_k}(\Omega)}^{\mu_{k+1}} \leq A^C C^{kCq} \max\left(1; \|u^*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k}; \|v^*\|_{L^{\mu_k}(\Omega)}^{\mu_k}\right)^C \quad (3.2.14)$$

As in [20], [26], [27], we obtain the iterative relation $E_{k+1} \leq r_k + CE_k$, where

$$(3.2.15) \quad \begin{cases} E_k = \ln \max\left(\|u^*\|_{L^{\lambda_k}(\Omega)}^{\lambda_k}; \|v^*\|_{L^{\mu_k}(\Omega)}^{\mu_k}\right) \\ r_k = ak + b \quad a = \ln C^{C \max(p,q)}, \quad b = \ln(A)^C \end{cases}$$

So, we get the iterative relation $E_k \leq dC^{k-1}$, where d denotes a positive constant.

Thus,

$$(3.2.16) \quad \|u^*\|_{L^{\lambda_k}(\Omega)} \leq \exp\left(\frac{E_k}{\lambda_k}\right) \leq \exp\left(\frac{d(C-1)}{pC\delta}\right)$$

$$\|v^*\|_{L^{\mu_k}(\Omega)} \leq \exp\left(\frac{d(C-1)}{qC\delta}\right)$$

then, u^* and v^* are bounded in $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$ independently of $k \in \mathcal{N}$

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