Commun. Math. Phys. 105, 547–570 (1986)

Communications in Mathematical Physics

Existence and Partial Regularity of Static Liquid Crystal Configurations

Robert Hardt*, David Kinderlehrer*, and Fang-Hua Lin**
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Abstract. We establish the existence and partial regularity for solutions of some boundary-value problems for the static theory of liquid crystals. Some related problems involving magnetic or electric fields are also discussed.

Introduction

The equilibrium configuration of a liquid crystal may be described in terms of its optical axis, a unit vector field n defined on the region Ω in $\mathbb{R}^3$ occupied by the material (see [E]). For a nematic liquid crystal, the Oseen–Frank free energy density W is given by

$$\begin{aligned} 2W(\nabla n, n) = &\kappa_1(\operatorname{div} n)^2 + \kappa_2(n\cdot\operatorname{curl} n)^2 + \kappa_3|n \times \operatorname{curl} n|^2 \\ &+ (\kappa_2 + \kappa_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2], \end{aligned} \tag{0.1}$$

where the constants κ_1, κ_2, κ_3, and κ_4 are generally assumed to satisfy

$$\kappa_1 > 0, \quad \kappa_2 > 0, \quad \kappa_3 > 0, \quad \kappa_2 \geqq |\kappa_4|, \quad \text{and} \quad 2\kappa_1 \geqq \kappa_2 + \kappa_4.$$

(Here we will assume only that κ_1, κ_2, and κ_3 are positive.)

The principal questions we shall discuss are the existence and partial regularity of a vectorfield n with the property that

$$\mathscr{W}(n) = \inf \mathscr{W}(u), \quad \text{where} \quad \mathscr{W}(u) = \int_\Omega W(\nabla u, u)dx, \tag{0.2}$$

and where the infimum is taken over all $u:\Omega \to \mathbb{S}^2$ having prescribed boundary values n_0 on $\partial\Omega$.

The existence of a minimizer $n \in H^1(\Omega, \mathbb{S}^2)$ by direct methods is presented in Sect. 1. The first ingredient of the proof is to establish that the class of competing functions is nonempty. The second involves certain coerciveness estimates for the functional $\mathscr{W}$. Of relevance here is the observation by C. Oseen, and later independently by J. L. Ericksen, that the last term in $\mathscr{W}$ is a surface energy in the

*Research partially supported by the National Science Foundation
**Research supported by an Alfred P. Sloan Graduate Felowship

sense that it depends only on the restriction of n and its tangential gradient to the boundary $\partial\Omega$. In Sect. 5, additional terms are added to $\mathscr{W}$ which allow, in particular, treatment of cholesteric liquid crystals and applied magnetic fields.

The partial regularity result we prove in Sect. 2 is that a minimizer n of $\mathscr{W}$ is real analytic on $\Omega \sim Z$ for some relatively closed subset Z of one dimensional Hausdorff measure zero. Recent work [HKL], involving a reverse Hölder inequality, shows that Z may be chosen to have Hausdorff dimension strictly less than one. In general, continuity of n on *all* of $\bar{\Omega}$ may be impeded by topological considerations; as for example, if Ω is the unit ball $\mathbb{B}$ and n_0 has nonzero degree as a mapping from $\partial\mathbb{B} = \mathbb{S}^2$ to $\mathbb{S}^2$. In the present paper, the set Z is defined as the set of points a in $\bar{\Omega}$ for which the normalized Dirichlet integral on the ball $\mathbb{B}_r(a)$,

$$r^{-1} \int_{\Omega \cap \mathbb{B}_r(a)} |\nabla u|^2 dx \tag{0.3}$$

fails to approach zero as $r \to 0$. Our main work involves establishing that, near points $a \in \Omega \sim Z$, this normalized integral decays like a positive power of r, as $r \to 0$. Local Hölder continuity of n on $\Omega \sim Z$ then follows by Morrey's lemma, and the higher regularity is established in 2.6.

A few words about the proof of Hölder continuity may be appropriate. In 2.3, an integral estimate is used to reduce the question of energy decay to estimating a normalized L^2 norm, a quantity more readily controlled under the "blowing-up" process. An analogous inequality was employed similarly for excess decay in [*HL*]. A suitable integral estimate may be obtained from a construction of R. Schoen and K. Uhlenbeck, [SU, 4.3]. However, here for completeness and the reader's convenience, we give a short proof of a slightly stronger estimate taken from [HL_3].

It is interesting to note that, in our proof by contradiction of energy decay, the constraint $|n| = 1$ implies that the image of any blow-up limit function v lies in a two dimensional plane. (This observation and the use of 2.3 seem to simplify somewhat the regularity theory of harmonic maps as well [SU, SU_2, GG].) Moreover, this v satisfies an elliptic system with constant coefficients, even though, unlike the situation with harmonic maps, the original minimizing problem is not necessarily elliptic.

In the presence of an electric field, one generally accounts for the effect of polarization, unlike the common assumption with magnetic fields. As a consequence, the electrostatic energy depends on an unknown electric field potential which competes with the bulk energy $\mathscr{W}$. In Sect. 4 we show, in a typical problem, how one obtains not only the optical axis n but also the electric field potential.

In all of these problems, the set Z defined above will be a compact subset of $\bar{\Omega}$ having one dimensional measure zero. The solution is regular on $\Omega \sim Z$, and, indeed, on $\bar{\Omega} \sim Z$ when given smooth boundary data (Sect. 5). For minimizing harmonic maps from Ω to $\mathbb{S}^2$ (the special case $\kappa_1 = \kappa_2 = \kappa_3, \kappa_4 = 0$) the work of R. Schoen and K. Uhlenbeck ([SU, SU_2]), using the monotonicity in r of the normalized Dirichlet integral (0.3), shows that Z is just a finite subset of Ω. Moreover, here the asymptotic behavior of a minimizer near a point of Z is describable [GW] by the work of [Si] and the elementary classification of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$. However the precise nature of Z is not well understood. Brézis, Coron, and Lieb [BCL, B] have begun a study of questions related to point defects and the stratification of their levels. For the liquid crystal problem, E. MacMillan obtained an existence result in [Ma] by assuming the κ_i satisfy certain inequalities to guarantee coercitivity of $\mathscr{W}$. He also treated regularity of planar solutions.

The study of stationary, not necessarily minimizing, harmonic maps is quite challenging. For two dimensional domains, R. Schoen [S] has shown the complete regularity of stationary points while Brézis and Coron [BC] have established the existence of "large" solutions. For higher dimensions, G. Liao [L] has shown the removability of an isolated singularity of a small energy stationary harmonic map.

Our partial regularity results for minimizers carry over to higher dimensions (where $\mathscr{H}^{\dim \Omega - 2}(Z) = 0$) for mappings between manifolds which minimize, e.g., the elliptic integrands treated in [GG]. This topic is not pursued here.

1. An Existence Theory

To simplify technical aspects of our presentation we shall assume some smoothness of the domain and the boundary data. Henceforth, let Ω be a bounded region in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and outward unit normal ν. To demonstrate the existence of a solution by direct methods, we are obliged to show that the Oseen–Frank energy functional (0.1), (0.2) has some lower semi-continuity and coerciveness properties.

To begin, we consider, in $H^1(\Omega, \mathbb{R}^3)$, the closed subset

$$H^1(\Omega, \mathbb{S}^2) = \{u \in H^1(\Omega, \mathbb{R}^3) \colon |u| = 1 \text{ almost everywhere in } \Omega\}.$$

Note, in particular, that the fuction $x/|x|$ belongs to $H^1(\Omega, \mathbb{S}^2)$. This example may be used to illustrate how $H^1(\Omega, \mathbb{S}^2)$ may be larger than the H^1 closure of smooth functions mapping Ω into $\mathbb{S}^2$ [SU_2, p. 267].

1.1. Lemma. *If $n_0 \colon \partial\Omega \to \mathbb{S}^2$ is Lipschitz, then the family*

$$\mathscr{A}(n_0) = \{u \in H^1(\Omega, \mathbb{S}^2) \colon n_0 = \text{trace of } u \text{ on } \partial\Omega\}$$

is nonempty.

Proof. If $\bar{\Omega}$ has any handles, then we can choose a smooth embedded closed disk $B \subset \bar{\Omega}$ so that $B \cap \partial\Omega = \partial B$, B is orthogonal to $\partial\Omega$, and B crosses transversely at one point some generator of $\Pi_1(\Omega)$. Using two distinct copies B_- and B_+ of B, we may form a new Lipschitz 3 manifold with boundary, $(\Omega \sim B) \cup B_- \cup B_+$, which has one less handle than $\bar{\Omega}$. Since $n_0 | \partial B$ is nul-homotopic in $\mathbb{S}^2$ we may extend n_0 to B_- and B_+. Continuing, we eventually reduce to the case where $\bar{\Omega}$ is bilipschitz homeomorphic to a closed ball. Then an elementary calculation shows that homogeneous degree 0 extension gives the desired finite energy extension. □

This is only a special case of a general result of B. White [W] on the existence of finite energy extensions. Moreover $\mathscr{A}(n_0)$ is nonempty even for $n_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^2)$ by [HL_3]. It is also interesting that the strong H^1 closure of the continuous functions in $\mathscr{A}(n_0)$ may be strictly smaller than $\mathscr{A}(n_0)$ even when n_0 has degree 0 [HL_2].

In most discussions of the liquid crystal equations, the last term in W is set to zero, namely $\kappa_4 = -\kappa_2$. This is because it is (in a formal sense) a divergence,

$$\operatorname{tr}(\nabla u)^2 - (\operatorname{div} u)^2 = \operatorname{div}[(\nabla u)u - (\operatorname{div} u)u],$$

and thus does not contribute to the equilibrium equations. It is sometimes called a surface energy. As Oseen, and independently, Ericksen [E, p. 238], cf. also [E_2], have observed, it is a surface energy in the strictest sense, for the expression $[(\nabla u)u-(\operatorname{div} u)u]\cdot v$ depends only on $u|\partial\Omega$ and its tangential derivatives.

1.2. Lemma. *For any Lipschitz function $n_0\colon \partial\Omega\to\mathbb{S}^2$, there is a number $\mathscr{S}(n_0)$ such that*

$$\mathscr{S}(n_0)=\tfrac{1}{2}\int_\Omega [\operatorname{tr}(\nabla u)^2-(\operatorname{div} u)^2]dx \quad \textit{for all } u\in\mathscr{A}(n_0).$$

Proof. First suppose $u\in\mathscr{A}(n_0)\cap\mathscr{C}^1(\bar\Omega)$. On $\partial\Omega$ we define

$$\nabla_{\tan}u=\nabla u-(\nabla u)v\otimes v,$$

and easily verify that $\nabla_{\tan}u$ depends only on n_0. Noting that, on $\partial\Omega$,

$$\begin{aligned}&[(\nabla u)u-(\operatorname{div} u)u]\cdot v-[(\nabla_{\tan}u)u-\operatorname{tr}(\nabla_{\tan}u)u]\cdot v=[(\nabla u)(v\otimes v)u-\operatorname{tr}(\nabla u\; v\otimes v)u]\cdot v\\ &=\Sigma_{i,j,k}u^i_{x_j}(v^jv^k)u^kv^i-u^i_{x_j}(v^jv^i)u^kv^k=0,\end{aligned}$$

we deduce from the divergence theorem that

$$2\mathscr{S}(n_0)=\int_{\partial\Omega}[(\nabla u)u-(\operatorname{div} u)u]\cdot v\,d\mathscr{H}^2=\int_{\partial\Omega}[(\nabla_{\tan}u)u-\operatorname{tr}(\nabla_{\tan}u)u]\cdot v\,d\mathscr{H}^2,$$

which depends only on $u|_{\partial\Omega}=n_0$.

For an arbitrary $u\in\mathscr{A}(n_0)$, one may check by the standard device of straightening $\partial\Omega$ locally and applying Fourier transforms, that the latter expression for $\mathscr{S}(n_0)$ is well-defined and depends only on n_0. □

For a given choice of positive constants κ_1, κ_2, and κ_3, let

$$\alpha=\min\{\kappa_1,\kappa_2,\kappa_3\},\quad \beta=3(\kappa_1+\kappa_2+\kappa_3),\text{ and } \tilde{\mathscr{W}}(u)=\int_\Omega \tilde W(\nabla u,u)dx,$$

where

$$\begin{aligned}2\tilde W(\nabla u,u)&=2W(\nabla u,u)+(\alpha-\kappa_2-\kappa_4)[\operatorname{tr}(\nabla u)^2-(\operatorname{div} u)^2]\\ &=\kappa_1(\operatorname{div} u)^2+\kappa_2(u\cdot\operatorname{curl} u)^2+\kappa_3|u\times\operatorname{curl} u|^2\\ &\quad+\alpha[\operatorname{tr}(\nabla u)^2-(\operatorname{div} u)^2]\text{ for } u\in H^1(\Omega,\mathbb{S}^2).\end{aligned}$$

1.3. Corollary. *For any Lipschitz function $n_0\colon\partial\Omega\to\mathbb{S}^2$ and $n\in\mathscr{A}(n_0)$,*

n minimizes $\mathscr{W}$ in $\mathscr{A}(n_0)$ if and only if n minimizes $\tilde{\mathscr{W}}$ in $\mathscr{A}(n_0)$.

Proof. For any $u\in\mathscr{A}(n_0)$, one sees from 1.2 that

$$\begin{aligned}\tilde{\mathscr{W}}(u)-\tilde{\mathscr{W}}(n)&=\mathscr{W}(u)+(\alpha-\kappa_2-\kappa_4)\mathscr{S}(n_0)-[\mathscr{W}(n)+(\alpha-\kappa_2-\kappa_4)\mathscr{S}(n_0)]\\ &=\mathscr{W}(u)-\mathscr{W}(n).\qquad\square\end{aligned}$$

The following indicates the reason for the particular choice of the coefficient $\frac{1}{2}(\alpha-\kappa_2-\kappa_4)$ in the definition of $\tilde W$.

1.4. Lemma. $\frac{1}{2}\alpha|\nabla u|^2\le\tilde W(\nabla u,u)\le\beta|\nabla u|^2$ *for $u\in H^1(\Omega,\mathbb{S}^2)$. Moreover, $\tilde{\mathscr{W}}$ is lower semicontinuous with respect to weak convergence in $H^1(\Omega,\mathbb{S}^2)$.*

Proof. For $u \in H^1(\Omega, \mathbb{S}^2)$, we have the identity $|\nabla u|^2 = \operatorname{tr}(\nabla u)^2 + |\operatorname{curl} u|^2$. To verify this, note that for any square matrix A, $|A|^2 = \operatorname{tr}(AA^t) = \operatorname{tr}(A^2) + \frac{1}{2}|A - A^t|^2$. By the above identity and the definitions of α and β,

$$\begin{aligned}\alpha|\nabla u|^2 &\leqq 2\tilde{W}(\nabla u, u)\\ &= (\kappa_1 - \alpha)(\operatorname{div} u)^2 + \kappa_2(u\cdot\operatorname{curl} u)^2 + \kappa_3|u \times \operatorname{curl} u|^2 + \alpha \operatorname{tr}(\nabla u)^2\\ &\leqq 2\beta|\nabla u|^2.\end{aligned}$$

Letting $\gamma = \min\{\kappa_2, \kappa_3\}$, we may write

$$\begin{aligned}\tilde{W}(\nabla u, u) = {}&\tfrac{1}{2}(\kappa_1 - \alpha)(\operatorname{div} u)^2 + \tfrac{1}{2}(\gamma - \alpha)|\operatorname{curl} u|^2 + \tfrac{1}{2}(\kappa_2 - \gamma)(u\cdot\operatorname{curl} u)^2\\ &+ \tfrac{1}{2}(\kappa_3 - \gamma)|u \times \operatorname{curl} u|^2 + \tfrac{1}{2}\alpha|\nabla u|^2 \quad \text{for } u \in H^1(\Omega, \mathbb{S}^2).\end{aligned}$$

Since each term has a non-negative coefficient, the lower semicontinuity of $\tilde{\mathscr{W}}$ on $\mathscr{A}(n_0)$ follows from the strong convergence in L^2 of any weakly convergent sequence in H^1. □

1.5. Theorem. *For any Lipschitz function n_0: $\partial\Omega \to \mathbb{S}^2$, there exists an $n \in \mathscr{A}(n_0)$ such that $\mathscr{W}(n) = \inf_{u \in \mathscr{A}(n_0)} \mathscr{W}(u)$.*

Proof. Let n_k be a minimizing sequence in $\mathscr{A}(n_0)$ for the functional $\tilde{\mathscr{W}}$ defined above. By 1.4, this sequence has bounded H^1 norm and so possesses a subsequence, weakly convergent to a limit $n \in H^1(\Omega, \mathbb{R}^3)$. From the strong convergence in L^2 and H^1 trace theory, $n \in \mathscr{A}(n_0)$. Finally by the lower semicontinuity 1.4, $\tilde{\mathscr{W}}(n) = \inf_{u \in \mathscr{A}(n_0)} \tilde{\mathscr{W}}(u)$, and the theorem follows from 1.3. □

A cholesteric liquid crystal [E, p. 246] has an energy density of the form

$$\begin{aligned}2W_{\text{cholesteric}}(\nabla n, n) &= \kappa_1(\operatorname{div} n)_2 + \kappa_2[(n\cdot\operatorname{curl} n) + \tau]^2 + \kappa_3|n \times \operatorname{curl} n|^2\\ &\quad + (\kappa_2 + \kappa_4)[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2]\\ &= 2W(\nabla n, n) + 2\kappa_2\tau(n\cdot\operatorname{curl} n) + \kappa_2\tau^2\end{aligned}$$

for some real constant τ. Unlike with nematic liquid crystals, a constant vector field is not a minimizer of the cholesteric free energy. When $\kappa_2 + \kappa_4 = 0$, this role is played, for example, by a mapping of the form

$$n(x) = (\cos \tau x_3, \sin \tau x_3, 0).$$

For a given smooth divergence free vector field H, representing a magnetic field, the contribution to energy may be described by adding to W or $W_{\text{cholesteric}}$ a term of the form

$$F(\nabla u, u) = \sum_{j,k,l} [a_{jkl} u^j_{x_k} u^l + b_{jk} u^j u^k + c_j u^j],$$

where a_{jkl}, b_{jk}, and c_j are bounded (or sufficiently integrable) functions on Ω. In general, let

$$\mathscr{F}(u) = \int_\Omega F(\nabla u, u)dx \quad \text{for } u \in H^1(\Omega, \mathbb{S}^2).$$

1.6. Theorem. *For F as above and $n_0 \colon \partial\Omega \to \mathbb{S}^2$ Lipschitz, there exists an $n \in \mathscr{A}(n_0)$ such that* $(\mathscr{W} + \mathscr{F})(n) = \inf_{u \in \mathscr{A}(n_0)} (\mathscr{W} + \mathscr{F})(u)$.

Proof. The proof is virtually identical to the previous one. With $\tilde{\mathscr{W}}$ defined as before, note that

$$\tfrac{1}{2}\alpha \int_\Omega |\nabla u|^2 dx \leqq (\tilde{\mathscr{W}} + \mathscr{F})(u) + \int_\Omega |F(\nabla u, u)| dx.$$

By Cauchy's inequality and the L^∞ bound on u, a minimizing sequence for $\mathscr{W} + \mathscr{F}$ remains bounded in H^1. Moreover $\mathscr{F}$ is continuous under H^1 bounded weak convergence. □

We complete this section with a discussion of the Euler–Lagrange equations satisfied by a stationary point of $\mathscr{W}$. These will be helpful in studying the blow-up limits of the next section.

Consider a stationary point n of $\mathscr{W}$ in $\mathscr{A}(n_0)$,

$$\delta \int_\Omega W(\nabla n, n) dx = 0.$$

To simplify the explanation, we shall proceed formaly, introducing a multiplier $-\frac{1}{2}\lambda(|u|^2 - 1)$. Then,

$$\delta \int_\Omega \{W(\nabla n, n) - \tfrac{1}{2}\lambda(|n|^2 - 1)\} dx = 0$$

for $n = n_0$ on $\partial\Omega$, but otherwise unconstrained, or

$$[d/dt]_{t=0} \int_\Omega \{W(\nabla(n + t\zeta), n + t\zeta) - \tfrac{1}{2}\lambda(|n + t\zeta|^2 - 1)\} dx = 0$$

for $\zeta \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty$. Thus

$$\int_\Omega \{W_p(\nabla n, n)\cdot\nabla\zeta + W_u(\nabla n, n)\cdot\zeta - \lambda n\cdot\zeta\} dx = 0, \tag{1.1}$$

where λ is an unknown function. The (weak) equation

$$-\operatorname{div}\{W_p(\nabla n, n)\} + W_u(\nabla n, n) = \lambda n \text{ in } \Omega, \quad n = n_0 \quad \text{on} \quad \partial\Omega,$$

with the unknown multiplier λ is useless to us as it stands. The multiplier may be found by choosing $\zeta = \eta n$ in (1.1) where η is a smooth cut-off function. This gives that

$$\int_\Omega \{\eta[W_p(\nabla n, n)\cdot\nabla n + W_u(\nabla n, n)\cdot n - \lambda] + W_p(\nabla n, n)\cdot n \otimes \nabla\eta\} dx = 0.$$

We now write $W(\nabla n, n) = \frac{1}{2}\alpha|\nabla n|^2 + V(\nabla n, n)$, where $\alpha = \min\{\kappa_1, \kappa_2, \kappa_3\}$, so $W_p(\nabla n, n) = \alpha\nabla n + V_p(\nabla n, n)$. Then, since $|n| = 1$ and $n\nabla n = 0$ a.e., we have that $nW_p(\nabla n, n) = nV_p(\nabla n, n)$, hence,

$$W_p(\nabla n, n)\cdot n \otimes \nabla\eta = nW_p(\nabla n, n)\cdot\nabla\eta = nV_p(\nabla n, n)\cdot\nabla\eta.$$

Deleting now the arguments in the symbols $W_p(\nabla n, n)$, $V_p(\nabla n, n)$, $W_u(\nabla n, n)$, and integrating by parts, we find that (in a weak sense) $\lambda = -\operatorname{div}(nV_p) + W_p\cdot\nabla n + W_u\cdot n$, hence,

$$-\operatorname{div} W_p + W_u = [-\operatorname{div}(nV_p)]n + (W_p\cdot\nabla n)n + (W_u\cdot n)n.$$

It is convenient to compare the first term on the right with $\operatorname{div}(n \otimes n V_p)$,

$$[-\operatorname{div}(n V_p)]n = \operatorname{div}(n \otimes n V_p) - \nabla n(n V_p).$$

We finally arrive at the weak equation

$$-\operatorname{div}(W_p - n \otimes n V_p) + (\mathbb{1} - n \otimes n) W_u - \nabla n(n V_p) - (W_p \cdot \nabla n) n = 0 \text{ in } \Omega. \tag{1.2}$$

This system may be abbreviated by writing

$$-\operatorname{div}(W_p - n \otimes n V_p) + Y(\nabla n, n) = 0 \text{ in } \Omega, \tag{1.3}$$

where $|Y(p,u)| \leq c|p|^2$.

For example, in the special case $W(\nabla u, u) = \frac{1}{2}|\nabla u|^2$, the equilibrium equation is

$$\Delta u + |\nabla u|^2 u = 0 \text{ in } \Omega.$$

But in general, λ involves the influence of second derivatives, and it is not clear that (1.2) is elliptic for every choice of the constants κ_i.

2. Partial Regularity of Minimizers

We first discuss *scaling*. For any n that minimizes $\mathscr{W}$ in the class $\mathscr{A}(n_0)$ and any ball $\mathbb{B}_R(a) \subset \Omega$, the formula,

$$n_{R,a}(x) = n(Rx + a) \quad \text{for } x \in \mathbb{B} = \mathbb{B}_1(0),$$

defines a function in $H^1(\mathbb{B}, \mathbb{S}^2)$ which is $\mathscr{W}$ minimizing in $\mathbb{B}$ with respect to its trace on $\partial\mathbb{B}$. We shall study "small energy" minimizers of $\mathscr{W}$ in $H^1(\mathbb{B}, \mathbb{S}^2)$. For $u \in H^1(\mathbb{B}, \mathbb{S}^2)$ and $0 < r < 1$, we define the normalized Dirichlet energy in $\mathbb{B}_r = \mathbb{B}_r(0)$ by

$$\mathbb{E}_r(u) = r^{-1} \int_{\mathbb{B}_r} |\nabla u|^2 dx.$$

Note that

$$\mathbb{E}_r(u_{R,a}) = (rR)^{-1} \int_{\mathbb{B}_{rR}(a)} |\nabla u|^2 dx.$$

Our regularity theorem is based on the behavior of blow-up sequences, obtained through rotation and scaling, and a "hybrid" integral inequality involving the L^2 norms of a minimizer and its gradient.

First we describe a few properties of interest concerning *blowing-up*. For any sequence $u_i \in H^1(\mathbb{B}, \mathbb{S}^2)$, the *associated normalized sequence*

$$v_i = \mathbb{E}_1(u_i)^{-1/2}(u_i - \bar{u}_i) \ \ (\text{where } \bar{u}_i = \fint_{\mathbb{B}} u_i dx = |\mathbb{B}|^{-1} \int_{\mathbb{B}} u_i dx)$$

satisfies $\|v_i\|_{H^1} \leq 1 + c_{\mathbb{B}}^{1/2}$. Here $c_{\mathbb{B}}$ is the best constant for the Poincaré inequality for $\mathbb{B}$,

$$\int_{\mathbb{B}} |u - \bar{u}|^2 dx \leq c_{\mathbb{B}} \int_{\mathbb{B}} |\nabla u|^2 dx \quad \text{for } u \in H^1(\mathbb{B}, \mathbb{R}^3).$$

A subsequence of v_i converges weakly in $H^1(\mathbb{B})$, strongly in $L^2(\mathbb{B})$, and pointwise almost everywhere to a function in $H^1(\mathbb{B}, \mathbb{R}^3)$. We will say that a sequence $u_i \in H^1(\mathbb{B}, \mathbb{S}^2)$ is a *special blow-up sequence* if

$$\bar{u}_i = (0,0,\lambda_i) \text{ for some } \lambda_i \geqq 0, \text{ and, as } i \to \infty,$$
$$\varepsilon_i^2 = \mathbb{E}_1(u_i) \to 0 \text{ and } v_i = \varepsilon_i^{-1}(u_i - \bar{u}_i) \text{ converges weakly in } H^1(\mathbb{B}).$$

Then $v = \lim_{i\to\infty} v_i$ is called the *blow-up limit function* for u_i.

2.1. Lemma (blow-up of constraint). *The image of any blow-up limit function v lies essentially in the $X-Y$ plane.*

Proof. Note that,

$$(1-|\bar{u}_i|)^2 = \fint_{\mathbb{B}} (1-|\bar{u}_i|)^2 dx \leqq \fint_{\mathbb{B}} |u_i - \bar{u}_i|^2 dx \leqq c_{\mathbb{B}} \varepsilon_i^2,$$

by the Poincaré inequality. Thus, as $i \to \infty$,

$$|1-|\bar{u}_i|| \leqq c_{\mathbb{B}}^{1/2} \varepsilon_i \to 0,$$

and for a subsequence of $\{i\}$,

$$\varepsilon_i^{-1}(1-|\bar{u}_i|) \to d \geqq 0 \quad \text{and} \quad \bar{u}_i \to \mathbf{e} = (0,0,1).$$

Observing that, almost everywhere,

$$|v_i|^2 + 2\varepsilon_i^{-1} v_i \cdot \bar{u}_i + \varepsilon_i^{-2} |\bar{u}_i|^2 = |v_i + \varepsilon_i^{-1} \bar{u}_i|^2 = |\varepsilon_i^{-1} u_i|^2 = \varepsilon_i^{-2},$$
$$\varepsilon_i |v_i|^2 + 2 v_i \cdot \bar{u}_i = \varepsilon_i^{-1}(1-|\bar{u}_i|^2) = (1+|\bar{u}_i|)\varepsilon_i^{-1}(1-|\bar{u}_i|),$$

we pass to the limit as $i \to \infty$, strongly in L^1, to conclude that $0 + 2v\cdot\mathbf{e} = 2d$ almost everywhere in $\mathbb{B}$.

Moreover, $d = \fint_{\mathbb{B}} v \cdot \mathbf{e}\, dx = \lim_{i\to\infty} \bar{v}_i \cdot \mathbf{e} = 0.$ □

2.2. Lemma (blow-up equation). *For any blow-up limit $v = (v^1, v^2, 0)$ of a special blow-up sequence of $\mathscr{W}$ minimizers, $v' = (v^1, v^2)$, is a solution of the constant coefficient elliptic system*

$$-\operatorname{div} \tilde{W}'_p(\nabla v', \mathbf{e}) = 0 \textit{ in } \mathbb{B},$$

where $\tilde{W}'_p$ denotes the first two rows of the matrix $\tilde{W}_p$. In particular, there is a positive constant c_0 (depending only on κ_1, κ_2, and κ_3) such that

$$\fint_{\mathbb{B}_r} |v|^2 dx \leqq c_0 r^2 \int_{\mathbb{B}} |v|^2 dx \quad \textit{for } 0 \leqq r \leqq 1. \tag{2.1}$$

Proof. In view of 1.3 and Eq. (1.3),

$$\int_{\mathbb{B}} \{[\tilde{W}_p(\nabla u_i, u_i) - u_i \otimes u_i \tilde{V}_p(\nabla u_i, u_i)] \cdot \nabla \zeta + \tilde{Y}(\nabla u_i, u_i) \cdot \zeta\} dx = 0$$

for any $\zeta \in H_0^1(\mathbb{B}, \mathbb{R}^3) \cap L^\infty$. Substituting $\nabla u_i = \varepsilon_i \nabla v_i$, dividing by ε_i, and letting $i \to \infty$, we obtain

$$\int_{\mathbb{B}} [\tilde{W}_p(\nabla v, \mathbf{e}) - \mathbf{e} \otimes \mathbf{e} V_p(\nabla v, \mathbf{e})] \cdot \nabla \zeta dx = 0$$

because $\tilde{Y}_n(\nabla u_i, u_i)$ is quadratic in ∇u_i,

$u_i \to \mathbf{e}$ strongly in L^2, $|u_i| = 1$ almost everywhere,
$\nabla v_i \to \nabla v$ weakly in L^2, and $\sup_i \|\nabla v_i\|_{L^2} < \infty$.

Moreover, by choosing ζ with $\zeta \cdot \mathbf{e} = 0$ [i.e., $\zeta = (\zeta^1, \zeta^2, 0)$], the equation simplifies to

$$\int_{\mathbb{B}} \tilde{W}_p(\nabla v, \mathbf{e}) \cdot \zeta dx = 0$$

because

$$(\mathbf{e} \otimes \mathbf{e}) \tilde{V}_p(\nabla v, \mathbf{e}) \cdot \nabla \zeta = [\tilde{V}_p(\nabla v, \mathbf{e}) \cdot \mathbf{e} \otimes \mathbf{e}] \nabla \zeta = [\tilde{V}_p(\nabla v, \mathbf{e}) \cdot \nabla(\mathbf{e} \cdot \zeta)] \mathbf{e} = 0.$$

Thus $v' = (v^1, v^2)$ is a weak solution of the system of two equations in two unknowns $-\operatorname{div} \tilde{W}'_p(\nabla v', \mathbf{e}) = 0$. This is elliptic because, writing $W_{p_{ij}}(\xi, \mathbf{e}) = \Sigma_{h,k} A_{ijhk} \xi_{hk}$, we have the inequality $\tilde{W}_p(\xi, \mathbf{e}) \cdot \xi \geqq \alpha |\xi|^2$ for all $\xi = (\xi_{ij})$, which implies, in the special case of ξ with vanishing third column, that

$$\sum_{i,k=1,2} \sum_{j,k=1,2,3} A_{hkij} \xi_{hk} \xi_{ij} \geqq \alpha |\xi|^2.$$

Finally since $\bar{v} = 0$, the L^2 estimate follows from standard linear elliptic theory (see e.g. [F, 5.2.5]). □

2.3. Lemma (Hybrid inequality). *There exists a positive constant c (depending only on κ_1, κ_2, and κ_3) so that if $0 < \lambda < 1$ and if u is a minimizer of $\mathscr{W}$ in $H^1(\mathbb{B}, \mathbb{S}^2)$, then*

$$\mathbb{E}_{1/2}(u) \leqq \lambda \mathbb{E}_1(u) + c\lambda^{-1} \int_{\mathbb{B}} |u - \bar{u}|^2 dx.$$

Proof. For an increasing fuction η on $[0, 1]$,

$$\{s : \eta'(s) \geqq 8[\eta(1) - \eta(0)]\}$$

has Lebesgue measure $\leqq 1/8$. In particular, there is an $r \in [\frac{1}{2}, 1]$ such that $u | \partial \mathbb{B}_r \in H^1(\partial \mathbb{B}_r, \mathbb{S}^2)$,

$$\int_{\partial \mathbb{B}_r} |\nabla_{\tan} u|^2 d\mathscr{H}^2 \leqq 8 \int_{\mathbb{B}} |\nabla u|^2 dx, \quad \int_{\partial \mathbb{B}_r} |u - \bar{u}|^2 d\mathscr{H}^2 \leqq 8 \int_{\mathbb{B}} |u - \bar{u}|^2 dx. \tag{2.2}$$

(A slightly weakened version of 2.3 (resulting from replacing λ^{-1} by λ^{-q} for some positive q, and assuming $\mathbb{E}_1(u)$ sufficiently small) may now be easily derived from [SU, 4.3]. The argument given below is taken from [HL_3].)

We claim that there exists a function $w \in H^1(\mathbb{B}_r, \mathbb{S}^2)$ satisfying

$$w | \partial \mathbb{B}_r = u | \partial \mathbb{B}_r \quad \text{and} \quad \int_{\mathbb{B}_r} |\nabla w|^2 dx \leqq 32 \Big(\int_{\partial \mathbb{B}_r} |\nabla_{\tan} u|^2 d\mathscr{H}^2\Big)^{1/2} \Big(\int_{\partial \mathbb{B}_r} |u - \bar{u}|^2 d\mathscr{H}^2\Big)^{1/2}. \tag{2.3}$$

With such a comparison function w, we may use 1.4 and the $\tilde{\mathscr{W}}$ minimality of $u|\mathbb{B}_r$ to infer that

$$\mathbb{E}_{1/2}(u) \leqq 2\mathbb{E}_r(u) \leqq 4\alpha^{-1}\tilde{\mathscr{W}}(u|\mathbb{B}_r) \leqq 4\alpha^{-1}\tilde{\mathscr{W}}(w|\mathbb{B}_r) \leqq 4\beta\alpha^{-1}\int_{\mathbb{B}_r}|\nabla w|^2 dx,$$

and then obtain the desired hybrid inequality by employing (2.2) and Cauchy's inequality $A\cdot B \leqq \frac{1}{2}\delta A^2 + \frac{1}{2}\delta^{-1}B^2$ with $\delta = \lambda/512\beta\alpha^{-1}$.

To obtain a function w satisfying (2.3), we first choose the harmonic function $h:\mathbb{B}_r \to \mathbb{R}^3$ with $h|\partial\mathbb{B}_r = u|\partial\mathbb{B}_r$. Using the divergence theorem, Schwarz's inequality, and the harmonic fuction identity

$$r\int_{\partial\mathbb{B}_r}|\nabla_{\tan}h|^2 d\mathscr{H}^2 = \int_{\mathbb{B}_r}|\nabla h|^2 dx + r\int_{\partial\mathbb{B}_r}|\partial h/\partial r|^2 d\mathscr{H}^2,$$

we obtain the desired inequality with w replaced by h,

$$\begin{aligned}\int_{\mathbb{B}_r}|\nabla h|^2 dx = \int_{\mathbb{B}_r}|\nabla(h-\bar{u})|^2 dx &= \int_{\partial\mathbb{B}_r}(h-\bar{u})\cdot(\partial h/\partial r)d\mathscr{H}^2 \\ &\leqq \Big(\int_{\partial\mathbb{B}_r}|h-\bar{u}|^2 d\mathscr{H}^2\Big)^{1/2}\Big(\int_{\partial\mathbb{B}_r}|\partial h/\partial r|^2 d\mathscr{H}^2\Big)^{1/2} \\ &\leqq \Big(\int_{\partial\mathbb{B}_r}|u-\bar{u}|^2 d\mathscr{H}^2\Big)^{1/2}\Big(\int_{\partial\mathbb{B}_r}|\nabla_{\tan}u|^2 d\mathscr{H}^2\Big)^{1/2}. \end{aligned} \tag{2.4}$$

Unfortunately, the image of h probably does not lie in $\mathbb{S}^2$ (although it does lie in $\mathbb{B}_1$). To correct this we consider, for $a \in \mathbb{B}_{1/2}$, the projection

$$\Pi_a(x) = (x-a)/|x-a|,$$

and note that, by Sard's theorem, the composition $\Pi_a \circ h \in H^1(\mathbb{B}_r, \mathbb{S}^2)$ for almost all a. Using Fubini's theorem, we estimate

$$\begin{aligned}\int_{\mathbb{B}_{1/2}}\int_{\mathbb{B}_r}|\nabla(\Pi_a\circ h)(x)|^2 dx\,da &\leqq 2\int_{\mathbb{B}_r}|\nabla h(x)|^2\int_{\mathbb{B}_{1/2}}|h(x)-a|^{-2}da\,dx \\ &\leqq 2\int_{\mathbb{B}_r}|\nabla h(x)|^2\int_{\mathbb{B}_1}|y|^{-2}dy\,dx = 8\pi\int_{\mathbb{B}_r}|\nabla h(x)|^2 dx.\end{aligned}$$

Thus we may choose $a \in \mathbb{B}_{1/2}$ so that $\int_{\mathbb{B}_r}|\nabla(\Pi_a\circ h)(x)|^2 dx \leqq 8\int_{\mathbb{B}_r}|\nabla h(x)|^2 dx$. Letting $w = (\Pi_a|\mathbb{S}^2)^{-1}\circ\Pi_a\circ h$, we conclude that $w|\partial\mathbb{B}_r = u|\partial\mathbb{B}_r$ and that

$$\int_{\mathbb{B}_r}|\nabla w(x)|^2 dx \leqq [\mathrm{Lip}(\Pi_a|\mathbb{S}^2)^{-1}]^2\int_{\mathbb{B}_r}|\nabla(\Pi_a\circ h)(x)|^2 dx \leqq 32\int_{\mathbb{B}_r}|\nabla h(x)|^2 dx,$$

which, along with (2.4), implies (2.3). □

2.4. Theorem (Energy improvement). *There are positive constants ε and $\theta < 1$ (depending only on κ_1, κ_2, and κ_3) so that if u is a minimizer of $\mathscr{W}$ in $H^1(\mathbb{B}, \mathbb{S}^2)$ with $\mathbb{E}_1(u) < \varepsilon^2$, then*

$$\mathbb{E}_\theta(u) \leqq \theta\mathbb{E}_1(u).$$

Proof. Were the theorem false, there would be, for each θ with $0<\theta<1$, a sequence $u_1, u_2, \ldots$, of $\mathscr{W}$ minimizers so that $\varepsilon_i^2 = \mathbb{E}_1(u_i) \to 0$ as $i \to \infty$ and $\mathbb{E}_\theta(u_i) > \theta\varepsilon_i^2$ for each i. Passing to a subsequence, we may assume that $v_i = \varepsilon_i^{-1}(u_i - \bar{u}_i)$ converges weakly in H^1 to a function $v \in H^1(\mathbb{B}, \mathbb{R}^3)$. Moreover, by choosing rotations Q_i of $\mathbb{R}^3$ so that the vectors $\overline{Q_i u_i}$ are proportional to $\mathbf{e}$, and by replacing u_i by $Q_i u_i$, we see that the u_i now form a special blow-up sequence (as considered in 2.1 and 2.2).

For $\theta \leqq r \leqq 1$, and i sufficiently large (depending on θ),

$$\left|\fint_{\mathbb{B}_r} |v_i|^2 dx - \fint_{\mathbb{B}_r} |v|^2 dx\right| \leqq \fint_{\mathbb{B}_r} |v_i - v|^2 dx \leqq c_0\theta^2 \leqq c_0 r^2.$$

For such i, it follows from (2.1) that

$$\fint_{\mathbb{B}_r} |u_i - \bar{u}_i|_2 dx \leqq \varepsilon_i^2 \fint_{\mathbb{B}_r} |v_i|^2 dx \leqq \varepsilon_i^2 \Big[c_0 r^2 + \fint_{\mathbb{B}_r} |v|^2 dx\Big] \leqq 2c_0 r^2 \varepsilon_i^2 \tag{2.5}$$

whenever $\theta \leqq r \leqq 1$. For each i, we may also apply the hybrid inequality 2.3 to the normalized function $(u_i)_{2\theta,0}$ to obtain

$$\mathbb{E}_\theta(u_i) \leqq \lambda \mathbb{E}_{2\theta}(u_i) + c\lambda^{-1} \fint_{\mathbb{B}_{2\theta}} |u_i - \bar{u}_i|^2 dx.$$

Choosing the positive integer $k = k(\theta)$ for which $0 < 2^k\theta \leqq 1$, we iterate $k-1$ more times and apply estimate (2.5). to obtain

$$\begin{aligned}
\mathbb{E}_\theta(u_i) &\leqq \lambda^k \mathbb{E}_{2^k\theta}(u_i) + \sum_{j=1}^{k} \lambda^{j-1} c\lambda^{-1} \fint_{\mathbb{B}_{2^j\theta}} |u_i - \bar{u}_i|^2 dx \\
&\leqq \cdots \leqq 2\lambda^k \varepsilon_i^2 + 2c_0 c\lambda^{-1} \sum_{j=1}^{k} \lambda^{j-1} (2^j\theta)^2 \varepsilon_i^2 \\
&\leqq 2[\lambda^k + (1-4\lambda)^{-1} c_0 c \lambda^{-2} \theta^2] \varepsilon_i^2,
\end{aligned}$$

for i sufficiently large (depending on θ and λ). Letting $\lambda = \theta^{3/k}$, we see that $\lambda \leqq 1/8$. Finally, since $k \to \infty$ as $\theta \to 0$, we may fix $\theta < 1/4$ small enough to guarantee that

$$16 c_0 c \theta < \theta^{6/k},$$

and conclude that, for i sufficiently large,

$$\mathbb{E}_\theta(u_i) < [2\theta^3 + 2\cdot 2\cdot (1/16)\theta] \varepsilon_i^2 \leqq (1/4\theta + 1/4\theta)\varepsilon_i^2 < \theta\varepsilon_i^2,$$

contradicting the choice of u_i. □

2.5. Corollary (Energy decay). *If $n \in H^1(\Omega, \mathbb{S}^2)$ is a minimizer of $\mathscr{W}$ (as in Sect. 1), if $\mathbb{B}_R(a) \subset \Omega$, and if $\int_{\mathbb{B}_R(a)} |\nabla n|^2 dx \leqq \varepsilon^2 R$, then*

$$\int_{\mathbb{B}_r(a)} |\nabla n|^2 dx \leqq \theta^{-2} R^{-1} \varepsilon^2 r^2 \quad \text{for } 0 \leqq r \leqq R,$$

where ε and θ are as in 2.4.

Proof. Apply 2.4 with $u(x) = n_{R,a}(x) = n(Rx + a)$, then with

$$u(x) = n_{\theta R,a}(x), \quad n_{\theta^2 R,a}(x), \quad n_{\theta^2 R,a}(x), \ldots$$

to infer inductively that

$$(\theta^k R)^{-1} \int_{\mathbb{B}_{\theta^k R}(a)} |\nabla n|^2 dx = \mathbb{E}_1(n_{\theta^k R,a}) = \mathbb{E}_\theta(n_{\theta^{k-1} R,a})$$
$$\leqq \theta \mathbb{E}_1(n_{\theta^k R,a}) \leqq \theta \cdot \theta^{k-1} \mathbb{E}_1(n_{R,a}) = \theta^k \varepsilon^2$$

for $k = 1, 2, 3, \ldots$. Given $0 < r \leqq R$, choose k so that $\theta^{k+1} R < r \leqq \theta^k R$ to conclude that

$$r^{-1} \int_{\mathbb{B}_r(a)} |\nabla n|^2 dx \leqq \theta^{-1} (\theta^k R)^{-1} \int_{\mathbb{B}_{\theta^k R}(a)} |\nabla n|^2 dx \leqq \theta^{-1} \theta^k \varepsilon^2 \leqq \theta^{-2} R^{-1} \varepsilon^2 r. \qquad \square$$

2.6. Theorem (Interior partial regularity). *If $n \in H^1(\Omega, \mathbb{S}^2)$ is a minimizer of $\mathscr{W}$ (as in Sect. 1), then n is analytic on $\Omega \sim Z$ for some relatively closed subset Z of Ω which has one dimensional Hausdorff measure zero.*

Proof. Let

$$Z = \{a \in \Omega : \limsup_{r \downarrow 0} r^{-1} \int_{\mathbb{B}_r(a)} |\nabla n|^2 dx > 0\}.$$

Since $\int_\Omega |\nabla n|^2 dx < \infty$, an elementary covering argument [F, 2.10.19(3)] shows that Z has one dimensional Hausdorff measure zero.

Fix a point $a \in \Omega \sim Z$ and choose $R > 0$ so that $\mathbb{B}_{2R}(a) \subset \Omega \sim Z$ and

$$R^{-1} \int_{\mathbb{B}_{2R}(a)} |\nabla n|^2 dx \leqq \varepsilon^2.$$

Then for any $b \in \mathbb{B}_R(a)$,

$$R^{-1} \int_{\mathbb{B}_R(b)} |\nabla n|^2 dx \leqq \varepsilon^2,$$

and so, by 2.3.

$$\int_{\mathbb{B}_r(b)} |\nabla n|^2 dx \leqq \theta^{-2} R^{-1} \varepsilon^2 r^2 \quad \text{for } 0 \leqq r \leqq R.$$

Thus $\mathbb{B}_R(a) \subset \Omega \sim Z$. We conclude that Z is relatively closed in Ω, and, by Morrey's Lemma [M, 3.5.2], that $n \in C^{0,1/2}[\mathbb{B}_R(a)]$.

To infer the higher regularity of n near a, we assume that $n(a) = \mathbf{e} = (0, 0, 1)$ and choose $0 < s < R$ so that $n[\mathbb{B}_S(a)]$ is contained in $\mathbb{S}^2 \cap \mathbb{B}_{1/2}(\mathbf{e})$. With $n = (n^1, n^2, n^3)$, we may, in $\mathbb{B}_S(a)$, substitute

$$n^3 = \sqrt{1 - (n^1)^2 - (n^2)^2}, \quad \nabla n^3 = (-n^1 \nabla n^1 - n^2 \nabla n^2) / \sqrt{1 - (n^1)^2 - (n^2)^2},$$

in the weak Eq. (1.3) with $\zeta = (\zeta^1, \zeta^2, 0)$ to infer that $(n^1, n^2) | \mathbb{B}_s(a)$ is a critical point for $\int_{\mathbb{B}_s(a)} J(\nabla u, u) dx$ where, for $p = (p^1, p^2) \in (\mathbb{R}^2)^2$ and $u = (u^1, u^2) \in \mathbb{R}^2$,

$$J(p, u) = \tilde{W}[(p^1, p^2, (-u^1 p^1 - u^2 p^2) / \sqrt{1 - (u^1)^2 - (u^2)^2}, [u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2}]].$$

Note that J is analytic on $\{(p, u) : |u| < 1\}$ and that $J(\cdot, u)$ is a quadratic polynomial for each u. Moreover, by 1.3,

$$J(p,0)=\tilde{W}[(p,0),\mathbf{e}]\geqq \tfrac{1}{2}\alpha|p|^2 \quad \text{for all } p\in(\mathbb{R}^2)^2.$$

Then, for some positive $\delta<\frac{1}{2}$, $J(p,u)\geqq\frac{1}{4}\alpha|p|^2$ on $\{(p,u)\colon |u|<\delta\}$. Choosing, by the continuity of n at a, a positive t so that $n[\mathbb{B}_t(a)]$ is contained in $\mathbb{B}_\delta(\mathbf{e})$, we conclude that $(n^1,n^2)|\mathbb{B}_t(a)$ satisfies a strongly elliptic system with analytic coefficients. By [M, 6.7], $(n^1,n^2)|\mathbb{B}_t(a)$, and hence $n|\mathbb{B}_t(a)$, is analytic. □

3. Partial Regularity with the Modified Functional

Let F and $\mathscr{F}$ be as in 1.5, and let

$$[F]=\sup_{\Omega}\Sigma_{j,k,l}|a_{jkl}|+\Sigma_{jk}|b_{j,k}|+\Sigma_j|c_j|.$$

Our discussion of partial regularity will closely follow Sect. 2.

3.1. *Scaling.* If n is a minimizer for the functional $\mathscr{W}+\mathscr{F}$ in $H^1(\Omega,\mathbb{S}^2)$, then, for any ball $\mathbb{B}_r(a)\in\Omega$, the scaled function $n_{r,a}$ (considered in 2.1) now minimizes $\mathscr{W}+\mathscr{F}_{r,a}$, where the coefficients of the corresponding integrand $F_{r,a}$ are defined, for $x\in\bar{\mathbb{B}}$, by the expressions

$$ra_{jkl}(rx+a),\quad r^2b_{jk}(rx+a),\quad r^2c_j(rx+a).$$

Note that $[F_{r,a}]\leqq r[F]$. We shall study "small energy" minimizers of $\mathscr{W}+\mathscr{F}$ in $H^1(\mathbb{B},\mathbb{S}^2)$ where $[F]$ is small.

3.2. Lemma (blow-up equation). *Suppose $v=(v^1,v^2,0)\in H^1(\mathbb{B},\mathbb{S}^2)$ is a blow-up limit for a special blow-up sequence u_i (as in 2.1) where each u_i minimizes some functional $\mathscr{W}+\mathscr{F}_i$ so that the integrand F_i corresponding to $\mathscr{F}_i$ satisfies*

$$\lim_{i\to\infty}[F_i]/\varepsilon_i=0.$$

Then, as in 2.2, $v'=(v_1,v_2)$ again satisfies the elliptic system $-\operatorname{div}\tilde{W}'_p(\nabla v',\mathbf{e})=0$ in $\mathbb{B}$, (and hence the L^2 estimate of 2.2).

Proof. With $\zeta=(\zeta^1,\zeta^2,0)$ as in 2.2, we again substitute $\nabla u_i=\varepsilon_i\nabla v_i$ in the weak equation for u_i and divide by ε_i. The new terms are

$$\Sigma_{j,k,l}a^1_{j,k,l}[(v_i)_{x_k}\zeta+\varepsilon_i^{-1}(u_i)^1(\zeta_{x_k})_i]+\varepsilon_i^{-1}\Sigma_{j,k}b_{j,k}[(u_i)^j\zeta^k+(u_i)^k\zeta^j]+\varepsilon_i^{-1}\Sigma_j c_j\zeta^j,$$

where $\zeta_i=(\mathbb{1}-u_i\otimes u_i)\zeta$, $(\nabla\zeta)_i=(\mathbb{1}-u_i\otimes u_i)\nabla\zeta$, and a_{jkl}, b_{jk}, and c_j are the coefficients of F_i. The integral of these terms approaches 0 as $i\to\infty$ because of the assumption on $[F_i]$. □

One may check that Lemma 3.2 remains true without the hypothesis $\lim_{i\to\infty}[F_i]/\varepsilon_i=0$ in case the F_i all come from a single cholesteric energy function.

3.3. Lemma (Hybrid inequality). *There exists a positive constant c (depending only on κ_1, κ_2, and κ_3) so that if $0<\lambda<1$, F is as in 1.5 with $\Omega=\mathbb{B}$, and u is a minimizer of*

$\mathscr{W}+\mathscr{F}$ in $H^1(\mathbb{B},\mathbb{S}^2)$, then

$$\mathbb{E}_{1/2}(u) \leqq \lambda \mathbb{E}_1(u) + c\lambda^{-1} \int_{\mathbb{B}} |u-\bar{u}|^2 dx + c\lambda^{-2}[F]^2.$$

Proof. To verify this, we argue as in 2.2 (with λ replaced by $\lambda/2$) and use again the function w as a comparison function. We now have various additional terms arising from $F(u)$ and $F(w)$. To handle these, note that $\mathbb{E}_r(w) \leqq c\lambda^{-1}\mathbb{E}_1(u)$. Thus we may employ the inequality $|\Sigma_{j,k,l} a_{jkl} n^j_{x_k} n^l| \leqq \mu |\nabla n|^2 + (2\mu)^{-2}[F]^2$ with $n=u$ or $n=w$ and with μ being a suitable multiple of λ^2 to guarantee that the square gradient terms may be absorbed as contributions to $\lambda\mathbb{E}_r(u)$. The remaining terms coming from $F(u)$ and $F(w)$ all can be bounded by $c\lambda^{-2}[F]^2$ because $|u|=|w|=1$ almost everywhere. □

3.4. Theorem (Energy improvement). *There are positive constants ε, η, and $\theta<1$ (depending only on κ_1, κ_2, and κ_3) so that if u is a minimizer of $\mathscr{W}+\mathscr{F}$ in $H^1(\mathbb{B},\mathbb{S}^2)$ with $\mathbb{E}_1(u)<\varepsilon^2$, then*

$$\mathbb{E}_\theta(u) \leqq \theta \max\{\mathbb{E}_1(u), \eta[F]^2\}.$$

Proof. We argue as in 2.4. If the theorem were false, then, for any fixed $0<\theta<\frac{1}{2}$, there would exist $\mathscr{F}_i$ as in 1.5 as well as minimizers u_i of $\mathscr{W}+\mathscr{F}_i$ for which $\varepsilon_i^2 = \mathbb{E}_1(u_i) \to 0$ and $\mathbb{E}_\theta(u_i)/\theta[F_i]^2 \to \infty$ as $i\to\infty$ while $\mathbb{E}_\theta(u_i) > \theta\varepsilon_i^2$ for each i. In particular, $\lim_{i\to\infty}[F_i]/\varepsilon_i = 0$, because $\varepsilon_i^2 = \mathbb{E}_1(u_i) \geqq \theta\mathbb{E}_\theta(u_i)$. A blow-up limit function v, chosen as in 2.4, now satisfies the conclusions of both 2.1 and 3.2. Using the same L^2 estimate for v and choosing k, θ, and λ as before, we deduce that

$$\mathbb{E}_\theta(u_i) \leqq \tfrac{1}{2}\theta\varepsilon_i^2 + c\lambda^{-2}[F_i]^2 \sum_{j=1}^{k} \lambda^{j-1}(2^j\theta)^2 < \theta\varepsilon_i^2,$$

for i sufficiently large, contradicting the choice of u_i. □

3.5. Corollary (Energy decay). *If $n \in H^1(\Omega,\mathbb{S}^2)$ is a minimizer of $\mathscr{W}+\mathscr{F}$ (as in 1.6), if $\mathbb{B}_R(a) \subset \Omega$, and if $\int_{\mathbb{B}_R(a)} |\nabla n|^2 dx \leqq \varepsilon^2 R$, then*

$$\int_{\mathbb{B}_r(a)} |\nabla n|^2 dx \leqq \theta^{-2} \max\{\varepsilon^2, \eta[F]^2R^2\} R^{-1} r^2 \quad \text{for } 0<r<R.$$

Proof. Recalling from 3.1 the scaling estimate for [F], we iterate 3.4 as in 2.5 with ε^2 replaced by $\max\{\varepsilon^2, \eta[F]^2R^2\}$. □

3.6. Theorem (Interior partial regularity). *If the coefficients of F belong to $\mathscr{C}^{k,\mu}_{\mathrm{loc}}(\Omega)$, where $k \in \{0,1,\dots,\infty,\omega\}$ and $0<\mu<1$, then each minimizer n of $\mathscr{W}+\mathscr{F}$ belongs to $\mathscr{C}^{k,\mu}_{\mathrm{loc}}(\Omega \sim Z, \mathbb{S}^2)$ for some relatively closed subset Z of Ω which has one dimensional Hausdorff measure zero.*

Proof. Taking Z as in 2.6, Hölder continuity on $\Omega \sim Z$ again follows from Morrey's Lemma. The argument for higher regularity using elliptic theory continues to hold in the presence of the additional lower order terms whose coefficients are locally bounded in $\mathscr{C}^{k,\mu}$ norm. □

4. Electric Fields

An electric field E impressed on the liquid crystal gives rise to a polarization density $P = \varepsilon_\perp E + \varepsilon_a(E\cdot u)u$ in Ω for some constants $\varepsilon_\perp$ and ε_a. For a positive constant ε_0, the displacement vector is given by $D = \varepsilon_0 E + P$ in Ω, and contributes the term $-\frac{1}{2}D\cdot E$ to the energy of the system. In the absence of free charge and in static equilibrium, Maxwell's equations for D and E are $\operatorname{div} D = 0$ and $\operatorname{curl} E = 0$. Because of the second equation, it is reasonable to consider an electric potential function φ for E, $E = -\nabla\varphi$. The field E, or, for our purposes, the potential φ, will be regarded as another dependent variable in the problem, unlike the case of a magnetic field, cf. [DeG, p. 99] or [HK].

In order to simplify notation and to avoid confusion, we set $\alpha_0 = \varepsilon_0 + \varepsilon_\perp$ and $\alpha_a = \varepsilon_a$. Then

$$D = D(\nabla\varphi, n) = -[\alpha_0 \mathbb{1} + \alpha_a n\otimes n)\nabla\varphi,$$
$$A(\nabla\varphi, n) = \tfrac{1}{2}D\cdot E = -\tfrac{1}{2}[\alpha_0 \mathbb{1} + \alpha_a n\otimes n)\nabla\varphi\cdot\nabla\varphi.$$

Assuming further that $|\alpha_a| < \alpha_0$, we obtain the coerciveness condition $A(\xi, u) \geqq \lambda|\xi|^2$ for $\xi\in\mathbb{R}^3$, where λ depends only on α_a and α_0. The total energy of a virtual director configuration u and field potential ψ is $\mathscr{E}^*(u,\psi) = \int_\Omega [W(\nabla u, u) - A(\nabla\psi, u)]dx$. This functional is not bounded below because the two energies complete. Nevertheless, we can obtain critical points by imposing Gauss's law as a constraint. We may then extend our partial regularity theorem to this case.

As a typical problem, we shall consider given fixed functions $n_0\colon \partial\Omega \to \mathbb{S}^2$ and $\varphi_0\colon \partial\Omega \to \mathbb{R}$, where n_0 is Lipschitz and $\varphi_0\in H^{1/2}(\partial\Omega)$. (Other boundary values and boundary value problems may be treated.) Let

$$\mathscr{A}^*(n_0) = \mathscr{A}(n_0) \times \{\psi\in H^1(\Omega)\colon \psi = \varphi_0 \text{ on } \partial\Omega\}.$$

We wish to find a critical point $(n,\varphi)\in\mathscr{A}^*(n_0)$ of $\mathscr{E}^*$, i.e., a solution of $\delta\mathscr{E}^*(n,\varphi) = 0$.

For any $u\in H^1(\Omega, \mathbb{S}^2)$, the Dirichlet problem

$$-\operatorname{div}(\alpha_0 \mathbb{1} + \alpha_a u\otimes u)\nabla\psi = 0 \text{ in } \Omega, \quad \psi = \varphi_0 \text{ on } \partial\Omega,$$

has a unique solution which we denote by $\Phi(u)$ [or by $\Phi_{\varphi_0}(u)$ to indicate the dependence on the boundary values φ_0]. Thus $\Phi(u)$ is the unique minimizer of $\int_\Omega A(\nabla\psi, u)dx$ among $\psi\in H^1(\Omega)$ with $\psi = \varphi_0$ on $\partial\Omega$. Then, if $\tilde\varphi$ is some fixed H^1 extension of φ_0 to Ω, we have the obvious estimate

$$\lambda\int_\Omega |\nabla\Phi(u)|^2 dx \leqq \int_\Omega A[\nabla\Phi(u), u]dx \leqq \int_\Omega A(\nabla\tilde\varphi, u)dx \leqq c\int_\Omega |\nabla\tilde\varphi|^2 dx \leqq c(\Omega, \varphi_0). \quad (4.1)$$

For $u\in H^1(\Omega, \mathbb{S}^2)$ we now let $\mathscr{E}(u)$ [or $\mathscr{E}_{\varphi_0}(u)$ to indicate the dependence on φ_0] denote the energy

$$\mathscr{E}(u) = \mathscr{E}^*[u, \Phi(u)] \text{ [i.e. } \mathscr{E}_{\varphi_0}(u) = \mathscr{E}^*[u, \Phi_{\varphi_0}(u)]]. \quad (4.2)$$

It is worthwhile noting immediately that the potential contribution $\varphi = \Phi(n)$ will not impede the partial regularity of n (obtained below) because this contribution occurs through $\nabla\varphi$ and, owing to the De Giorgi–Nash theorem,

$$\int_{\mathbb{B}_r(a)} |\nabla\varphi|^2 dx \leqq Cr^{1+\mu} \text{ for } a\in\Omega \text{ and } r \text{ sufficiently small.} \tag{4.3}$$

For our existence theory, we shall obviously minimize $\mathscr{E}(u)$ in $\mathscr{A}(n_0)$, but first let us verify that this gives the correct result.

4.1. Theorem. *A pair (n, φ) is a critical point of $\mathscr{E}^*(u, \psi)$,*

$$\delta\mathscr{E}^*(n, \varphi) = 0 \quad on \quad \mathscr{A}^*(n_0), \tag{4.4}$$

if and only if $\varphi = \Phi(n)$ and

$$\delta\mathscr{E}(n) = 0 \quad \text{on} \quad \mathscr{A}(n_0). \tag{4.5}$$

In this case, (n, φ) is a (weak) solution of the field equations [see (1.11)]

$$-\operatorname{div}\{W_p - n\otimes nV_p\} + Y(\nabla n, n) + (\mathbb{1} - n\otimes n)A_u(\nabla\varphi, n) = 0, \tag{4.6}$$

$$\operatorname{div} D(\nabla\varphi, n) = 0, \tag{4.7}$$

subject to $n = n_0$ and $\varphi = \varphi_0$ on $\partial\Omega$.

First we prove the following lemma.

4.2. Lemma. *Let $\gamma\in H^{1/2}(\partial\Omega)$ and let $a(x,t) = [a_{ij}(x,t)]$ be a square matrix-valued function on $\Omega\times\mathbb{R}$ which is uniformly positive definite, $a(x,t)\xi\cdot\xi \geqq \lambda|\xi|^2$ for $(x,t)\in\Omega\times\mathbb{R}$ and all ξ, measurable for $x\in\Omega$, and Lipschitz for $t\in\mathbb{R}$. Let ψ_t and Ψ denote the solutions of the problems*

$$\begin{aligned}&-\operatorname{div}[a(\cdot,t)\nabla\psi_t] = 0 \text{ in } \Omega, \quad \psi_t = \gamma \text{ on } \partial\Omega, \quad \text{and}\\ &-\operatorname{div}[a(\cdot,0)\nabla\Psi] = -\operatorname{div}[(\partial a/\partial t)(\cdot,0)\nabla\varphi_0] \text{ in } \Omega, \quad \Psi = 0 \text{ on } \partial\Omega.\end{aligned} \tag{4.8}$$

Then,

$$[\psi_t - \psi_0]/t \to \Psi \text{ in } H^1(\Omega) \quad \text{as } t\to 0.$$

Proof. For any $\eta\in H_0^1(\Omega)$,

$$\int_\Omega [a(x,t)\nabla(\psi_t - \psi_0)]\cdot\nabla\eta dx = -\int_\Omega [a(x,t) - a(x,0)]\nabla\psi_0\cdot\nabla\eta dx.$$

Choosing $\eta = \psi_t - \psi_0$ and applying Schwarz's inequality, we find that

$$\lambda\int_\Omega |\nabla(\psi_t - \psi_0)|^2 dx \leqq \|\partial a/\partial t\|_{L^\infty}\cdot\|\nabla\psi_0\|_{L^2}\cdot\|\nabla(\psi_t - \psi_0)\|_{L^2}\cdot|t|,$$

hence,

$$\sup_t \|\nabla[(\psi_t - \psi_0)/t]\|_{L^2} < \infty.$$

We conclude that $(\psi_t - \psi_0)/t \to \Psi$ weakly in $H^1(\Omega)$ as $t\to 0$ because any limit of an H^1 weakly convergent subsequence $(\psi_{t_i} - \psi_0)/t_i$ must satisfy (4.8) and hence equal Ψ.

To obtain strong convergence in $H^1(\Omega)$, we note that

$$\begin{aligned} I(t) &= \int_\Omega [a(x,t)\nabla(\psi_t - \psi_0 - t\psi)]\cdot\nabla(\psi_t - \psi_0 - t\psi)dx \\ &= \int_\Omega \{[a(x,t)\nabla(\psi_t - \psi_0)]\cdot\nabla(\psi_t - \psi_0) \\ &\quad - 2t[a(x,t)\nabla(\psi_t - \psi_0)]\cdot\nabla\Psi + t^2[a(x,t)\nabla\Psi]\cdot\nabla\Psi\}dx \\ &= \int_\Omega \{a(x,t) - a(x,0)]\nabla\psi_0\cdot\nabla(\psi_t - \psi_0) \\ &\quad - 2t[a(x,t)\nabla(\psi_t - \psi_0)]\cdot\nabla\Psi + t^2[a(x,t)\nabla\Psi]\cdot\nabla\Psi\}dx \end{aligned}$$

We conclude from the weak convergence and (4.8) that, as $t\to 0$,

$$\begin{aligned} \lambda(\|\nabla(\psi_t - \psi_0)/t - \nabla\Psi\|_{L^2})^2 \leqq t^{-2}I(t) \to \int_\Omega [(\partial a/\partial t)(x,0)\nabla\psi_0 - 2a(x,0)\nabla\Psi \\ + a(x,0)\nabla\Psi]\cdot\nabla\Psi dx = 0. \quad \square \end{aligned}$$

Proof of Theorem 4.1. The argument for this is standard. One need only account for the constraint $|n| = 1$ and the variation of the field as a function of n, which is the motivation for Lemma 4.2. For example, let $v\in H^1(\Omega,\mathbb{R}^3)\cap L^\infty$ be given and set

$$n_t = (n + tv)/|n + tv| \quad \text{and} \quad \psi_t = \Phi(n_t) \text{ for } |t| < 1/\|v\|_{L^\infty}.$$

Then

$$\begin{aligned} n_0 &= n, & \psi_0 &= \varphi = \Phi(n), \\ \zeta &= [d/dt]_{t=0} n_t = (\mathbb{1} - n\otimes n)v, & \Psi &= [d/dt]_{t=0}\psi_t \in H_0^1(\Omega), \end{aligned}$$

$$[d/dt]_{t=0}\mathscr{E}(n_t) = \int_\Omega \{W_p(\nabla n, n)\nabla\zeta + W_u(\nabla n, n)\zeta + A_u(\nabla\varphi, n)\zeta + D(\nabla\varphi, n)\nabla\Psi\}dx.$$

So (4.5) implies (4.6). Likewise (4.6) and (4.7) imply (4.5).

Similarly, one shows that (4.4) is equivalent to (4.6) and (4.7). $\square$

The existence of a critical point of $\mathscr{E}^*$ on $\mathscr{A}^*(n_0)$ is now a consequence of 4.2 and the following:

4.3. Theorem *For n_0 and φ_0 as above, there exists an $n\in\mathscr{A}(n_0)$ such that*

$$\mathscr{E}(n) = \inf_{u\in\mathscr{A}(n_0)} \mathscr{E}(u).$$

Proof. Let $n_i\in\mathscr{A}(n_0)$ be an $\mathscr{E}$ minimizing sequence, and set $\varphi_i = \Phi(n_i)$. Using the energy bound (4.1) and arguing as in 1.5 and 1.6, we obtain bounds

$$\int_\Omega |\nabla\varphi_i|^2 dx \leqq \lambda^{-1}c(\Omega,\varphi_0),$$

$$\int_\Omega |\nabla n_i|^2 dx \leqq 2\alpha^{-1}[\sup_i \mathscr{E}(n_i) + \mathscr{S}(n_0) + c(\Omega,\gamma) + c_1|\Omega|]$$

(for some choice of constant c_1 in the cholesteric case). Passing to subsequences, we may suppose that $n_i\to n\in\mathscr{A}(n_0)$ and $\varphi_i\to\varphi\in H^1(\Omega)$, weakly in H^1, strongly in L^2, and pointwise almost everywhere. Now, as before,

$$\int_\Omega W(\nabla n, n)dx \leqq \liminf_{i\to\infty} \int_\Omega W(\nabla n_i, n_i)dx.$$

On the other hand, since φ_i is the solution of a minimum problem,

$$\int_\Omega A(\nabla \varphi_i, n_i)dx \leqq \int_\Omega A(\nabla \varphi, n_i)dx.$$

By the uniform bound, $|n_i| \leqq 1$, and Lebesgue's theorem,

$$\lim_{i\to\infty} \int_\Omega A(\nabla\varphi, n_i)dx = \int_\Omega A(\nabla\varphi, n)dx.$$

Thus

$$\limsup_{i\to\infty} \int_\Omega A(\nabla\varphi_i, n_i)dx \leqq \int_\Omega A(\nabla\varphi, n)dx,$$

and

$$\mathscr{E}(n) = \inf_{u\in\mathscr{A}(n_0)} \mathscr{E}(u). \qquad \square$$

We now turn to the partial regularity of a critical point (n, φ), obtained by minimizing $\mathscr{E}$. To simplify technical aspects, our attention is confined to the nematic case. We first discuss *scaling*. Suppose that $\varphi = \Phi(n)$, where n is, as above, a minimizer of $\mathscr{E}$ in $\mathscr{A}(n_0)$. For any ball $\mathbb{B}_r(a) \subset \Omega$, one may consider the functions $\varphi_{r,a}$ and $n_{r,a}$, defined by

$$\varphi_{r,a}(x) = \varphi(rx + a) \quad \text{and} \quad n_{r,a}(x) = n(rx + a) \quad \text{for } x\in\mathbb{B}.$$

Then, with respect to their own boundary values on $\partial\mathbb{B}$,

$$\varphi_{r,a} = \Phi(n_{r,a}) \quad \text{and} \quad n_{r,a} \text{ minimizes } \mathscr{E}.$$

Note that

$$\int_{\mathbb{B}} |\nabla\varphi_{r,a}|^2 dx \leqq r^{-1} \int_{\mathbb{B}_r(a)} |\nabla\varphi|^2 dx.$$

We shall obtain estimates for small energy minimizers $u\in H^1(\mathbb{B}, \mathbb{S}^2)$ of $\mathscr{E}$ involving the quantity

$$\|\nabla\Phi(u)\|^2 = \int_{\mathbb{B}} |\nabla\Phi(u)|^2 dx,$$

as well as on $\mathbb{E}_1(u)$.

4.4. Lemma (blow-up equation). *Suppose $v\in H^1(\mathbb{B}, \mathbb{S}^2)$ is a blow-up limit for a special blow-up sequence u_i, where each u_i minimizes some functional $\mathscr{E}_i = \mathscr{E}_{\gamma_i}$ [see (4.2)] with $\gamma_i\in H^{1/2}(\partial\mathbb{B})$ and where $\lim_{i=\infty} \|\nabla\Phi(u_i)\|^2/\varepsilon_i = 0$. Then, as in 2.2, $v' = (v^1, v^2)$ again satisfies the elliptic system*

$$-\operatorname{div} \tilde{W}'_p(\nabla v', \mathrm{e}) = 0 \text{ in } \mathbb{B},$$

(and hence the L^2 estimate of 2.2).

Proof. The function u_i satisfies the weak form of Eq. (4.5) with φ replaced by $\varphi_i = \Phi(u_i)$. Using a test function $\zeta = (\zeta^1, \zeta^2, 0)$ as in 2.2, we again substitute $\nabla u_i = \varepsilon_i \nabla v_i$ and divide by ε_i. The integral of the new term $\varepsilon_i^{-1} A_u(\nabla\varphi_i, \bar{u}_i + \varepsilon_i v_i)\cdot\zeta$ approaches 0 as $i \to \infty$ because of the assumption on $\|\nabla\varphi_i\|^2$. □

4.5. Lemma (Hybrid inequality). *There exists a positive constant c (depending only on κ_1, κ_2, and κ_3) so that if $0 < \lambda < 1$ and u is a minimizer of $\mathscr{E} = \mathscr{E}_{\varphi_0}(u)$ in $H^1(\mathbb{B}, \mathbb{S}^2)$ with $\varphi_0 \in H^{1/2}(\partial\mathbb{B})$, then*

$$\mathbb{E}_{1/2}(u) \leqq \lambda \mathbb{E}_1(u) + c\lambda^{-1} \int_{\mathbb{B}} |u - \bar{u}|^2 dx + c\|\nabla\Phi(u)\|^2.$$

Proof. First note that u is, by the argument of 1.3, a minimizer for the corresponding adjusted energy $\tilde{\mathscr{E}} = \mathscr{E} - \mathscr{W} + \tilde{\mathscr{W}}$,

$$\tilde{\mathscr{E}}(u) = \int_{\Omega} [\tilde{W}(\nabla u, u) - A[\nabla\Phi(u), u]]dx.$$

Then choosing $r \in [\frac{1}{2}, 1]$, w, and λ as in 2.2 and changing c suitably, we may use the $\tilde{\mathscr{E}}$ minimality of $u|\mathbb{B}_r$ to infer that

$$\begin{aligned}\mathbb{E}_{1/2}(u) &\leqq 2\mathbb{E}_r(u) \leqq 4\alpha^{-1}\tilde{\mathscr{W}}(u|\mathbb{B}_r) \leqq 4\alpha^{-1}\tilde{\mathscr{E}}(u|\mathbb{B}_r) + c\|\nabla\Phi(u)\|^2 \\ &\leqq 4\alpha^{-1}\tilde{\mathscr{E}}(w|\mathbb{B}_r) + c\|\nabla\Phi(u)\|^2 \leqq 4\beta\alpha^{-1}\int_{\mathbb{B}_r}|\nabla w|^2 dx + c\|\Phi(u)\|^2 \\ &\leqq \lambda\mathbb{E}_1(u) + c\lambda^{-1}\int_{\mathbb{B}}|u - \bar{u}|^2 dx + c\|\nabla\Phi(u)\|^2. \quad \square\end{aligned}$$

4.6. Theorem (Energy improvement). *There are positive constants ε, η, and $\theta < 1$ (depending only on κ_1, κ_2, and κ_3) so that if u is a minimizer of $\mathscr{E} = \mathscr{E}_\gamma$ with $\gamma \in H^{1/2}(\partial\mathbb{B})$ and $\mathbb{E}_1(u) < \varepsilon^2$, then $\mathbb{E}_\theta(u) \leqq \theta \max\{\mathbb{E}_1(u), \eta\|\nabla\Phi(u)\|^2\}$.*

Proof. We argue just as in 3.4 with $[F]^2$ replaced by $\|\nabla\Phi(u)\|^2$. □

4.7. Corollary (Energy decay). *If n is a minimizer of $\mathscr{E} = \mathscr{E}_\gamma$ in $H^1(\Omega, \mathbb{S}^2)$ with $\gamma \in H^{1/2}(\partial\Omega)$, if $\mathbb{B}_R(a) \subset \Omega$, and if $\int_{\mathbb{B}_R(a)} |\nabla n|^2 dx \leqq \varepsilon^2 R$, then*

$$\int_{\mathbb{B}_r(a)} |\nabla n|^2 dx \leqq c\theta^{-2} \max\{\varepsilon^2, \eta\|\nabla\Phi(n)\|^2 R^{-1}\} R^{-1} r^{1+\mu} \quad \textit{for } 0 < r < R/2,$$

for some positive constants c and μ, depending only on the electric field constants $\varepsilon_0, \varepsilon_\perp$, and ε_a.

Proof. From the inequality (4.3) and scaling, we obtain the estimate

$$r^{-1}\int_{\mathbb{B}_r(a)} |\Phi(n)|^2 dx \leqq c\|\Phi(n)\|^2 R^{-1} r^\mu \quad \text{for} \quad 0 < r < R/2.$$

Recalling from 4.3 the scaling estimate for $\|\nabla\Phi(n)\|^2$, we now iterate 4.6 as in 3.5. □

4.8. Theorem (Interior partial regularity). *If n is a minimizer of $\mathscr{E} = \mathscr{E}_{\varphi_0}$ in $H^1(\Omega, \mathbb{S}^2)$ with $\varphi_0 \in H^{1/2}(\partial\Omega)$, then $\varphi = \Phi(n)$ is locally Hölder continuous on Ω, and both n and φ*

are analytic on $\Omega \sim Z$ *for some relatively closed subset* Z *of* Ω *which has one dimensional Hausdorff measure zero.*

Proof. The Hölder continuity of φ follows from De Giorgi's theorem [D]. Taking Z as in 2.6, the Hölder continuity of n on $\Omega \sim Z$ follows from Morrey's Lemma and 4.7.

To verify the higher regularity of n and φ near a point $a \in \Omega \sim Z$, we assume, for convenience, that $n(a) = \mathbf{e} = (0, 0, 1)$. Recalling the argument for higher regularity in 2.5 and using the Euler equations for (n, φ) obtained in 4.2, we readily verify that on a small neighborhood of a, the triple (n^1, n^2, φ) satisfies a strongly elliptic system with analytic coefficients. Thus n and φ are analytic near a by [M, 6.7]. □

5. Partial Regularity at the Boundary

Suppose n minimizes $\mathscr{W}$ in the family $\mathscr{A}(n_0)$ as in Sect. 1. Since n extends to an H^1 function defined in a neighborhood of $\bar{\Omega}$, the set

$$Y = \{a \in \partial\Omega : \limsup_{r \downarrow 0} r^{-1} \int_{\mathbb{B}_r(a)} |\nabla n|^2 dx > 0\}$$

has, as in 2.6, one dimensional measure zero. Here, assuming that $a \in \partial\Omega \sim Y$, $k \in \{1, 2, \ldots, \infty, \omega\}$, $0 < \mu < 1$, and both $\partial\Omega$ and n_0 are $\mathscr{C}^{k,\mu}$ near a, we show that n is $\mathscr{C}^{k,\mu}$ near a. Our discussion below, which involves modifying Sect. 2, can easily be adapted to handle boundary regularity for the problem of minimizing $\mathscr{W} + \mathscr{F}$ (Sects. 1.5, 3) or for the electric field problem (Sect. 4).

5.1. *Scaling.* For any $\mathscr{C}^1$ function ψ: $\mathbb{R}^2 \to \mathbb{R}$ with $\psi(0) = 0 = |\nabla\psi(0)|$, let

$$\Omega_\psi = \{(x_1, x_2, x_3) \in \mathbb{B}_2 : x_3 < \psi(x_1, x_2)\}.$$

For Ω, a, and n_0 as above, there exists a positive number R, a rotation $h \in \mathbb{SO}(3)$, and a function $\psi_{R,a} \in \mathscr{C}^1(\mathbb{R}^2)$, so that

$$\psi_{R,a}(0) = 0 = |\nabla\psi_{R,a}(0)|, \quad \mathrm{Lip}(\psi_{R,a}) \leqq 1,$$
$$\Omega_{\psi_{R,a}} = \{h^{-1}[(y-a)/R] : y \in \mathbb{B}_{2R}(a) \cap \Omega\},$$

and n_0 is $\mathscr{C}^1$ on $\mathbb{B}_{2R}(a) \cap \partial\Omega$. For $0 < r \leqq R$, let $\psi_{r,a}(x) = \psi_{R,a}(rx/R)$. Then, for n as above, the expression $n_{r,a}(x) = n[rh(x) + a]$ defines a function in $H^1(\Omega_{\psi_{r,a}}, \mathbb{S}^2)$ which is $\mathscr{W}$ minimizing. Its trace on $\mathbb{B}_2 \cap \partial\Omega_{\psi_{r,a}}$ is given by $(n_0)_{r,a}(x) = n_0[rh(x) + a]$. Note that

$$\mathbb{E}_1(n_{r,a}) = r^{-1} \int_{\mathbb{B}_r(a)} |\nabla n|^2 dx, \quad \mathrm{Lip}(\psi_{r,a}) = \mathrm{Lip}(\psi_{R,a}) R^{-1} r,$$
$$\mathrm{Lip}[(n_0)_{r,a}] = \mathrm{Lip}[(n_0)_{R,a}] R^{-1} r.$$

We will study the behavior of small energy minimizers of $\mathscr{W}$ whose traces on $\mathbb{B}_2 \cap \partial\Omega_\psi$ have small Lipschitz norms.

To treat *blowing-up* at a boundary point, suppose that, for $i = 1, 2, \ldots,$

$\psi_i : \mathbb{R}^2 \to \mathbb{R}$ is $\mathscr{C}^1$ with $\psi_i(0) = 0 = |\nabla\psi_i(0)|$ and $\mathrm{Lip}(\psi_i) \leqq 1$,

u_i belongs to $H^1(\Omega_{\psi_i}, \mathbb{S}^2)$, $g_i = u_i | \mathbb{B}_2 \cap \partial\Omega_{\psi_i}$ is Lipschitz, and, as $i \to \infty$,

$$\mathbb{E}_1(u_i) = \int_{\mathbb{B}\cap\Omega_{\psi_i}} |\nabla u_i|^2 dx \to 0 \quad \text{and} \quad [\mathrm{Lip}(\psi_i) + \mathrm{Lip}(g_i)]^2/\mathbb{E}_1(u_i) \to 0.$$

It is convenient to have u_i defined on the whole ball $\mathbb{B} = \mathbb{B}_1$ by letting

$$u_i(x) = -u_i[x_1, x_2, -x_3 + 2\psi_i(x_1, x_2)] \quad \text{for } x \in \mathbb{B} \sim \Omega_{\psi_i}.$$

Then

$$\lim_{i\to\infty} \varepsilon_i^{-2} \int_{\mathbb{B}\cap\Omega_{\psi_i}} |\nabla u_i|^2 dx = 0 \quad \text{where} \quad \varepsilon_i^2 = \int_{\mathbb{B}} |\nabla u_i|^2 dx.$$

As in Sect. 2, we can compose with rotations and then pass to a subsequence to insure that

$$\mathbf{e}\cdot\bar{u}_i = |\bar{u}_i| \text{ for all } i \; [\text{where } \bar{u}_i = \textstyle\fint_{\mathbb{B}} u_i dx] \fint \text{ and}$$

$$v_i = \varepsilon_i^{-1}(u_i - \bar{u}_i)|\mathbb{B} \text{ converges weakly in } H^1.$$

Under all these conditions, we say that u_i is a *special boundary blow-up sequence*, and we again call $v = \lim_{i\to\infty} v_i$ a *blow-up limit.*

5.2. Lemma (blow-up equation). *Suppose v is a blow-up limit for a special boundary blow-up sequence u_i of $\mathscr{W}$ minimizers, as above. Then, for almost all $x \in \mathbb{B}$,*

$$v(x)\cdot\mathbf{e} = 0 \quad \textit{and} \quad v(x_1, x_2, -x_3) = -v(x_1, x_2, x_3).$$

Moreover, $v' = (v^1, v^2)$ is a solution of the elliptic system

$$-\mathrm{div}\, \tilde{W}'_p(\nabla v', \mathbf{e}) = 0 \quad \textit{on} \quad \{(x_1, x_2, x_3) \in \mathbb{B} : x_3 > 0\},$$

and satisfies the L^2 estimate of 2.2.

Proof. The first conclusion was established in 2.1. To obtain the second, we note that $\lim_{i\to\infty} \varepsilon_i^{-2}\mathrm{Lip}(\psi_i) = 0$ and use the strong L^1 convergence of v_i to v to verify that

$$\int_{\mathbb{B}} v\cdot\zeta dx = \lim_{i\to\infty} \int_{\mathbb{B}} v_i\cdot\zeta dx = 0$$

for any $\zeta \in \mathscr{C}^0(\mathbb{B}, \mathbb{R}^3)$ satisfying $\zeta(x_1, x_2, -x_3) = \zeta(x_1, x_2, x_3)$.

Next we note that a function with support in $\mathbb{B} \cap \{x_3 > 0\}$ has, for i sufficiently large, support in $\mathbb{B}\cap\Omega_{\psi_i}$. Using such a function as a variation, we find that, in $\{(x_1, x_2, x_3) \in \mathbb{B} : x_3 > 0\}$, $v' = (v^1, v^2)$ satisfies (in a weak sense) the above system. Moreover, since $v' \in H^1(\mathbb{B}, \mathbb{R}^2)$ and is odd in x_3, v' has zero trace on $\{(x_1, x_2, x_3) \in \mathbb{B} : x_3 = 0\}$ and so satisfies the L^2 estimate of 2.2 by the linear elliptic boundary estimate [M$_2$, 6.3]. □

5.3. Lemma (Hybrid inequality). *There exist positive constants c and q (depending only on κ_1, κ_2, and κ_3) so that if $0 < \lambda < 1$, ψ is as in 5.1, $\mathrm{Lip}(\psi) \leq 1$, u is a minimizer of $\mathscr{W}$ in $H^1(\Omega_\psi, \mathbb{S}^2)$, g is the trace of u on $\mathbb{B}\cap\partial\Omega_\psi$, and $\mathrm{Lip}(\psi) \leq c^{-1}\lambda^{1/2}$, then*

$$\mathbb{E}_{1/2}(u) \leqq \lambda \mathbb{E}_1(u) + c\lambda^{-1} \int_{\Omega_\psi \cap \mathbb{B}} |u - \bar{u}|^2 dx + c\lambda^{-1}(\operatorname{Lip} g)^2.$$

[*Here* $\mathbb{E}_r(u) = r^{-1} \int_{\Omega_\psi \cap \mathbb{B}_r} |\nabla u|^2 dx.$]

Proof. First one may find a universal constant Λ so that, for any $r \in [0, 1]$ and any ψ: $\mathbb{R}^2 \to \mathbb{R}$ with $\operatorname{Lip}(\psi) \leqq 1$, there exists a bilipschitz map $\Upsilon = \Upsilon(r, \psi)$: $\mathbb{B}_r \cap \Omega_\psi \to \overline{\mathbb{B}_r}$ with $\sup\{\operatorname{Lip}\Upsilon, \operatorname{Lip}\Upsilon^{-1}\} \leqq \Lambda$. Next we choose $r \in [\frac{1}{2}, 1]$ exactly as in the proof of 2.3. Using the extension $G: \overline{\Omega_\psi} \to \mathbb{S}^2$ of g given by $G(x_1, x_2, x_3) = g[x_1, x_2, \psi(x_1, x_2)]$, we define the map $\omega: \partial\mathbb{B}_r \to \mathbb{S}^2$ by

$$\omega = (u - G) \circ \Upsilon^{-1} \text{ on } \Upsilon(\Omega_\psi \cap \partial\mathbb{B}_r), \quad \omega = 0 \text{ on } \Upsilon(\mathbb{B}_r \cap \partial\Omega_\psi).$$

We now construct $w \in H^1(\mathbb{B}_r, \mathbb{S}^2)$ exactly as in 2.3 with $u|\partial\mathbb{B}_r$ replaced by ω and $\bar{u}$ replaced by $\bar{u} + \bar{G}$, where $\bar{G} = \fint_{\mathbb{B}} G dx$. Note that

$$\int_{\partial\mathbb{B}_r} |G - \bar{G}|^2 d\mathscr{H}^2 \leqq 16\pi(\operatorname{Lip} g)^2.$$

Since $w \circ \Upsilon + G$ and $u|\Omega_\psi \cap \mathbb{B}_r$ now have the same trace on $\partial(\Omega_\psi \cap \mathbb{B}_r)$, we conclude from the $\tilde{\mathscr{W}}$ minimality of $u|\Omega_\psi \cap \mathbb{B}_r$, Cauchy's inequality, and (2.2) that,

$$\begin{aligned}
\mathbb{E}_{1/2}(u) &\leqq 2\mathbb{E}_r(u) \leqq 4\alpha^{-1}\mathscr{W}(u|\Omega_\psi \cap \mathbb{B}r) \leqq 4\alpha^{-1}\mathscr{W}(w \circ \Upsilon + G) \\
&\leqq 4\beta\alpha^{-1} \int_{\mathbb{B}_r} |\nabla(w \circ \Upsilon + G)|^2 dx \leqq 4\beta\alpha^{-1}\Lambda^5 \int_{\Omega_\psi \cap \mathbb{B}_r} |\nabla(w + G \circ \Upsilon^{-1})|^2 dx \\
&\leqq 8\beta\alpha^{-1}\Lambda^5 \int_{\mathbb{B}_r} |\nabla w|^2 dx + c \int_{\mathbb{B}_r} |\nabla G|^2 dx \\
&\leqq 256\beta\alpha^{-1}\Lambda^5 \Big(\int_{\partial\mathbb{B}_r} |\nabla_{\tan}\omega|^2 d\mathscr{H}^2\Big)^{1/2} \Big(\int_{\partial\mathbb{B}_r} |\omega - \bar{u} - \bar{G}|^2 d\mathscr{H}^2\Big)^{1/2} + c(\operatorname{Lip} g)^2 \\
&\leqq 128\beta\alpha^{-1}\Lambda^5 \Big[\delta \int_{\partial\mathbb{B}_r} |\nabla_{\tan}\omega|^2 d\mathscr{H}^2 + \delta^{-1} \int_{\partial\mathbb{B}_r} |\omega - \bar{u} - \bar{G}|^2 d\mathscr{H}^2\Big] + c(\operatorname{Lip} g)^2 \\
&\leqq 128\beta\alpha^{-1}\Lambda^{10} \Big[\delta \int_{\partial\mathbb{B}_r} |\nabla_{\tan} u|^2 d\mathscr{H}^2 + \delta^{-1} \int_{\partial\mathbb{B}_r} |u - \bar{u}|^2 d\mathscr{H}^2\Big] + c\delta^{-1}(\operatorname{Lip} g)^2 \\
&\leqq \lambda \mathbb{E}_1(u) + c\lambda^{-1} \int_{\Omega_\psi \cap \mathbb{B}} |u - \bar{u}|^2 dx + c\lambda^{-1}(\operatorname{Lip} g)^2,
\end{aligned}$$

for $\delta = \lambda/1024\beta\alpha^{-1}\Lambda^{10}$ and an appropriate choice of c. □

5.4. Theorem (Energy improvement). *There are positive constants ε, η, and $\theta < 1$ depending only on κ_1, κ_2, and κ_3) so that if ψ is as in 5.1, u is a minimizer of $\mathscr{W}$ in $H^1(\Omega_\psi, \mathbb{S}^2)$, g is the trace of u on $\mathbb{B} \cap \partial\Omega_\psi$, and $\mathbb{E}_1(u) < \varepsilon^2$, then*

$$\mathbb{E}_\theta(u) \leqq \theta \max\{\mathbb{E}_1(u), \eta(\operatorname{Lip} g + \operatorname{Lip}\psi)^2\}.$$

Proof. If the theorem were false, then, for any $0 < \theta < 1$, there would exist C^1 functions ψ_i: $\mathbb{R}^2 \to \mathbb{R}$ with $\psi_i(0) = 0 = |\nabla\psi_i(0)|$ and $\mathscr{W}$ minimizers u_i in $H^1(\Omega_{\psi_i}, \mathbb{S}^2)$

having Lipschitz traces g_i on $\mathbb{B} \cap \partial\Omega_{\psi_i}$ so that

$$\mathbb{E}_\theta(u_i) > \theta \mathbb{E}_1(u_i) \text{ for all } i \text{ and, as } i \to \infty,$$
$$\mathbb{E}_1(u_i) \to 0 \text{ and } [\mathrm{Lip}(\psi_i) + \mathrm{Lip}(g_i)]^2/\mathbb{E}_1(u_i) \to 0.$$

As in 2.4, we may, after composing with rotations and passing to a subsequence, assume that u_i is a special boundary blow-up sequence. We define u_i on the whole ball $\mathbb{B}$ as in 5.1, and let ε_i, v_i, and v be as in 5.1 and 5.2. Using 5.2 and repeated application of 5.3, the argument now proceeds as in 3.3 [with $[F_i]^2$ replaced by $\mathrm{Lip}(\psi_i) + \mathrm{Lip}(g_i)$]. Since

$$\varepsilon_i^2/\mathbb{E}_1(u_i) \to 2 \quad \text{as} \quad i \to \infty,$$

we now find that, for i sufficiently large,

$$\mathbb{E}_\theta(u_i) \leqq \tfrac{1}{2}\theta \mathbb{E}_1(u_i) + \tfrac{1}{4}\theta\varepsilon_i^2 < \theta \mathbb{E}_1(u_i),$$

the desired contradiction. □

5.5. Corollary (Energy decay). *Suppose Ω is a domain in $\mathbb{R}^3$, $a \in \partial\Omega$, $n_0 \colon \partial\Omega \to \mathbb{S}^2$, and both $\partial\Omega$ and n_0 are both $\mathscr{C}^1$ near a. Then, if $R > 0$ is sufficiently small (depending on the $\mathscr{C}^1$ norms of $\partial\Omega$ and n_0 near a) and if $n \in H^1(\Omega, \mathbb{S}^2)$ is a minimizer of $\mathscr{W}$ with*

$$n|\partial\Omega = n_0 \quad \textit{and} \quad \int_{\mathbb{B}_R(a)} |\nabla n|^2 dx \leqq \varepsilon^2 R,$$

then

$$\int_{\mathbb{B}_r(a)} |\nabla n|^2 dx \leqq \theta^{-2} \max\{\varepsilon^2, \eta R^2\} R^{-2} r^2 \quad \textit{for}\ 0 < r < R.$$

Proof. Recalling 5.1, one uses 5.4 and argues as in 2.5 or 3.5. □

5.6. Theorem (Partial regularity at the boundary). *Suppose Ω, n_0, n, and Y are as in Sect. 1 and 5.0, Z is as in 2.6, and X is a relatively closed subset of $\partial\Omega$ such that n_0 and $\partial\Omega$ are $\mathscr{C}^{k,\mu}$ off of X for some $k \in \{1, 2, \ldots, \infty, \omega\}$ and $0 < \mu < 1$. Then n belongs to $\mathscr{C}^{k,\mu}_{\mathrm{loc}}(\bar{\Omega} \sim X \sim Y \sim Z)$.*

Proof. To obtain local Hölder continuity on $\bar{\Omega} \sim X \sim Y \sim Z$, we note that trivially

$$\int_{\mathbb{B}_r(b) \cap \Omega} |\nabla n|^2 dx \leqq \int_{\mathbb{B}_R(a) \cap \Omega} |\nabla n|^2 dx \quad \text{whenever } \mathbb{B}_r(b) < \mathbb{B}_r(a),$$

and apply 5.0, 5.1, 5.5, and 2.5 as in the proof of 2.6.

To prove higher regularity near a point $a \in \partial\Omega \sim X \sim Y$, we assume that $n(a) = (0, 0, 1)$, and observe as in the proof of 2.6 that, near a, the pair (n^1, n^2) satisfies an elliptic system with analytic coefficients on a $\mathscr{C}^{k,\mu}$ domain and has $\mathscr{C}^{k,\mu}$ boundary values near a. The theorem now follows from [M, Sect. 6]. □

Acknowledgement. The authors are pleased to acknowledge many stimulating and fruitful discussions with J. L. Ericksen. We also thank H. Brezis for many interesting conversations. Moreover we benefited from our contact with the participants of the I.M.A. Workshop on the Theory and Applications of Liquid Crystals, January, 1985. Many open mathematical questions are described in the Abstracts and Proceedings of this conference.

References

[B] Brézis, Singularities of liquid crystals, To appear in the Proceedings-I.M.A. Workshop on the Theory and Applications of Liquid-Ericksen J., Kinderlehrer D. (eds) Crystals, January 1985

[BC] Brézis, H., Coron, J. M.: Large solutions for harmonic maps in two dimensions. Commun. Math. Physics, **92**, 203–215 (1983)

[BCL] Brézis, H., Coron, J. M., Lieb, E.: Minimization problems with point defects, In preparation

[D] De Giorgi, E.: Sulla differenziabilita e l'analiticita delle estremali degli integrali multipli regolari. Mem. Acad. Ser. Torino **143**, 25–43 (1957)

[deG] De Gennes, P.: The physics of liquid crystals. Oxford: Clarendon Press 1974

[E] Ericksen, J. L.: Equilibrium theory of liquid crystals. In: Advances in Liquid Crystals, vol. 2, pp. 233–299; Brown, G. H. (ed.) New York: Academic Press 1976

[E_2] ———, Nilpotent energies in liquid crystal theory. Arch. Ration, Mech. Anal. **10**, pp. 189–196 (1962)

[F] Federer, H.: Geometric measure theory. Berlin, Heidelberg, New York: Springer 1969

[GG] Giaquinta, M., Giusti, E.: The singular set of the minima of certain quadratic functionals. Ann. Sci. Norm. Super: Pisa (IV), **XI**, 1, 45–55 (1984)

[GW] Gulliver, R., White, B.: In preparation

[HK] Hardt, R., Kinderlehrer, D.: Mathematical questions of liquid crystal theory. To appear in the Proceedings-I.M.A. Workshop on the theory and applications of liquid crystals. Ericksen J., Kinderlehrer D. (eds.) 1985

[HKL] Hardt, R., Kinderlehrer, D., Lin, F.H.: In preparation

[HL] Hardt, R., Lin, F. H.: Tangential regularity near the C^1 boundary. Geometric measure theory. Am. Math. Soc. Proc. Sym. Pure Math. **43**, 245–253

[HL_2] ———, A remark on H^1 mappings. To appear in Manuscripta Math.

[HL_3] ———, Mappings that minimize the p^{th} power of the gradient. In preparation

[L] Liao, G.: A regularity theory for harmonic maps with small energy. To appear in J. Differ. Geom.

[M] Morrey, C. B. Jr.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966

[M_2] ———, Second order elliptic systems of differential equations. Ann. Math. Stud. **33**, 101–159 (1854)

[Ma] MacMillan, E.: The statics of liquid crystals. Thesis, master of science. Johns Hopkins University 1982

[S] Schoen, R.: Analytic aspects of the harmonic map problem. To appear in J. Differ. Geom.

[Si] Simon, L.: Isolated singularities of extrema of geometric variational problems. Preprint, Australian National University 1984

[SU] Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Differ. Geom. **17**, 307–335 (1982)

[SU_2] ———, Boundary regularity and the Dirichlet problem for harmonic maps. J. Differ. Geom. **18**, 253–268 (1983)

[W] White, B.: In preparation

Communicated by S.-T. Yau

Received August 15, 1985

Note added in proof. Leon Simon has recently informed us that Stefan Luckhaus has found a result similar to Lemma 2.3.