Existence and Optimality of Cournot-Nash Equilibria in a Bilateral Oligopoly with Atoms and an Atomless Part*

Francesca Busetto[†], Giulio Codognato[‡], Sayantan Ghosal[§], Ludovic Julien[¶], Simone Tonin[‖]

Abstract

We consider a bilateral oligopoly version of the Shapley window model with large traders, represented as atoms, and small traders, represented by an atomless part. For this model, we provide a general existence proof of a Cournot-Nash equilibrium that allows one of the two commodities to be held only by atoms. Then, we show, using a corollary proved by Shitovitz (1973), that a Cournot-Nash allocation is Pareto optimal if and only if it is a Walras allocation.
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1 Introduction

Gabszewicz and Michel (1997) introduced the so-called model of bilateral oligopoly, which consists of a two-commodity exchange economy where each

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[†]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy.

[‡]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy, and Economix, UPL, Univ Paris Nanterre, CNRS, F92000 Nanterre, France.

[§]Adam Smith Business School, University of Glasgow, Glasgow G12 8QQ, United Kingdom.

[¶]Economix, UPL, Univ Paris Nanterre, CNRS, F92000 Nanterre, France.

[‖]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy.

trader holds only one of the two commodities available for trade. In this framework, strategic interaction among traders was modeled as in strategic market games à la Shapley and Shubik (see Giraud (2003) for a survey of this literature). This model was analyzed, in the case of a finite number of traders, by Bloch and Ghosal (1997), Bloch and Ferrer (2001), Dickson and Hartley (2008), Amir and Bloch (2009), among others.

In this paper, we consider the mixed bilateral oligopoly model introduced by Codognato et al. (2015). Following Shitovitz (1973), this model analyzes a mixed economy with large traders, represented as atoms, and small traders, represented by an atomless part. Noncooperative exchange is formalized as in the Shapley window model, a strategic market game which was first proposed informally by Lloyd S. Shapley and further analyzed by Sahi and Yao (1989), Codognato and Ghosal (2000), Busetto et al. (2011), Busetto et al. (2018), among others.

The first goal of the paper is to prove the existence of a Cournot-Nash equilibrium for the mixed bilateral oligopoly version of the Shapley window model. Busetto et al. (2011) provided an existence proof for the mixed version of the Shapley window model with any finite number of commodities. Their proof is based on the same assumptions as the proof provided by Sahi and Yao (1989) for the case of exchange economies with a finite number of traders. In particular, it requires that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space: these restrictions are stated by Busetto et al. (2011) in their Assumption 4. Clearly, this proof does not apply to the bilateral oligopoly case where all atoms hold only one of the two commodities. Busetto et al. (2018) proposed an alternative existence proof which is essentially based on restrictions on endowments and preference of the atomless part of the economy rather than on atoms. In particular, they kept all the assumptions made by Busetto et al. (2011) with the exception of their Assumption 4, which was replaced by a new restriction requiring that the set of commodities is strongly connected through the characteristics of traders in the atomless part. The existence proof in Busetto et al. (2018) used a theorem which shows that any sequence of prices corresponding to a sequence of Cournot-Nash equilibria has a subsequence which converges to a strictly positive price vector and it requires that each commodity is held by a subset of the atomless part with positive measure. An appealing feature of our existence result is that it allows a commodity to be held only by atoms. The case of atoms, oligopolists, on one side of the market, facing an atomless part, on the other side, is the one closest to the basic oligopoly model considered by Cournot (1838). In order to cover this case, we cannot directly use the price convergence theorem shown by Busetto et al. (2018) but we have to combine that proof with the price convergence result proved by Dubey and Shubik (1978), which holds for a strategic market game with a finite number of traders, i.e., for a purely atomic exchange economy. This is one reason why our existence proof is not just a two-commodity case of that provided by Busetto et al. (2018). There is one more difference between the two results. A step in the proof of both existence theorems consists in showing that the aggregate bid matrix, obtained as the limit of a sequence of perturbed Cournot-Nash equilibria, is irreducible. In Busetto et al. (2018), obtaining this result requires that the two ordered pairs generated by the two traded commodities are connected through traders' characteristics. Here, instead, we impose that there is a coalition of traders in the atomless part with differentiable and additively separable utility functions which have infinite partial derivatives along the boundary of the consumption set. This can be seen as an atomless version of an assumption on utility functions made by Bloch and Ferrer (2001) to prove the existence of a Cournot-Nash equilibrium in their finite, purely atomic, bilateral framework.

In the second main theorem of the paper, we provide a characterization of the Pareto optimality of Cournot-Nash allocations. The issue of Pareto optimality in strategic market games was raised since the seminal paper by Shapley and Shubik (1977). This first analysis was mainly an intuitive discussion of Pareto optimality in the Edgeworth box. Then, more formal results on this issue were obtained by Dubey (1980), Dubey et al. (1980), Aghion (1985), Dubey and Rogawski (1990), among others. These results were obtained using the approach to general equilibrium based on differential topology and hold generically. Our second theorem is a general result stating that a Cournot-Nash allocation is Pareto optimal if and only if it is a Walras allocation. This result establishes a relationship among the Cournotian tradition of oligopoly, the Walrasian tradition of perfect competition, and the Paretian analysis of optimality. Some examples computed by Codognato et al. (2015) provide evidence that this characterization holds non-vacuously.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In section 3, we prove the existence of a Cournot-Nash equilibrium. In Section 4, we characterize the Pareto optimality of Cournot-Nash equilibria. In Section 5, we discuss the model. In Section 6, we draw some conclusions and we sketch some further lines of research.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and T_0 the atomless part of T. We assume that $\mu(T_1) > 0$ and $\mu(T_0) > 0$.[1] A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are 2 different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. *There is a coalition S such that $\mathbf{w}^1(t) > 0$, $\mathbf{w}^2(t) = 0$, for each $t \in S$, $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, for each $t \in S^c$. Moreover, card$(S \cap T_1) \geq 2$, whenever $\mu(S \cap T_0) = 0$, and card$(S^c \cap T_1) \geq 2$, whenever $\mu(S^c \cap T_0) = 0$.*[2]

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T$.*

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

The following assumption is an atomless version of the second assumption on utility functions made by Bloch and Ferrer (2001) in their finite, purely atomic, bilateral framework.

[1] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

[2] card(A) denotes the cardinality of a set A.

Assumption 4. *There is a coalition $\bar{T}$, with $\bar{T} \subset T_0$, such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v_t^1(x^1) + v_t^2(x^2)$, for each $x \in R_+^2$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, $j \in \{1, 2\}$, for each $t \in \bar{T}$.*[3]

A price vector is a nonnull vector $p \in R_+^2$. We say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R_+^2 : \sum_{i=1}^2 p^i = 1\}$.

Let $\mathbf{X}^0 : T_0 \times R_{++}^2 \to \mathcal{P}(R_+^2)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R_{++}^2$, $\mathbf{X}^0(t, p) = \text{argmax}\{u_t(x) : x \in R_+^2$ and $px \leq p\mathbf{w}(t)\}$. For each $p \in R_{++}^2$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}^0(t, p)\, d\mu : \mathbf{x}^0(\cdot, p)$ is integrable and $\mathbf{x}^0(t, p) \in \mathbf{X}^0(t, p)$, for each $t \in T_0\}$. Finally, let $\mathbf{Z}^0 : R_{++}^2 \to \mathcal{P}(R^2)$ be a correspondence which associates with each $p \in R_{++}^2$ the Minkowski difference between the set $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu$ and the set $\{\int_{T_0} \mathbf{w}(t)\, d\mu\}$.[4] According to Debreu (1982), let $|x| = \sum_{i=1}^2 |x^i|$, for each $x \in R^2$, and let $d[0, V] = \inf_{x \in V} |x|$, for each $V \subset R^2$. The next proposition, which we shall use in the proof of the existence theorem in Section 3, is based on Property (iv) in Debreu (1982), p. 728.

Proposition 1. *Under Assumptions 1, 2, and 3, let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \gg 0$, for each $n \in \{1, 2, \ldots\}$, which converges to a normalized price vector $\bar{p}$. Then, $\bar{p}^1 = 0$ and $\mu(S^c \cap T_0) > 0$, or, $\bar{p}^2 = 0$ and $\mu(S \cap T_0) > 0$, imply that the sequence $\{d[0, \mathbf{Z}^0(p^n)]\}$ diverges to $+\infty$.*

Proof. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \gg 0$, for each $n \in \{1, 2, \ldots\}$, which converges to a normalized price vector $\bar{p}$. Suppose that $\bar{p}^1 = 0$ and $\mu(S^c \cap T_0) > 0$. Then, we have that $\bar{p}^2 = 1$. But then, the sequence $\{d[0, \mathbf{X}^0(t, p^n)]\}$ diverges to $+\infty$ as $\bar{p}^2\mathbf{w}^2(t) > 0$, for each $t \in S^c \cap T_0$, by Lemma 4 in Debreu (1982), p. 721. Therefore, $\{d[0, \mathbf{Z}^0(p^n)]\}$ diverges to $+\infty$ as $\mu(S^c \cap T_0) > 0$, by the argument used in the proof of Property (iv) in Debreu (1982), p. 728. Suppose that $\bar{p}^2 = 0$ and $\mu(S \cap T_0) > 0$. Then, $\{d[0, \mathbf{Z}^0(p^n)]\}$ diverges to $+\infty$, by using symmetrically the previous argument. Hence, $\bar{p}^1 = 0$ and $\mu(S^c \cap T_0) > 0$, or, $\bar{p}^2 = 0$ and $\mu(S \cap T_0) > 0$, imply that the sequence $\{d[0, \mathbf{Z}^0(p^n)]\}$ diverges to $+\infty$. ■

A Walras equilibrium is a pair $(p, \mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $p\mathbf{x}(t) = p\mathbf{w}(t)$ and $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in$

[3]In this paper, differentiability means continuous differentiability and is to be understood to include the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

[4]For a discussion of the properties of the correspondences introduced above and their proofs see, for instance, Debreu (1982), Section 4.

$\{x \in R^2_+ : px = p\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}$ for which there exists a price vector p such that the pair $(p, \mathbf{x})$ is a Walras equilibrium.

Borrowing from Codognato et al. (2015), we introduce now the two-commodity version of the Shapley window model. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R^4_+ : \sum_{j=1}^2 b_{ij} \leq \mathbf{w}^i(t),\ i \in \{1,2\}\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j \in \{1,2\}$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we call "aggregate matrix" the matrix $\bar{\mathbf{B}}$ such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j \in \{1,2\}$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$ and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

Consider the following two further definitions (see Sahi and Yao (1989)).

Definition 1. *A nonnegative square matrix A is said to be irreducible if, for every pair (i,j), with $i \neq j$, there is a positive integer k such that $a^{(k)}_{ij} > 0$, where $a^{(k)}_{ij}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 2. *Given a strategy selection $\mathbf{b}$, a price vector p is said to be market clearing if*

$$p \in R^2_{++}, \sum_{i=1}^2 p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^2 \bar{\mathbf{b}}_{ji}),\ j \in \{1,2\}. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise. For each strategy selection $\mathbf{b}$ such that $p(\mathbf{b}) \gg 0$, we assume that the price vector $p(\mathbf{b})$ is normalized.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^2 \mathbf{b}_{ji}(t) + \sum_{i=1}^2 \mathbf{b}_{ij}(t)\frac{p^i}{p^j}, \text{ if } p \in R^2_{++}, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise},
\end{aligned}$$

$j \in \{1,2\}$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.[5] It is straightforward to show that this assignment is an allocation.

We are now able to define a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

Definition 3. *A strategy selection $\hat{\mathbf{b}}$ such that $\hat{\bar{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

Finally, we introduce the same type of perturbation of the strategic market game which was used by Sahi and Yao (1989) and Busetto et al. (2011) to prove their existence theorems. Given $\epsilon > 0$ and a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}_\epsilon$ as the matrix such that $\bar{\mathbf{b}}_{\epsilon ij} = (\bar{\mathbf{b}}_{ij} + \epsilon)$, $i, j \in \{1, 2\}$. Clearly, the matrix $\bar{\mathbf{B}}_\epsilon$ is irreducible. The interpretation is that an outside agency places fixed bids of ϵ for each pair of commodities 1 and 2. Given $\epsilon > 0$, we denote by $p^\epsilon(\mathbf{b})$ the function which associates, with each strategy selection $\mathbf{b}$, the unique, up to a scalar multiple, price vector which satisfies

$$\sum_{i=1}^{2} p^i(\bar{\mathbf{b}}_{ij} + \epsilon) = p^j(\sum_{i=1}^{2}(\bar{\mathbf{b}}_{ji} + \epsilon)),\ j \in \{1, 2\}. \tag{2}$$

For each strategy selection $\mathbf{b}$, we assume that the price vector $p^\epsilon(\mathbf{b})$ is normalized.

Definition 4. *Given $\epsilon > 0$, a strategy selection $\hat{\mathbf{b}}^\epsilon$ is an ϵ-Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^\epsilon(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_t(\mathbf{x}(t, b(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

[5] In order to save in notation, with some abuse we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

3 Existence

In this section, we state and prove our first main theorem, establishing the existence of a Cournot-Nash equilibrium for the two-commodity version of the Shapley window model presented above.

Theorem 1. *Under Assumptions 1, 2, 3, and 4, there exists a Cournot-Nash equilibrium* $\hat{\mathbf{b}}$.

Proof. The first step in the proof of Theorem 1 requires that we show the existence of an ϵ-Cournot-Nash equilibrium. To this end, we use a result already proved by Busetto et al. (2011) by applying the Kakutani-Fan-Glicksberg theorem. It is stated in the following lemma.

Lemma 1. *For each* $\epsilon > 0$, *there exists an* ϵ*-Cournot-Nash equilibrium* $\hat{\mathbf{b}}^{\epsilon}$.

Proof. See the proof of Lemma 3 in Busetto et al. (2011). ■

Then, we have to show that there exists the limit of a sequence of ϵ-Cournot-Nash equilibria and that this limit is a Cournot-Nash equilibrium. Let $\epsilon_n = \frac{1}{n}$, $n \in \{1, 2, \ldots\}$. By Lemma 1, for each $n \in \{1, 2, \ldots\}$, there is an ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_n}$. The next step consists in showing that any sequence of normalized prices generated by the sequence of ϵ-Cournot-Nash equilibria corresponding to the sequence $\{\epsilon_n\}$ has a convergent subsequence whose limit is a strictly positive normalized price vector. In order to prove this result, we cannot use the price convergence theorem proved by Busetto et al. (2018) as the proof of this theorem requires that each commodity is held by a subset of the atomless part with positive measure whereas, in our framework, one of the two commodities may be held only by atoms. Therefore, we have to combine the price convergence proof provided by Busetto et al. (2018) with another price convergence proof, proposed by Dubey and Shubik (1978), which holds for a purely atomic exchange economy. To this end, we need to introduce the following preliminary lemma, based on the uniform monotonicity lemma proved by Dubey and Shubik (1978).

Lemma 2. *Consider an atom* $\tau \in T_1$ *and a commodity* $j \in \{1, 2\}$. *For each real number* $H > 0$, *there is a real number* $0 < h(u_\tau(\cdot), j, H) < 1$, *depending on* $u_\tau(\cdot)$, j, *and* H, *such that if* $x \in R^2_+$, $\|x\| \leq H$, $y \in R^2_+$ *and* $\|y - x\| \leq h(u_\tau(\cdot), j, H)$, *then* $u_\tau(y + e^j) > u_\tau(x)$.[6]

[6] $\|\cdot\|$ denotes the Euclidean norm and e^j denotes the vector whose jth coordinate is 1 and whose other coordinate vanishes.

Proof. It is an immediate consequence of Lemma C (the uniform monotonicity lemma) in Dubey and Shubik (1978) as $u_\tau(\cdot)$ is continuous and strongly monotone, by Assumption 2. ■

We can now state and prove the price convergence lemma.

Lemma 3. *Let $\{\hat{p}^{\epsilon_n}\}$ be a sequence of normalized prices such that $\{\hat{p}^{\epsilon_n}\} = p(\hat{\mathbf{b}}^{\epsilon_n})$ where $\hat{\mathbf{b}}^{\epsilon_n}$ is an ϵ-Cournot-Nash equilibrium, for each $n \in \{1, 2, \ldots\}$. Then, there exists a subsequence $\{\hat{p}^{\epsilon_{k_n}}\}$ of the sequence $\{\hat{p}^{\epsilon_n}\}$ which converges to a normalized price vector $\hat{p} \gg 0$.*

Proof. Let $\{\hat{p}^{\epsilon_n}\}$ be a sequence of normalized prices such that $\{\hat{p}^{\epsilon_n}\} = p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})$, where $\hat{\mathbf{b}}^{\epsilon_n}$ is an ϵ-Cournot-Nash equilibrium, for each $n \in \{1, 2, \ldots\}$. Then, there is a subsequence $\{\hat{p}^{\epsilon_{k_n}}\}$ of the sequence $\{\hat{p}^{\epsilon_n}\}$ which converges to a price vector $\hat{p} \in \Delta$, as the unit simplex Δ is a compact set. Consider the case where $\mu(S \cap T_0) > 0$ and $\mu(S^c \cap T_0) > 0$. Suppose, without loss of generality, that $\hat{p}^1 = 0$. Then, the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_n}})]\}$ diverges to $+\infty$ as $\mu(S^c \cap T_0) > 0$, by Proposition 1. We adapt now to our framework the argument used by Busetto et al. (2018) to prove their Theorem 1. Let $\hat{\mathbf{x}}^{\epsilon_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{\epsilon_n}(t), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))$, for each $t \in T$, and for each $n \in \{1, 2, \ldots\}$. Then, $\hat{\mathbf{x}}^{\epsilon_n}(t) \in \mathbf{X}^0(t, p^{\epsilon_n})$, for each $t \in T_0$, and for each $n \in \{1, 2, \ldots\}$, by the same argument used by Codognato and Ghosal (2000) to prove their Theorem 2. But then, $(\int_{T_0} \hat{\mathbf{x}}^{\epsilon_n}(t)\, d\mu - \int_{T_0} \mathbf{w}(t)\, d\mu) \in \mathbf{Z}^0(\hat{p}^{\epsilon_n})$, for each $n \in \{1, 2, \ldots\}$. We have that

$$\int_{T_0} \hat{\mathbf{x}}^{\epsilon_n}(t)\, d\mu \leq \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu + e^1 + e^2$$

as $\int_T \hat{\mathbf{x}}^{\epsilon_n}(t)\, d\mu \leq \int_T \mathbf{w}(t)\, d\mu + \epsilon_n e^1 + \epsilon_n e^2$, for each $n \in \{1, 2, \ldots\}$. Then,

$$\left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + 1$$

as $-\int_{T_1} \mathbf{w}^i(t)\, d\mu - 1 \leq \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_n}(t)\, d\mu \leq 2\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + 1$, $i \in \{1, 2\}$, for each $n \in \{1, 2, \ldots\}$. But then,

$$\sum_{i=1}^{2} \left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \sum_{i=1}^{2} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + 1 \right),$$

for each $n \in \{1, 2, \ldots\}$. Moreover, there exists an n_0 such that

$$d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_n}})] > \sum_{i=1}^{2} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + 1 \right),$$

for each $n \geq n_0$, as the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_n}})]\}$ diverges to $+\infty$. Then,

$$\sum_{i=1}^{2} \left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_{k_n}}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| > \sum_{i=1}^{2} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + 1 \right)$$

as $\sum_{i=1}^{2} | \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_{k_n}}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu| \geq d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_n}})]$, for each $n \geq n_0$, a contradiction. Therefore, we must have that $\hat{p}^1 > 0$. Consider the case where $\mu(S \cap T_0) = 0$ or $\mu(S^c \cap T_0) = 0$. Suppose, without loss of generality, that $\mu(S \cap T_0) = 0$. Then, we have that $\mu(S^c \cap T_0) > 0$ as $\mu(T_0) > 0$. Moreover, there are at least two atoms $\tau, \rho \in S \cap T_1$, by Assumption 1. We have that $\hat{p}^1 > 0$, as $\mu(S^c \cap T_0) > 0$, by the same argument used in the proof of the previous case. In order to prove that $\hat{p}^2 > 0$, we now show that there is a real number $\eta > 0$ such that

$$\frac{\hat{p}^{2\epsilon_n}}{\hat{p}^{1\epsilon_n}} > \eta, \tag{3}$$

for each $n \in \{1, 2, \ldots\}$. To this end, we adapt to our framework the proof of Lemma 2 in Dubey and Shubik (1978). In what follows, we shall use the fact that (2) and the normalization rule imply straightforwardly that (3) holds if and only if

$$\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n} > \eta,$$

for each $n \in \{1, 2, \ldots\}$. Consider any n. We now prove that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$ or $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\rho) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$. Suppose, by way of contradiction, that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) > \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$ and $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\rho) > \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$. Then, we have that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) + \hat{\mathbf{b}}^{\epsilon_n}_{12}(\rho) > \bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}$, a contradiction. Therefore, we must have that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$ or $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\rho) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$. Suppose that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$. Moreover, suppose that $\mathbf{w}^1(\tau) - \hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \geq \frac{\mathbf{w}^1(\tau)}{2}$. Let $0 < \gamma < \min\{\epsilon_n, \frac{\mathbf{w}^1(\tau)}{2}, 2\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}\}$ and let $b^\gamma(\tau) = \hat{\mathbf{b}}^{\epsilon_n}(\tau) + \gamma e^2$. Then, we have

$$\begin{aligned}
&\mathbf{x}^1(\tau, b^\gamma(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n} \setminus b^\gamma(\tau))) - \mathbf{x}^1(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) \\
&= (\mathbf{w}^1 - \hat{\mathbf{b}}^{\epsilon_n}_{12} - \gamma) - (\mathbf{w}^1 - \hat{\mathbf{b}}^{\epsilon_n}_{12}) \\
&= -\gamma,
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbf{x}^2(\tau, b^\gamma(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n} \setminus b^\gamma(\tau))) - \mathbf{x}^2(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))\\
&= (\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) + \gamma)\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n + \gamma} - \hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau)\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\\
&= \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n - \hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau)}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n + \gamma}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\gamma\\
&> \frac{\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2} + \frac{\epsilon_n}{2} + \frac{\gamma}{2}}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n + \gamma}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\gamma,
\end{aligned}
$$

as $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \le \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$ and $\gamma < \epsilon_n$. Then, we obtain

$$
\mathbf{x}^2(\tau, b^\gamma(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n} \setminus b^\gamma(\tau))) - \mathbf{x}^2(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) > \frac{1}{2}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\gamma. \quad (4)
$$

Let us define

$$
z = -2\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}e^1.
$$

Then, we have the vector inequality

$$
\begin{aligned}
&\mathbf{x}(\tau, b^\gamma(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n} \setminus b^\gamma(\tau)))\\
&\ge \mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) + \frac{1}{2}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\gamma(z + e^2), \quad (5)
\end{aligned}
$$

where the inequality is strict for the second component by (4). Let $H = \sqrt{2}\max\{\int_T \mathbf{w}^1(t)\, d\mu + 1, \int_T \mathbf{w}^2(t)\, d\mu + 1\}$ and let $y = \mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) + z$. It is straightforward to verify that $\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) \in R^2_+$ and $\|\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))\| \le H$. Suppose that $y \in R^2_+$ and $\|z\| \le h(u_\tau(\cdot), 2, H)$. Then, by Lemma 2, we obtain that

$$
u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) + z + e^2) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))).
$$

But then, we have that

$$
u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) + \frac{1}{2}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}\gamma(z + e^2)) \ge u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))),
$$

as $0 < \frac{1}{2}\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21}+\epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}+\epsilon_n}\gamma < 1$ and the function $u_\tau(\cdot)$ is quasi-concave, by Assumption 2. Therefore, it follows that

$$u_\tau(\mathbf{x}(\tau, b^\gamma(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n} \setminus b^\gamma(\tau)))) > u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))),$$

as (5) holds strictly for its second component and $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, a contradiction. Thus, it must be that $y \notin R^2_+$ or $\|z\| > h(u_\tau(\cdot), 2, H)$. Suppose that $y \notin R^2_+$. Then, it follows that

$$\mathbf{x}^1(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) - 2\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n} < 0,$$

as $y = \mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) + z$ and $\mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) \in R^2_+$. But then, it must be that

$$\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n} > \frac{\mathbf{w}^1(\tau)}{4},$$

as $\mathbf{x}^1(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})) = \mathbf{w}^1(\tau) - \hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) \geq \frac{\mathbf{w}^1(\tau)}{2}$. Suppose that $\|z\| > h(u_\tau(\cdot), 2, H)$. Then, we have that

$$\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n} > \frac{h(u_\tau(\cdot), 2, H)}{2}.$$

Suppose now that $\mathbf{w}^1(\tau) - \hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) < \frac{\mathbf{w}^1(\tau)}{2}$. Then, we have that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\tau) > \frac{\mathbf{w}^1(\tau)}{2}$. But then, it must be that

$$\frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12} + \epsilon_n}{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{21} + \epsilon_n} > \frac{\mathbf{w}^1(\tau)}{2(\int_T \mathbf{w}^2(t)\, d\mu + 1)}.$$

Let

$$\alpha = \min\left\{\frac{\mathbf{w}^1(\tau)}{4}, \frac{h(u_\tau(\cdot), 2, H)}{2}, \frac{\mathbf{w}^1(\tau)}{2(\int_T \mathbf{w}^2(t)\, d\mu + 1)}\right\}.$$

Thus, we have that

$$\frac{\hat{p}^{2\epsilon_n}}{\hat{p}^{1\epsilon_n}} > \alpha.$$

Suppose that $\hat{\mathbf{b}}^{\epsilon_n}_{12}(\rho) \leq \frac{\bar{\hat{\mathbf{b}}}^{\epsilon_n}_{12}}{2}$. Let

$$\beta = \min\left\{\frac{\mathbf{w}^1(\rho)}{4}, \frac{h(u_\rho(\cdot), 2, H)}{2}, \frac{\mathbf{w}^1(\rho)}{2(\int_T \mathbf{w}^2(t)\, d\mu + 1)}\right\}.$$

Thus, by the same argument used in the previous case, we have that

$$\frac{\hat{p}^{2\epsilon_n}}{\hat{p}^{1\epsilon_n}} > \beta.$$

Let $\eta = \min\{\alpha, \beta\}$. Therefore, we can conclude that

$$\frac{\hat{p}^{2\epsilon_n}}{\hat{p}^{1\epsilon_n}} > \eta,$$

for each $n \in \{1, 2, \ldots\}$. Consider the sequence $\{\hat{p}^{\epsilon_{k_n}}\}$. From (3), we obtain that

$$\hat{p}^{2\epsilon_{k_n}} > \eta \hat{p}^{1\epsilon_{k_n}},$$

for each $n \in \{1, 2, \ldots\}$. Then, we obtain that

$$\hat{p}^2 > \eta \hat{p}^1,$$

as the sequence $\{\hat{p}^{\epsilon_{k_n}}\}$ converges to $\hat{p}$. But then, we have that $\hat{p}^2 > 0$ as $\eta > 0$ and $\hat{p}^1 > 0$. Hence, having considered all possible cases, we can conclude that $\hat{p} \gg 0$. ■

We now follow the argument used by Busetto et al. (2018) to prove their Theorem 2. In the next part of the proof, we apply a generalization of the Fatou lemma in several dimensions provided by Artstein (1979). By Lemma 1, there is an ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_n}$, for each $n \in \{1, 2, \ldots\}$. The fact that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ belongs to the compact set $\{(b_{ij}) \in R^4_+ : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j \in \{1, 2\}\}$ and the sequence $\{\hat{p}^{\epsilon_n}\}$, where $\hat{p}^{\epsilon_n} = p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n})$, belongs to the unit simplex Δ, for each $n \in \{1, 2, \ldots\}$, implies that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_n}\}$ which converges to an element of the set $\{(b_{ij}) \in R^4_+ : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j \in \{1, 2\}\}$ and a subsequence $\{\hat{p}^{\epsilon_{k_n}}\}$ of the sequence $\{\hat{p}^{\epsilon_n}\}$ which converges to a price vector $\hat{p} \in \Delta$, with $\hat{p} \gg 0$, by Lemma 3. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), by this theorem there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}(t)\}$, for each $t \in T$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Moreover, $\hat{p}$ and $\bar{\hat{\mathbf{B}}}$ satisfy (1) as $\hat{p}^{\epsilon_{k_n}}$ and $\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}_{\epsilon_{k_n}}$ satisfy (2), for each $n \in \{1, 2, \ldots\}$, the sequence $\{\hat{p}^{\epsilon_{k_n}}\}$ converges to $\hat{p}$, the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\epsilon_{k_n}\}$ converges to 0. Suppose, without loss of generality, that $\mu(S^c \cap \bar{T}) > 0$. We now show that $\bar{\hat{\mathbf{b}}}_{21} > 0$. Suppose that $\bar{\hat{\mathbf{b}}}_{21} = 0$. Then, we have that

$\int_{S^c \cap \bar{T}} \hat{\mathbf{b}}_{21}(t)\, d\mu = 0$ as $\mu(S^c \cap \bar{T}) > 0$. Consider a trader $\tau \in S^c \cap \bar{T}$. We can suppose that $\hat{\mathbf{b}}_{21}(\tau) = 0$ as we ignore null sets. Since $\hat{\mathbf{b}}(\tau)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_n}}(\tau)\}$, there is a subsequence $\{\hat{\mathbf{b}}^{\epsilon_{h_{k_n}}}(\tau)\}$ of this sequence which converges to $\hat{\mathbf{b}}(\tau)$. Let $\hat{\mathbf{x}}^{\epsilon_n}(\tau) = \mathbf{x}(\tau, \hat{\mathbf{b}}^{\epsilon_n}(\tau), p^{\epsilon_n}(\hat{\mathbf{b}}^{\epsilon_n}))$, for each $n \in \{1, 2, \ldots\}$, and $\hat{\mathbf{x}}(\tau) = \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p})$. Then, the subsequence $\{\hat{\mathbf{x}}^{\epsilon_{h_{k_n}}}(\tau)\}$ of the sequence $\{\hat{\mathbf{x}}^{\epsilon_n}(\tau)\}$ converges to $\hat{\mathbf{x}}(\tau)$ as the sequence $\{\hat{\mathbf{b}}^{\epsilon_{h_{k_n}}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$ and the sequence $\{\hat{p}^{\epsilon_{h_{k_n}}}\}$ converges to $\hat{p}$, with $\hat{p}^{\epsilon_{h_{k_n}}} \gg 0$, for each $n \in \{1, 2, \ldots\}$, and $\hat{p} \gg 0$. But then, it must be that $\hat{\mathbf{x}}^1(\tau) = 0$ as $\hat{\mathbf{b}}_{21}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) \in \mathbf{X}^0(\tau, \hat{p})$ as $\hat{\mathbf{x}}^{\epsilon_{h_{k_n}}}(\tau) \in \mathbf{X}^0(\tau, \hat{p}^{\epsilon_{h_{k_n}}})$, for each $n \in \{1, 2, \ldots\}$, and the correspondence $\mathbf{X}^0(\tau, \cdot)$ is upper hemicontinuous, by the argument used in Debreu (1982), p. 721. Therefore, we have that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1} = +\infty$ by Assumption 4 and $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^1} \leq \lambda \hat{p}^1$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\hat{\mathbf{x}}^2(\tau) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $\hat{p}\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = \lambda \hat{p}^2$, by the necessary conditions of the Kuhn-Tucker theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, we can conclude that $\bar{\hat{\mathbf{b}}}_{21} > 0$. It is straightforward to verify that (2) and the normalization rule imply that

$$\frac{\bar{\hat{\mathbf{b}}}_{12}^{\epsilon_{k_n}} + \epsilon_{k_n}}{\bar{\hat{\mathbf{b}}}_{21}^{\epsilon_{k_n}} + \epsilon_{k_n}} = \frac{\hat{p}^{2\epsilon_{k_n}}}{\hat{p}^{1\epsilon_{k_n}}},$$

for each $n \in \{1, 2, \ldots\}$. Then, we must have that $\bar{\hat{\mathbf{b}}}_{12} > 0$ as the sequence $\{\hat{p}^{\epsilon_{k_n}}\}$ converges to $\hat{p}$, the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_n}}\}$ converges to $\bar{\hat{\mathbf{B}}}$, the sequence $\{\epsilon_{k_n}\}$ converges to 0, $\hat{p} \gg 0$, and $\bar{\hat{\mathbf{b}}}_{21} > 0$. But then, $\bar{\hat{\mathbf{B}}}$ is irreducible. Consider a trader $\tau \in T_1$. The matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is irreducible as $\bar{\hat{\mathbf{b}}}_{21} \setminus b(\tau) > 0$, by the previous argument. Consider a trader $\tau \in T_0$. The matrix $\bar{\hat{\mathbf{B}}} \setminus b(\tau)$ is irreducible as $\bar{\hat{\mathbf{B}}} = \bar{\hat{\mathbf{B}}} \setminus b(\tau)$. Then, the matrix $\bar{\hat{\mathbf{B}}} \setminus b(t)$ is irreducible, for each $t \in T$. But then, from the same argument used by Busetto et al. (2011) in their existence proof (Cases 1 and 3), it follows that $u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t))))$, for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$. Hence, $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. ■

4 Optimality

In this section, we study the Pareto optimality properties of a Cournot-Nash allocation for the two-commodity version of the Shapley window model.

Shapley and Shubik (1977) first raised the question of the Pareto optimality features of Cournot-Nash allocations in the context of the prototypical strategic market games they proposed in that work. Nevertheless, their analysis was mainly based on examples drawn in an Edgeworth box. Then, some more general results about the Pareto optimality properties of Cournot-Nash allocations of strategic market games were obtained, both for exchange economies with a finite number of traders and with an atomless continuum of traders, by Dubey (1980), Dubey et al. (1980), Aghion (1985), Dubey and Rogawski (1990), among others. Their theorems were obtained in a framework of differential topology and hold generically.[7] Here, we extend the Pareto optimality analysis to our mixed version of the Shapley window model and we obtain a general result which characterizes Pareto optimal Cournot-Nash allocations as Walras allocations. To this end, we need to introduce the following further definitions. An allocation $\mathbf{x}$ is said to be individually rational if $u_t(\mathbf{x}(t)) \geq u_t(\mathbf{w}(t))$, for each $t \in T$. An allocation $\mathbf{x}$ is said to be Pareto optimal if there is no allocation $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for each $t \in T$. An efficiency equilibrium is a pair $(p, \mathbf{x})$, consisting of a price vector p and an allocation $\mathbf{x}$, such that $u_t(\mathbf{x}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : px = p\mathbf{x}(t)\}$, for each $t \in T$. We can now state and prove our optimality theorem, which establishes an equivalence between the set of Pareto optimal Cournot-Nash allocations and the set of Cournot-Nash allocations, whenever the latter are also Walrasian.

Theorem 2. *Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is Pareto optimal. We adapt to our framework the argument used by Shitovitz (1973) to prove the corollary to his Lemma 2. It is straightforward to verify that $\hat{\mathbf{x}}$ is individually rational. Let $\hat{\mathbf{G}} \to \mathcal{P}(R^2)$ be a correspondence such that $\hat{\mathbf{G}}(t) = \{x - \hat{\mathbf{x}}(t) : x \in R^2_+ \text{ and } u_t(x) > u_t(\hat{\mathbf{x}}(t))\}$, for each $t \in T$. Moreover, let $\int_T \hat{\mathbf{G}}(t)\, d\mu = \{\int_T \hat{\mathbf{g}}(t)\, d\mu : \hat{\mathbf{g}}(t) \text{ is integrable and } \hat{\mathbf{g}}(t) \in \hat{\mathbf{G}}(t), \text{ for each } t \in T\}$. The set $\{x \in R^2_+ : u_t(x) \geq u_t(\hat{\mathbf{x}})\}$ is convex as $u_t(\cdot)$ is quasi-concave, by Assumption 2, for each $t \in T_1$. Then, it is straightforward to verify that the set $\hat{\mathbf{G}}(t)$ is convex, for each $t \in T_1$. But then, $\int_T \hat{\mathbf{G}}(t)\, d\mu$ is convex, by Theorem 1

[7] For a discussion of this literature, see Giraud (2003), p. 359 and p. 365.

in Shitovitz (1973). We now prove that $0 \notin \int_T \hat{\mathbf{G}}(t)\,d\mu$. Suppose that $0 \in \int_T \hat{\mathbf{G}}(t)\,d\mu$. Then, there is an assignment $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) > u_t(\hat{\mathbf{x}}(t))$, for each $t \in T$, which is an allocation as $\int_T \mathbf{y}(t)\,d\mu = \int_T \hat{\mathbf{x}}(t)\,d\mu = \int_T \mathbf{w}(t)\,d\mu$. But then, $\hat{\mathbf{x}}$ is not Pareto optimal, a contradiction. Therefore, it must be that $0 \notin \int_T \hat{\mathbf{G}}(t)\,d\mu$. Then, there exists a vector $\tilde{p}$ such that $\tilde{p} \in R^2$, $(\tilde{p} \neq 0)$, and $\tilde{p}\int_T \hat{\mathbf{G}}(t)\,d\mu \geq 0$, by the supporting hyperplane theorem. But then, the pair $(\tilde{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium, by Lemma 2 in Shitovitz (1973). We have that $\hat{\mathbf{x}}(t) \in \mathbf{X}^0(t, \hat{p})$, by the same argument used by Codognato and Ghosal (2000) to prove their Theorem 2, for each $t \in T_0$. Consider a trader $\tau \in \bar{T}$ and suppose that either $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$. Then, the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of our Theorem 1, by Assumption 4. But then, we have that $\hat{\mathbf{x}}(t) \gg 0$. Therefore, it must be that

$$\frac{\frac{\partial u_t(\hat{\mathbf{x}}(t))}{\partial x^1}}{\frac{\partial u_t(\hat{\mathbf{x}}(t))}{\partial x^2}} = \frac{\hat{p}^1}{\hat{p}^2},$$

for each $t \in \bar{T}$. It must also be that

$$\frac{\frac{\partial u_t(\hat{\mathbf{x}}(t))}{\partial x^1}}{\frac{\partial u_t(\hat{\mathbf{x}}(t))}{\partial x^2}} = \frac{\tilde{p}^1}{\tilde{p}^2},$$

as the pair $(\tilde{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium, for each $t \in \bar{T}$. Then, there exists a real number $\theta > 0$ such that $\hat{p}^1 = \theta\tilde{p}^1$ and $\hat{p}^2 = \theta\tilde{p}^2$. But then, $\hat{\mathbf{x}}$ is such that $\hat{p}\hat{\mathbf{x}}(t) = \hat{p}\mathbf{w}(t)$ and $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T$. Therefore, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Suppose now that the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Then, $\hat{\mathbf{x}}$ is Pareto optimal, by the first fundamental theorem of welfare economics. Hence, $\hat{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. ■

We study now the relationship between the set of Cournot-Nash allocations, the core, and the set of Walras allocations.

We say that an allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition S if $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for each $t \in S$, and $\int_S \mathbf{y}(t)\,d\mu = \int_S \mathbf{w}(t)\,d\mu$. The core is the set of all allocations which are not dominated via any coalition. The following corollary is a straightforward consequence of Theorem 2.

Corollary 1. *Under Assumptions 1, 2, 3, and 4, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is in the core if and only if the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is in the core. Then, $\hat{\mathbf{x}}$ is Pareto optimal. But then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium, by Theorem 2. Suppose that the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Then, $\hat{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. Hence, $\hat{\mathbf{x}}$ is in the core if and only if the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. ■

The next proposition provides a characterization of Pareto optimal Cournot-Nash allocations. To prove it, we use a result obtained in Codognato et al. (2015) which provides a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation. This characterization result requires the following assumption.

Assumption 5. *$u_t : R^2_+ \to R$ is differentiable, for each $t \in T_1$.*

Our characterization of Pareto optimal Cournot-Nash allocations is the following.

Proposition 2. *Under Assumptions 1, 2, 3, 4, and 5, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is Pareto optimal if and only if $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is Pareto optimal. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium, by Theorem 2. But then, $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$, by Theorem 4 in Codognato et al. (2015). Suppose that $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium, by Theorem 4 in Codognato et al. (2015). But then, $\hat{\mathbf{x}}$ is Pareto optimal, by Theorem 2. Hence, $\hat{\mathbf{x}}$ is Pareto optimal if and only if $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$. ■

The following corollary provides a characterization of Cournot-Nash allocations which are in the core.

Corollary 2. *Under Assumptions 1, 2, 3, 4, and 5, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, $\hat{\mathbf{x}}$ is the core if and only if $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$.*

Proof. Let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that $\hat{\mathbf{x}}$ is in the core. Then, $\hat{\mathbf{x}}$

is Pareto optimal. But then, $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$, by Proposition 2. Suppose that $\hat{\mathbf{x}}^1(t) = 0$ or $\hat{\mathbf{x}}^2(t) = 0$, for each $t \in T_1$. Then the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium, by Theorem 4 in Codognato et al. (2015). But then, $\hat{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. ■

Examples 6, 7, 8, and 9 in Codognato et al. (2015) show that Theorem 2, Proposition 2, and Corollaries 1 and 2 hold non-vacuously.

5 Discussion of the model

This section is devoted to a discussion of some issues related to the existence and optimality of Cournot-Nash equilibria.

It is straightforward to show, using Theorem 5 in Codognato et al. (2015), that, in our mixed bilateral oligopoly framework, under Assumptions 1, 2, and 3, the set of Cournot-Nash allocations of the Shapley window model coincides with the set of the Cournot-Nash allocations of the model with commodity money proposed by Dubey and Shubik (1978) and of its generalization to complete markets proposed by Amir et al. (1990). Therefore, all the results obtained in this paper also hold for these models.

Let us further discuss now some features of the models in this class, when considered in a bilateral oligopoly framework, and their relationships with the results obtained in this paper.

Busetto et al. (2018) considered a mixed version of the Shapley window model for exchange economies with any finite number of commodities and, in their Theorem 2, they proved the existence of a Cournot-Nash equilibrium. In this paper, we have considered a bilateral oligopoly version of the model analyzed by Busetto et al. (2018). Nevertheless, our existence result, Theorem 1, is not just a two-commodity case of Theorem 2 in Busetto et al. (2018). While Assumptions 2 and 3 are the same in Busetto et al. (2018) and here, Assumptions 1 and 4 differ. Let us analyze in more detail the difference between the two versions of Assumptions 1 and 4 and the role they play in the two existence proofs.

To be applied to our bilateral framework, Assumption 1 in Busetto et al. (2018) could be restated as follows.

Assumption 1′. *There is a coalition S such that $\mu(S \cap T) > 0$, $\mu(S^c \cap T) > 0$, $\mathbf{w}^1(t) > 0$, $\mathbf{w}^2(t) = 0$, for each $t \in S$, $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, for each $t \in S^c$. Moreover, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.*

It is clear that if an initial assignment $\mathbf{w}$ satisfies Assumption $1'$, then it also satisfies our Assumption 1. Assumption $1'$ rules out the cases where $\mu(S \cap T_0) = 0$ or $\mu(S^c \cap T_0) = 0$, that is those in which only the atoms hold one of the two commodities. Therefore, the price convergence theorem proved by Busetto et al. (2018), their Theorem 1, that holds only under Assumption $1'$, cannot be used in the proof of our existence theorem. In order to cover the case where one of the two commodities is held only by atoms, we have proved a price convergence lemma which combines the argument used by Busetto et al. (2018) in the proof of their Theorem 1 with another argument used by Dubey and Shubik (1978) in the proof of their convergence result for the purely atomic case, their Lemma 2. Moreover, we had to extend the proof of Dubey and Shubik (1978), which holds under the assumption of concave utility functions, to cover the case of quasi-concave utility functions, as required by our Assumption 2.

Assumption 4 in Busetto et al. (2018) imposes a relation between commodities based on traders' characteristics, called relation C, and it requires that the set of commodities is strongly connected in terms of relation C, i.e., through traders' characteristics. It could be recasted, in the bilateral framework, as follows. We say that commodities i, j stand in relation C if there is a coalition T^i such that $T^i \subset \{t \in T_0 : \mathbf{w}^i(t) > 0,\, \mathbf{w}^j(t) = 0\}$, $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v_t^i(x^i) + v_t^j(x^j)$, for each $x \in R_+^2$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, for each $t \in T^i$. Assumption 4 in Busetto et al. (2018) can then be restated as follows.

Assumption 4′. *Commodities* 1 *and* 2 *and commodities* 2 *and* 1 *stand in relation* C.

It is immediate to verify that neither our Assumption 4 implies Assumption $4'$ nor the converse holds. Therefore, we can conclude that our existence theorem is not a special case of the existence theorem in Busetto et al. (2018).

Finally, let us notice that, in Section 2, we have provided a definition of a Cournot-Nash equilibrium referring explicitly to irreducible matrices. This definition applies only to active Cournot-Nash equilibria according to the definition of Sahi and Yao (1989). Nevertheless, in the Shapley window model, as in all other strategic market games, the strategy selection $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t) = 0$, for each $t \in T$, is a Cournot-Nash equilibrium, usually called trivial equilibrium. This raises the question whether, under Assumptions 1-4, the allocation corresponding to the trivial Cournot-Nash equilibrium, namely the initial assignment $\mathbf{w}$, may be Pareto optimal. The following proposition provides a negative answer to this question.

Proposition 3. *Under Assumptions 1, 2, 3, and 4, the allocation* $\mathbf{w}$ *is not Pareto optimal.*

Proof. Suppose that $\mathbf{w}$ is Pareto optimal. Then, there exists a price vector $\tilde{p}$ such that the pair $(\tilde{p}, \mathbf{w})$ is an efficiency equilibrium, by the same argument used in the proof of Theorem 2. But then, the pair $(\tilde{p}, \mathbf{w})$ is a Walras equilibrium. Therefore, the necessary Kuhn-Tucker conditions lead to the same contradiction as in the proof of Theorem 1. Hence, the allocation $\mathbf{w}$ is not Pareto optimal. ■

6 Conclusion

In Theorem 1, we have shown the existence of a Cournot-Nash equilibrium for the mixed bilateral oligopoly version of the Shapley window model first analyzed by Codognato et al. (2015). Then, in Theorem 2, we have proved that a Cournot-Nash allocation is Pareto optimal if and only if it is a Walras allocation. The proof of this theorem is crucially based on a corollary in Shitovitz (1973), showing that the first and second welfare theorem still hold in mixed exchange economies. In their main theorem, Codognato et al. (2015) proved that, under a further differentiability assumption on atoms' utility functions, the condition which characterizes the nonempty intersection of the sets of Walras and Cournot-Nash allocations requires that each atom demands a null amount of one commodity. Combining this result with our Theorem 2 we have obtained, as a proposition, a characterization of the optimality property of Cournot-Nash equilibria, which requires that at a Pareto optimal Cournot-Nash equilibrium each atom demands a null amount of one commodity. Recasting antitrust analysis in the bilateral oligopoly framework, we could use these results in further research as a first step to analyze competition policy in a general equilibrium framework.

In the previous section, we have already stressed that the results we have obtained for the bilateral version of the Shapley window model also hold for other prototypes of strategic market games in the line inspired by Shapley and Shubik (1977). Moreover, we have pointed out some issues connected with existence and Pareto optimality which in our opinion deserve to be considered for future research. A further question we propose to answer in forthcoming work is if the results obtained in this paper hold for another type of strategic market game, that is the one with fiat money first introduced by Postlewaite and Schmeidler (1978) and further analyzed by Peck et al. (1992), Koutsougeras and Ziros (2008), Koutsougeras (2009), among others.

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