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Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian

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Abstract

In this article, we study a class of fractional coupled systems with Riemann-Stieltjes integral boundary conditions and generalized p-Laplacian which involves two different parameters. Based on the Guo-Krasnosel'skii fixed point theorem, some new results on the existence and nonexistence of positive solutions for the fractional system are received, the impact of the two different parameters on the existence and nonexistence of positive solutions is also investigated. An example is then given to illuminate the application of the main results.

MSC: 26A33; 34B18

Keywords: positive solutions; fractional coupled system; Riemann-Stieltjes integral conditions; generalized p-Laplacian operator

1 Introduction

In this paper, our main research is the existence and nonexistence of positive solutions for the following fractional coupled system with generalized p-Laplacian involving Riemann-Stieltjes integral conditions.

$$\begin{cases} D_{0^+}^{\beta_1}(\phi(D_{0^+}^{\alpha_1}u(t))) + \lambda_1 f_1(t,u(t),v(t)) = 0, \\ D_{0^+}^{\beta_2}(\phi(D_{0^+}^{\alpha_2}v(t))) + \lambda_2 f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad \phi(D_{0^+}^{\alpha_1}u(0)) = (\phi(D_{0^+}^{\alpha_1}u(1)))' = 0, \\ v(0) = v'(0) = \cdots v^{(m-2)} = 0, \qquad \phi(D_{0^+}^{\alpha_2}v(0)) = (\phi(D_{0^+}^{\alpha_2}v(1)))' = 0, \\ u(1) = \mu_1 \int_0^1 g_1(s)v(s)\,dA_1(s), \qquad v(1) = \mu_2 \int_0^1 g_2(s)u(s)\,dA_2(s), \end{cases} \tag{1}$$

where $\lambda_i > 0$ $(i = 1,2)$ is a parameter, $1 < \beta_i \le 2$, $n-1 < \alpha_1 \le n$, $m-1 < \alpha_2 \le m$, $n,m \ge 2$, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i}$ are the standard Riemann-Liouville derivatives. $\mu_i > 0$ is a constant, $g_i : (0,1) \to [0,+\infty)$ is continuous with $g_i \in L^1(0,1)$, A_i is right continuous on $[0,1)$, left continuous at $t = 1$, and nondecreasing on $[0,1]$, $A_i(0) = 0$, $\int_0^1 x(s)\,dA_i(s)$ denotes the Riemann-Stieltjes integrals of x with respect to A_i, ϕ is a generalized p-Laplacian operator and satisfies the following condition $(\mathbf{H}_0)$.

The positive solution (u,v) of system (1) means that $(u,v) \in C[0,1] \times C[0,1]$, (u,v) satisfies system (1) and $u(t) > 0$, $v(t) > 0$ for all $t \in (0,1]$.

($\mathbf{H}_0$) $\phi : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism, and there exist two increasing homeomorphisms $\psi_1, \psi_2 : (0, +\infty) \to (0, +\infty)$ such that

$$\psi_1(x)\phi(y) \leq \phi(xy) \leq \psi_2(x)\phi(y), \quad x, y > 0.$$

Moreover, $\phi, \phi^{-1} \in C^1(\mathbb{R})$, where ϕ^{-1} denotes the inverse of ϕ and $\mathbb{R} = (-\infty, +\infty)$.

Lemma 1.1 ([1]) *Assume that* ($\mathbf{H}_0$) *holds. Then*

$$\psi_2^{-1}(x)y \leq \phi^{-1}\bigl(x\phi(y)\bigr) \leq \psi_1^{-1}(x)y, \quad x, y > 0.$$

For ϕ satisfying ($\mathbf{H}_0$), we call it a generalized p-Laplacian operator, it contains two important special cases: $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$ ($p > 1$) (see [1]). Many researchers have studied the existence of positive solutions for two above cases due to their great application background (see [2–15]). Combined with the fractional calculus, the application of the above two kinds of special circumstances becomes more extensive and practical. For the sake of considering the turbulent flow in a porous medium, the governing equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u^m}{\partial x}\left|\frac{\partial u^m}{\partial x}\right|^{p-1}\right) = g\left(t, u, \frac{\partial u}{\partial t}\right), \quad m \leq 2, \frac{1}{2} \leq p \leq 1 \tag{2}$$

was presented by Leibenson (see [2]). If $p = 1$, $m > 0$, it is used as a nonlinear model for the dispersion of animals and insects (see [3]).

In [4], Lu et al. studied the existence of positive solution for the fractional boundary value problem with a p-Laplacian operator:

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \qquad D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = 0, \end{cases} \tag{3}$$

where $2 < \alpha \leq 3, 1 < \beta \leq 2, D_{0^+}^{\alpha}, D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1, f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function. By the properties of Green's function and the Guo-Krasnosel'skii fixed point theorem, some results on the existence of positive solutions are obtained.

In [5], Wang et al. investigated the same equation as (3) for $1 < \alpha \leq 2$, $0 < \beta \leq 1$, with boundary value condition $u(0) = 0$, $D_{0^+}^{\alpha}u(0) = 0$, $u(1) = au(\xi)$, where $0 \leq a \leq 1$, $0 < \xi < 1$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function. Through the application of the Guo-Krasnosel'skii fixed point theorem and the Leggett-Williams theorem, sufficient conditions for the existence of positive solutions are received.

In system (1), $\int_0^1 g_1(s)v(s)\,dA_1(s)$, $\int_0^1 g_2(s)u(s)\,dA_2(s)$ denote the Riemann-Stieltjes integrals, and A_i is a function of bounded variation, which implies that dA_i can be a signed measure. Then, a multipoint boundary value problem and an integral boundary value problem are included in our study, that is to say, system (1) includes more generalized boundary value conditions. Henderson and Luca in [16] considered the following system:

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ D_{0^+}^{\alpha_2} v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad u(1) = \sum_{i=1}^{p} a_i u(\xi_i), \\ v(0) = v'(0) = \cdots v^{(m-2)} = 0, \qquad v(1) = \sum_{i=1}^{q} b_i v(\eta_i), \end{cases} \tag{4}$$

where $\lambda_i > 0$ ($i = 1, 2$) is a parameter, $n - 1 < \alpha_1 \leq n$, $m - 1 < \alpha_2 \leq m$, $n, m \geq 2$, $D_{0^+}^{\alpha_i}$, $D_{0^+}^{\beta_i}$ are the standard Riemann-Liouville derivatives. $a_i > 0$, $b_i > 0$ are constants, $f_i : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function. By the Guo-Krasnosel'skii fixed point theorem, the authors in [16] got the existence of positive solutions on system (4). System (4) with uncoupled and coupled multi-point boundary value conditions

$$\begin{cases} u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad u(1) = \mu_1 \int_0^1 u(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots v^{(m-2)} = 0, \qquad v(1) = \mu_2 \int_0^1 v(s)\, dA_2(s), \end{cases}$$

$$\begin{cases} u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad u(1) = \mu_1 \int_0^1 v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots v^{(m-2)} = 0, \qquad v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s) \end{cases}$$

has been studied in many papers, where $\mu_i > 0$ is a constant, for $\mu_i = 1$ as an exceptional case (see [17–23] and the references therein). However, these articles only study the existence of positive solutions for the system, and do not relate to the nonexistence of positive solutions.

Up to now, coupled boundary value conditions for a fractional differential system with generalized p-Laplacian like system (1) have seldom been considered when λ_1, λ_2 are different. Motivated by the results mentioned above, in this paper, we obtain several new existence and nonexistence results for positive solutions in terms of different values of the parameter λ_i by using the properties of Green's function and the Guo-Krasnosel'skii fixed point theorem on cone. Especially, paying attention to the nonlinear operator $D_{0^+}^{\beta}(\phi(D_{0^+}^{\alpha}))$ with the discussion in (1), we can convert it to the linear operator $D_{0^+}^{\beta} D_{0^+}^{\alpha}$, if $\phi(u) = u$, and the additive index law

$$D_{0^+}^{\beta} D_{0^+}^{\alpha} u(t) = D_{0^+}^{\alpha+\beta} u(t)$$

holds under some reasonable constraints on the function u (see [24]). Therefore, our article promotes, includes and improves the previous results in a certain degree.

2 Preliminaries and lemmas

For convenience of the reader, we present some necessary definitions about fractional calculus theory.

Definition 2.1 ([24, 25]) Let $\alpha > 0$ and u be piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$. Then, for $t > 0$, we call

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s)\, ds,$$

the Riemann-Liouville fractional integral of u of order α.

Definition 2.2 ([24, 25]) The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 \le \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $\mathbb{N}$ denotes the natural number set, the function $u(t)$ is n times continuously differentiable on $[0, +\infty)$.

Lemma 2.1 ([24, 25]) *Let $\alpha > 0$, if the fractional derivatives $D_{0^+}^{\alpha-1} u(t)$ and $D_{0^+}^{\alpha} u(t)$ are continuous on $[0, +\infty)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty)$, n is the smallest integer greater than or equal to α.

Similarly to the proof in [22], it enables us to obtain the following Lemmas 2.2, 2.3 and Remark 2.1.

Lemma 2.2 *Assume that the following condition* $(\mathbf{H}_1)$ *holds.*

$(\mathbf{H}_1)$

$$k_1 = \int_0^1 g_1(t) t^{\alpha_2 - 1}\, dA_1(t) > 0, \qquad k_2 = \int_0^1 g_2(t) t^{\alpha_1 - 1}\, dA_2(t) > 0,$$

$$1 - \mu_1 \mu_2 k_1 k_2 > 0.$$

Let $h_i \in C(0,1) \cap L(0,1)$ $(i = 1,2)$, then the system with the coupled boundary conditions

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) + h_1(t) = 0, \qquad D_{0^+}^{\alpha_2} v(t) + h_2(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad u(1) = \mu_1 \int_0^1 g_1(s) v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots v^{(n-2)} = 0, \qquad v(1) = \mu_2 \int_0^1 g_2(s) u(s)\, dA_2(s) \end{cases} \tag{5}$$

has a unique integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,s) h_1(s)\, ds + \int_0^1 H_1(t,s) h_2(s)\, ds, \\ v(t) = \int_0^1 K_2(t,s) h_2(s)\, ds + \int_0^1 H_2(t,s) h_1(s)\, ds, \end{cases} \tag{6}$$

where

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1 \mu_2 k_1 t^{\alpha_1 - 1}}{1 - \mu_1 \mu_2 k_1 k_2} \int_0^1 g_2(t) \overline{G}_1(t,s)\, dA_2(t) + \overline{G}_1(t,s), \\ H_1(t,s) &= \frac{\mu_1 t^{\alpha_1 - 1}}{1 - \mu_1 \mu_2 k_1 k_2} \int_0^1 g_1(t) \overline{G}_2(t,s)\, dA_1(t), \\ K_2(t,s) &= \frac{\mu_2 \mu_1 k_2 t^{\alpha_2 - 1}}{1 - \mu_1 \mu_2 k_1 k_2} \int_0^1 g_1(t) \overline{G}_2(t,s)\, dA_1(t) + \overline{G}_2(t,s), \\ H_2(t,s) &= \frac{\mu_2 t^{\alpha_2 - 1}}{1 - \mu_1 \mu_2 k_1 k_2} \int_0^1 g_2(t) \overline{G}_1(t,s)\, dA_2(t), \end{aligned} \tag{7}$$

and

$$\overline{G}_i(t,s) = \frac{1}{\Gamma(\alpha_i)} \begin{cases} [t(1-s)]^{\alpha_i-1} - (t-s)^{\alpha_i-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha_i-1}, & 0 \le t \le s \le 1, \end{cases} \quad i = 1,2.$$

Lemma 2.3 *For $t, s \in [0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$ $(i = 1,2)$ defined as* (7) *satisfy*

$$\begin{aligned} &K_1(t,s), H_2(t,s) \le \rho s(1-s)^{\alpha_1-1}, \qquad K_2(t,s), H_1(t,s) \le \rho s(1-s)^{\alpha_2-1}, \\ &K_1(t,s), H_1(t,s) \le \rho t^{\alpha_1-1}, \qquad K_2(t,s), H_2(t,s) \le \rho t^{\alpha_2-1}, \\ &K_1(t,s) \ge \varrho t^{\alpha_1-1} s(1-s)^{\alpha_1-1}, \qquad H_2(t,s) \ge \varrho t^{\alpha_2-1} s(1-s)^{\alpha_1-1}, \\ &K_2(t,s) \ge \varrho t^{\alpha_2-1} s(1-s)^{\alpha_2-1}, \qquad H_1(t,s) \ge \varrho t^{\alpha_1-1} s(1-s)^{\alpha_2-1}, \end{aligned}$$

where

$$\begin{aligned} \rho = \max\Bigg\{ &\frac{1}{\Gamma(\alpha_1-1)} \left(\frac{\mu_1\mu_2 k_1}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 g_2(t)\, dA_2(t) + 1 \right), \\ &\frac{\mu_1}{\Gamma(\alpha_2-1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_1(t)\, dA_1(t), \\ &\frac{1}{\Gamma(\alpha_2-1)} \left(\frac{\mu_2\mu_1 k_2}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 g_1(t)\, dA_1(t) + 1 \right), \\ &\frac{\mu_2}{\Gamma(\alpha_1-1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_2(t)\, dA_2(t) \Bigg\}, \\ \varrho = \min\Bigg\{ &\frac{\mu_1\mu_2 k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_2(t)(1-t)t^{\alpha_1-1}\, dA_2(t), \\ &\frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_1(t)(1-t)t^{\alpha_2-1}\, dA_1(t), \\ &\frac{\mu_2\mu_1 k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_1(t)(1-t)t^{\alpha_2-1}\, dA_1(t), \\ &\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 g_2(t)(1-t)t^{\alpha_1-1}\, dA_2(t) \Bigg\}. \end{aligned}$$

Remark 2.1 From Lemma 2.3, for $t, \widetilde{t}, s \in [0,1]$, we have

$$\begin{aligned} &K_i(t,s) \ge \omega t^{\alpha_i-1} K_i(\widetilde{t},s), \qquad H_i(t,s) \ge \omega t^{\alpha_i-1} H_i(\widetilde{t},s), \quad i = 1,2, \\ &K_1(t,s) \ge \omega t^{\alpha_1-1} H_2(\widetilde{t},s), \qquad H_2(t,s) \ge \omega t^{\alpha_2-1} K_1(\widetilde{t},s), \\ &K_2(t,s) \ge \omega t^{\alpha_2-1} H_1(\widetilde{t},s), \qquad H_1(t,s) \ge \omega t^{\alpha_1-1} K_2(\widetilde{t},s), \end{aligned}$$

where $\omega = \frac{\varrho}{\rho}$, ϱ, ρ are defined as in Lemma 2.3, $0 < \omega < 1$.

From Lemmas 2.1 and 2.2, we obtain the following Lemma 2.4.

Lemma 2.4 *Let $1 < \beta_i \le 2$, $n-1 < \alpha_1 \le n$, $m-1 < \alpha_2 \le m$, $h_i \in C(0,1) \cap L(0,1)$ $(i = 1,2)$, the following system of fractional differential equations*

$$\begin{cases} D_{0^+}^{\beta_1}(\phi(D_{0^+}^{\alpha_1}u(t))) + \lambda_1 h_1(t) = 0, \qquad D_{0^+}^{\beta_2}(\phi(D_{0^+}^{\alpha_2}v(t))) + \lambda_2 h_2(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots u^{(n-2)} = 0, \qquad \phi(D_{0^+}^{\alpha_1}u(0)) = (\phi(D_{0^+}^{\alpha_1}u(1)))' = 0, \\ v(0) = v'(0) = \cdots v^{(m-2)} = 0, \qquad \phi(D_{0^+}^{\alpha_2}v(0)) = (\phi(D_{0^+}^{\alpha_2}v(1)))' = 0, \\ u(1) = \mu_1 \int_0^1 v(s)\,dA_1(s), \qquad v(1) = \mu_2 \int_0^1 u(s)\,dA_2(s) \end{cases}$$

has a unique integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)\phi^{-1}(\lambda_1 \int_0^1 G_1(s,\tau)h_1(\tau)\,d\tau)\,ds \\ \qquad + \int_0^1 H_1(t,s)\phi^{-1}(\lambda_2 \int_0^1 G_2(s,\tau)h_2(\tau)\,d\tau)\,ds, \\ v(t) = \int_0^1 K_2(t,s)\phi^{-1}(\lambda_2 \int_0^1 G_2(s,\tau)h_2(\tau)\,d\tau)\,ds \\ \qquad + \int_0^1 H_2(t,s)\phi^{-1}(\lambda_1 \int_0^1 G_1(s,\tau)h_1(\tau)\,d\tau)\,ds, \end{cases}$$

where

$$G_i(s,\tau) = \frac{1}{\Gamma(\beta_i)} \begin{cases} s[s(1-\tau)]^{\beta_i-2} - (s-\tau)^{\beta_i-1}, & 0 \le \tau \le s \le 1, \\ s[s(1-\tau)]^{\beta_i-2}, & 0 \le s \le \tau \le 1, \end{cases} \quad i = 1,2. \tag{8}$$

Lemma 2.5 ([26]) *The function $G_i(s,\tau)$ defined as* (8) *is continuous on* $[0,1] \times [0,1]$, *and for $s,\tau \in [0,1]$, $G_i(s,\tau)$ satisfies*

$$(1) \quad G_i(s,\tau) \ge 0; \qquad (2) \quad G_i(s,\tau) \le G_i(\tau,\tau); \qquad (3) \quad G_i(s,\tau) \ge s^{\beta_i-1}G_i(1,\tau).$$

In the rest of the paper, we always suppose that the following assumption holds:

($\mathbf{H}_2$) *$f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous.*

Let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}, \qquad \|u\| = \max_{t\in[0,1]}|u(t)|, \qquad \|v\| = \max_{t\in[0,1]}|v(t)|.$$

Denote

$$K = \{(u,v) \in X : u(t) \ge \omega t^{\alpha_1-1}\|(u,v)\|, v(t) \ge \omega t^{\alpha_2-1}\|(u,v)\|, t \in [0,1]\},$$

where ω is defined as Remark 2.1. It is easy to see that K is a positive cone in X. Under the above conditions ($\mathbf{H}_0$)($\mathbf{H}_1$)($\mathbf{H}_2$), for any $(u,v) \in K$, we can define an integral operator $T : K \to X$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad 0 \le t \le 1, \tag{9}$$

$$
\begin{aligned}
T_1(u,v)(t) &= \int_0^1 K_1(t,s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 H_1(t,s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds, \\
T_2(u,v)(t) &= \int_0^1 K_2(t,s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 H_2(t,s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds,
\end{aligned}
\tag{10}
$$

we know that (u,v) is a positive solution of system (1) if and only if (u,v) is a fixed point of T in K.

Lemma 2.6 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold. Then* $T: K \to K$ *is a completely continuous operator.*

Proof By the routine discussion, we know that $T: K \to X$ is well defined, so we only prove $T(K) \subseteq K$. For any $(u,v) \in K$, $0 \le t, \widetilde{t} \le 1$, by Remark 2.1, we have

$$
\begin{aligned}
T_1(u,v)(t) &= \int_0^1 K_1(t,s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 H_1(t,s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\ge \int_0^1 \omega t^{\alpha_1-1}K_1(\widetilde{t},s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 \omega t^{\alpha_1-1}H_1(\widetilde{t},s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\ge \omega t^{\alpha_1-1}\left(\int_0^1 K_1(\widetilde{t},s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\right. \\
&\quad \left. + \int_0^1 H_1(\widetilde{t},s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\right) \\
&\ge \omega t^{\alpha_1-1}T_1(u,v)(\widetilde{t}).
\end{aligned}
\tag{11}
$$

On the other hand,

$$
\begin{aligned}
T_1(u,v)(t) &\ge \int_0^1 \omega t^{\alpha_1-1}H_2(\widetilde{t},s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 \omega t^{\alpha_1-1}K_2(\widetilde{t},s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds \\
&\ge \omega t^{\alpha_1-1}\left(\int_0^1 H_2(\widetilde{t},s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\right. \\
&\quad \left. + \int_0^1 K_2(\widetilde{t},s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\right) \\
&\ge \omega t^{\alpha_1-1}T_2(u,v)(\widetilde{t}).
\end{aligned}
\tag{12}
$$

Then we have

$$T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\|T_1(u,v)\|, \qquad T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\|T_2(u,v)\|,$$

i.e.,

$$T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\|(T_1(u,v),T_2(u,v))\|.$$

In the same way as (11) and (12), we can prove that

$$T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\|(T_1(u,v),T_2(u,v))\|.$$

Therefore, we have $T(K) \subseteq K$.

According to the Ascoli-Arzela theorem, we can easily get that $T : K \to K$ is completely continuous. The proof is completed. □

In order to obtain the existence of the positive solutions of system (1), we will use the following cone compression and expansion fixed point theorem.

Lemma 2.7 ([27]) *Let P be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : P \cap \overline{\Omega}_2 \backslash \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied*:

$$\|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2,$$

or

$$\|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2,$$

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

Denote

$$f_{10} = \liminf_{x\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(x)}, \qquad f_1^0 = \limsup_{x\to 0^+} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(x)},$$

$$f_{20} = \liminf_{y\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(y)}, \qquad f_2^0 = \limsup_{y\to 0^+} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(y)},$$

$$f_{1\infty} = \liminf_{x\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(x)}, \qquad f_1^\infty = \limsup_{x\to +\infty} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(x)},$$

$$f_{2\infty} = \liminf_{y\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(y)}, \qquad f_2^\infty = \limsup_{y\to +\infty} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(y)},$$

$$L_1 = \max\left\{2\rho\varphi_1^{-1}\left(\int_0^1 G_1(\tau,\tau)\,d\tau\right), 2\rho\varphi_1^{-1}\left(\int_0^1 G_2(\tau,\tau)\,d\tau\right)\right\},$$

$$L_2=\min\biggl\{2\varrho\omega\theta^2\int_0^1\varphi_2^{-1}\bigl(s^{\beta_1-1}\bigr)s^{\beta_1-1}\varphi_2^{-1}\biggl(\int_0^1 G_1(1,\tau)\,d\tau\biggr)\,ds,$$
$$2\varrho\omega\theta^2\int_0^1\varphi_2^{-1}\bigl(s^{\beta_2-1}\bigr)s^{\beta_2-1}\varphi_2^{-1}\biggl(\int_0^1 G_2(1,\tau)\,d\tau\biggr)\,ds\biggr\},$$
$$L_3=\min\biggl\{2\varrho\theta\int_0^1\varphi_2^{-1}\bigl(s^{\beta_1-1}\bigr)s(1-s)^{\alpha_1-1}\varphi_2^{-1}\biggl(\int_0^1 G_1(1,\tau)\,d\tau\biggr)\,ds,$$
$$2\varrho\theta\int_0^1\varphi_2^{-1}\bigl(s^{\beta_2-1}\bigr)s(1-s)^{\alpha_2-1}\varphi_2^{-1}\biggl(\int_0^1 G_2(1,\tau)\,d\tau\biggr)\,ds\biggr\},\quad \theta=\min_{t\in[a,b]}\bigl\{t^{\alpha_1-1},t^{\alpha_2-1}\bigr\}.$$

3.1 Existence of system (1)

Theorem 3.1 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{i\infty}\varphi_1(L_1^{-1})>f_i^0\varphi_2(L_2^{-1})$, *then system* (1) *has at least one positive solution for*

$$\lambda_i\in\biggl(\frac{\varphi_2(L_2^{-1})}{f_{i\infty}},\frac{\varphi_1(L_1^{-1})}{f_i^0}\biggr),\tag{13}$$

where we impose $\frac{1}{f_{i\infty}}=0$ *if* $f_{i\infty}=+\infty$ *and* $\frac{1}{f_i^0}=+\infty$ *if* $f_i^0=0$ $(i=1,2)$.

Proof For any λ_i satisfying (13), there exists $\varepsilon_0>0$ such that

$$\frac{\varphi_2(L_2^{-1})}{f_{i\infty}-\varepsilon_0}\le\lambda_i\le\frac{\varphi_1(L_1^{-1})}{f_i^0+\varepsilon_0}.\tag{14}$$

By the definition of f_i^0, there exists $r_1>0$ such that

$$f_i(t,x,y)\le\bigl(f_i^0+\varepsilon_0\bigr)\max\bigl\{\phi(x),\phi(y)\bigr\},\quad 0\le x,y\le r_1,t\in[0,1].\tag{15}$$

Let $K_{r_1}=\{(u,v)\in K:\|(u,v)\|<r_1\}$. For any $(u,v)\in\partial K_{r_1}$, $t\in[0,1]$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned}&u(t)\le\bigl|u(t)\bigr|\le\|u\|\le\bigl\|(u,v)\bigr\|\le r_1,\\&v(t)\le\bigl|v(t)\bigr|\le\|v\|\le\bigl\|(u,v)\bigr\|\le r_1,\quad t\in[0,1].\end{aligned}\tag{16}$$

Thus, for any $(u,v)\in\partial K_{r_1}$, by (15), (16) and $(\mathbf{H}_0)$, we have

$$f_i\bigl(t,u(t),v(t)\bigr)\le\bigl(f_i^0+\varepsilon_0\bigr)\phi(r_1),\quad t\in[0,1].\tag{17}$$

Hence, for any $(u,v)\in\partial K_{r_1}$, by Lemmas 1.1, 2.3, 2.5 and (17), we conclude that

$$\begin{aligned}\bigl\|T_1(u,v)(t)\bigr\|&=\max_{t\in[0,1]}\biggl|\int_0^1 K_1(t,s)\phi^{-1}\biggl(\lambda_1\int_0^1 G_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\\&\quad+\int_0^1 H_1(t,s)\phi^{-1}\biggl(\lambda_2\int_0^1 G_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\biggr)\,ds\biggr|\\&\le\max_{t\in[0,1]}\biggl|\int_0^1\rho t^{\alpha_1-1}\phi^{-1}\biggl(\lambda_1\int_0^1 G_1(\tau,\tau)\bigl(f_1^0+\varepsilon_0\bigr)\phi(r_1)\,d\tau\biggr)\,ds\\&\quad+\int_0^1\rho t^{\alpha_1-1}\phi^{-1}\biggl(\lambda_2\int_0^1 G_2(\tau,\tau)\bigl(f_2^0+\varepsilon_0\bigr)\phi(r_1)\,d\tau\biggr)\,ds\biggr|\end{aligned}$$

$$\begin{aligned}
&\le \rho r_1\varphi_1^{-1}\big(\lambda_1\big(f_1^0+\varepsilon_0\big)\big)\varphi_1^{-1}\left(\int_0^1 G_1(\tau,\tau)\,d\tau\right)\\
&\quad+\rho r_1\varphi_1^{-1}\big(\lambda_2\big(f_2^0+\varepsilon_0\big)\big)\varphi_1^{-1}\left(\int_0^1 G_2(\tau,\tau)\,d\tau\right)\\
&\le r_1\varphi_1^{-1}\big(\lambda_1\big(f_1^0+\varepsilon_0\big)\big)\frac{L_1}{2}+r_1\varphi_1^{-1}\big(\lambda_2\big(f_2^0+\varepsilon_0\big)\big)\frac{L_1}{2}\\
&\le r_1=\big\|(u,v)\big\|.
\end{aligned}\tag{18}$$

Similarly to (18), for any $(u,v)\in\partial K_{r_1}$, we also have

$$\big\|T_2(u,v)\big\|\le r_1=\big\|(u,v)\big\|.$$

Consequently, we have

$$\big\|T(u,v)\big\|=\max\big\{\big\|T_1(u,v)\big\|,\big\|T_2(u,v)\big\|\big\}\le r_1=\big\|(u,v)\big\|,\quad (u,v)\in\partial K_{r_1}.\tag{19}$$

On the other hand, by the definition of $f_{i\infty}$, there exist $r_1',r_2'>0$ such that

$$\begin{aligned}
&f_1(t,x,y)\ge(f_{1\infty}-\varepsilon_0)\phi(x),\quad x\ge r',y\ge 0,t\in[a,b]\subset(0,1),\\
&f_2(t,x,y)\ge(f_{2\infty}-\varepsilon_0)\phi(y),\quad y\ge r',x\ge 0,t\in[a,b]\subset(0,1).
\end{aligned}\tag{20}$$

Choose $r_2=\max\{\frac{r_1'}{\omega\theta},\frac{r_2'}{\omega\theta},2r_1\}$. Let $K_{r_2}=\{(u,v)\in K:\|(u,v)\|<r_2\}$. For any $(u,v)\in\partial K_{r_2}$, by the definition of $\|\cdot\|$, we have

$$\begin{aligned}
&u(t)\ge\omega t^{\alpha_1-1}\big\|(u,v)\big\|\ge\omega\theta r_2\ge r_1',\\
&v(t)\ge\omega t^{\alpha_2-1}\big\|(u,v)\big\|\ge\omega\theta r_2\ge r_2',\quad t\in[a,b]\subset(0,1).
\end{aligned}\tag{21}$$

Thus, for any $(u,v)\in\partial K_{r_2}$, by (20), (21) and $(\mathbf{H}_0)$, we have

$$\begin{aligned}
&f_1\big(t,u(t),v(t)\big)\ge(f_{1\infty}-\varepsilon_0)\phi\big(u(t)\big)\ge(f_{1\infty}-\varepsilon_0)\phi(\omega\theta r_2),\quad t\in[a,b]\subset(0,1),\\
&f_2\big(t,u(t),v(t)\big)\ge(f_{2\infty}-\varepsilon_0)\phi\big(v(t)\big)\ge(f_{2\infty}-\varepsilon_0)\phi(\omega\theta r_2),\quad t\in[a,b]\subset(0,1).
\end{aligned}\tag{22}$$

Hence, for any $(u,v)\in\partial K_{r_2}$, by Lemmas 1.1, 2.3, 2.5 and (22), we have

$$\begin{aligned}
&\big\|T_1(u,v)(t)\big\|\\
&\quad\ge\min_{t\in[a,b]}\bigg|\int_0^1\varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1}\phi^{-1}\left(\lambda_1\int_0^1 s^{\beta_1-1}G_1(1,\tau)(f_{1\infty}-\varepsilon_0)\phi(\omega\theta r_2)\,d\tau\right)ds\\
&\quad+\int_0^1\varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1}\phi^{-1}\left(\lambda_2\int_0^1 s^{\beta_2-1}G_2(1,\tau)(f_{2\infty}-\varepsilon_0)\phi(\omega\theta r_2)\,d\tau\right)ds\bigg|\\
&\quad\ge\varrho\omega\theta^2 r_2\int_0^1\varphi_2^{-1}\big(s^{\beta_1-1}\big)s^{\beta_1-1}\varphi_2^{-1}\left(\int_0^1 G_1(1,\tau)\,d\tau\right)ds\,\psi_2^{-1}\big(\lambda_1(f_{1\infty}-\varepsilon_0)\big)\\
&\quad+\varrho\omega\theta^2 r_2\int_0^1\varphi_2^{-1}\big(s^{\beta_2-1}\big)s^{\beta_2-1}\varphi_2^{-1}\left(\int_0^1 G_2(1,\tau)\,d\tau\right)ds\,\psi_2^{-1}\big(\lambda_2(f_{2\infty}-\varepsilon_0)\big)\\
&\quad\ge r_2\varphi_2^{-1}\big(\lambda_1(f_{1\infty}-\varepsilon_0)\big)\frac{L_2}{2}+r_2\varphi_2^{-1}\big(\lambda_2(f_{2\infty}-\varepsilon_0)\big)\frac{L_2}{2}\\
&\quad\ge r_2=\big\|(u,v)\big\|.
\end{aligned}\tag{23}$$

Therefore, we obtain

$$\|T(u,v)\| = \max\{\|T_1(u,v)\|, \|T_2(u,v)\|\} \geq r_2 = \|(u,v)\| \quad \text{for any } (u,v) \in \partial K_{r_2}. \tag{24}$$

It follows from the above discussion, (18), (24), Lemmas 2.6 and 2.7 that, for any $\lambda_i \in (\frac{\varphi_2(L_2^{-1})}{f_{i\infty}}, \frac{\varphi_1(L_1^{-1})}{f_i^0})$, T has a fixed point $(u,v) \in \overline{K}_{r_2} \setminus K_{r_1}$, so system (1) has at least one positive solution (u,v); moreover, (u,v) satisfies $r_1 \leq \|(u,v)\| \leq r_2$. The proof is completed. □

Remark 3.1 From the proof of Theorem 3.1, if we choose

$$\overline{L}_2 = \varrho\omega\theta^2 \int_0^1 \varphi_2^{-1}(s^{\beta_1-1})s^{\beta_1-1}\varphi_2^{-1}\left(\int_0^1 G_1(1,\tau)\,d\tau\right)ds, \quad \theta = \min_{t\in[a,b]}\{t^{\alpha_1-1}, t^{\alpha_2-1}\}, \tag{25}$$

then for $\lambda_1 \in (\frac{\varphi_2(\overline{L}_2^{-1})}{f_{1\infty}}, \frac{\varphi_1(L_1^{-1})}{f_1^0})$, $\lambda_2 \in (0, \frac{\varphi_1(L_1^{-1})}{f_2^0})$, the conclusion of Theorem 3.1 is valid.
Or we choose

$$\widetilde{L}_2 = \varrho\omega\theta^2 \int_0^1 \varphi_2^{-1}(s^{\beta_2-1})s^{\beta_2-1}\varphi_2^{-1}\left(\int_0^1 G_2(1,\tau)\,d\tau\right)ds, \quad \theta = \min_{t\in[a,b]}\{t^{\alpha_1-1}, t^{\alpha_2-1}\}, \tag{26}$$

then, for $\lambda_1 \in (0, \frac{\varphi_1(L_1^{-1})}{f_1^0})$, $\lambda_2 \in (\frac{\varphi_2(\widetilde{L}_2^{-1})}{f_{2\infty}}, \frac{\varphi_1(L_1^{-1})}{f_2^0})$, the conclusion of Theorem 3.1 is valid.

Theorem 3.2 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{i0}\varphi_1(L_1^{-1}) > f_i^\infty\varphi_2(L_2^{-1})$, *then system* (1) *has at least one positive solution for*

$$\lambda_i \in \left(\frac{\varphi_2(L_2^{-1})}{f_{i0}}, \frac{\varphi_1(L_1^{-1})}{f_i^\infty}\right),$$

where we impose $\frac{1}{f_{i0}} = 0$ *if* $f_{i0} = +\infty$ *and* $\frac{1}{f_i^\infty} = +\infty$ *if* $f_i^\infty = 0$, $i = 1, 2$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and so we omit it.

Remark 3.2 Similar to Remark 3.1, if we choose $\overline{L}_2$ as (25), then for $\lambda_1 \in (\frac{\varphi_2(\overline{L}_2^{-1})}{f_{10}}, \frac{\varphi_1(L_1^{-1})}{f_1^\infty})$, $\lambda_2 \in (0, \frac{\varphi_1(L_1^{-1})}{f_2^\infty})$, the conclusion of Theorem 3.2 is valid.
Or we choose $\widetilde{L}_2$ as (26), then for $\lambda_1 \in (0, \frac{\varphi_1(L_1^{-1})}{f_1^\infty})$, $\lambda_2 \in (\frac{\varphi_2(\widetilde{L}_2^{-1})}{f_{20}}, \frac{\varphi_1(L_1^{-1})}{f_2^\infty})$, the conclusion of Theorem 3.2 is valid.

Theorem 3.3 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and there exist* $R > r > 0$ *such that*

$$\lambda_i \min_{\substack{t\in[a,b]\subset(0,1)\\ \omega\theta r\leq x,y\leq r}} f_i(t,x,y) \geq \phi\left(\frac{r}{L_3}\right), \qquad \lambda_i \max_{\substack{t\in[0,1]\\ 0\leq x,y\leq R}} f_i(t,x,y) \leq \phi\left(\frac{R}{L_1}\right), \quad i = 1, 2. \tag{27}$$

Then system (1) *has at least one positive solution* (u,v); *moreover,* (u,v) *satisfies* $r \leq \|(u,v)\| \leq R$.

Proof Set $K_r = \{(u,v) \in K : \|(u,v)\| < r\}$. For any $(u,v) \in \partial K_r$, by the definition of $\|\cdot\|$, we have

$$\omega\theta r \leq \omega t^{\alpha_1-1} r = \omega t^{\alpha_1-1}\|(u,v)\| \leq u(t) \leq r,$$
$$\omega t^{\alpha_2-1} r = \omega t^{\alpha_2-1}\|(u,v)\| \leq v(t) \leq r, \quad t \in [a,b] \subset (0,1).$$

Thus, for any $(u,v) \in \partial K_r$, by the first inequality of (27), we have

$$\lambda_i \min_{\substack{t\in[a,b]\subset(0,1) \\ \omega\theta r\le u(t),v(t)\le r}} f_i\bigl(t,u(t),v(t)\bigr) \ge \phi\left(\frac{r}{L_3}\right), \quad i=1,2. \tag{28}$$

Hence, for any $(u,v) \in \partial K_r$, by Lemmas 1.1, 2.3, 2.5 and (28), we have

$$\begin{aligned}
&\bigl\|T_1(u,v)(t)\bigr\| \\
&\ge \min_{t\in[a,b]}\biggl|\int_0^1 \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1}\phi^{-1}\left(\lambda_1\int_0^1 s^{\beta_1-1}G_1(1,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 \varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1}\phi^{-1}\left(\lambda_2\int_0^1 s^{\beta_2-1}G_2(1,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds\biggr| \\
&\ge \varrho\theta\int_0^1 s(1-s)^{\alpha_1-1}\phi^{-1}\left(\lambda_1\int_0^1 s^{\beta_1-1}G_1(1,\tau)\min_{\substack{\tau\in[a,b]\subset(0,1) \\ \omega r\le u(\tau),v(\tau)\le r}} f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \varrho\theta\int_0^1 s(1-s)^{\alpha_2-1}\phi^{-1}\left(\lambda_2\int_0^1 s^{\beta_2-1}G_2(1,\tau)\min_{\substack{\tau\in[a,b]\subset(0,1) \\ \omega r\le u(\tau),v(\tau)\le r}} f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\ge \phi^{-1}\Bigl(\lambda_1 \min_{\substack{\tau\in[a,b]\subset(0,1) \\ \omega r\le u(\tau),v(\tau)\le r}} f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\Bigr)\frac{L_3}{2} \\
&\quad + \phi^{-1}\Bigl(\lambda_2 \min_{\substack{\tau\in[a,b]\subset(0,1) \\ \omega r\le u(\tau),v(\tau)\le r}} f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\Bigr)\frac{L_3}{2} \\
&\ge r = \bigl\|(u,v)\bigr\|.
\end{aligned} \tag{29}$$

Therefore, we obtain

$$\bigl\|T(u,v)\bigr\| = \max\bigl\{\bigl\|T_1(u,v)\bigr\|, \bigl\|T_2(u,v)\bigr\|\bigr\} \ge r = \bigl\|(u,v)\bigr\| \quad \text{for any } (u,v)\in\partial K_r. \tag{30}$$

Choose $K_R = \{(u,v)\in K : \|(u,v)\| < R\}$. For any $(u,v)\in\partial K_R$, $t\in[0,1]$, by the definition of $\|\cdot\|$, we know that

$$\begin{aligned}
&u(t) \le \bigl|u(t)\bigr| \le \|u\| \le \bigl\|(u,v)\bigr\| \le R, \\
&v(t) \le \bigl|v(t)\bigr| \le \|v\| \le \bigl\|(u,v)\bigr\| \le R, \quad t\in[0,1].
\end{aligned} \tag{31}$$

Thus, for any $(u,v)\in\partial K_R$, by the first inequality of (27) and (31), we have

$$\lambda_i \max_{\substack{t\in[0,1] \\ 0\le u(t),v(t)\le R}} f_i\bigl(t,u(t),v(t)\bigr) \le \phi\left(\frac{R}{L_1}\right), \quad i=1,2. \tag{32}$$

Hence, for any $(u,v)\in\partial K_R$, by Lemmas 1.1, 2.3, 2.5 and (32), we can gain

$$\begin{aligned}
\bigl\|T_1(u,v)(t)\bigr\| &\le \max_{t\in[0,1]}\biggl|\int_0^1 \rho t^{\alpha_1-1}\phi^{-1}\left(\lambda_1\int_0^1 G_1(\tau,\tau)\max_{\substack{\tau\in[0,1] \\ 0\le u(\tau),v(\tau)\le R}} f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\
&\quad + \int_0^1 \rho t^{\alpha_1-1}\phi^{-1}\left(\lambda_2\int_0^1 G_2(\tau,\tau)\max_{\substack{\tau\in[0,1] \\ 0\le u(\tau),v(\tau)\le R}} f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds\biggr|
\end{aligned}$$

$$\begin{aligned}
&\le \phi^{-1}\Big(\lambda_1 \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_1\big(\tau,u(\tau),v(\tau)\big)\Big)\frac{L_1}{2}\\
&\quad + \phi_1^{-1}\Big(\lambda_2 \max_{\substack{\tau\in[0,1]\\ 0\le u(\tau),v(\tau)\le R}} f_i\big(\tau,u(\tau),v(\tau)\big)\Big)\frac{L_1}{2}\\
&\le R = \big\|(u,v)\big\|.
\end{aligned}\tag{33}$$

Similarly to (33), for any $(u,v)\in\partial K_R$, we also have

$$\big\|T_2(u,v)\big\| < R = \big\|(u,v)\big\|.$$

Consequently, we have

$$\big\|T(u,v)\big\| = \max\big\{\big\|T_1(u,v)\big\|,\big\|T_2(u,v)\big\|\big\} < R = \big\|(u,v)\big\|,\quad (u,v)\in\partial K_R. \tag{34}$$

It follows from the above discussion, (30), (34), Lemmas 2.6 and 2.7 that T has a fixed point $(u,v)\in\overline{K}_R\setminus K_r$, so system (1) has at least one positive solution (u,v); moreover, (u,v) satisfies $r\le\|(u,v)\|\le R$. The proof is completed. □

Remark 3.3 From the proof of Theorem 3.3, if we choose

$$\begin{aligned}
&\overline{L}_3 = \varrho\theta\int_0^1 \varphi_2^{-1}\big(s^{\beta_1-1}\big)s(1-s)^{\alpha_1-1}\varphi_2^{-1}\bigg(\int_0^1 G_1(1,\tau)\,d\tau\bigg)\,ds,\\
&\quad\theta = \min_{t\in[a,b]}\big\{t^{\alpha_1-1},t^{\alpha_2-1}\big\},
\end{aligned}\tag{35}$$

then for

$$\lambda_1 \min_{\substack{t\in[a,b]\subset(0,1)\\ \omega r\le x,y\le r}} f_1(t,x,y)\ge\phi\bigg(\frac{r}{\overline{L}_3}\bigg),\qquad \lambda_i \max_{\substack{t\in[0,1]\\ 0\le x,y\le R}} f_i(t,x,y)\le\phi\bigg(\frac{R}{L_1}\bigg),\quad i=1,2,$$

the conclusion of Theorem 3.3 is valid.

Or we choose

$$\begin{aligned}
&\widetilde{L}_3 = \varrho\theta\int_0^1 \varphi_2^{-1}\big(s^{\beta_2-1}\big)s(1-s)^{\alpha_2-1}\varphi_2^{-1}\bigg(\int_0^1 G_2(1,\tau)\,d\tau\bigg)\,ds,\\
&\quad\theta = \min_{t\in[a,b]}\big\{t^{\alpha_1-1},t^{\alpha_2-1}\big\}.
\end{aligned}\tag{36}$$

Then, for

$$\lambda_2 \min_{\substack{t\in[a,b]\subset(0,1)\\ \omega r\le x,y\le r}} f_2(t,x,y)\ge\phi\bigg(\frac{r}{\widetilde{L}_3}\bigg),\qquad \lambda_i \max_{\substack{t\in[0,1]\\ 0\le x,y\le R}} f_i(t,x,y)\le\phi\bigg(\frac{R}{L_1}\bigg),\quad i=1,2,$$

the conclusion of Theorem 3.3 is valid.

Theorem 3.4 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{i0}=f_{i\infty}=+\infty$, *then there exists* $\lambda_i^*>0$ *such that system* (1) *has at least two positive solutions for* $\lambda_i\in(0,\lambda_i^*)$, $i=1,2$.

Proof Choose $r > 0$, define

$$\chi_i(r) = \sup_{r>0} \frac{\phi(r)}{\varphi_2(L_1)\max_{\substack{t\in[0,1]\\ 0\le x,y\le r}} f_i(t,x,y)}, \quad i = 1,2.$$

In view of the continuity of f_i and $f_{i0} = f_{i\infty} = +\infty$, we know $\chi_i(r) : (0,+\infty) \to (0,+\infty)$ is continuous and

$$\lim_{t\to 0^+} \chi_i(r) = \lim_{t\to +\infty} \chi_i(r) = 0.$$

So, there exists $r^* \in (0,+\infty)$ such that $\chi_i(r^*) = \sup_{r>0} \chi_i(r) = \lambda_i^*$. Therefore, for $\lambda_i \in (0,\lambda_i^*)$, we can find r_1, r_2 $(0 < r_1 < r^* < r_2 < +\infty)$ satisfying $\chi_i(r_1) = \lambda_1$, $\chi_i(r_2) = \lambda_2$. Thus, by $(\mathbf{H}_0)$, we have

$$\lambda_1 \max_{\substack{t\in[0,1]\\ 0\le x,y\le r_1}} f_i(t,x,y) \le \frac{\phi(r_1)}{\varphi_2(L_1)} \le \phi\left(\frac{r_1}{L_1}\right), \tag{37}$$

$$\lambda_2 \max_{\substack{t\in[0,1]\\ 0\le x,y\le r_2}} f_i(t,x,y) \le \frac{\phi(r_2)}{\varphi_2(L_1)} \le \phi\left(\frac{r_2}{L_1}\right). \tag{38}$$

From the condition $f_{i0} = f_{i\infty} = +\infty$, there exist R_1, R_2 $(0 < R_1 < r_1 < r^* < r_2 < R_2 < +\infty)$ satisfying

$$\frac{f_1(t,x,y)}{\phi(x)} \ge \frac{1}{\lambda_1\varphi_1(\omega\theta)\varphi_1(L_3)}, \quad (x,y) \in [0,R_1] \cup [R_2,+\infty), t \in [a,b] \subset (0,1),$$
$$\frac{f_2(t,x,y)}{\phi(y)} \ge \frac{1}{\lambda_2\varphi_1(\omega\theta)\varphi_1(L_3)}, \quad (x,y) \in [0,R_1] \cup [R_2,+\infty), t \in [a,b] \subset (0,1).$$

Hence, by $(\mathbf{H}_0)$, we get

$$\lambda_i \min_{\substack{t\in[a,b]\subset(0,1)\\ \omega\theta R_1\le x,y\le R_1}} f_i(t,x,y) \ge \phi\left(\frac{R_1}{L_3}\right), \tag{39}$$

$$\lambda_i \min_{\substack{t\in[a,b]\subset(0,1)\\ \omega\theta R_2\le x,y\le R_2}} f_i(t,x,y) \ge \phi\left(\frac{R_1}{L_3}\right). \tag{40}$$

By (37) and (39), (38) and (40), combining with Lemmas 2.6, 2.7 and Theorem 3.3, system (1) has at least two positive solutions for $\lambda_i \in (0,\lambda_i^*)$, $i = 1,2$. The proof is completed. □

Remark 3.4 From the proof of Theorem 3.4, assume that $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ hold, if $f_{i0} = +\infty$ or $f_{i\infty} = +\infty$, then there exists $\lambda_i^* > 0$ such that system (1) has at least one positive solution for $\lambda_i \in (0,\lambda_i^*)$, $i = 1,2$.

3.2 Nonexistence of system (1)

Theorem 3.5 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_i^\infty < +\infty$, $f_i^0 < +\infty$, *then there exists* $\lambda_{i0} > 0$ *such that for* $\lambda_i \in (0,\lambda_{i0})$ $(i = 1,2)$, *system* (1) *has no positive solution.*

Proof From the definitions of f_i^{∞}, f_i^0, which are finite, there exist positive constants M_i^1, M_i^2 and R_1, R_2 ($R_1 < R_2$) such that

$$\begin{aligned} f_i(t,x,y) &\leq M_i^1 \max\{\phi(x),\phi(y)\}, \quad 0 \leq x,y \leq R_1, t \in [0,1], \\ f_i(t,x,y) &\leq M_i^2 \max\{\phi(x),\phi(y)\}, \quad x,y \geq R_2, t \in [0,1]. \end{aligned}$$

Set $M_i^0 = \max\{M_i^1, M_i^2, \max_{t\in[0,1],R_1\leq x,y\leq R_2} \frac{f_i(t,x,y)}{\max\{\phi(x),\phi(y)\}}\}$, we have

$$f_i(t,x,y) \leq M_i^0 \max\{\phi(x),\phi(y)\}, \quad x,y \geq 0, t \in [0,1].$$

Assume that (u,v) is a positive solution of system (1), we will show that this leads to a contradiction. Define $\lambda_{i0} = (M_i^0)^{-1}\varphi_1(L_1^{-1})$, since $\lambda_i \in (0,\lambda_{i0})$, by Lemmas 1.1, 2.3 and 2.5, we conclude that

$$\begin{aligned} u(t) &= T_1(u,v)(t) \\ &= \int_0^1 K_1(t,s)\phi^{-1}\left(\lambda_1 \int_0^1 G_1(s,\tau)f_1\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\quad + \int_0^1 H_1(t,s)\phi^{-1}\left(\lambda_2 \int_0^1 G_2(s,\tau)f_2\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\leq \int_0^1 \rho t^{\alpha_1-1}\phi^{-1}\left(\lambda_1 \int_0^1 G_1(\tau,\tau)M_1^0 \max\bigl\{\phi\bigl(u(\tau)\bigr),\phi\bigl(v(\tau)\bigr)\bigr\}\,d\tau\right)ds \\ &\quad + \int_0^1 \rho t^{\alpha_1-1}\phi^{-1}\left(\lambda_2 \int_0^1 G_2(\tau,\tau)M_2^0 \max\bigl\{\phi\bigl(u(\tau)\bigr),\phi\bigl(v(\tau)\bigr)\bigr\}\,d\tau\right)ds \\ &\leq \rho\bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_1 M_1^0\bigr)\varphi_1^{-1}\left(\int_0^1 G_1(\tau,\tau)\,d\tau\right) \\ &\quad + \rho\bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_2 M_2^0\bigr)\varphi_1^{-1}\left(\int_0^1 G_2(\tau,\tau)\,d\tau\right) \\ &\leq \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_1 M_1^0\bigr)\frac{L_1}{2} + \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_2 M_2^0\bigr)\frac{L_1}{2}. \end{aligned} \tag{41}$$

Therefore, we conclude

$$\begin{aligned} \|u\| &\leq \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_1 M_1^0\bigr)\frac{L_1}{2} + \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_2 M_2^0\bigr)\frac{L_1}{2} \\ &< \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_{10} M_1^0\bigr)\frac{L_1}{2} + \bigl\|(u,v)\bigr\|\varphi_1^{-1}\bigl(\lambda_{20} M_2^0\bigr)\frac{L_1}{2} = \bigl\|(u,v)\bigr\|. \end{aligned} \tag{42}$$

Similarly to (41) (42), we also have

$$\|v\| < \bigl\|(u,v)\bigr\|. \tag{43}$$

Hence, by (42) (43), we get

$$\bigl\|(u,v)\bigr\| = \max\bigl\{\|u\|,\|v\|\bigr\} < \bigl\|(u,v)\bigr\|, \tag{44}$$

which is a contradiction. Therefore, system (1) has no positive solution. The proof is completed. □

Theorem 3.6 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{i\infty} > 0$, $f_{i0} > 0$, $f_i(t,x,y) > 0$ *for* $t \in [a,b] \subset (0,1)$, $x \geq 0$, $y > 0$ *or* $t \in [a,b] \subset (0,1)$, $x > 0$, $y \geq 0$, *then there exists* $\lambda_{i*} > 0$ *such that for* $\lambda_i \in (\lambda_{i*}, +\infty)$ $(i = 1,2)$, *system* (1) *has no positive solution.*

Proof From the definitions of $f_{i\infty}, f_{i0}$, which are finite, there exist positive constants m_i^1, m_i^2 and R_3, R_4 ($R_3 < R_4$) such that

$$f_1(t,x,y) \geq m_1^1\phi(x), \quad 0 \leq x,y \leq R_3, t \in [a,b] \subset (0,1),$$
$$f_1(t,x,y) \geq m_1^2\phi(x), \quad x,y \geq R_4, t \in [a,b] \subset (0,1).$$

Set $m_1^0 = \min\{m_1^1, m_1^2, \min_{t\in[a,b]\subset(0,1),R_3\leq x,y\leq R_4} \frac{f_1(t,x,y)}{\phi(x)}\}$, we have

$$f_1(t,x,y) \geq m_1^0\phi(x), \quad x,y \geq 0, t \in [a,b] \subset (0,1).$$

Similarly, set $m_2^0 = \min\{m_2^1, m_2^2, \min_{t\in[a,b]\subset(0,1),R_3\leq x,y\leq R_4} \frac{f_2(t,x,y)}{\phi(y)}\}$, we have

$$f_2(t,x,y) \geq m_2^0\phi(y), \quad x,y \geq 0, t \in [a,b] \subset (0,1).$$

Assume that (u,v) is a positive solution of system (1), we will show that this leads to a contradiction. Define $\lambda_{i*} = (m_i^0)^{-1}\varphi_2(L_2^{-1})$, since $\lambda_i \in (\lambda_{i*}, +\infty)$, by Lemmas 1.1, 2.3 and 2.5, we conclude that

$$\begin{aligned}
\|u\| &= \big\|T_1(u,v)\big\| \\
&\geq \min_{t\in[a,b]} \int_0^1 \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1}\phi^{-1}\left(\lambda_1\int_0^1 s^{\beta_1-1}G_1(1,\tau)m_1^0\phi\big(u(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 \varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1}\phi^{-1}\left(\lambda_2\int_0^1 s^{\beta_2-1}G_2(1,\tau)m_2^0\phi\big(v(\tau)\big)\,d\tau\right)ds \\
&\geq \varrho\omega\theta^2\big\|(u,v)\big\|\int_0^1 \varphi_2^{-1}\big(s^{\beta_1-1}\big)s^{\beta_1-1}\varphi_2^{-1}\left(\int_0^1 G_1(1,\tau)\,d\tau\right)ds\psi_2^{-1}\big(\lambda_1 m_1^0\big) \\
&\quad + \varrho\omega\theta^2\big\|(u,v)\big\|\int_0^1 \varphi_2^{-1}\big(s^{\beta_2-1}\big)s^{\beta_2-1}\varphi_2^{-1}\left(\int_0^1 G_2(1,\tau)\,d\tau\right)ds\psi_2^{-1}\big(\lambda_2 m_2^0\big) \\
&\geq \big\|(u,v)\big\|\varphi_2^{-1}\big(\lambda_1 m_1^0\big)\frac{L_2}{2} + \big\|(u,v)\big\|\varphi_2^{-1}\big(\lambda_2 m_2^0\big)\frac{L_2}{2} > \big\|(u,v)\big\|.
\end{aligned} \tag{45}$$

Similarly to (45), we also have

$$\|v\| > \big\|(u,v)\big\|. \tag{46}$$

Hence, by (45) (46), we get

$$\big\|(u,v)\big\| = \max\big\{\|u\|, \|v\|\big\} > \big\|(u,v)\big\|, \tag{47}$$

which is a contradiction. Therefore, system (1) has no positive solution. The proof is completed. □

Similar to the proof of Theorem 3.6, we obtain the following Theorems 3.7 and 3.8.

Theorem 3.7 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{1\infty} > 0$, $f_{10} > 0$, $f_1(t,x,y) > 0$ *for* $t \in [a,b] \subset (0,1)$, $x \geq 0$, $y > 0$ *or* $t \in [a,b] \subset (0,1)$, $x > 0$, $y \geq 0$, *then there exists* $\lambda_{1*} > 0$ *such that for* $\lambda_1 \in (\lambda_{1*}, +\infty)$, $\lambda_2 \in (0,+\infty)$, *system* (1) *has no positive solution.*

Theorem 3.8 *Assume that* $(\mathbf{H}_0)(\mathbf{H}_1)(\mathbf{H}_2)$ *hold and* $f_{2\infty} > 0$, $f_{20} > 0$, $f_2(t,x,y) > 0$ *for* $t \in [a,b] \subset (0,1)$, $x \geq 0$, $y > 0$ *or* $t \in [a,b] \subset (0,1)$, $x > 0$, $y \geq 0$, *then there exists* $\lambda_{2*} > 0$ *such that for* $\lambda_2 \in (\lambda_{2*}, +\infty)$, $\lambda_1 \in (0,+\infty)$, *system* (1) *has no positive solution.*

Remark 3.5 From the proof of Theorems 3.1-3.8, if we choose

$$f_{10} = \liminf_{y\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(y)}, \qquad f_1^0 = \limsup_{y\to 0^+} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(y)},$$

$$f_{20} = \liminf_{x\to 0^+} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(x)}, \qquad f_2^0 = \limsup_{x\to 0^+} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(x)},$$

$$f_{1\infty} = \liminf_{y\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(y)}, \qquad f_1^\infty = \limsup_{y\to +\infty} \sup_{\substack{t\in[0,1]\\ x\in[0,+\infty)}} \frac{f_1(t,x,y)}{\phi(y)},$$

$$f_{2\infty} = \liminf_{x\to +\infty} \inf_{\substack{t\in[a,b]\subset(0,1)\\ y\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(x)}, \qquad f_2^\infty = \limsup_{x\to +\infty} \sup_{\substack{t\in[0,1]\\ y\in[0,+\infty)}} \frac{f_2(t,x,y)}{\phi(x)},$$

then all the conclusion of Theorems 3.1-3.8 are valid.

4 Example

Consider the fractional differential system

$$\begin{cases} D^{\frac{3}{2}}(D_{0^+}^{\frac{5}{2}}u(t)) + \lambda_1 f_1(t,u(t),v(t)) = 0, \\ D^{\frac{3}{2}}(D_{0^+}^{\frac{5}{2}}v(t)) + \lambda_2 f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad D_{0^+}^{\frac{5}{2}}u(0) = (D_{0^+}^{\frac{5}{2}}u(1))' = 0, \qquad u(1) = \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}v(t)\,dt, \\ v(0) = v'(0) = 0, \qquad D_{0^+}^{\frac{5}{2}}v(0) = (D_{0^+}^{\frac{5}{2}}v(1))' = 0, \qquad v(1) = \int_0^1 u(t)\,dt^{\frac{1}{2}}, \end{cases} \tag{48}$$

where $\lambda_i > 0$ $(i = 1,2)$ is a parameter, $\alpha_1 = \alpha_2 = \frac{5}{2}$, $\beta_1 = \beta_2 = \frac{3}{2}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = 1$, $A_1(t) = t$, $A_2(t) = t^{\frac{1}{2}}$, $g_1(t) = t^{-\frac{1}{2}}$, $g_2(t) = 1$, $\phi(x) = x$, choose $\varphi_1(x) = \varphi_2(x) = x$. Then we have

$$\begin{aligned} k_1 &= \int_0^1 g_1(t)t^{\alpha_2-1}\,dA_1(t) = \int_0^1 t^{-\frac{1}{2}}t^{\frac{3}{2}}\,dt = \frac{1}{2} > 0, \\ k_2 &= \int_0^1 g_2(t)t^{\alpha_1-1}\,dA_2(t) = \int_0^1 t^{\frac{3}{2}}\,dt^{\frac{1}{2}} = \frac{1}{2}\int_0^1 t\,dt = \frac{1}{4} > 0, \\ 1 - \mu_1\mu_2k_1k_2 &= \frac{15}{16} > 0. \end{aligned}$$

So, condition $(\mathbf{H}_1)$ holds. Next, in order to demonstrate the application of our main results obtained in Section 3, we choose two different sets of functions f_i $(i = 1,2)$ such that f_i satisfies the conditions of Theorems 3.1 and 3.5.

Case 1. Let $f_1(t,x,y) = \frac{x^2}{1+t} + x\sin y$, $f_2(t,x,y) = \frac{y^2}{e^t} + y\sin x$, choose $[\frac{1}{4},\frac{2}{2}] \subset [0,1]$, we know $f_{i\infty} = +\infty$, $f_i^0 = 0$. Then, by Theorem 3.1, system (48) has at least one positive solution for $\lambda_i \in (0,+\infty)$ $(i = 1,2)$.

Case 2. Let $f_1(t,x,y) = \frac{(10x^2+x)(3+\sin y)}{(1+t)(x+1)}$, $f_2(t,x,y) = \frac{(10y^2+y)(2+\sin x)}{e^t(y+1)}$, therefore, we have $f_1^\infty = 40$, $f_1^0 = 3$, $f_2^\infty = 30$, $f_2^0 = 2$, and for $x, y \leq 0$, we get $x \leq f_1(t,x,y) \leq 40x$, $y \leq f_2(t,x,y) \leq 30y$. By calculation, we obtain $L_1 = 2.2445 \int_0^1 \frac{\tau^{\frac{1}{2}}(1-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}\, d\tau \approx 0.6632$. Then, by Theorem 3.5, system (48) has no positive solution for $\lambda_1 \in (0, 0.0377)$, $\lambda_2 \in (0, 0.0503)$.

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The authors declare that they have no competing interests.

Authors' contributions
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