

# *Existence and Nonexistence of Positive Radial Solutions of Neumann Problems with Critical Sobolev Exponents*

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*Communicated by J. SERRIN*

## 1. Introduction

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary and let  $\alpha \in C^\infty(\Omega)$ . For  $\lambda > 0$ ,  $p > 1$ ,  $n \geq 3$  we consider the following problem

$$\begin{aligned} -\Delta u &= u^p + \lambda \alpha(x) u & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

When  $p < \frac{n+2}{n-2}$  and  $\alpha(x) = -1$ , this problem has been discussed extensively in the works of NI [12], LIN & NI [10] and LIN, NI & TAKAGI [11]. They have proved that there exist positive constants  $\lambda_0$  and  $\lambda_1$ , with  $\lambda_0 \leq \lambda_1$ , such that (1.1) admits a non-constant solution for  $\lambda \geq \lambda_1$  and does not admit any non-constant solution for  $\lambda < \lambda_0$ . In view of their results, it was conjectured by LIN & NI [10] that a similar result holds even for  $p \geq \frac{n+2}{n-2}$ .

When  $p = \frac{n+2}{n-2}$ , BREZIS [7] posed the question of finding conditions on  $\alpha$  and  $\Omega$  for which (1.1) admits a solution. Clearly when  $\alpha(x) \geq 0$ , (1.1) does not admit any solution. Therefore we have to consider two cases: (i)  $\alpha(x)$  changes sign, (ii)  $\alpha(x) \leq 0$ .

In case (i) some partial results have been obtained in [3] by using the variational methods of BREZIS & NIRENBERG [8]. To describe the results of [3], we further assume that  $\int_{\Omega} \alpha(x) dx < 0$ , that there exists an  $x_0 \in \partial\Omega$  such that  $\alpha(x_0) > 0$ , and that  $\partial\Omega$  is flat at  $x_0$  of order at least four. Under these assumptions, it was shown that for  $n \geq 4$  there exists a  $\lambda^* > 0$  such that (1.1) admits a solution if and only if  $\lambda \in (0, \lambda^*)$ .

In case (ii) the standard variational arguments do not seem to work. On the other hand, in this situation it is easy to construct an example (see Remark 2 at the end of Section 4) such that for any  $\Omega$  we can find a negative function  $\alpha(x)$  for which (1.1) admits a solution. In view of this and the results of LIN, NI & TAKAGI [11], we shall consider the very restricted case of problem (1.1) when  $\lambda\alpha(x) \equiv -1$ ,  $\Omega$  is a ball and the solution is radial.

Let  $B(R)$  denote the ball of radius  $R$  with center at the origin and let  $\mu_1(R)$  be the first non-zero eigenvalue of the radial problem

$$\begin{aligned} -\Delta\varphi &= \mu\varphi \quad \text{in } B(R), \\ \frac{\partial\varphi}{\partial\nu} &= 0 \quad \text{on } \partial B(R). \end{aligned} \tag{1.2}$$

We consider the problem

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} - u \quad \text{in } B(R), \\ u &> 0, \quad u \text{ is radial in } B(R), \\ \frac{\partial u}{\partial\nu} &= 0 \quad \text{on } \partial B(R) \end{aligned} \tag{1.3}$$

and prove the following

**Theorem.** *Let  $p = (n+2)/(n-2)$ . The following conclusions hold:*

- (a) *If  $n \geq 3$  and  $p-1 > \mu_1(R)$ , then (1.3) admits a solution which is radially increasing.*
- (b) *If  $n \in \{4, 5, 6\}$  and  $p-1 < \mu_1(R)$ , then (1.3) admits a solution which is radially decreasing.*
- (c) *If  $n = 3$ , then there exists an  $R^* > 0$  such that for  $0 < R < R^*$ , (1.3) does not admit any nonconstant solution.*

Here we remark that part (a) of the theorem has been proved by NI [12] and LIN & NI [10], and that part (b) gives a counter-example to a part of the conjecture of LIN & NI [10].

Since we are looking for radial solutions, (1.3) reduces to studying the first turning point  $R_1(\gamma)$  of  $v(r, \gamma)$ , where  $v$  satisfies

$$\begin{aligned} -v'' - \frac{n-1}{r}v' &= \frac{n+2}{v^{n-2}} - v, \\ v'(0) &= 0, \quad v(0) = \gamma > 0 \end{aligned} \tag{1.4}$$

and  $R_1(\gamma)$  is defined by

$$R_1(\gamma) = \sup \{r; v'(s, \gamma) \neq 0 \quad \forall s \in (0, r)\}. \tag{1.5}$$

Because of the continuity of  $\gamma \rightarrow R_1(\gamma)$ , we shall be able to deduce the theorem from knowledge of the behavior of  $R_1(\gamma)$  as  $\gamma \rightarrow 0, 1$  and  $\infty$ . Information about the behavior of  $R_1(\gamma)$  as  $\gamma \rightarrow 0, 1$  is available in the literature. Therefore the main difficulty lies in understanding its behavior at  $\infty$ . We illustrate this for  $n = 6$ .

Let  $n = 6$ ,  $\gamma > 1$ ,  $\eta = v(R_1(\gamma), \gamma)$  and  $w = v - \eta$ . Then  $w$  satisfies

$$-\Delta w = w^2 + (2\eta - 1)w + \eta(\eta - 1) \quad \text{in } B(R_1(\gamma)),$$

$$w > 0 \quad \text{in } B(R_1(\gamma)),$$

$$w = \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \partial B(R_1(\gamma)).$$

Hence by Pohožaev's identity we have

$$2(2\eta - 1) \int_{B(R_1(\gamma))} w^2 dx + 8\eta(\eta - 1) \int_{B(R_1(\gamma))} w dx = 0.$$

This implies that  $\eta > 1/2$  and hence  $v(r, \gamma) > 1/2$  for all  $r \in (0, R_1(\gamma))$ . Now the asymptotic analysis of ATKINSON & PELETIER [5] suggests that we can find positive constants  $\delta, C_1, C_2, C_3$  and  $\gamma_0$  such that, for  $\gamma > \gamma_0$  and  $R(\gamma) = C_1\gamma^{-1/6}$ ,

$$R(\gamma) < R_1(\gamma), \quad (1.6)$$

$$1 - v(R(\gamma), \gamma) \geq \delta, \quad (1.7)$$

$$C_1/\gamma^{1/6} \leq |v'(R(\gamma), \gamma)| \leq C_2/\gamma^{1/6}. \quad (1.8)$$

Integrating (1.4) from  $R(\gamma)$  to  $R_1(\gamma)$  and using (1.6)–(1.8), we obtain for  $C = C_1^5 C_2$  that

$$C/\gamma \geq -R(\gamma)^5 v'(R(\gamma), \gamma) = \int_{R(\gamma)}^{R_1(\gamma)} r^5 v(1 - v) dr \geq \delta/12(R_1(\gamma)^6 - C_1/\gamma).$$

Hence

$$R_1(\gamma)^6 \leq \left( \frac{12C}{\delta} + C_1 \right) / \gamma \rightarrow 0 \quad \text{as } \gamma \rightarrow \infty. \quad (1.9)$$

When  $n \leq 5$  it may not be true that  $v(R_1(\gamma), \gamma)$  is bounded away from zero as  $\gamma \rightarrow \infty$ , whereas estimates similar to (1.6)–(1.8) still hold. Therefore in this case we have to adopt a different procedure to study  $R_1(\gamma)$  as  $\gamma \rightarrow \infty$ .

The paper is divided into two parts. In the first part (Section 3), we study the behavior of  $R_1(\gamma)$  as  $\gamma \rightarrow 0, 1$ . In the second part (Section 4), following the techniques developed in ATKINSON & PELETIER [5], we obtain estimates similar to (1.6)–(1.8). Using these (see Section 2) we obtain the proof of the theorem.

In a forthcoming paper we shall study problem (1.3) when  $-\Delta$  is replaced by the  $p$ -Laplacian for  $p \leq n$ .

While revising this paper, we learned of a recent result of BUDD, KNAAP & PELETIER [9], which discusses the question of existence and non-existence of solutions of (1.3) when  $u^{(n+2)/(n-2)} - u$  is replaced by  $u^{(n+2)/(n-2)} - u^q$  for  $1 < q < 4/(n-2)$ . This problem, for  $q = 4/(n-2)$ , has also been treated by ADIMURTHI, KNAAP & YADAVA [4].

Recently, ADIMURTHI & MANCINI [1] have tackled this problem in an arbitrary domain using variational techniques. We learned from Prof. J. SERRIN that X. J. WANG [13] has also found related results.

## 2. Proof of the Theorem

In order to prove the theorem, we make use of the standard substitutions,

$$t = \left( \frac{n-2}{r} \right)^{n-2}, \quad k = \frac{2(n-1)}{n-2}, \quad p = \frac{n+2}{n-2} = 2k-3, \quad y(t, \gamma) = v(r, \gamma),$$

introduced in [5]. Then from (1.4),  $y$  satisfies the Emden-Fowler equation

$$\begin{aligned} -y'' &= t^{-k}(y^p - y), \\ y(\infty) &= \gamma > 0, \quad y'(\infty) = 0. \end{aligned} \quad (2.1)$$

Let  $S_1(\gamma)$  be the first turning point of  $y(t, \gamma)$ , defined by

$$S_1(\gamma) = \inf \{t; y'(s, \gamma) \neq 0 \quad \forall s \in (t, \infty)\}. \quad (2.2)$$

Let  $\varphi$  be the solution of

$$\begin{aligned} -\varphi'' &= t^{-k} \varphi \quad \text{in } (0, \infty), \\ \varphi(\infty) &= 1, \quad \varphi'(\infty) = 0 \end{aligned} \quad (2.3)$$

and let  $\tau_0$  and  $\tau_1$  respectively be the first zero and first turning point of  $\varphi$ , i.e.,

$$\begin{aligned} \tau_0 &= \inf \{t; \varphi(s) > 0 \quad \text{for } s > t\}, \\ \tau_1 &= \inf \{t; \varphi'(s) > 0 \quad \text{for } s > t\}. \end{aligned} \quad (2.4)$$

Then we have

**Lemma A.** *Let  $\gamma \neq 0, 1$ . Then*

- (i)  $S_1(\gamma)$  exists and  $y(S_1(\gamma), \gamma) > 0$ .
- (ii) If  $\gamma \in (0, 1)$ , then  $y$  is decreasing, with

$$\lim_{\gamma \rightarrow 0} S_1(\gamma) = 0, \quad (2.5)$$

$$\lim_{\gamma \rightarrow 1} S_1(\gamma) = (p-1)^{1/(k-2)} \tau_1. \quad (2.6)$$

- (iii) If  $\gamma > 1$ , then  $y$  is increasing, with

$$\lim_{\gamma \rightarrow 1} S_1(\gamma) = (p-1)^{1/(k-2)} \tau_1. \quad (2.7)$$

This result is contained in the works of NI [12] and LIN & NI [10]. For the sake of completeness, we present the proof in Section 3.

**Lemma B.** *Let  $\gamma \in (1, \infty)$ . Then*

- (i) For  $t \geq S_1(\gamma)$ ,

$$y(t, \gamma) \geq Z_1(t, \gamma), \quad (2.8)$$

where

$$Z_1(t, \gamma) = \frac{\gamma t}{\{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)\}^{1/(k-2)}}.$$

(ii) If  $3 \leq n \leq 6$ , there exist positive constants  $\delta$ ,  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  and  $\gamma_0$  such that, for all  $\gamma \geq \gamma_0$  and  $S(\gamma) = C_1 \gamma^{1/(k-1)}$ ,

$$S_1(\gamma) < S(\gamma), \quad (2.9)$$

$$1 - \gamma(S(\gamma), \gamma) \geq \delta, \quad (2.10)$$

$$C_3/\gamma \leq \gamma'(S(\gamma), \gamma) \leq C_2/\gamma, \quad (2.11)$$

$$\lim_{\gamma \rightarrow \infty} S_1(\gamma) \geq C_4. \quad (2.12)$$

Assuming the validity of Lemmas A and B, we first complete the proof of the theorem. Since Lemma A gives the behavior of  $S_1(\gamma)$  as  $\gamma \rightarrow 0, 1$ , to prove the theorem we must study its behavior at  $\infty$ . For this we need three further lemmas.

**Lemma 2.1.** Let  $Z_1$  be as defined in (2.8). Then

$$-Z_1'' = \left( \frac{\gamma^p - \gamma}{\gamma^p} \right) t^{-k} Z_1^p \text{ in } (0, \infty), \quad (2.13)$$

$$\lim_{t \rightarrow \infty} Z_1 = \gamma, \quad (2.14)$$

$$\gamma - Z_1(t, \gamma) + tZ_1'(t, \gamma) = \left( \frac{\gamma^p - \gamma}{\gamma^p} \right) \int_t^\infty Z_1^p s^{-k+1} ds, \quad (2.15)$$

$$tZ_1'(t, \gamma) - Z_1(t, \gamma) = \frac{-\gamma t^{k-1}}{\{t^{k-2} + (\gamma^{p-1} - 1)/(k-1)\}^{(k-1)/(k-2)}}. \quad (2.16)$$

This lemma follows easily from the definition of  $Z_1$ .

**Lemma 2.2.** If  $n = 3$  ( $k = 4$ ), then

$$\overline{\lim}_{\gamma \rightarrow \infty} S_1(\gamma) < \infty.$$

**Proof.** Let  $\beta(t) = t \cosh \frac{1}{t}$ . It is easy to verify that  $\beta$  satisfies

$$\beta'' = t^{-4} \beta \quad \text{in } (0, \infty), \quad (2.17)$$

$$\lim_{t \rightarrow 0} \beta(t) = \infty, \quad \beta(t) = t + C(t), \quad (2.18)$$

where  $C(t) \geq 0$ . Let  $T_0$  be such that  $\beta'(T_0) = 0$ . Then the lemma follows if we can show that

$$\overline{\lim}_{\gamma \rightarrow \infty} S_1(\gamma) \leq T_0. \quad (2.19)$$

Let  $W = (\gamma\beta' - \beta\gamma')$ . Then  $W(\infty) = \gamma$  and  $W'(t) = t^{-4} \gamma^5 \beta$ . Integrating  $W'$  from  $S_1(\gamma)$  to  $\infty$  and using (2.8), (2.18), (2.15) and (2.16), we obtain

$$\begin{aligned} \gamma(S_1(\gamma), \gamma) \beta'(S_1(\gamma)) &= \gamma - \int_{S_1(\gamma)}^{\infty} t^{-4} \gamma^5 \beta \, dt \\ &\leq \gamma - \int_{S_1(\gamma)}^{\infty} t^{-3} Z_1 \, dt \leq \gamma - \frac{\gamma^5}{(\gamma^5 - \gamma)} [\gamma - Z_1 + S_1(\gamma) Z_1'] \\ &= -\frac{\gamma^2}{\gamma^5 - \gamma} + \left( \frac{\gamma^5}{(\gamma^5 - \gamma)} \right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{3}(\gamma^4 - 1)\}^{3/2}}. \quad (2.20) \end{aligned}$$

From (2.9) it follows that  $S_1(\gamma) = O(\gamma^{1/3})$  as  $\gamma \rightarrow \infty$ ; hence we have

$$\left( \frac{\gamma^5}{(\gamma^5 - \gamma)} \right) \frac{\gamma S_1(\gamma)^3}{\{S_1(\gamma)^2 + \frac{1}{3}(\gamma^4 - 1)\}^{3/2}} = O\left(\frac{1}{\gamma^4}\right)$$

as  $\gamma \rightarrow \infty$ . This together with (2.20) and (i) of Lemma A implies that  $\beta'(S_1(\gamma)) < 0$  for  $\gamma$  large, and so  $S_1(\gamma) \leq T_0$ . This proves (2.19) and hence the lemma.

**Lemma 2.3.** *If  $n \in \{4, 5, 6\}$ , then*

$$\lim_{\gamma \rightarrow \infty} S_1(\gamma) = \infty. \quad (2.21)$$

**Proof.** Suppose (2.21) is not true. Then for a sequence of values  $\gamma \rightarrow \infty$ , we have

$$\lim_{\gamma \rightarrow \infty} S_1(\gamma) < \infty. \quad (2.22)$$

For the sequel we use  $C, C_1, C_2$ , etc., to denote positive constants independent of  $\gamma$ . Now from (2.8), (2.9) we have for  $t \in (S_1(\gamma), S(\gamma))$ ,

$$\gamma(t, \gamma) \geq Z_1(t, \gamma) \geq C \frac{t}{\gamma}. \quad (2.23)$$

Let

$$H(t) = \frac{1}{2} t \gamma'^2 - \frac{1}{2} \gamma \gamma' + t^{1-k} \left( \frac{\gamma^{p+1}}{p+1} - \frac{\gamma^2}{2} \right).$$

Then  $H(\infty) = 0$  and  $H'(t) = \frac{p-1}{2} t^{-k} \gamma^2$ . Hence  $H(t) \leq 0$ . Now integrating  $H'(t)$  from  $S_1(\gamma)$  to  $S(\gamma)$  and using (2.23), we obtain

$$\begin{aligned} -H(S_1(\gamma)) &\geq \frac{p-1}{2} \int_{S_1(\gamma)}^{S(\gamma)} \gamma^2 t^{-k} \, dt \\ &\geq \frac{C}{\gamma^2} \int_{S_1(\gamma)}^{S(\gamma)} t^{-k+2} \, dt = C \frac{\varrho(\gamma)}{\gamma^2}, \quad (2.24) \end{aligned}$$

where

$$\varrho(\gamma) = \begin{cases} \log \frac{S(\gamma)}{S_1(\gamma)} & \text{if } k = 3, \\ (S(\gamma)^{3-k} - S_1(\gamma)^{3-k}) & \text{if } k < 3. \end{cases}$$

From (2.10), (2.11) and (2.22) we have

$$\begin{aligned} C_2/\gamma &\geq y'(S(\gamma), \gamma) = \int_{S_1(\gamma)}^{S(\gamma)} y(1 - y^{p-1}) t^{-k} dt \\ &\geq \frac{\delta}{k-1} y(S_1(\gamma), \gamma) \left( \frac{1}{S_1(\gamma)^{k-1}} - \frac{1}{S(\gamma)^{k-1}} \right) \\ &\geq Cy(S_1(\gamma), \gamma). \end{aligned}$$

Hence

$$\begin{aligned} -H(S_1(\gamma)) &= S_1(\gamma)^{1-k} y(S_1(\gamma), \gamma)^2 \left( \frac{1}{2} - \frac{y(S_1(\gamma), \gamma)^{p-1}}{p+1} \right) \\ &\leq C_3 \frac{S_1(\gamma)^{1-k}}{\gamma^2}. \end{aligned}$$

This combined with (2.24) gives

$$S_1(\gamma)^{k-1} \leq C_4/\varrho(\gamma). \quad (2.25)$$

Since  $S_1(\gamma)$  is bounded by assumption, it follows that  $\varrho(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow \infty$ . Therefore from (2.25),  $S_1(\gamma) \rightarrow 0$  as  $\gamma \rightarrow \infty$ , contradicting (2.12). This proves the lemma.

**Proof of the Theorem.** For  $\gamma \neq 0, 1$ , let  $R_1(\gamma)$  and  $u(r, \gamma)$  be defined by

$$\begin{aligned} t &= \left( \frac{n-2}{r} \right)^{n-2}, \quad S_1(\gamma) = \left( \frac{n-2}{R_1(\gamma)} \right)^{n-2}, \\ u(r, \gamma) &= y(t, \gamma). \end{aligned}$$

Then  $u$  satisfies

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} - u \quad \text{in } B(R_1(\gamma)), \\ u &> 0 \quad \text{in } B(R_1(\gamma)), \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B(R_1(\gamma)). \end{aligned}$$

Define  $R_1 = [(n-2)/\tau_1]^{n-2}$ . It is easy to see that  $\mu_1(R_1(\gamma)) = (R_1/R_1(\gamma))^2$ . Since  $\gamma \rightarrow R_1(\gamma)$  is continuous, (a) follows from (2.5) and (2.6), (b) follows from (2.7) and (2.21), and (c) follows from Lemma 2.2. This proves the theorem.

### 3. Proof of Lemma A

Let  $k > 2$  and let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^1$ -function. For  $\gamma > 0$ , let  $Y(t, \gamma)$  be the solution of

$$\begin{aligned} -Y'' &= t^{-k} f(Y), \\ Y(\infty) &= \gamma, \quad Y'(\infty) = 0. \end{aligned} \quad (3.1)$$

Let

$$F(s) = \int_0^s f(r) dr,$$

$$H(t) = \frac{1}{2} t Y'^2 - \frac{1}{2} Y Y' + t^{1-k} F(Y), \quad (3.2)$$

$$H_1(t) = \frac{1}{2} t Y'^2 - \frac{1}{2} Y Y' + \frac{t^{1-k}}{2(k-1)} Y f(Y). \quad (3.3)$$

It is then easy to see that  $Y$  satisfies

$$\lim_{t \rightarrow \infty} H(t) = \lim_{t \rightarrow \infty} H_1(t) = 0, \quad (3.4)$$

$$\lim_{t \rightarrow \infty} Y'(t, \gamma) t^{k-1} = \frac{f(\gamma)}{(k-1)}, \quad (3.5)$$

$$H'(t) = \frac{1}{2} t^{-k} [Y f(Y) - 2(k-1) F(Y)], \quad (3.6)$$

$$H_1'(t) = \frac{Y' t^{1-k}}{2(k-1)} [Y f'(Y) - (2k-3) f(Y)], \quad (3.7)$$

$$(Y' Y^{1-k} t^{k-1})' = -2(k-1) t^{k-2} Y^{-k} H_1(t). \quad (3.8)$$

From now on, we assume that  $f(0) = f(1) = 0$  and  $f'(1) > 0$ . Furthermore, we assume that

$$(s-1)f(s) > 0 \quad \text{for } s > 0 \text{ and } s \neq 1. \quad (3.9)$$

For  $\gamma > 0$ ,  $\gamma \neq 1$ , put

$$S_0(\gamma, f) = \inf \{t; Y(s, \gamma) \neq 1, Y'(s, \gamma) \neq 0 \quad \forall s > t\}, \quad (3.10)$$

$$S_1(\gamma, f) = \inf \{t; Y(s, \gamma) > 0, Y'(s, \gamma) \neq 0 \quad \forall s > t\}. \quad (3.11)$$

We then have the following

**Lemma 3.1.** For  $s \geq 0$ , assume that  $f$  satisfies

$$s f'(s) - (2k-3) f(s) \geq 0. \quad (3.12)$$

Then

$$Y(t, \gamma) \geq \eta_1(t, \gamma) \quad (3.13)$$

for  $\gamma > 1$  and  $t \geq S_1(\gamma, f)$ , where

$$\eta_1(t, \gamma) = \frac{\gamma t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1)\gamma} \right\}^{1/(k-2)}}.$$

**Proof.** Let  $t > S_1(\gamma, f)$ . Since  $\gamma > 1$ , it follows from (3.9) that  $Y'(t, \gamma) > 0$ . Therefore from (3.12) and (3.7),  $H_1'(t) \geq 0$ . Hence  $H_1$  is increasing and from (3.4),  $H_1(t) \leq 0$ . From (3.8), we have

$$(Y' Y^{1-k} t^{k-1})' \geq 0.$$

Integrating this twice from  $t$  to  $\infty$  and using (3.5), we obtain

$$\frac{1}{Y^{k-2}} - \frac{1}{\gamma^{k-2}} \leq \frac{\gamma^{1-k} f(\gamma)}{(k-1) t^{k-2}},$$

which gives

$$Y(t, \gamma) \geq \frac{\gamma t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1)\gamma} \right\}^{1/(k-2)}}.$$

This proves the lemma.

**Lemma 3.2.** For  $s \geq 0$ , assume that  $f$  satisfies

$$sf'(s+1) - (2k-3)f(s+1) \leq 0. \quad (3.14)$$

Then

$$Y(t, \gamma) \leq 1 + \eta_2(t, \gamma), \quad (3.15)$$

for  $\gamma > 1$  and  $t \geq S_1(\gamma, f)$ , where

$$\eta_2(t, \gamma) = \frac{(\gamma-1)t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1)(\gamma-1)} \right\}^{1/(k-2)}}.$$

**Proof.** Let  $V = Y - 1$ ,  $f_1(s) = f(s+1)$ . Then  $V$  satisfies

$$-V'' = t^{-k} f_1(V),$$

$$V(\infty) = \gamma - 1, \quad V'(\infty) = 0. \quad (3.16)$$

Since  $\gamma > 1$ , from (3.9), we get  $Y(t, \gamma) \geq 1$  and  $Y'(t, \gamma) > 0$  for  $t \geq S_0(\gamma, f)$ . Hence  $V(t) \geq 0$  and  $V'(t) > 0$ . Therefore for  $t \geq S_0(\gamma, f)$ , we have from (3.16), (3.7) and (3.14) that  $H_1'(t) \leq 0$ . So we deduce that  $H_1(t) \geq 0$  from (3.4) and that

$$(V' V^{1-k} t^{k-1})' \leq 0$$

from (3.8). Integrating twice and using (3.5) we obtain for all  $t \geq S_0(\gamma, f)$  that

$$V(t, \gamma) \leq \frac{(\gamma-1)t}{\left\{ t^{k-2} + \frac{f(\gamma)}{(k-1)(\gamma-1)} \right\}^{1/(k-2)}} = \eta_2(t, \gamma),$$

that is, for  $t \geq S_0(\gamma, f)$ ,

$$Y(t, \gamma) \leq 1 + \eta_2(t, \gamma). \quad (3.17)$$

Since  $Y(t, \gamma) \leq 1$  for  $t \in [S_1(\gamma, f), S_0(\gamma, f)]$ , inequality (3.17) continues to hold for  $t \geq S_1(\gamma, f)$ . This proves the lemma.

As an immediate consequence of these lemmas we have the following

**Lemma 3.3.** *Let  $\gamma > 1$  and let  $y(t, \gamma)$  satisfy (2.1). For  $t \geq S_1(\gamma)$ ,*

$$(i) \ y(t, \gamma) \geq Z_1(t, \gamma) \quad \text{if } n \geq 3. \quad (3.18)$$

$$(ii) \ y(t, \gamma) \leq 1 + Z_2(t, \gamma) \quad \text{if } 3 \leq n \leq 6, \quad (3.19)$$

where

$$Z_1(t, \gamma) = \frac{\gamma t}{\left\{ t^{k-2} + \frac{\gamma^{2(k-2)} - 1}{(k-1)} \right\}^{1/(k-2)}},$$

$$Z_2(t, \gamma) = \frac{(\gamma - 1)t}{\left\{ t^{k-2} + \frac{\gamma(\gamma^{2(k-2)} - 1)}{(k-1)(\gamma - 1)} \right\}^{1/(k-2)}}.$$

**Proof.** Let  $p = 2k - 3$  and  $f(s) = s^p - s$  for  $s \geq 0$ . Extend  $f$  as a  $C^1$ -function to  $\mathbb{R}$ . Then clearly  $f$  satisfies (3.9), and for  $s \geq 0$ ,

$$sf'(s) - (2k - 3)f(s) = 2(k - 2)s \geq 0.$$

Hence, (3.18) follows from Lemma 3.1.

For  $s \geq 1$ ,  $n \leq 6$ , let  $h(s) = -ps^{p-1} + (p - 1)s + 1$ . Since  $n \leq 6$  we have  $p \geq 2$ . Therefore  $h''(s) = -p(p - 1)(p - 2)s^{p-3} \leq 0$  and hence  $h$  is concave. Since  $h(1) = 0$  and  $h'(1) = -(p - 1)^2$ , we have  $h(s) \leq -(p - 2)^2(s - 1) \leq 0$ .

For  $s \geq 0$ , we have

$$\begin{aligned} sf'(s + 1) - (2k - 3)f(s + 1) &= -p(s + 1)^{p-1} + (p - 1)(s + 1) + 1 \\ &= h(s + 1) \leq 0. \end{aligned}$$

Hence (3.19) follows from Lemma 3.2. This proves the lemma.

For  $i = 1, 2$ , and  $\gamma_i > 0$  let  $q_i: \mathbb{R}_+ \rightarrow \mathbb{R}$  be continuous functions. Let  $\varphi_i$  satisfy

$$\begin{aligned} -\varphi_i'' &= t^{-k} q_i(t) \varphi_i, \\ \varphi_i(\infty) &= \gamma_i, \quad \varphi_i'(\infty) = 0. \end{aligned} \quad (3.20)$$

Denote by  $T_{0,i}$  and  $T_{1,i}$  respectively the first zero and first turning point of  $\varphi_i$  (see (2.4)). Then

**Lemma 3.4.** (i) *Assume that  $T_{0,1}$  exists and also that  $q_2(t) \geq q_1(t)$  for  $t \geq T_{0,2}$ . Then  $T_{0,2} > 0$  and  $T_{0,2} \geq T_{0,1}$ .*

(ii) *Assume that  $T_{0,1}$  and  $T_{1,1}$  exist and also that  $q_2(t) \geq q_1(t)$  for  $t \geq T_{1,2}$ . Then  $T_{1,2} > 0$  and  $T_{1,2} \geq T_{1,1}$ .*

**Proof.** Let  $W = \varphi_1' \varphi_2 - \varphi_1 \varphi_2'$ . Then  $W(\infty) = 0$  and

$$W'(t) = t^{-k}(\varrho_2 - \varrho_1) \varphi_1 \varphi_2. \quad (3.21)$$

Suppose that (i) is not true. Then  $T_{0,2} < T_{0,1}$  and hence from (3.21),  $W'(t) \geq 0$  for all  $t \geq T_{0,1}$ . Therefore  $W(T_{0,1}) \leq 0$ . But  $W(T_{0,1}) = \varphi_1'(T_{0,1}) \varphi_2(T_{0,1}) > 0$ , which is a contradiction. This proves (i).

Suppose that (ii) is not true. Then  $T_{1,2} < T_{1,1}$ . From (i) it follows that  $T_{0,1} < T_{0,2}$ . Using (3.21), we obtain  $W'(t) \geq 0$  for  $t \in [T_{1,1}, T_{0,1}]$ . Therefore we have

$$\begin{aligned} 0 < -\varphi_1(T_{1,1}) \varphi_2'(T_{1,1}) &= W(T_{1,1}) \leq W(T_{0,1}) \\ &= \varphi_1'(T_{0,1}) \varphi_2(T_{0,1}) < 0, \end{aligned}$$

which is a contradiction. This proves (ii) and hence the lemma.

Let  $\varphi, \tau_0, \tau_1$  be as in (2.3) and (2.4). For  $a > 0$ , denote  $\varphi(t, a) = \varphi(at)$  and let  $\tau_{0,a}$  and  $\tau_{1,a}$  be the first zero and first turning point of  $\varphi(\cdot, a)$ . Then we have

$$\begin{aligned} \tau_{0,a} &= \frac{\tau_0}{a}, \quad \tau_{1,a} = \frac{\tau_1}{a}, \\ -\varphi''(\cdot, a) &= a^{2-k} t^{-k} \varphi(\cdot, a), \\ \varphi(\infty, a) &= 1, \quad \varphi'(\infty, a) = 0. \end{aligned} \quad (3.22)$$

Let  $y(t, \gamma)$  and  $S_1(\gamma)$  be as in (2.1) and (2.2). Define

$$S_0(\gamma) = \inf \{t, y(s, \gamma) \neq 1, y'(s, \gamma) \neq 0 \quad \forall s > t\}. \quad (3.23)$$

We then have

**Lemma 3.5.** *If  $\gamma \neq 0, 1$ , then  $S_0(\gamma)$  exists and*

$$\lim_{\gamma \rightarrow 0} S_0(\gamma) = 0.$$

**Proof.** First consider the case  $\gamma > 1$ . Let

$$\begin{aligned} \varphi_2(t) &= y(t, \gamma) - 1, \\ \varrho_2(t) &= \frac{(\varphi_2 + 1)^p - (\varphi_2 - 1)^p}{\varphi_2}. \end{aligned}$$

Then  $\varphi_2$  satisfies

$$\begin{aligned} -\varphi_2'' &= t^{-k} \varrho_2 \varphi_2, \\ \varphi_2(\infty) &= \gamma - 1, \quad \varphi_2'(\infty) = 0. \end{aligned}$$

From (3.23) it follows that  $S_0(\gamma)$  is the first zero of  $\varphi_2$  and that  $\varrho_2(t) \geq (p-1)$  for  $t \geq S_0(\gamma)$ . Taking  $\varrho_1 = (p-1)$ ,  $\varphi_1(t) = \varphi(t, (p-1)^{-1/(k-2)})$  in (i) of Lemma 3.4, we conclude that  $S_0(\gamma)$  exists and

$$(p-1)^{1/(k-2)} \tau_0 \leq S_0(\gamma). \quad (3.24)$$

Now consider the case  $0 < \gamma < 1$ . Let

$$\varphi_2 = 1 - y(t, \gamma),$$

$$\varrho_2(t) = \frac{(1 - \varphi_2) - (1 - \varphi_2)^p}{\varphi_2}.$$

Then  $\varphi_2$  satisfies

$$-\varphi_2'' = t^{-k} \varrho_2(t) \varphi_2,$$

$$\varphi_2(\infty) = 1 - \gamma, \quad \varphi_2'(\infty) = 0,$$

with  $S_0(\gamma)$  as its first zero. By taking  $\varrho_1 = \min \{\varrho_2(t), t \geq S_0(\gamma)\}$ ,  $\varphi_1(t) = \varphi(t, \varrho_1^{1/(k-2)})$  in (i) of Lemma 3.4, we obtain the existence of  $S_0(\gamma)$ . Since  $\varrho_2(t) \leq (p-1)$  for  $t \geq S_0(\gamma)$ , again from (i) of Lemma 3.4, we obtain

$$S_0(\gamma) \leq (p-1)^{1/(k-2)} \tau_0. \quad (3.25)$$

Now suppose that  $S_0(\gamma)$  does not tend to zero as  $\gamma$  approaches zero. Then by going to a subsequence and using (3.25), we have

$$\lim_{\gamma \rightarrow 0} S_0(\gamma) = S_0 > 0. \quad (3.26)$$

Since the boundedness of  $y$  implies that  $y'$  and  $y''$  are uniformly bounded in  $(S_0(\gamma), \infty)$ , the Arzelà-Ascoli Theorem implies that there exists a subsequence such that  $y(t, \gamma) \rightarrow y_0(t)$  uniformly on compact sets and that  $y_0$  satisfies

$$-y_0'' = t^{-k}(y_0^p - y_0) \quad \text{in } (S_0, \infty), \quad (3.27)$$

$$y_0(\infty) = y_0'(\infty) = 0.$$

From the uniqueness of the solution of (3.27),  $y_0 \equiv 0$ . But  $y_0(S_0) = 1$ . This contradiction proves the lemma.

**Proof of the Lemma A.** From (2.1), it follows that  $y$  is increasing for  $\gamma > 1$ , and  $y$  is decreasing for  $\gamma < 1$ .

First consider the case  $\gamma > 1$ . Suppose  $S_1(\gamma) = 0$ . Then  $y(t, \gamma) > 0$  for  $t > 0$  by (3.18), and  $y(t, \gamma)$  is an increasing function by (2.1). Since  $y(S_0(\gamma), \gamma) = 1$ , from Lemma 3.5 we can find a  $C > 0$  such that for  $t \in (0, (S_0(\gamma)/2)$ ,

$$1 - y^{p-1}(t, \gamma) \geq C. \quad (3.28)$$

From (3.18), we can find a  $C_1 > 0$  such that for  $t \in (0, S_0(\gamma)/2)$ ,

$$y(t, \gamma) \geq C_1 t. \quad (3.29)$$

Integrating (2.1) and using (3.28) and (3.29), we have

$$\begin{aligned} \infty &> y'(S_0(\gamma)/2, \gamma) \geq \int_0^{S_0(\gamma)/2} t^{-k} y(1 - y^{p-1}) dt \\ &\geq CC_1 \int_0^{S_0(\gamma)/2} t^{-k+1} dt = \infty, \end{aligned}$$

which is a contradiction. Hence  $S_1(\gamma) > 0$ ; from (3.18), we have  $y(S_1(\gamma), \gamma > 0)$ .

Let  $v = y - 1$  and  $f_1(s) = (s + 1)^p - (s + 1)$ . Then  $v(\infty) = \gamma - 1$ ,  $v'(\infty) = 0$ , and  $S_0(\gamma)$  and  $S_1(\gamma)$  respectively are the first zero and first turning points of  $v$ . Moreover,  $v$  satisfies

$$-v'' = t^{-k} \left( \frac{f_1(v)}{v} \right) v. \quad (3.30)$$

Now integrating (3.30) and using (3.24), we can find a  $C > 0$  such that, for all  $1 < \gamma \leq 2$ ,

$$v'(S_0(\gamma)) = \int_{S_0(\gamma)}^{\infty} t^{-k} \left( \frac{f_1(v)}{v} \right) v dt \leq C(\gamma - 1). \quad (3.31)$$

Since

$$\sup \left\{ \frac{f_1(v)}{v} ; t \geq S_0(\gamma), \gamma \in (1, 2] \right\} < \infty,$$

as a consequence of (i) of Lemma 3.4,  $S_0(\gamma)$  is bounded for  $\gamma \in (0, 2]$ . From this and from (3.31), we can find a  $C_1 > 0$  such that for  $1 < \gamma \leq 2$ ,

$$|v(S_1(\gamma))| \leq v'(S_0(\gamma)) (S_0(\gamma) - S_1(\gamma)) \leq C(\gamma - 1). \quad (3.32)$$

This inequality implies that for any  $\varepsilon > 0$ , we can find a  $\delta > 0$  such that whenever  $\gamma - 1 \leq \delta$ ,  $t \geq S_1(\gamma)$ ,

$$(1 - \varepsilon)(p - 1) \leq \frac{f_1(v(t))}{v(t)} \leq (1 + \varepsilon)(p - 1). \quad (3.33)$$

From (3.22), (3.33) and (ii) of Lemma 3.4, we obtain

$$[(1 - \varepsilon)(p - 1)]^{1/(k-2)} \tau_1 \leq S_1(\gamma) \leq [(1 + \varepsilon)(p - 1)]^{1/(k-2)} \tau_1$$

for  $\gamma \leq 1 + \delta$ . This inequality implies that

$$\lim_{\gamma \rightarrow 1} S_1(\gamma) = (p - 1)^{1/(k-2)} \tau_1. \quad (3.34)$$

Now consider the case in which  $0 < \gamma < 1$ . Suppose  $S_1(\gamma) = 0$ . From Lemma 3.5,  $S_0(\gamma)$  exists and  $y(S_0(\gamma), \gamma) = 1$ . Hence from (2.1),

$$-y'(t, \gamma) \leq -y'(S_0(\gamma), \gamma) \quad (3.35)$$

for all  $t \in (0, S_0(\gamma))$ . Also we can find a  $C > 0$  such that for  $t \in (0, S_0(\gamma)/2)$ ,

$$y^p(t, \gamma) - y(t, \gamma) \geq C. \quad (3.36)$$

Integrating (2.1) and using (3.35) and (3.36), we have

$$\begin{aligned} -y'((S_0(\gamma), \gamma) &\geq -y'(t, \gamma) \geq \int_t^{S_0(\gamma)/2} s^{-k}(y^p - y) dt \\ &\geq C \left[ \frac{1}{t^{k-1}} - \left( \frac{2}{S_0(\gamma)} \right)^{k-1} \right] \rightarrow \infty \end{aligned}$$

as  $t \rightarrow 0$ , which is a contradiction. This implies that  $S_1(\gamma)$  exists.

Let  $v = 1 - \gamma$  and  $f_1(s) = (1 - s) - (1 - s)^p$ . Then  $v(\infty) = 1 - \gamma$ ,  $v'(\infty) = 0$ ,  $S_0(\gamma)$  and  $S_1(\gamma)$  are the first zero and first turning points of  $v$ . Moreover,  $v$  satisfies

$$-v'' = t^{-k} \left( \frac{f_1(v)}{v} \right) v. \quad (3.37)$$

Since

$$\inf \left\{ \frac{f_1(v)}{v} ; t \geq S_0(\gamma), \frac{1}{2} \leq \gamma < 1 \right\} > 0,$$

by Lemma 3.4(i) and by (3.22) we have

$$\inf \{ S_0(\gamma) ; \frac{1}{2} \leq \gamma < 1 \} > 0.$$

Therefore by integrating (3.37), we have for some constant  $C > 0$ ,

$$v'(S_0(\gamma)) = \int_{S_0(\gamma)}^{\infty} t^{-k} \left( \frac{f_1(v)}{v} \right) v dt \leq C(1 - \gamma). \quad (3.38)$$

From (3.25), (3.38) and the mean value theorem, we can find a  $C_1 > 0$  such that

$$|v(S_1(\gamma))| \leq |v'(S_0(\gamma))| (S_0(\gamma) - S_1(\gamma)) \leq C_1(1 - \gamma).$$

This implies that for every  $\varepsilon > 0$ , we can find a  $\delta > 0$  such that

$$(1 - \varepsilon)(p - 1) \leq \frac{f_1(v)}{v} \leq (1 + \varepsilon)(p - 1) \quad (3.39)$$

whenever  $1 - \gamma \leq \delta$  and  $t \geq S_1(\gamma)$ . From (3.22), (3.39), and Lemma 3.4(ii) we obtain

$$[(1 - \varepsilon)(p - 1)]^{1/(k-2)} \tau_1 \leq S_1(\gamma) \leq [(1 + \varepsilon)(p - 1)]^{1/(k-2)} \tau_1$$

for  $1 - \gamma \leq \delta$ . This inequality implies that

$$\lim_{\gamma \rightarrow 1} S_1(\gamma) = (p - 1)^{1/(k-2)} \tau_1. \quad (3.40)$$

Since  $S_1(\gamma) < S_0(\gamma)$ , from Lemma 3.5 we have  $\lim_{\gamma \rightarrow 0} S_1(\gamma) = 0$ . Now the lemma follows from (3.34) and (3.40).

#### 4. Proof of Lemma B

Let  $n \leq 6$  and  $\gamma > 1$ . Let  $y(t, \gamma)$ ,  $S_1(\gamma)$ , and  $S_0(\gamma)$  be as in (2.1), (2.2), (3.23), respectively. For the sequel we use  $C, C_1, C_2$ , etc., to denote positive constants independent of  $\gamma$ , but which may be different in different inequalities. We have the following

**Lemma 4.1.** For  $\gamma$  large,

$$S_0(\gamma) = O(\gamma), \quad (4.1)$$

$$y(t, \gamma) \leq 1 + Ct/\gamma \quad \text{for } t \geq S_0(\gamma), \quad (4.2)$$

$$y(\gamma^2, \gamma) \geq C\gamma, \quad (4.3)$$

$$C_1/\gamma \leq y'(2S_0(\gamma), \gamma) \leq C_2/\gamma. \quad (4.4)$$

For  $t \in (2S_0(\gamma), \gamma^2)$

$$1 + \frac{C_1(t - S_0(\gamma))}{\gamma} \leq y(t, \gamma) \leq 1 + \frac{C_2(t - S_0(\gamma))}{\gamma}. \quad (4.5)$$

**Proof.** By Lemma (3.5),  $S_0(\gamma)$  exists and from (3.18),

$$Z_1(S_0(\gamma), \gamma) \leq y(S_0(\gamma), \gamma) = 1. \quad (4.6)$$

This implies that

$$S_0(\gamma)^{k-2} \leq \frac{\gamma^{p-1}}{(k-1)(\gamma^{k-2} - 1)}. \quad (4.7)$$

Since  $p = 2k - 3$ , it follows from (4.7) that  $S_0(\gamma) = O(\gamma)$  as  $\gamma \rightarrow \infty$ . This proves (4.1).

For large  $\gamma$  we have,  $\frac{\gamma^p - \gamma}{(k-1)(\gamma-1)} \geq C\gamma^{2(k-2)}$  and hence from (3.19),

$$y(t, \gamma) \leq 1 + \frac{\gamma t}{\left\{ t^{k-2} + \frac{\gamma^p - \gamma}{(k-1)(\gamma-1)} \right\}^{1/(k-2)}} \leq 1 + \frac{Ct}{\gamma}.$$

for all  $t \geq S_0(\gamma)$ . This proves (4.2).

Again from (3.18), we have

$$y(\gamma^2, \gamma) \geq Z_1(\gamma^2, \gamma) \geq C\gamma$$

for  $\gamma$  large. This proves (4.3).

From the concavity of  $y$  in  $[S_0(\gamma), 2S_0(\gamma)]$  and from (4.2) we have for large  $\gamma$  that

$$y'(2S_0(\gamma), \gamma) \leq \frac{y(2S_0(\gamma), \gamma) - 1}{S_0(\gamma)} \leq \frac{C_2 S_0(\gamma)}{\gamma S_0(\gamma)} = \frac{C_2}{\gamma}. \quad (4.8)$$

Again, from the concavity of  $y$  in  $[2S_0(\gamma), \gamma^2]$  and from (4.1)–(4.3), we have for large  $\gamma$  that

$$y'(2S_0(\gamma), \gamma) \geq \frac{y(\gamma^2, \gamma) - y(2S_0(\gamma), \gamma)}{\gamma^2 - 2S_0(\gamma)} \geq \frac{C\gamma + O(1)}{\gamma^2 + O(\gamma)} \geq \frac{C_1}{\gamma}.$$

This together with (4.8) proves (4.4).

Let  $t \in [2S_0(\gamma), \gamma^2]$ . From (4.2), we then have

$$y(t, \gamma) \leq 1 + \frac{Ct}{\gamma} = 1 + \frac{C(t - S_0(\gamma))}{\gamma} \frac{t}{(t - S_0(\gamma))} \leq 1 + C_2 \frac{(t - S_0(\gamma))}{\gamma}. \quad (4.9)$$

From the concavity of  $y$  in  $[S_0(\gamma), \gamma^2]$  and from (4.1)–(4.3), it follows that

$$\begin{aligned} \frac{y(t, \gamma) - y(S_0(\gamma), \gamma)}{t - S_0(\gamma)} &\geq \frac{y(\gamma^2, \gamma) - y(S_0(\gamma), \gamma)}{\gamma^2 - S_0(\gamma)} \\ &\geq \frac{C\gamma + O(1)}{\gamma^2 + O(\gamma)} \geq \frac{C_1}{\gamma} \end{aligned}$$

for  $\gamma$  large and for  $t \in [S_0(\gamma), \gamma^2]$ . Hence

$$y(t, \gamma) \geq 1 + \frac{C_1(t - S_0(\gamma))}{\gamma}.$$

This together with (4.9) proves (4.5) and hence the lemma.

**Lemma 4.2.**  $\lim_{\gamma \rightarrow \infty} S_0(\gamma) > 0$ .

**Proof.** Integrating (2.1) and using (4.4) and (4.5) we obtain for  $\gamma$  large that

$$\begin{aligned} \frac{C_2}{\gamma} &\geq y'(2S_0(\gamma), \gamma) = \int_{2S_0(\gamma)}^{\infty} t^{-k}(y^p - y) dt \\ &\geq (p-1) \int_{2S_0(\gamma)}^{\gamma^2} t^{-k}(y-1) dt \geq \frac{C}{\gamma} \int_{2S_0(\gamma)}^{\gamma^2} t^{-k}(t - S_0(\gamma)) dt \\ &\geq \frac{CS_0(\gamma)^{2-k}}{\gamma} \int_2^{\gamma^2/S_0(\gamma)} t^{-k}(t-1) dt \geq \frac{CS_0(\gamma)^{2-k}}{\gamma}, \end{aligned}$$

which implies that  $\lim_{\gamma \rightarrow \infty} S_0(\gamma) \geq C_4 > 0$ , since  $\gamma^2/S_0(\gamma) \rightarrow \infty$  as  $\gamma \rightarrow \infty$  and  $k > 2$ . This proves the lemma.

**Lemma 4.3.** For  $\gamma$  large,

$$C_1\gamma \leq S_0(\gamma) \leq C_2\gamma, \quad (4.10)$$

$$C_1/\gamma \leq y'(S_0(\gamma), \gamma) \leq C_2/\gamma. \quad (4.11)$$

**Proof.** Let  $v = y - 1$  and  $f_1(s) = (s+1)^p - (s+1)$ . Then  $v$  satisfies

$$\begin{aligned} -v'' &= t^{-k}f_1(v), \\ v(\infty) &= \gamma - 1, \quad v'(\infty) = 0 \end{aligned} \quad (4.12)$$

and  $S_0(\gamma)$  is the first zero of  $v$ . Let  $F_1(s)$  be the primitive of  $f_1$  and let

$$H(t) = \frac{1}{2}tv'^2 - \frac{1}{2}vv' + t^{1-k}F_1(v).$$

Then from (3.6) we have

$$-H'(t) = \frac{t^{-k}}{2}h(v+1), \quad (4.13)$$

where

$$h(s) = s^p - \frac{p-1}{2}s^2 - s + \left(\frac{p-1}{2}\right).$$

Since  $n \leq 6$ , we have  $p \geq 2$ , and therefore we obtain that  $h$  is convex for  $s \geq 1$  and satisfies

$$h(s) \geq C(s-1)^p \quad (4.14)$$

for  $s \geq 1$ . Integrating (4.13) and using (4.14) and (4.5) we obtain

$$\begin{aligned}
 H(2S_0(\gamma)) &= \int_{2S_0(\gamma)}^{\infty} t^{-k} h(v+1) dt \\
 &\geq C \int_{S_0(\gamma)}^{\gamma^2} t^{-k} \frac{(t - S_0(\gamma))^p}{\gamma} dt \\
 &= \frac{CS_0(\gamma)^{p-k+1}}{\gamma^p} \int_{\gamma^2/(2S_0(\gamma))}^{\gamma^2/S_0(\gamma)} t^{-k+p} dt = C/\gamma,
 \end{aligned} \tag{4.15}$$

since  $p = 2k - 3$ . On the other hand, we have from (4.1) and (4.4), that

$$\begin{aligned}
 H(2S_0(\gamma)) &\leq S_0(\gamma) v'(S_0(\gamma))^2 + 2^{1-k} S_0(\gamma)^{1-k} F_1(v(2S_0(\gamma))) \\
 &\leq C_1 \left\{ \frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k} F_1 \left( C_2 \frac{S_0(\gamma)}{\gamma} \right) \right\}.
 \end{aligned} \tag{4.16}$$

Now we assert that

$$\lim_{\gamma \rightarrow \infty} \frac{S_0(\gamma)}{\gamma} > 0. \tag{4.17}$$

Suppose (4.17) is not true. Then for a subsequence  $\gamma \rightarrow \infty$ , we can find  $C_3 > 0$  such that

$$F_1 \left( C_2 \frac{S_0(\gamma)}{\gamma} \right) \leq C_3 \left( \frac{S_0(\gamma)}{\gamma} \right)^2. \tag{4.18}$$

From (4.15), (4.16) and (4.18) we have

$$\begin{aligned}
 C/\gamma &\leq H(2S_0(\gamma)) \leq C_4 \left\{ \frac{S_0(\gamma)}{\gamma^2} + S_0(\gamma)^{1-k} \left( \frac{S_0(\gamma)}{\gamma} \right)^2 \right\} \\
 &\leq \frac{C_4}{\gamma} \left( \frac{S_0(\gamma)}{\gamma} \right) \left( 1 + \frac{1}{S_0(\gamma)^{k-2}} \right).
 \end{aligned}$$

This, together with Lemma (4.2), implies that

$$0 < C_5 \leq \left( \frac{S_0(\gamma)}{\gamma} \right) \rightarrow 0$$

as  $\gamma \rightarrow \infty$ , which is a contradiction. This proves (4.17). Now (4.10) follows from (4.1) and (4.17).

From the concavity of  $y$  and (4.4), we have

$$y'(S_0(\gamma), \gamma) \geq y'(2S_0(\gamma), \gamma) \geq \frac{C_1}{\gamma}. \tag{4.19}$$

For  $\gamma$  large it follows from (4.2) and (4.1) that  $y(t, \gamma) \leq C$  for  $t \in [S_0(\gamma), 2S_0(\gamma)]$ . Hence from (4.4) and (4.10) we have

$$\begin{aligned} y'(S_0(\gamma), \gamma) &= y'(2S_0(\gamma), \gamma) + \int_{S_0(\gamma)}^{2S_0(\gamma)} t^{-k}(y^p - y) dt \\ &\leq \frac{C}{\gamma} + O\left(\frac{1}{\gamma^{k-1}}\right) \leq \frac{C_2}{\gamma}. \end{aligned}$$

This, together with (4.19), proves (4.11) and the lemma.

**Proof of Lemma B.** Inequality (2.8) follows from (3.18).

Let  $t_0 \in [S_1(\gamma), S_0(\gamma)]$  be such that

$$y'(t_0, \gamma) = \frac{y'(S_0(\gamma), \gamma)}{2}. \quad (4.20)$$

Then from (4.11), (4.20), and the restriction that  $0 < \gamma \leq 1$ , we obtain for  $\gamma$  large that

$$C_1/\gamma \leq \frac{y'(S_0(\gamma), \gamma)}{2} = \int_{t_0}^{S_0(\gamma)} t^{-k}(y - y^p) dt \leq \frac{C}{t_0^{k-1}},$$

that is,

$$t_0 \leq C_1 \gamma^{1/(k-1)}. \quad (4.21)$$

Let  $S(\gamma) = C_1 \gamma^{1/(k-1)}$ ; then clearly from (4.20) we have

$$S_1(\gamma) \leq S(\gamma), \quad (4.22)$$

$$\begin{aligned} C_3/\gamma &\leq \frac{y'(S_0(\gamma), \gamma)}{2} = y'(t_0, \gamma) \\ &\leq y'(S(\gamma), \gamma) \leq y'(S_0(\gamma), \gamma) \leq C_4/\gamma. \end{aligned} \quad (4.23)$$

This, together with (4.21) and (4.22), proves (2.9) and (2.11).

Now from the convexity of  $y$  in  $[S_1(\gamma), S_0(\gamma)]$  and (4.23) we have

$$\frac{1 - y(S(\gamma), \gamma)}{S_0(\gamma) - S(\gamma)} \geq y'(S(\gamma), \gamma) \geq C_3/\gamma. \quad (4.24)$$

From (4.24) and (4.10), we have

$$1 - y(S(\gamma), \gamma) \geq C - O\left(\frac{1}{\gamma^{(k-2)/(k-1)}}\right).$$

Hence we can find a  $\delta > 0$  such that for  $\gamma$  large,  $1 - y(S(\gamma), \gamma) \geq \delta$  and this proves (2.10).

Since  $S(\gamma) = O\left(\gamma^{\frac{1}{(k-1)}}$ , from (3.18) we get

$$y(t, \gamma) \geq Z_1(t, \gamma) \geq Ct/\gamma. \quad (4.25)$$

for all  $t \in [S_1(\gamma), S(\gamma)]$ . From (2.9), (2.10), (2.11) and (4.25) we have

$$\begin{aligned} C/\gamma &\geq y'(S(\gamma), \gamma) = \int_{S_1(\gamma)}^{S(\gamma)} t^{-k} y(1 - y^{p-1}) dt \geq \frac{C_1}{\gamma} \delta \int_{S_1(\gamma)}^{S(\gamma)} t^{-k+1} dt \\ &\geq \frac{C_2}{\gamma} \left( \frac{1}{S_1(\gamma)^{k-2}} - \frac{1}{S(\gamma)^{k-2}} \right). \end{aligned}$$

This implies that

$$\lim_{\gamma \rightarrow \infty} S_1(\gamma) > 0.$$

This proves (2.12) and hence the lemma.

**Remark 1.** Let  $n \geq 3$  and  $p > 1$ . Then there exists an  $R_0 > 0$  such that for  $0 < R < R_0$ , the problem

$$\begin{aligned} -\Delta u &= u^p - u \quad \text{in } B(R), \\ u &> 0, \text{ } u \text{ is radial} \quad \text{in } B(R), \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{in } \partial B(R) \end{aligned} \tag{4.26}$$

does not admit any solution  $u$  such that  $u'$  changes sign.

**Proof.** We consider two cases:  $1 < p < \frac{n+2}{n-2}$  and  $p \geq \frac{n+2}{n-2}$ .

Case 1.  $1 < p < \frac{n+2}{n-2}$ . In this situation, by a result of LIN, NI & TAGAKI [11] there exists an  $R_0 > 0$  such that for  $0 < R < R_0$ , problem (4.26) does not admit a nonconstant solution. This proves the remark.

Case 2.  $p \geq \frac{n+2}{n-2}$ . Let  $v(r, \gamma)$  denote the solution of

$$\begin{aligned} -\left(v'' + \frac{n-1}{r}v'\right) &= v^p - v \quad \text{in } (0, \infty), \\ v(0) &= \gamma > 0, \quad v'(0) = 0. \end{aligned}$$

Let  $R_1(\gamma) < R_2(\gamma) < \dots$  be the turning points (i.e.,  $v'(R_i(\gamma), \gamma) = 0$ ) of  $v(r, \gamma)$ . From the result of NI [12], we know that  $v(r, \gamma) > 0$  for all  $\gamma > 0$ .

Now the remark follows from the following

**Assertion.** There exists a constant  $C > 0$  such that

$$\sup_{\gamma \in (0, \infty)} R_2(\gamma) \geq C. \tag{4.27}$$

To prove this we adopt the method used in ATKINSON, BREZIS & PELETIER [6] and in ADIMURTHI & YADAVA [2]. Proceeding as in Lemma A, we obtain

$$\lim_{\gamma \rightarrow 0} R_1(\gamma) = \infty, \quad \lim_{\gamma \rightarrow 1} R_1(\gamma) > 0.$$

Therefore it is sufficient to prove that

$$\sup_{\gamma \in (1, \infty)} R_2(\gamma) \geq C. \quad (4.28)$$

Let  $w(r, \gamma) = v(r, \gamma) - 1$  and let  $T_1(\gamma)$  and  $T_2(\gamma)$  respectively be the first and second zeros of  $w(r, \gamma)$ . Then

$$T_1(\gamma) < R_1(\gamma) < T_2(\gamma) < R_2(\gamma).$$

Therefore, in order to prove (4.28), it is sufficient to show that

$$\sup_{\gamma \in (1, \infty)} T_2(\gamma) \geq C. \quad (4.29)$$

Since  $v(r, \gamma) > 0$  for all  $\gamma > 1$ , we get

$$\sup_{\gamma \in (1, \infty)} \{ |w(r, \gamma)|; T_1(\gamma) < r < T_2(\gamma) \} \leq 1. \quad (4.30)$$

Let  $Z(r) = \left( \frac{n-2}{r} \right)^{\frac{n-2}{2}}$ . Then  $Z$  satisfies

$$Z'' + \left( \frac{n-1}{r} \right) Z' + \frac{1}{4} Z^{4/(n-2)} = 0 \quad \text{in } (0, \infty), \quad (4.31)$$

$$\lim_{r \rightarrow 0} Z(r) = \infty.$$

From (4.30) and (4.31) we can choose an  $r_0 > 0$  such that for all  $\gamma > 1$  and  $r \in (0, r_0) \cap [T_1(\gamma), T_2(\gamma)]$ ,

$$\frac{(w+1)^p - (w+1)}{w} < \frac{1}{4} Z(r)^{4/(n-2)}.$$

Now by Sturm's comparison theorem, there exists a  $C > 0$  such that (4.29) holds. This completes the proof of the remark.

**Remark 2.** Given any  $\Omega$ , we can construct a negative function  $\alpha \in C^\infty(\Omega)$  such that the problem

$$\begin{aligned} -\Delta u &= u^p + \alpha(x) u \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \quad (4.32)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

admits a solution.

The construction of  $\alpha$  is similar to the construction given by BREZIS [7] for the Dirichlet problem.

Let  $a \in C^\infty(\Omega)$ , be such that  $a$  changes sign in  $\Omega$  and  $\int_{\Omega} a(x) dx < 0$ . By the result of HESS & SENN [14] there exists a  $\lambda_1(\Omega) > 0$  such that

$$-\Delta v = \lambda_1(\Omega) a(x) v \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega \text{ and}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

admits a solution. Define

$$\alpha(x) = \lambda_1(\Omega) a(x) - \mu^{p-1} v^{p-1}, \quad u = \mu v,$$

where  $\mu$  is a positive real number. Obviously  $u$  satisfies (4.32). By choosing  $\mu$  large, we get  $\alpha < 0$ .

*Acknowledgement.* We thank Dr. VEERAPPA GOWDA for assisting us in doing some numerical computation for this problem and also for several discussions.

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(Received January 2, 1991)