

## EXISTENCE AND REPRESENTATION THEOREMS FOR A SEMILINEAR SOBOLEV EQUATION IN BANACH SPACE\*

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**Abstract.** An existence theory is developed for a semilinear evolution equation in Banach space which is modeled on boundary value problems for partial differential equations of Sobolev type. The operators are assumed to be measurable and to satisfy coercive estimates which are not necessarily uniform in their time dependence, and to satisfy Lipschitz conditions on the nonlinear term. Applications are briefly indicated.

**1. Introduction.** We shall consider the abstract Cauchy problem for the nonlinear evolution equation

$$\mathcal{M}(t)u'(t) + \mathcal{L}(t)u(t) = f(t, u(t))$$

in a separable and reflexive Banach space. The linear operators  $\mathcal{M}(t)$  are assumed to be weakly measurable in  $t$  and to satisfy nonuniform coercive estimates over the Banach space which permit them to degenerate for certain values of  $t$ . The family of linear operators  $\mathcal{L}(t)$  are assumed to be weakly measurable in  $t$ . The nonlinear term  $f(t, u)$  is measurable in  $t$  and Lipschitz in  $u$ .

Three types of solution are considered: weak, mild, and strong. A mild solution is (essentially) a weak solution which permits a certain integral representation, and we shall prove that these two notions differ by a measurability assumption. A strong solution is a weak solution for which each term in the equation belongs to a specified Hilbert space for almost every  $t$ .

The plan of the paper is as follows. Section 2 contains some technical results and notation we shall use. These include measurability of vector- and operator-valued functions, Gronwall's inequality, and an elementary fixed-point theorem for Banach space-valued functions.

The weak solution is defined in § 3, where we obtain results on uniqueness, local existence and global existence under various hypotheses. These results are used in § 4 to construct the linear propagator (which resolves the linear equation with  $f \equiv 0$ ) and thereby to introduce the notion of a mild solution. We prove that mild solutions (local and global) exist with the same hypotheses as used for existence of weak solutions.

Strong solutions are introduced in § 5. We give sufficient conditions for a mild solution to be strong; these conditions are essentially that the operators  $\mathcal{M}(t)$  dominate the operators  $\mathcal{L}(t)$ . Finally we obtain independently a sufficient condition for the existence (and uniqueness) of a strong solution; this condition requires that the function  $f$  be dominated by the operators  $\mathcal{M}(t)$ .

**2. Preliminaries.** For notation and standard material in functional analysis except as noted below, we shall refer to [11]. The space of continuous linear operators from the normed linear space  $X$  to the normed linear space  $Y$  will be

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denoted by  $L(X, Y)$ , and  $L(X)$  means  $L(X, X)$ . The space  $L(X, Y)$  with the uniform, strong and weak operator topologies is indicated by  $L_u(X, Y)$ ,  $L_s(X, Y)$  and  $L_w(X, Y)$ , respectively. Absolutely continuous (strongly, weakly) is abbreviated by AC (respectively, SAC, WAC). An operator-valued function  $h: [0, 1] \rightarrow L(X, Y)$  is called SAC from  $[0, 1]$  to  $L_s(X, Y)$  if for each  $x \in X$  the map  $t \rightarrow h(t)x$  is SAC from  $[0, 1]$  to  $Y$ , and  $h$  is SAC from  $[0, 1]$  to  $L_u(X, Y)$  if it is SAC from  $[0, 1]$  to the normed linear space  $L_u(X, Y)$  (see [11, pp. 40–41, 52–53]).

All linear spaces will be over the field  $C$  of complex numbers. Each of our results will hold if the spaces are over the real field  $R$  and if conjugate-linearity is replaced by linearity. The modifications will be obvious.

The antidual of the normed linear space  $X$  is the Banach space  $X'$  of conjugate-linear continuous maps from  $X$  to  $C$ . If  $x \in X$ , the map  $\phi \rightarrow \phi(x): X' \rightarrow C$  is continuous and conjugate-linear and hence determines an element  $Jx \in X''$ . This defines a linear isometry  $J: X \rightarrow X''$  by the identity  $\langle Jx, \phi \rangle_{X''-X'} = \langle \phi, x \rangle_{X'-X}$ . We say  $X$  is reflexive if  $J$  is onto, and we identify each  $Jx \in X''$  with  $x \in X$  (see [11, pp. 32–33]).

We shall need to discuss the adjoint of a map  $T \in L(X, X')$ . If  $x \in X$ , the map  $y \rightarrow \langle Ty, x \rangle_{X'-X}: X \rightarrow C$  is continuous and linear, so this determines a  $T'x \in X'$  for which  $\langle T'y, x \rangle_{X'-X} = \langle Ty, x \rangle_{X'-X}$  for all  $y \in X$ . This defines the map  $T' \in L(X, X')$ . The adjoint of the map  $T \in L(X, X')$  is the map  $T^* \in L(X'', X')$  defined by  $\langle T^*y, x \rangle_{X'-X} = \langle y, Tx \rangle_{X''-X'}$  for  $y \in X''$ ,  $x \in X$ . Comparing this with the above, we have for  $x, y \in X$ ,  $\langle T^* \circ Jy, x \rangle_{X'-X} = \langle Jy, Tx \rangle_{X''-X'} = \langle Tx, y \rangle_{X'-X} = \langle T'y, x \rangle_{X'-X}$ . This shows that  $T^* \circ J = T'$ , so when we identify  $X$  and  $J(X)$  we see that  $T^*$  is an extension of  $T'$ . When  $X$  is reflexive, we have  $T^* = T'$  under the indicated identification, and this will simplify many of the duality arguments to follow (see [11, pp. 42–43]).

For strongly measurable functions  $t \rightarrow x(t)$  from the real interval  $I$  to the Banach space  $X$  we shall use exclusively the Bochner integral with respect to Lebesgue measure on  $I$ . If  $1 \leq p < \infty$ ,  $L^p(I, X)$  is the Banach space of strongly measurable functions  $x(\cdot): I \rightarrow X$  for which

$$\|x\|_{L^p(X)} = \left( \int_I \|x(t)\|_X^p dt \right)^{1/p} < \infty,$$

and  $\|x\|_{L^p(X)}$  is the norm. Similarly,  $L^\infty(I, X)$  is the Banach space of strongly measurable functions  $x(\cdot): I \rightarrow X$  for which the norm  $\|x\|_{L^\infty(X)} = \text{ess sup } \{\|x(t)\|_X: t \in I\}$  is finite (see [11, pp. 71–89]).

When the Banach space  $X$  is separable, the notions of weak measurability and strong measurability of  $X$ -valued functions are equivalent. The following similar result for operator-valued functions will be useful.

**PROPOSITION 2.1.** *Let  $X$  and  $Y$  be separable Banach spaces and  $h: [0, 1] \rightarrow L_s(X, Y)$  a bounded function. Then  $h$  is measurable (in the strong operator topology) if and only if there is a sequence of countably-valued measurable functions  $h_n: [0, 1] \rightarrow h([0, 1]) \subseteq L(X, Y)$  such that  $h_n(t) \rightarrow h(t)$  in  $L_s(X, Y)$ , uniformly in  $t \in [0, 1]$ .*

**LEMMA 2.2.** *Let  $X$  and  $Y$  be separable Banach spaces and  $\{T_\alpha: \alpha \in A\} \subseteq L(X, Y)$ . There is a countable subset  $\{T_{\alpha_n}: n \geq 1\}$  which is strongly dense in  $\{T_\alpha: \alpha \in A\}$ .*

*Proof.* By considering subsets of the form  $\{T_\alpha : n-1 \leq \|T_\alpha\|_{L(X,Y)} < n\}$ , we may assume the  $T_\alpha$  are uniformly bounded. Consider the space  $l^1(Y) \equiv \{(y_n) : y_n \in Y \text{ and } \sum_{n=1}^\infty \|y_n\|_Y < \infty\}$ . Since  $Y$  is separable there is a sequence  $\{\eta_n\}$  dense in  $Y$ ; those sequences in  $l^1(Y)$  of the form  $(\eta_{n_1}, \eta_{n_2}, \dots, \eta_{n_k}, 0, 0, \dots)$  are dense in  $l^1(Y)$ , so  $l^1(Y)$  is separable.

Let the sequence  $\{\xi_n\}$  be dense in  $X$  and define a map  $\phi : L(X, Y) \rightarrow l^1(Y)$  by  $\phi(T) = (T(\xi_k)/\|\xi_k\|_X 2^k : k \geq 1)$  for  $T \in L(X, Y)$ . Since  $\{\phi(T_\alpha) : \alpha \in A\}$  is a subset of the separable  $l^1(Y)$ , it is separable, and hence has a dense subset of the form  $\{\phi(T_{\alpha_n}) : n \geq 1\}$ . Thus for any  $T_\beta$ ,  $\beta \in A$ , there is a sequence  $(\phi(T_m) : T_m \in \{T_{\alpha_n}\})$  such that  $\phi(T_m) \rightarrow \phi(T_\beta)$  in  $l^1(Y)$ . Then  $T_m(\xi_k) \rightarrow T_\beta(\xi_k)$  in  $Y$  for every  $k \geq 1$ . But  $\{\xi_k\}$  dense in  $X$  and  $\{T_\alpha\}$  bounded imply that  $T_m \rightarrow T_\beta$  in  $L_s(X, Y)$ .

*Proof of Proposition 2.1.* Let  $\{T_n : n \geq 1\}$  be a strongly dense subset of the range  $h([0, 1])$ . Since  $h([0, 1])$  is bounded, the topology induced on it by  $L_s(X, Y)$  is metrizable, and the metric is given by  $\rho(T, U) = \sum_{j=1}^\infty |(T - U)x_j|_Y / (1 + |(T - U)x_j|_Y) 2^j$ , where  $\{x_j : j \geq 1\}$  is dense in the unit sphere of  $X$ . If  $h$  is measurable in  $L_s(X, Y)$ , then each of the maps  $t \rightarrow |(h(t) - T_n)x_j|_Y$  is measurable [11, p. 72] and so then is  $t \rightarrow \rho(h(t), T_n)$ . For any  $\varepsilon > 0$ , each of the sets  $E_n = \{t \in [0, 1] : \rho(h(t), T_n) < \varepsilon\}$  is measurable with  $\bigcup \{E_n : n \geq 1\} = [0, 1]$ . The function defined on  $[0, 1]$  by  $h_\varepsilon(t) = T_n$  for  $t \in E_n \sim \bigcup \{E_j : 1 \leq j < n\}$  is measurable, countably-valued in  $h([0, 1])$  and  $\rho(h(t), h_\varepsilon(t)) < \varepsilon$  on  $[0, 1]$ . The converse is clear.

Finally we cite an elementary inequality and corresponding fixed-point theorem [4], [9].

LEMMA 2.3. Let  $Z(\cdot) \in L^\infty([0, 1], R)$  satisfy for some  $\alpha \geq 0$  the inequality

$$(2.1) \quad 0 \leq Z(t) \leq \alpha + \int_0^t K(\tau) Z(\tau) d\tau$$

for  $t \in [0, 1]$ , where  $K(\cdot) \in L^1([0, 1], R)$ ,  $K(t) \geq 0$ . Then

$$(2.2) \quad Z(t) \leq \alpha \exp \left\{ \int_0^t K(\tau) d\tau \right\}$$

for  $t \in [0, 1]$ .

LEMMA 2.4. Let  $X$  be a Banach space and  $F$  a map of the closed and bounded subset  $M$  of  $L^\infty([0, 1], X)$  into itself satisfying

$$\|(Fu)(t) - (Fv)(t)\|_X \leq \int_0^t K(\tau) \|u(\tau) - v(\tau)\|_X d\tau$$

for  $t \in [0, 1]$ , where  $K(\cdot) \in L^1([0, 1], R)$  and each  $K(t) \geq 0$ . Then there exists exactly one solution in  $M$  of the equation  $F(u) = u$ .

**3. The weak solution.** Let  $V$  be a reflexive and separable Banach space; the norm is given by  $\|v\|_V$  and the  $V' - V$  antiduality by  $\langle \phi, v \rangle$ . Let  $a > 0$  and assume that for each  $t \in I_a = [0, a]$  we are given a continuous sesquilinear form  $m(t; \cdot, \cdot)$  on  $V$ . This defines a family of operators  $\mathcal{M}(t) \in L(V, V')$  by the identity

$$(3.1) \quad m(t; x, y) = \langle \mathcal{M}(t)x, y \rangle, \quad x, y \in V.$$

Let  $b > 0$ ,  $x_0 \in V$  and  $B_b(x_0) = \{x \in V : \|x - x_0\|_V \leq b\}$ . Assume that we are given a function  $f : I_a \times B_b(x_0) \rightarrow V'$ .

DEFINITION. A function  $x: I_a \rightarrow V$  is a *weak solution* of the Cauchy problem

$$(3.2) \quad \mathcal{M}(t)x'(t) = f(t, x(t)), \quad x(0) = x_0$$

if it is SAC with range in  $B_b(x_0)$ , weakly differentiable a.e. on  $I_a$  and (3.2) is satisfied for a.e.  $t \in I_a$ .

*Remark.* It follows [11, p. 88] that  $x' \in L^1(I_a, V)$  is a strong derivative a.e. with  $x(t) - x(s) = \int_s^t x'(\tau) d\tau$ . It suffices to require that  $x$  be WAC and a.e. have a weak derivative  $x' \in L^1(I_a, V)$ .

The results of this section on weak solutions of (3.2) are obtained from combinations of the following assumptions listed here for reference.

- (I) There is a measurable function  $k: I_a \rightarrow (0, \infty)$  such that  $|m(t; x, x)| \geq k(t)|x|_V^2$  for  $x \in V$ , a.e. on  $I_a$ .
- (II) There is a measurable function  $Q: I_a \rightarrow [1, \infty)$  such that  $|f(t, x) - f(t, y)|_{V'} \leq Q(t)|x - y|_V$  for  $x, y \in B_b(x_0)$  a.e. on  $I_a$ , and  $Q/k \in L^1(I_a, R)$ .
- (III) For each pair,  $x, y \in V$ , the function  $t \rightarrow m(t; x, y): I_a \rightarrow C$  is measurable.
- (IV) For each  $x \in B_b(x_0)$ , the function  $t \rightarrow f(t, x): I_a \rightarrow V'$  is (weakly) measurable. For a.e.  $t \in I_a$ , the function  $x \rightarrow f(t, x)$  is continuous from  $B_b(x_0)$  with the norm topology to  $V'$  with the weak (= weak\*) topology.

Suppose (I) holds; then for a.e.  $t \in I_a$  the operator  $\mathcal{M}(t): V \rightarrow V'$  is an isomorphism with  $\|\mathcal{M}^{-1}(t)\|_{L(V', V)} \leq k(t)^{-1}$ . To see this, note from (3.1) and (I) that  $k(t)|x|_V^2 \leq |\mathcal{M}(t)x|_{V'}|x|_V$ , and hence  $k(t)|x|_V \leq |\mathcal{M}(t)x|_{V'}$ . This shows that  $\mathcal{M}(t)$  is injective with closed range in  $V'$ . Hence the range of  $\mathcal{M}(t)$  is the annihilator in  $V'$  of the null space of the adjoint  $\mathcal{M}(t)'$  [4, pp. 180–181], [11, p. 44]. But  $\mathcal{M}(t)'$  satisfies the same conditions as  $\mathcal{M}(t)$ , so it has a trivial null space. Thus  $\mathcal{M}(t)$  is onto  $V'$  and the result follows from the inequalities above.

Let  $x_1$  and  $x_2$  be weak solutions of (3.2) on  $I_a$  and assume (I) holds. Then we obtain the estimate

$$|x'_1(t) - x'_2(t)|_V \leq k(t)^{-1}|f(t, x_1(t)) - f(t, x_2(t))|_{V'}.$$

Since  $x_1 - x_2: I_a \rightarrow V$  is SAC with summable derivative, we have

$$(3.3) \quad x_1(t) - x_2(t) = x_1(0) - x_2(0) + \int_0^t (x'_1(s) - x'_2(s)) ds$$

on  $I_a$ . If we also assume (II), then we obtain the estimate

$$(3.4) \quad |x_1(t) - x_2(t)|_V \leq |x_1(0) - x_2(0)|_V + \int_0^t k(s)^{-1}Q(s)|x_1(s) - x_2(s)|_V ds.$$

This yields the following.

**THEOREM 1.** *Assume (I) and (II). Then there is at most one weak solution of (3.2) on  $I_a$ . If  $\{x_n(\cdot)\}$  is a sequence of weak solutions of the equation (3.2) with initial conditions  $x_n(0)$ ,  $n \geq 0$ , then  $x_n(0) \rightarrow x_0(0)$  in  $V$  implies that  $x_n(t) \rightarrow x_0(t)$  in  $V$ , uniformly on  $I_a$ .*

*Proof.* These results follow from the preceding inequality (3.4) and Lemma 2.3 with  $Z(t) = |x_1(t) - x_2(t)|_V$ . That this function is bounded follows from (3.3), since  $x'_1, x'_2 \in L^1(I_a, V)$ .

We consider next the existence of solutions.

**LEMMA 3.1.** *Assume (I) and (III). Then the operator-valued map  $t \rightarrow \mathcal{M}^{-1}(t): I_a \rightarrow L_s(V', V)$  is measurable.*

*Proof.* Since  $k(t) > 0$  on  $I_a$  and  $k$  is measurable, the sets defined by  $J_n = \{t \in I_a : k(t) \geq 1/n\}$ ,  $n \geq 1$ , are measurable and  $\bigcup \{J_n : n \geq 1\} = I_a$ .

The function  $\mathcal{M} : I_a \rightarrow L_w(V, V')$  is measurable by (III) and  $V'' = V$ ; hence it is measurable  $I_a \rightarrow L_s(V, V')$  since  $V'$  is separable [11, pp. 34, 74–75]. Let  $m \geq 1$ ; the restriction of  $\mathcal{M}$  to  $J_m$  is strongly measurable, so by Proposition 2.1 there is a sequence of countably-valued measurable functions  $\mathcal{M}_k : J_m \rightarrow \mathcal{M}(J_m) \subseteq L_s(V, V')$  such that, for  $t \in J_m$ ,  $\mathcal{M}_k(t) \rightarrow \mathcal{M}(t)$  in  $L_s(V, V')$  as  $k \rightarrow \infty$ . Since each  $\mathcal{M}_k(t) \in \mathcal{M}(J_m)$ , we have  $\|\mathcal{M}_k^{-1}(t)\|_{L(V', V)} \leq k(t)^{-1} \leq m$ , so for  $\phi \in V'$ ,  $\|\mathcal{M}_k^{-1}(t)\phi - \mathcal{M}^{-1}(t)\phi\|_V = \|\mathcal{M}_k^{-1}(t)[\mathcal{M}(t)x - \mathcal{M}_k(t)x]\|_V \leq m\|\mathcal{M}(t)x - \mathcal{M}_k(t)x\|_V \rightarrow 0$  as  $k \rightarrow \infty$ . Hence  $\mathcal{M}^{-1} : J_m \rightarrow L_s(V', V)$  is measurable for every  $m \geq 1$ , and this yields the desired result.

LEMMA 3.2. Assume (I), (III) and (IV). If  $x : I_a \rightarrow B_b(x_0)$  is measurable,  $I_a \rightarrow V$ , then the function  $t \rightarrow \mathcal{M}^{-1}(t) \circ f(t, x(t)) : I_a \rightarrow V$  is measurable.

*Proof.* For every  $\phi \in V'$  we have

$$\langle \phi, \mathcal{M}^{-1}(t)f(t, x(t)) \rangle = \overline{\langle f(t, x(t)), \mathcal{M}^{-1}(t)'\phi \rangle},$$

where  $\mathcal{M}^{-1}(t)' : V' \rightarrow V$  is the adjoint of  $\mathcal{M}^{-1}(t)$ , so it suffices to show that  $f(t, x(t))$  and  $\mathcal{M}^{-1}(t)'\phi$  are measurable in  $V'$  and  $V$ , respectively.

By Lemma 3.1,  $\mathcal{M}^{-1}(t)$  is measurable, so the identity  $\langle \psi, \mathcal{M}^{-1}(t)'\phi \rangle = \langle \phi, \mathcal{M}^{-1}(t)\psi \rangle$  for  $\psi$  in  $V'$  implies that  $\mathcal{M}^{-1}(t)'\phi$  is weakly (hence strongly) measurable.

Since  $V'$  is separable and  $V$  is reflexive, the measurability of  $f(t, x(t))$  will follow from that of  $t \rightarrow \langle f(t, x(t)), v \rangle$  for every  $v \in V$ . Suppose first that  $x : I_a \rightarrow V$  is a countably-valued function assuming the value  $x_j$  on  $G_j$ , where  $\{G_j : j \geq 1\}$  is a measurable partition of  $I_a$ . Let  $\phi_j(t)$  be the characteristic function of  $G_j$ . Then we have  $f(t, x(t)) = \sum \{f(t, x_j)\phi_j(t) : j \geq 1\}$  on  $I_a$ ; each term is measurable by (IV), so  $f(t, x(t))$  is measurable when  $x(t)$  is countably-valued. But any measurable function is a strong limit of countably-valued functions, so the result follows from the continuity requirement in (IV).

THEOREM 2. Assume (I), (II), (III) and (IV). Let  $x_0 \in V$  be such that  $\|f(t, x_0)\|_{V'} \leq Q(t)b_0$  on  $I_c$ , where  $c \in I_a$  is chosen so that  $\int_0^c Q(t)k(t)^{-1} dt \leq b(b_0 + b)^{-1}$ . Then there exists a (unique) weak solution of (3.2) on  $I_c$ .

*Proof.* Define  $M$  to be those continuous functions  $x \in L^\infty(I_c, V)$  for which  $x(t) \in B_b(x_0)$  on  $I_c$ . For any  $x \in M$  the function  $t \rightarrow \mathcal{M}^{-1}(t)f(t, x(t))$  is measurable  $I_c \rightarrow V$  by Lemma 3.2, and we have the estimate  $\|\mathcal{M}^{-1}(t)f(t, x(t))\|_V \leq k(t)^{-1}Q(t) \cdot (b_0 + b)$  on  $I_c$ . Hence the function is integrable, and we can define on  $I_c$  the function

$$(3.5) \quad [Fx](t) = x_0 + \int_0^t \mathcal{M}^{-1}(s)f(s, x(s)) ds.$$

It follows that  $\|[Fx](t) - x_0\|_V \leq b$ , so  $F$  maps  $M$  into itself. Finally we have from (II) that  $\|[Fx](t) - [Fy](t)\|_V \leq \int_0^t Q(s)k(s)^{-1}\|x(s) - y(s)\|_V ds$ , so Lemma 2.4 asserts that there is a unique  $x \in M$  for which

$$(3.6) \quad x(t) = x_0 + \int_0^t \mathcal{M}^{-1}(s)f(s, x(s)) ds$$

on  $I_c$ . But this is equivalent to being a weak solution of (3.2) (see [11, p. 88]), so the result follows.

**THEOREM 3.** Assume (I), (II), (III), (IV) and  $B_b(x_0) = V$ . Let  $x_0 \in V$  be such that  $\|f(t, x_0)\|_{V'} \leq Q(t)b_0$  on  $I_a$ . Then there exists a unique weak solution of (3.2) on  $I_a$ .

*Proof.* Let  $u(t) = b_0 \exp \int_0^t k(s)^{-1} Q(s) ds$  and define  $M$  to be the continuous functions in  $L^\infty(I_a, V)$  for which  $\|x(t) - x_0\|_V \leq u(t) - b_0$  for all  $t \in I_a$ . For any  $x \in M$  we have

$$(3.7) \quad \|\mathcal{M}^{-1}(s)f(s, x(s))\|_V \leq k(s)^{-1}Q(s)(\|x(s) - x_0\|_V + b_0),$$

so the boundedness of  $x$  implies that we can define  $Fx \in L^\infty(I_a, V)$  by (3.5). Also we have from (3.7) the estimate

$$\|[Fx](t) - x_0\|_V \leq \int_0^t k(s)^{-1}Q(s)u(s) ds = u(t) - b_0$$

so  $Fx \in M$ . Lemma 2.4 applies again to give the result.

*Remark.* The estimate (3.7) is a growth condition on the second term in  $f(t, x)$  and results from the Lipschitz condition in (II) and the above estimate on  $f(t, x_0)$ . This combination of hypotheses has advantages in applications. (See, for example, the discussion following (6.9).) In particular, it applies directly to linear equations.

**4. The mild solution.** In addition to the forms  $\{m(t; \cdot, \cdot) : t \in I_a\}$ , the function  $f$ , and the space  $V$  as in § 3, suppose we are given a second family  $\{l(t; \cdot, \cdot) : t \in I_a\}$  of continuous sesquilinear forms on  $V$ . As before each of these determines an operator  $\mathcal{L}(t) \in L(V, V')$  by the identity

$$l(t; x, y) = \langle \mathcal{L}(t)x, y \rangle, \quad x, y \in V.$$

We shall consider weak solutions of the equation

$$(4.1) \quad \mathcal{M}(t)x'(t) + \mathcal{L}(t)x(t) = f(t, x(t))$$

and its linear homogeneous counterpart

$$(4.2) \quad \mathcal{M}(t)x'(t) + \mathcal{L}(t)x(t) = 0$$

under assumptions like the following.

- (V) For each pair  $x, y \in V$ , the function  $t \rightarrow l(t; x, y) : I_a \rightarrow \mathbb{C}$  is measurable, and there is a measurable function  $K : I_a \rightarrow \mathbb{R}$  such that  $|l(t; x, y)| \leq K(t)\|x\|_V\|y\|_V$ ,  $x, y \in V$  a.e. on  $I_a$ , and  $K/k \in L^1(I_a, \mathbb{R})$ .

Our purpose in considering (4.1) is to separate the nonlinear term and characterize those weak solutions which have an integral representation sharper than (3.6). With the assumption (V), the equation (4.1) is certainly no more general than (3.2), since the assumptions (II) and (IV) hold for  $f(t, x) - \mathcal{L}(t)x$  whenever they hold for  $f(t, x)$ . Hence the results of § 3 apply to (4.1) when we assume (V).

Consider the linear equation (4.2). If we assume (I), (III) and (V), then Theorem 3 asserts that for each  $x_0 \in V$  and  $s \in I_a$  there is a unique weak solution  $x(t)$  of (4.2) which satisfies  $x(s) = x_0$ . This solution is characterized by the integral equation

$$(4.3) \quad x(t) = x_0 - \int_s^t \mathcal{M}^{-1}(\xi)\mathcal{L}(\xi)x(\xi) d\xi, \quad t \in I_a.$$

From Lemma 2.3 it follows that

$$(4.4) \quad |x(t)|_V \leq |x_0|_V \exp \left| \int_s^t (K(\xi)/k(\xi)) d\xi \right|.$$

For each  $t \in I_a$ , we see from Theorem 1 that the dependence of  $x(t)$  on  $x_0$  is linear and from (4.4) that it is continuous from  $V$  to  $V$ . Hence, for each  $t, s \in I_a$  there is a unique  $G(t, s) \in L(V)$  defined by  $G(t, s)x_0 = x(t)$ , where  $x(t)$  is given by (4.3). We summarize this construction as the following result.

**PROPOSITION 4.1.** *Assume (I), (III) and (V). Then there is a function  $G: I_a \times I_a \rightarrow L(V)$  for which:*

- (i) *for each  $x_0 \in V$  the function  $x(t) = G(t, s)x_0$  is the unique solution of (4.3);*
- (ii)  *$G$  is a linear propagator [5]:  $G(t, s) = G(t, \xi)G(\xi, s)$ ,  $G(t, t) = I$  for  $t, s, \xi \in I_a$ ;*

$$(iii) \quad \|G(t, s)\|_{L(V)} \leq \exp \left| \int_s^t (K(\xi)/k(\xi)) d\xi \right|;$$

(iv) *for each  $s \in I_a$ ,  $G(\cdot, s): I_a \rightarrow L_s(V)$  is SAC;*

(v) *for each  $t \in I_a$ ,  $G(t, \cdot): I_a \rightarrow L_s(V)$  is continuous.*

**COROLLARY.** *In addition to the above, assume that both of the functions  $\mathcal{M}$  and  $\mathcal{L}: I_a \rightarrow L_u(V, V')$  are a.e. separably-valued. Then for each  $s \in I_a$  the function  $G(\cdot, s): I_a \rightarrow L_u(V)$  is the unique continuous solution of*

$$G(t, s) = I - \int_s^t \mathcal{M}^{-1}(\xi) \mathcal{L}(\xi) G(\xi, s) d\xi,$$

(vi)  *$G(\cdot, s): I_a \rightarrow L_u(V)$  is SAC, and*

(vii) *for each  $t \in I_a$ ,  $G(t, \cdot): I_a \rightarrow L_u(V)$  is SAC.*

*Proof.*  $\mathcal{M}$  and  $\mathcal{L}$  are weakly measurable and a.e. separably-valued so they are uniformly measurable [11, p. 75]. Thus the map  $t \rightarrow \mathcal{M}^{-1}(t) \mathcal{L}(t): I_a \rightarrow L_u(V)$  is summable, and Lemma 2.4, with  $X = L_u(V)$  and  $M$  the set of continuous  $x \in X$  for which  $\|x(t) - I\| \leq \exp(|\int_s^t k(\xi)^{-1} K(\xi) d\xi|) - 1$ , shows there is an operator-valued function which satisfies

$$M(t, s) = I - \int_s^t \mathcal{M}^{-1}(\xi) \mathcal{L}(\xi) M(\xi, s) d\xi.$$

But for  $x_0 \in V$ , the function  $t \rightarrow M(t, s)x_0$  is the unique solution of (4.3), so  $M = G$ . That each  $G(\cdot, s)$  is SAC in  $L_u(V)$  follows from the integral representation above, and this fact, the identity

$$G(t, s_1) - G(t, s_2) = G(t, s_2) \{G(s_2, 0) - G(s_1, 0)\} G(0, s_1)$$

and the uniform boundedness of  $G$  imply the last result.

Assume (I), (III) and (V), and let  $x(t)$  be a weak solution of (4.1). Since  $s \rightarrow G(s, 0)$  is a.e. strongly differentiable and  $G(0, s) = G^{-1}(s, 0)$  is strongly continuous, it follows [4, pp. 136-137] that  $G(0, s)$  is differentiable and  $(d/ds)G(0, s) = -G(0, s)[(d/ds)G(s, 0)]G(0, s) = +G(0, s)\mathcal{M}^{-1}(s)\mathcal{L}(s)$  a.e. on  $I_a$ , where  $d/ds$  denotes the strong derivative. Since  $x(t)$  is differentiable a.e. we have

$$(4.5) \quad (d/ds)[G(0, s)x(s)] = G(0, s)[x'(s) + \mathcal{M}^{-1}(s)\mathcal{L}(s)x(s)]$$

and hence from (4.1) follows

$$(4.6) \quad (d/ds)[G(0, s)x(s)] = G(0, s)\mathcal{M}^{-1}(s)f(s, x(s))$$

a.e. on  $I_a$ . Since  $x(t)$  is a weak solution it follows that the right side of (4.5) is in  $L^1(I_a, V)$  so we may integrate (4.6). If  $\mathcal{M}$  and  $\mathcal{L}$  are a.e. separably-valued in  $L_u(V)$ , then SAC of  $G(0, s)$  in  $L_u(V)$  and that of  $x: I_a \rightarrow V$  imply that  $G(0, s)x(s)$  is SAC in  $V$  and we integrate (4.6) to obtain (after operating with  $G(t, 0)$ ) [11, p. 88]

$$(4.7) \quad x(t) = G(t, 0)x_0 + \int_0^t G(t, s)\mathcal{M}^{-1}(s)f(s, x(s))ds.$$

This is the desired integral representation.

DEFINITION. Assume (I), (III) and (V). A *mild solution* of (4.1) is a continuous function  $x: I_a \rightarrow V$  which satisfies (4.7). (In particular, the integrand belongs to  $L^1(I_a, V)$  for each  $t \in I_a$ .)

In the special case of equation (3.1), which is obtained from setting  $\mathcal{L} = 0$  and hence  $G(t, s) = I$ , it follows by comparing (4.7) with (3.6) that mild solutions are equivalent to weak solutions. Our next result states the relation between weak and mild solutions in the general case.

THEOREM 4. Assume (I), (III) and (V). Then a mild solution of (4.1) is a weak solution of (4.1); a weak solution is a mild solution if  $\mathcal{M}$  and  $\mathcal{L}$  are a.e. separably-valued.

*Proof.* The second statement was proved in the discussion preceding the definition of mild solution. If  $x: I_a \rightarrow V$  is a mild solution, then from

$$x(t) = G(t, 0) \left\{ x_0 + \int_0^t G(0, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds \right\}$$

it follows that  $x$  is strongly differentiable a.e., satisfies (4.1) a.e. and  $x' \in L^1(I_a, V)$ . Thus we need only to verify that  $x$  is WAC (see Remark following definition of weak solution).

Let  $v \in V$  and  $\phi \in V'$ . Applying  $\phi$  to the identity

$$G(t, 0)v = v - \int_0^t \mathcal{M}^{-1}(\xi)\mathcal{L}(\xi)G(\xi, 0)v d\xi$$

and then taking the indicated adjoints give us the weak integral identity

$$\langle G^*(t, 0)\phi, v \rangle = \langle \phi, v \rangle - \int_0^t \langle G^*(\xi, 0)\mathcal{L}^*(\xi)\mathcal{M}^{-1}(\xi)^*\phi, v \rangle d\xi.$$

From this we obtain the strong integral

$$G^*(t, 0)\phi = \phi - \int_0^t G^*(\xi, 0)\mathcal{L}^*(\xi)\mathcal{M}^{-1}(\xi)^*\phi d\xi$$

in  $V'$  from estimates like (iii) of Proposition 4.1 and the measurability of adjoints. From this we see that  $t \rightarrow G^*(t, 0)\phi: I_a \rightarrow V'$  is SAC. But we already know  $t \rightarrow x_0 + \int_0^t G(0, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds: I_a \rightarrow V$  is SAC, so it follows from  $\langle \phi, x(t) \rangle = \langle G^*(t, 0)\phi, x_0 + \int_0^t G(0, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds \rangle$  that  $x(t)$  is WAC on  $I_a$ .

COROLLARY. Assume (I), (II), (III) and (V). Then there is at most one mild solution of (4.1).

*Proof.* Every mild solution is a weak solution and there is at most one weak solution.



**THEOREM 5.** Assume (I), (II), (III), (IV) and (V). Let  $x_0 \in V$  be such that  $\|f(t, x_0)\|_{V'} \leq Q(t)b_0$  on  $I_a$ . Then there is a  $c$ ,  $0 < c \leq a$ , such that there exists a (unique) mild solution of (4.1) on  $I_c$ . If additionally  $B_b(x_0) = V$ , then there is a mild solution on  $I_a$ .

*Proof.* Let  $x(\cdot)$  be strongly continuous from  $I_a$  to  $V$  and  $x(t) \in B_b(x_0)$  for all  $t \in I_a$ . For any  $\phi \in V'$ , the map  $s \rightarrow \langle \phi, G(t, s)\mathcal{M}^{-1}(s)f(s, x(s)) \rangle = \langle G^*(t, s)\phi, \mathcal{M}^{-1}(s)f(s, x(s)) \rangle$  is measurable by Lemma 3.2 and Proposition 4.1(v). Also we have the estimates of Proposition 4.1(iii) and (II), which show that the map  $s \rightarrow G(t, s)\mathcal{M}^{-1}(s)f(s, x(s))$  is in  $L^1(I_a, V)$  for any  $t \in I_a$ . Also, for  $t, \tau \in I_a$  we have

$$\begin{aligned} & \int_0^t G(t, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds - \int_0^\tau G(\tau, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds \\ (4.8) \quad &= [G(t, 0) - G(\tau, 0)] \int_0^\tau G(0, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds \\ &+ \int_\tau^t G(t, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds \end{aligned}$$

and this difference converges to zero in  $V$  as  $t \rightarrow \tau$ . Thus, for any  $x$  as above we define a continuous function by

$$[Fx](t) = G(t, 0)x_0 + \int_0^t G(t, s)\mathcal{M}^{-1}(s)f(s, x(s)) ds.$$

Finally, the estimate

$$\|[Fx](t) - x_0\|_V \leq \|G(t, 0)x_0 - x_0\|_V + \exp \left\{ \int_0^a (K/k) \right\} (b_0 + b) \int_0^t (Q(s)/k(s)) ds$$

shows that for  $c$  sufficiently small,  $F$  maps the set  $M$  of those continuous functions  $x: I_c \rightarrow V$  with every  $x(t) \in B_b(x_0)$  into itself. The estimate of Lemma 2.4 follows from (II), so  $F$  has a unique fixed point. When  $B_b(X_0) = V$ , we may proceed as in Theorem 3.

**5. The strong solution.** Let  $V$  be the reflexive and separable Banach space of § 3. Let  $H$  be a Hilbert space with norm and inner product given by  $\|h\|_H$  and  $(h_1, h_2)_H$ , respectively. Assume  $V$  is a dense subset of  $H$  (so  $H$  is separable) and the injection  $V \hookrightarrow H$  is continuous. Thus we have  $\|v\|_H \leq c\|v\|_V$  on  $V$  for some  $c > 0$ . If we identify  $H$  and its antidual  $H'$  by the theorem of F. Riesz [4, pp. 43-44], we then have  $V \hookrightarrow H \hookrightarrow V'$ , the second injection following by duality from the first, and also  $\langle x, y \rangle_H = \langle x, y \rangle$  on  $H \times V$  under the indicated identification.

Let  $\{\mathcal{M}(t): t \in I_a\}$  and  $\{\mathcal{L}(t): t \in I_a\}$  be the families of operators in  $L(V, V')$  constructed in §§ 3 and 4. Define  $M(t)$  and  $L(t)$  to be the respective restrictions of  $\mathcal{M}(t)$  and  $\mathcal{L}(t)$  to  $H$ . These restrictions are unbounded operators on  $H$  with respective domains given by  $D(M(t)) = \{x \in V: \mathcal{M}(t)x \in H\}$  and  $D(L(t)) = \{x \in V: \mathcal{L}(t)x \in H\}$ . Note that an element  $x \in V$  is in  $D(M(t))$  if and only if the conjugate-linear map  $y \rightarrow m(t; x, y): V \rightarrow \mathbb{C}$  is continuous with respect to the topology induced by  $H$  on  $V$  (which is weaker than that of  $V$ ). (See [4, pp. 62-67] for an elementary discussion and references.)

DEFINITION. A *strong solution* of (4.1) is a weak solution for which each term of the equation is in  $H$  a.e. on  $I_a$ . This is equivalent to writing

$$(5.1) \quad M(t)x'(t) + L(t)x(t) = f(t, x(t)) \quad \text{a.e.}$$

We note that (5.1) is an equation in  $H$  whereas (4.1) is an equation in  $V'$ .

Our first result is a sufficient condition for the linear propagator to generate strong solutions.

PROPOSITION 5.1. Assume (I), (III), (V) and in addition:

(VI) for a.e.  $t \in I_a$ , we have  $D(M(s)) \subseteq D(L(t))$  for  $s \in I_t$ , and there is a  $K_1 \in L^1(I_a, R)$  such that  $\|L(t)M^{-1}(s)\|_{L(H)} \leq K_1(s)$  when  $0 \leq s \leq t \leq a$ .

Then for  $x_0 \in D(M(0))$  the function  $x(t) = G(t, 0)x_0$  is the strong solution of the homogeneous equation

$$(5.2) \quad M(t)x'(t) + L(t)x(t) = 0.$$

*Proof.* Consider the linear space  $X$  of all elements  $x \in L^\infty(I_a, V)$  for which

$$x(t) \in D(L(t)) \quad \text{a.e. on } I_a$$

and

$$\|L(\cdot)x(\cdot)\|_{L^\infty(I_a, H)} = \text{ess sup } \{ \|L(t)x(t)\|_H : t \in I_a \} < \infty.$$

(Note that for each  $v \in V$ , the map  $t \rightarrow (L(t)x(t), v)_H = \langle \mathcal{L}'(t)v, x(t) \rangle$  is measurable by (V) and the measurability of  $x(\cdot)$ ;  $V$  is dense in  $H$ , so this means  $t \rightarrow L(t)x(t): I_a \rightarrow H$  is measurable.) Since each  $L(t)$  is closed, it follows that  $X$  with the norm  $\|x\|_X = \max \{ \|x\|_{L^\infty(I_a, V)}, \|L(\cdot)x(\cdot)\|_{L^\infty(I_a, H)} \}$  is a Banach space.

Let  $x \in X$  and  $t \in [0, a]$ . For  $\xi \in I_t = [0, t]$  we have  $M^{-1}(\xi)L(\xi)x(\xi) \in D(M(\xi)) \subseteq D(L(t))$  by (VI), and from (I) the estimate  $\|M^{-1}(\xi)L(\xi)x(\xi)\|_H \leq c\|M^{-1}(\xi)L(\xi)x(\xi)\|_V \leq ck(\xi)^{-1}\|L(\xi)x(\xi)\|_H \leq ck(\xi)^{-1}\|x\|_X$ . In (V) we may assume  $K \geq 1$ , and hence  $k(\xi)^{-1}$  is in  $L^1(I_a, R)$ , without loss of generality. Since Lemma 3.2 implies  $\xi \rightarrow M^{-1}(\xi)L(\xi)x(\xi)$  is measurable  $I_a \rightarrow V$  and since  $V \hookrightarrow H$  is continuous, the map is measurable  $I_a \rightarrow H$ , hence in  $L^1(I_a, H)$ . Also from (VI) follows the estimate  $\|L(t)M^{-1}(\xi)L(\xi)x(\xi)\|_H \leq K_1(\xi)\|L(\xi)x(\xi)\|_H \leq K_1(\xi)\|x\|_X$ . For each  $v \in V$  the map  $\xi \rightarrow (L(t)M^{-1}(\xi)L(\xi)x(\xi), v)_H = \langle \mathcal{L}'(t)v, M^{-1}(\xi)\mathcal{L}(\xi)x(\xi) \rangle$  is measurable and  $V$  is dense in  $H$ , so the map  $\xi \rightarrow L(t)M^{-1}(\xi)L(\xi)x(\xi)$  is in  $L^1(I_t, H)$ . Since  $L(t)$  is closed we have  $\int_0^t M^{-1}(\xi)L(\xi)x(\xi) d\xi$  belongs to  $D(L(t))$  and

$$(5.3) \quad L(t) \int_0^t M^{-1}(\xi)L(\xi)x(\xi) d\xi = \int_0^t L(t)M^{-1}(\xi)L(\xi)x(\xi) d\xi$$

(see [11, p. 83] for a proof).

Consider the function defined a.e. on  $I_a$  by (5.3). From the estimates

$$\left\| L(t) \int_0^t M^{-1}(\xi)L(\xi)x(\xi) d\xi \right\|_H \leq \|K_1\|_{L^1(I_a, R)} \|x\|_X$$

and

$$\left\| \int_0^t M^{-1}(\xi)L(\xi)x(\xi) d\xi \right\|_V \leq \|(K/k)\|_{L^1(I_a, R)} \|x\|_X,$$

it follows that this function is in  $X$ . Finally, since  $x_0 \in D(M(0))$  and  $|L(t)x_0|_H \leq K_1(0)|M(0)x_0|_H$  the function  $F$  defined by

$$(5.4) \quad [Fx](t) = x_0 - \int_0^t M^{-1}(\xi)L(\xi)x(\xi) d\xi, \quad t \in I_a,$$

maps  $X$  into itself and satisfies

$$\|Fx_1 - Fx_2\|_X \leq \max \{ \|(K/k)\|_{L^1(I_a, R)}, \|K_1\|_{L^1(I_a, R)} \} \|x_1 - x_2\|_X.$$

By the usual arguments, it follows that there is a unique  $x \in X$  for which  $Fx = x$ , and this is the strong solution of (5.2). By the uniqueness of weak solutions it follows that  $x(t) = G(t, 0)x_0$ .

**COROLLARY.** Assume (I), (III), (V) and (VI). Then for  $x_0 \in D(M(s))$ ,  $s \in [0, a]$ , the function  $x(t) = G(t, s)x_0$  is the strong solution of (5.2) on  $[s, a]$ .

**THEOREM 6.** Assume (I), (III), (V) and (VI). Let  $x_0 \in D(M(0))$  and  $f: I_a \times B_b(x_0) \rightarrow H$  be given with  $|f(t, x)|_H \leq Q_1(t)g(|x|_V)$ , where  $g: [0, \infty] \rightarrow [0, \infty]$  maps bounded sets into bounded sets and the measurable function  $Q_1(\cdot)$  is such that  $Q_1K_1 \in L^1(I_a, R)$ . Then any mild solution of (4.1) is a strong solution.

*Proof.* Let  $u(\cdot)$  be a mild solution. For  $s \in [0, a]$ , define  $y(s) = M^{-1}(s)f(s, u(s))$ . The function  $t \rightarrow x(t, s) = G(t, s)y(s): [s, a] \rightarrow D(L(t))$  is the strong solution of (5.2), and hence has the representation  $x(t, s) = y(s) - \int_s^t M^{-1}(\xi)L(\xi)x(\xi, s) d\xi$ . This follows from the previous corollary since  $y(s) \in D(M(s))$ . From (5.3) we obtain the estimates

$$\begin{aligned} |L(t)x(t, s)|_H &\leq |L(t)y(s)|_H + \int_s^t K_1(\xi)|L(\xi)x(\xi, s)|_H \\ &\leq K_1(s)Q_1(s)g(|u(s)|_V) + \int_s^t K_1(\xi)|L(\xi)x(\xi, s)|_H d\xi \end{aligned}$$

and Lemma 2.3 thus gives

$$|L(t)x(t, s)|_H \leq K_1(s)Q_1(s)g(|u(s)|_V) \exp \left\{ \int_s^t K_1(\xi) d\xi \right\}$$

for  $0 \leq s \leq t \leq a$ . By an argument like that preceding (5.3), one can use Proposition 4.1(v) to show that  $s \rightarrow L(t)x(t, s): [0, t] \rightarrow H$  is measurable and the above estimate shows that

$$\int_0^t |L(t)G(t, s)M^{-1}(s)f(s, u(s))|_H ds < \infty.$$

(We have used the fact that  $\{g(|u(s)|_V): s \in [0, a]\}$  is bounded since  $u: I_a \rightarrow V$  is bounded.) Finally, the map  $s \rightarrow G(t, s)M^{-1}(s)f(s, u(s)) = G(t, s)\mathcal{M}^{-1}(s)f(s, u(s))$  is in  $L^1(I_t, V) \subseteq L^1(I_t, H)$  by the definition of mild solution, so we have (see [11, p. 83])

$$\int_0^t G(t, s)M^{-1}(s)f(s, u(s)) ds \in D(L(t)).$$

The result is now immediate from (4.7) and Proposition 5.1.

*Remark.* The requirement that  $x_0$  belong to  $D(M(0))$  is unnecessarily restrictive when  $D(L(t))$  is independent of  $t$ . It is only necessary to have the map  $t \rightarrow L(t)x_0$  in  $L^1(I_a, H)$  (see the argument preceding (5.4)).

Our final result is a sufficient condition for the existence of a strong solution of (3.2). In applications the function  $f$  will contain spatial derivatives of order as high as those of the leading operators, whereas Theorem 6 requires that  $f$  contain spatial derivatives of order at most half of the order of those of the leading operator.

**THEOREM 7.** *Assume (I) and (III), with  $V$  and  $H$  as given above. Suppose there is a separable and reflexive Banach space  $D$ , dense and continuously imbedded in  $H$  for which  $D(M(t)) \subseteq D$  and  $\|M(t)x\|_H \geq k(t)\|x\|_D$  for  $x \in D(M(t))$  and a.e.  $t \in I_a$ . For each  $x \in D$ , the function  $f(\cdot, x)$  is measurable from  $I_a$  to  $H$ , and for each  $t \in I_a$  we have  $\|f(t, x) - f(t, y)\|_H \leq Q(t)\|x - y\|_D$ . Assume that  $Q(t)/k(t)$  is in  $L^1(I_a, \mathbb{R})$ . Then for each  $x_0$  in  $D$  such that  $\|f(t, x_0)\|_H \leq b_0 Q(t)$ , there is a unique SAC function  $x: I_a \rightarrow D$  for which the strong derivative  $x'(t)$  exists a.e.,  $x' \in L^1(I_a, D)$ ,  $x'(t) \in D(M(t))$  a.e.,  $x(0) = x_0$  and*

$$M(t)x'(t) = f(t, x(t)) \quad \text{a.e. } t \in I_a.$$

The proof of Theorem 7 can be patterned after the techniques above. There are certain measurability results that must be obtained, but these can also be handled as above. If  $D$  is continuously imbedded in  $V$ , this function is a strong solution.

**6. Applications.** We shall present a rather general realization of the abstract evolution equations (4.1) and (5.1) as a mixed initial and boundary value problem for a partial differential equation of third order. The same technique yields similar results for higher order equations [4], [15]. Problems of this type arise in the flow of fluid through fissured rocks [3], thermodynamics [6], shear in second order fluids [8], [12], consolidation of clay [23], and others [10]. Certain examples of a linear and time-independent version of our model have been studied [1], [7], [16], [17], [18], [24]. Time-dependent and nonlinear variations have also been studied. In particular, [26] contains results for a linear equation like (4.1) in which the operators are strongly-differentiable, and [13] applies to the linearized form of equation (5.1) when the operators are realizations of regular elliptic boundary value problems and the time-dependence is continuous in the *uniform* operator topology. The nonlinear equation (3.2) is considered in [10] with the added assumption that the linear operators are independent of time. The Lipschitz assumptions in [10] imply ours, and "solution" in [10] means "weak solution" in our notation, so the existence results of [10] are contained in ours.

Our abstract results imply that each of the boundary value problems in the preceding applications is well-posed in an appropriate function space. The intent in the following is to display the types of nonlinear problems to which the abstract results apply, so we do not consider properties of solutions. Such properties as regularity [13], [19], [20] and asymptotic behavior [7], [14], [17], [21] have been discussed for linear equations. We refer to [2], [4], [15], [19] for references to unsupported results on regular elliptic boundary value problems and additional models like those below. Finally we remark that we assume no continuity in the time-dependence of the operators in our models below. In fact we simply require

that the coefficients be measurable in the space and time variables and not "too degenerate" in time. The third example does not require ellipticity of the leading operator.

Let  $\Omega$  be an open set in  $R^n$  with boundary  $\partial\Omega$  an  $(n - 1)$ -dimensional manifold with  $\Omega$  on one side of  $\partial\Omega$ .  $\Gamma_0$  is a measurable subset of  $\partial\Omega$  and  $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ .  $H^m$  will denote the space of (equivalence classes of) functions  $\phi \in H \equiv L^2(\Omega)$  such that  $D^\alpha \phi \in L^2(\Omega)$  when  $|\alpha| \leq m$ , where  $D^\alpha$  is a partial derivative of order  $|\alpha|$ . Then  $H^m$  is a Hilbert space with inner product

$$(\phi, \psi)_m = \sum \left( \int_{\Omega} D^\alpha \phi \overline{D^\alpha \psi} dx : |\alpha| \leq m \right).$$

Let  $V$  be the closed subspace of  $H^1(\Omega)$  consisting of those  $\phi \in H^1(\Omega)$  for which (the trace of)  $\phi$  vanishes on  $\Gamma_0$ . Then  $V$  is a reflexive and separable Banach space with the norm  $|\phi|_V = \sqrt{(\phi, \phi)_1}$ . We shall assume that  $\partial\Omega$  is sufficiently smooth for the divergence theorem to apply: there is a unit outward normal  $n(s) = (n_1(s), \dots, n_n(s))$  at each  $s \in \partial\Omega$  for which

$$\int_{\Omega} D_j \phi dx = \int_{\partial\Omega} n_j(s) \phi(s) ds, \quad j = 1, 2, \dots, n,$$

for all smooth functions  $\phi$ , where  $D_j = \partial/\partial x_j$ .

Let  $I = [0, 1]$  and functions  $m_0 \in L^\infty(\Omega \times I)$ ,  $\alpha \in L^\infty(\Gamma_1 \times I)$  be given, for which  $\operatorname{Re} \alpha(s, t) \geq 0$  and  $\operatorname{Re} m_0(x, t) \geq k$  for some number  $k \in (0, 1]$ . Let  $\eta: I \rightarrow I$  be measurable and assume  $\int_0^1 (\eta(t))^{-1} dt < \infty$ . Then for  $\phi, \psi$  in  $V$  we define

$$m(t; \phi, \psi) = \eta(t) \int_{\Omega} \sum_{j=1}^n D_j \phi \overline{D_j \psi} dx + \int_{\Omega} m_0(x, t) \phi \overline{\psi} dx + \int_{\Gamma_1} \alpha(s, t) \phi \overline{\psi} ds.$$

The assumptions (I) and (III) are satisfied with  $k(t) = k \cdot \eta(t)$ . The restriction of  $\mathcal{M}(t)\phi \in V'$  to  $C_0^\infty(\Omega)$  is the distribution given by

$$(6.1) \quad \mathcal{M}(t)\phi = -\eta(t) \sum_{j=1}^n D_j D_j \phi + m_0(\cdot, t)\phi.$$

By the regularity theory of elliptic operators, the domain of the restriction to  $H = L^2(\Omega)$  is given by

$$D(M(t)) = \{\phi \in V \cap H^2 : m(t; \phi, \psi) = (M(t)\phi, \psi)_{L^2(\Omega)}, \psi \in V\}.$$

The condition that  $\phi \in V$  means  $\phi$  vanishes on  $\Gamma_0$ , while from the second condition we see that

$$\eta(t) \int_{\Omega} \left( \sum_{j=1}^n D_j \phi \cdot \overline{D_j \psi} + \sum_{j=1}^n D_j D_j \phi \cdot \overline{\psi} \right) dx + \int_{\Gamma_1} \alpha \phi \overline{\psi} ds = 0$$

for all  $\psi$  in  $V$ . But elements of  $V$  are (essentially) arbitrary on  $\Gamma_1$ , so the divergence theorem asserts that this is a variational boundary condition

$$(6.2) \quad \eta(t) D_n \phi(s) + \alpha(s, t) \phi(s) = 0, \quad s \in \Gamma_1.$$

Here  $D_n = \sum_{j=1}^n n_j(s)D_j$  denotes the normal (directional) derivative on  $\partial\Omega$ . Thus we have

$$(6.3) \quad D(M(t)) = \{\phi \in H^2 : \phi = 0 \text{ on } \Gamma_0, \eta(t)D_n\phi + \alpha\phi = 0 \text{ on } \Gamma_1\},$$

where the equations on  $\partial\Omega$  are interpreted as above.

Assume we are given functions  $l_{ij}, l_j, l_0 \in L^\infty(\Omega \times I)$ ,  $i, j = 1, 2, \dots, n$ , and  $\beta \in L^\infty(\Gamma_1 \times I)$ . For  $\phi, \psi \in V$  define

$$\begin{aligned} l(t; \phi, \psi) = & \int_{\Omega} \left\{ \sum_{i,j=1}^n l_{ij}(x, t) D_j \phi \cdot \overline{D_i \psi} + \sum_{j=1}^n l_j(x, t) D_j \phi \cdot \overline{\psi} + l_0(x, t) \phi \cdot \overline{\psi} \right\} dx \\ & + \int_{\Gamma_1} \beta(s, t) \phi(s) \overline{\psi(s)} ds. \end{aligned}$$

As above we have

$$(6.4) \quad \mathcal{L}(t)\phi = - \sum_{i,j=1}^n D_i(l_{ij}(\cdot, t)D_j\phi) + \sum_{j=1}^n l_j(\cdot, t)D_j\phi + l_0(\cdot, t)\phi$$

and the domain of the restriction to  $H$  is given by

$$D(L(t)) = \left\{ \phi \in V : \sum_{i,j=1}^n D_i(l_{ij}(\cdot, t)D_j\phi) \in L^2(\Omega), l(t; \phi, \psi) = (L(t)\phi, \psi)_{L^2(\Omega)} \right. \\ \left. \text{for all } \psi \in V \right\}.$$

The second condition is the variational boundary condition

$$\sum_{i,j=1}^n n_i(s)l_{ij}(s, t)D_j\phi(s) + \beta(s, t)\phi(s) = 0, \quad s \in \Gamma_1.$$

The assumption (V) is satisfied with  $K(t) = K$  depending on the  $L^\infty(\Omega \times I)$  norms of the coefficients in (6.4) and the norm of  $\beta$  in  $L^\infty(\Gamma_1 \times I)$ .

A sufficient condition that the condition (VI) hold is that

$$(6.5) \quad \eta(t) \equiv 1, \quad l_{ij}(x, t) = \delta_{ij}, \quad \alpha = \beta \text{ is independent of } t.$$

In this case we see that  $D(M(t)) = D(L(t))$  is independent of  $t$  though the operators may vary with  $t$  through the lower order terms. A second sufficient condition for (VI) is that

$$(6.6) \quad l_{ij}(x, t) = 0 \quad \text{and} \quad \beta = 0.$$

Then  $D(L(t)) = V$  for  $t \in I$ . The estimate in (VI) is easily obtained and  $K_1$  depends on  $k, \eta$  and the coefficients in  $\mathcal{L}(t)$ . Other variations are possible; we may require (6.5) to hold for  $t \in [0, 1/2]$  and (6.6) for  $t \in [1/2, 1]$ , but we cannot interchange the order of these requirements.

*Example 1.* We consider the semilinear equation

$$(6.7a) \quad \mathcal{M}(t)D_t u(x, t) + \mathcal{L}(t)u(x, t) = F(x, t, u(x, t), D_j u(x, t)), \quad (x, t) \in \Omega \times I,$$

with the linear conditions

$$(6.7b) \quad \begin{aligned} u(s, t) = 0, \quad s \in \Gamma_0; \quad \eta(t)D_n D_t u(s, t) + \alpha(s, t)D_t u(s, t) + \beta(s, t)u(s, t) \\ + \sum_{i,j=1}^n n_i(s)l_{ij}(s, t)D_j u(s, t) = 0, \quad s \in \Gamma_1; \quad t \in I, \end{aligned}$$

and the initial condition

$$(6.7c) \quad u(x, 0) = u_0(x), \quad x \in \Omega.$$

The measurable function  $F: \Omega \times I \times C^{n+1} \rightarrow C$  is assumed to satisfy the Lipschitz condition

$$(6.8) \quad |F(x, t, \xi) - F(x, t, \eta)| \leq Q(t) \sum_{i=0}^n |\xi_i - \eta_i|, \quad \xi, \eta \in C^{n+1},$$

where  $Q(t) \geq 1$  is measurable and  $Q(\cdot)/\eta(\cdot) \in L^1(I, R)$ . From the Cauchy-Schwarz inequality we then obtain the estimate

$$\|F(\cdot, t, \phi, D_j \phi) - F(\cdot, t, \psi, D_j \psi)\|_{L^2(\Omega)} \leq Q(t) \sqrt{n+1} \|\phi - \psi\|_V$$

for  $\phi, \psi$  in  $V$ . Similarly, if  $F$  satisfies

$$(6.9) \quad |F(x, t, \xi)| \leq Q(t) \left( q(x) + \sum_{j=0}^n |\xi_j| \right), \quad x \in \Omega, \quad \xi \in C^{n+1},$$

where  $q(x) \geq 0$  and  $q \in L^2(\Omega)$ , then we obtain the estimate

$$\|F(\cdot, t, \phi, D_j \phi)\|_{L^2(\Omega)} \leq Q(t) g(\|\phi\|_V),$$

where  $g(x) = [(n+2)(\|q\|_{L^2(\Omega)}^2 + x^2)]^{1/2}$ ,  $x \geq 0$ .

From Theorem 5 we obtain the following. Let the spaces  $V$  and  $H$  and sesquilinear forms  $m(t; \cdot, \cdot)$  and  $l(t; \cdot, \cdot)$  be given as above. Let the measurable function  $F$  be given and satisfy (6.8). Thus  $f(t, \phi)(x) = F(x, t, \phi(x), D_j \phi(x))$  defines a function  $f: I \times V \rightarrow H$ ; since  $H$  is continuously embedded in  $V'$ ,  $f$  satisfies (II) and (IV). Assume that a  $u_0$  is given in  $V$  for which  $\|f(t, u_0)\|_{L^2(\Omega)} \leq Q(t)b_0$  on  $I$  for some  $b_0 > 0$ . (This estimate is automatically true if (6.9) holds.) Then there exists exactly one mild solution  $u(t)$  of (4.1) on  $I$ . This mild solution satisfies the partial differential equation (6.7a) a.e. on  $I$  in the sense of distributions, the initial condition (6.7c) is satisfied a.e. on  $\Omega$ , and the first boundary condition in (6.7b) is satisfied in the sense of traces on  $\Gamma_0$ . Finally, we have the identity

$$m(t; u'(t), \phi) + l(t; u(t), \phi) = (\mathcal{M}(t)u'(t) + \mathcal{L}(t)u(t), \phi)_H$$

for all  $\phi$  in  $V$ , and this is a variational boundary condition on  $\Gamma_1$  which by the divergence theorem implies the second condition in (6.7b). If we furthermore assume (6.5) and (6.9) and that  $u_0$  is given in  $D(M(0))$ , then Theorem 6 asserts that  $u(t)$  is a strong solution, so the boundary conditions (6.7b) are strengthened to require that  $u(t) \in D(L(t)) = D(M(t))$  for every  $t \in I$ . This also implies a regularity result, i.e., that  $u(t) \in H^2(\Omega)$  for  $t \in I$  (see (6.3)).

*Example 2.* The techniques above are applicable to solutions of the quasilinear equation

$$(6.10a) \quad \mathcal{M}(t)D_t u(x, t) + \sum_{i=1}^n D_i(F_i(x, t, u(x, t), D_j u(x, t))) = F_0(x, t, u(x, t), D_j u(x, t))$$

with the nonlinear boundary conditions

$$(6.10b) \quad \begin{aligned} u(s, t) &= 0, \quad s \in \Gamma_0; \quad \eta(t)D_n D_t u(s, t) + \alpha(s, t)D_t u(s, t) \\ &- \sum_{i=1}^n n_i(s)F_i(s, t, u(s, t), D_j u(s, t)) = G(s, t, u(s, t)), \quad s \in \Gamma_1, \end{aligned}$$

and an initial condition (6.7c). Here we assume  $F_i: \Omega \times I \times C^{n+1} \rightarrow C$ ,  $i = 0, 1, 2, \dots, n$ , are given which are measurable and satisfy (6.8).  $G: \Gamma_1 \times I \times C \rightarrow C$  is measurable and satisfies

$$|G(s, t, \xi) - G(s, t, \eta)| \leq Q(t)|\xi - \eta|, \quad \xi, \eta \in C.$$

Then we define  $f: I \times V \rightarrow V'$  by

$$\langle f(t, \phi), \psi \rangle = \int_{\Omega} \sum_{i=0}^n F_i(x, t, \phi(x), D_j \phi(x)) \overline{D_i \psi(x)} dx + \int_{\Gamma_1} G(s, t, \phi(s)) \overline{\psi(s)} ds,$$

where  $D_0 = 1$ . Weak (= mild) solutions are obtained from Theorem 3 (Theorem 5) for  $u_0$  appropriately chosen. Strong solutions are obtained from Theorem 7 if  $u_0 \in D \equiv V \cap H^2$ , (6.5) holds, and  $G = 0$ .

*Example 3.* Let  $I = \{0, 1\}$  and  $\Omega = I \times I$ .  $V$  is the closure of  $C_0^\infty(\Omega)$  in the norm  $|\cdot|_V$ , where  $|\phi|_V^2 = \int_{\Omega} (|D_1^2 \phi|^2 + |D_2 \phi|^2) dx$ . For any  $\phi$  in  $C_0^\infty$  we have

$$\begin{aligned} \int_0^1 \{|\phi(x_1, x_2)|^2 + x_1 D_1 |\phi(x_1, x_2)|^2\} dx_1 &= x_1 |\phi(x_1, x_2)|^2|_0^1 = 0, \\ \int_0^1 \{|\phi(x_1, x_2)|^2 dx_1\} &\leq 2 \int_0^1 x_1 |\phi(x_1, x_2)| |D_1 \phi(x_1, x_2)| dx_1 \\ &\leq \frac{1}{2} \int_0^1 |\phi(x_1, x_2)|^2 dx_1 + 2 \int_0^1 |D_1 \phi(x_1, x_2)|^2 dx_1, \end{aligned}$$

and hence,

$$\int_0^1 |\phi(x_1, x_2)|^2 dx_1 \leq 4 \int_0^1 |D_1 \phi(x_1, x_2)|^2 dx_1.$$

Integrating this with respect to  $x_2$  on  $I$  we obtain

$$(6.11) \quad \|\phi\|_{L^2(\Omega)} \leq 2 \|D_1 \phi\|_{L^2(\Omega)}$$

for all  $\phi$  in  $C_0^\infty(\Omega)$ . Thus  $V = \{\phi \in L^2(\Omega): D_1 \phi, D_1^2 \phi, D_2 \phi \in L^2(\Omega) \text{ and } \phi = 0 \text{ on } \partial\Omega, D_1 \phi = 0 \text{ when } x_1 = 0 \text{ or } 1\}$ . Define  $m(t; \phi, \psi) = \int_{\Omega} (D_1^2 \phi \overline{D_1^2 \psi} + D_2 \phi \overline{D_2 \psi}) dx$  on  $V$  and

$$l(t; \phi, \psi) = \sum_{j=0}^2 \sum_{k=0}^1 l_{jk}(x, t) D_1^j \phi D_2^k \psi dx,$$

where  $l_{jk} \in L^\infty(\Omega \times I)$ . Then (I), (III) and (V) are satisfied; we use (6.11) to verify the boundedness of  $l(t; \phi, \psi)$ . Thus we have

$$\mathcal{M}(t)\phi = D_1^4 \phi - D_2^2 \phi \quad \text{and} \quad \mathcal{L}(t)\phi = \sum_{j=0}^2 \sum_{k=0}^1 (-1)^k D^k (l_{jk}(\cdot, t) D^j \phi).$$

Nonlinear terms and coefficients in  $\mathcal{L}(t)$  could be added as above. Theorem 2 asserts the existence of a weak solution  $u(t)$  of (4.2) which satisfies the partial differential equation (4.2) in the sense of distributions on  $\Omega$ , the boundary conditions built into the space  $V$  at each  $t \in I$ , and the initial condition  $u(x, 0) = u_0(x)$  a.e. on  $\Omega$ , where  $u_0$  is given in  $V$ . (We note that  $\mathcal{M}(t)$  is not elliptic.)



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