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EXISTENCE AND STABILITY FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. The existence and stability properties of a class of partial functional differential equations are investigated. The problem is formulated as an abstract ordinary functional differential equation of the form $du(t)/dt = Au(t) + F(u_t)$, where A is the infinitesimal generator of a strongly continuous semigroup of linear operators $T(t)$, $t \geq 0$, on a Banach space X and F is a Lipschitz operator from $C = C([-r, 0]; X)$ to X. The solutions are studied as a semigroup of linear or nonlinear operators on C. In the case that F has Lipschitz constant L and $|T(t)| \leq e^{\omega t}$, then the asymptotic stability of the solutions is demonstrated when $\omega + L < 0$. Exact regions of stability are determined for some equations where F is linear.

1. **Introduction and preliminaries.** The purpose of this paper is to investigate existence and stability properties for a class of partial functional differential equations. As a model for this class one may take the equation

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) + f(t, w(x, t - r)), && 0 \leqslant x \leqslant \pi, \quad t \geqslant 0, \\ (1.1)\quad &w(0, t) = w(\pi, t) = 0, && t \geqslant 0, \\ &w(x, t) = \varphi(x, t), && 0 \leqslant x \leqslant \pi, \quad -r \leqslant t \leqslant 0, \end{aligned}$$

where f is a linear or nonlinear scalar-valued function, r is a positive number, and φ is a given initial function. In our development the second derivative term in (1.1) will correspond to a strongly continuous semigroup of linear operators on a Banach space of functions determined by the boundary conditions in (1.1). Accordingly, our approach will rely primarily on semigroup methods and the treatment of (1.1) as an abstract ordinary functional differential equation in a Banach space.

Our first objective will be to develop an existence theory for the nonlinear nonautonomous case and this will be done in §2. In the case that f is autonomous the solutions give rise to a strongly continuous semigroup of nonlinear operators on a Banach space of initial function values. This semigroup has been extensively studied for ordinary linear functional differential equations by J.

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Hale in [6] and recently for ordinary nonlinear functional differential equations by G. Webb in [13]. We will investigate the properties of this semigroup and its infinitesimal generator in §3. In the case of ordinary linear functional differential equations the spectral analysis of the infinitesimal generator of this semigroup gives considerable information about the behavior of solutions. We will give an analogue to such a development in the linear partial functional differential equations case in §4, where our approach will follow closely that of J. Hale in [6]. Lastly, we will apply our theory to some specific examples in §5, where we will give particular attention to the stability of solutions.

Before proceeding we shall set forth some notation and terminology that will be used throughout the paper. X will denote a Banach space over a real or complex field. $C = C([-r, 0]; X)$ will denote the Banach space of continuous X-valued functions on $[-r, 0]$, with supremum norm, where $r > 0$. If u is a continuous function from $[a-r, b]$ to X and $t \in [a, b]$, then u_t denotes the element of C given by $u_t(\theta) = u(t+\theta)$, $-r \leqslant \theta \leqslant 0$. If A is a linear or nonlinear operator from X to X, then $D(A)$, $R(A)$, $N(A)$ denote its domain, range, and null space, respectively. If A is linear then $\rho(A)$, $\sigma(A)$, $P\sigma(A)$ denote the resolvent, spectrum, and point spectrum of A, respectively. $B(X, X)$ will denote the space of bounded linear everywhere defined operators from X to X and if $A \in B(X, X)$, then $|A|$ is the norm of A. If A is linear and $\lambda \in \rho(A)$, then $R(\lambda; A)$ is $(A - \lambda I)^{-1} \in B(X, X)$, and if $\lambda \in \sigma(A)$, then $M_\lambda(A)$ is the generalized eigenspace of λ (that is, the smallest subspace of X containing $N(A - \lambda I)^k$, $k = 1, 2, \cdots$).

By a strongly continuous semigroup on X we shall mean a family $T(t)$, $t \geqslant 0$, of everywhere defined (possibly nonlinear) operators from X to X satisfying $T(t+s) = T(t)T(s)$ for $s, t \geqslant 0$, and $T(t)x$ is continuous as a function from $[0, \infty)$ to X for each fixed $x \in X$. The infinitesimal generator A_T of $T(t)$, $t \geqslant 0$, is the function from X to X defined by $A_T x = \lim_{t \to 0^+} t^{-1}(T(t)x - x)$ with $D(A_T)$ all x for which this limit exists. Finally, in the case that the semigroup is linear we shall require the following facts.

(1.2) A necessary and sufficient condition that a closed densely defined linear operator A_T be the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geqslant 0$, of operators in $B(X, X)$ such that $|T(t)| \leqslant e^{\omega t}$ for some real number ω is that $|R(\lambda; A_T)| \leqslant (\lambda - \omega)^{-1}$ for all $\lambda > \omega$ (see [4, Corollary 14, p. 626]).

(1.3) If X is complex and $T(t)$, $t \geqslant 0$, is as in (1.2), then for all λ such that $\operatorname{Re} \lambda > \omega$, $\lambda \in \rho(A_T)$ and $|R(\lambda; A_T)| \leqslant (\operatorname{Re} \lambda - \omega)^{-1}$ (see [14, Corollary 1, p. 241]).

2. Existence of solutions in the nonlinear case. We prove our main existence theorem in an integrated form using a method derived from the fundamental results of I. Segal in [11].

PROPOSITION 2.1. *Let* $F: [a, b] \times C \to X$ *such that* F *is continuous and satisfies*

$$\|F(t, \psi) - F(t, \hat{\psi})\|_X \leqslant L \|\psi - \hat{\psi}\|_C \quad \text{for } a \leqslant t \leqslant b, \psi, \hat{\psi} \in C, \tag{2.1}$$

where L *is a positive constant. Let* $T(t)$, $t \geqslant 0$, A_T, ω *be as in* (1.2). *If* $\varphi \in C$ *there is a unique continuous function* $u(t)$: $[a - r, b] \to X$ *which solves*

$$\begin{aligned} u(t) &= T(t-a)\varphi(0) + \int_a^t T(t-s)F(s, u_s)ds, \qquad a \leqslant t \leqslant b, \\ u_a &= \varphi. \end{aligned} \tag{2.2}$$

PROOF. First observe that if $w(s)$ is any continuous function from $[a - r, b]$ to X, then $T(t-s)F(s, w_s)$ is continuous in $s \in [a, t]$ by virtue of the continuity of F, the continuity of w_s as a function in s from $[a, t]$ to C, and the strong continuity of $T(t)$, $t \geqslant 0$. Define $u^0(t) = \varphi(t-a)$ for $a - r \leqslant t \leqslant a$ and $u^0(t) = T(t-a)\varphi(0)$ for $a \leqslant t \leqslant b$. In general, for each positive integer n, define

$$u^n(t) = \varphi(t-a) \qquad \text{for } a - r \leqslant t \leqslant a,$$

$$u^n(t) = T(t-a)\varphi(0) + \int_a^t T(t-s)F(s, u_s^{n-1})ds \quad \text{for } a \leqslant t \leqslant b.$$

Since F is continuous there exists M such that $\|F(s, u_s^0)\|_X \leqslant M$ for $a \leqslant s \leqslant b$. Then for $a \leqslant t \leqslant b$,

$$\|u^1(t) - u^0(t)\|_X \leqslant (t-a)e^{\omega(b-a)}M,$$

and, in general,

$$\|u^n(t) - u^{n-1}(t)\|_X \leqslant ML^{n-1}e^{n\omega(b-a)}(t-a)^n/n!.$$

Thus, $\lim_{n\to\infty} u^n(t) \stackrel{\text{def}}{=} u(t)$ exists uniformly on $[a - r, b]$ and $u(t)$ is continuous on $[a - r, b]$.

To establish that $u(t)$ satisfies (2.2) use

$$\begin{aligned} &\left\|u(t) - T(t-a)\varphi(0) - \int_a^t T(t-s)F(s, u_s)ds\right\|_X \\ &\qquad \leqslant \|u(t) - u^{n+1}(t)\|_X + \left\|\int_a^t T(t-s)(F(s, u_s) - F(s, u_s^n))ds\right\|_X \\ &\qquad \leqslant (1 + L(t-a)e^{\omega(b-a)})M \sum_{k=n+1}^{\infty} L^{k-1}e^{k\omega(b-a)}(t-a)^k/k!. \end{aligned}$$

To establish the uniqueness assertion suppose $v(t)$ satisfies (2.2) and let K be a constant such that $\|v(t) - u(t)\|_X \leq (t-a)K$. Then

$$\|v(t) - u^n(t)\|_X \leq KL^n e^{(n+1)\omega(b-a)}(t-a)^{n+1}/(n+1)!,$$

whereupon $v(t) = \lim_{n\to\infty} u^n(t)$ and the proof is complete.

COROLLARY 2.2. *Suppose the hypothesis of Proposition* 2.1 *and let* $u(t)$, $\hat{u}(t)$ *solve* (2.2) *for* $\varphi, \hat{\varphi} \in C$, *respectively. Then for* $a \leq t \leq b$

$$(2.3)\quad \begin{aligned} &\|u_t - \hat{u}_t\|_C \leq \|\varphi - \hat{\varphi}\|_C e^{(\omega+L)(t-a)} && \textit{if } \omega \geq 0, \textit{ and} \\ &\|u_t - \hat{u}_t\|_C \leq \|\varphi - \hat{\varphi}\|_C e^{-\omega r} e^{(\omega + Le^{-\omega r})(t-a)} && \textit{if } \omega < 0. \end{aligned}$$

PROOF. From (2.1) and (2.2) we have that for $a - r \leq t \leq b$

$$\|u(t) - \hat{u}(t)\|_X \leq e^{\omega(t-a)}\|\varphi(0) - \hat{\varphi}(0)\|_X + L\int_a^t e^{\omega(t-s)}\|u_s - \hat{u}_s\|_C\,ds.$$

If $\omega \geq 0$, then for $a \leq t \leq b$

$$\|u_t - \hat{u}_t\|_C \leq e^{\omega(t-a)}\|\varphi - \hat{\varphi}\|_C + L\int_a^t e^{\omega(t-s)}\|u_s - \hat{u}_s\|_C\,ds,$$

and if $\omega < 0$, then for $a \leq t \leq b$

$$\|u_t - \hat{u}_t\|_C \leq e^{-\omega r}e^{\omega(t-a)}\|\varphi - \hat{\varphi}\|_C + Le^{-\omega r}\int_a^t e^{\omega(t-s)}\|u_s - \hat{u}_s\|_C\,ds.$$

By Gronwall's lemma (2.3) follows.

PROPOSITION 2.3. *Suppose the hypothesis of Proposition* 2.1 *and in addition suppose that* F *is continuously differentiable from* $[a, b] \times C$ *to* X *and* F_1, F_2 *satisfy for* $a \leq t \leq b$, $\psi, \hat{\psi} \in C$, *and positive constants* β, γ,

$$(2.4)\quad \|F_1(t, \psi) - F_1(t, \hat{\psi})\|_X \leq \beta\|\psi - \hat{\psi}\|_C,$$

$$(2.5)\quad |F_2(t, \psi) - F_2(t, \hat{\psi})| \leq \gamma\|\psi - \hat{\psi}\|_C.$$

Then, for $\varphi \in C$ *such that* $\varphi(0) \in D(A_T)$, $\dot{\varphi} \in C$, *and* $\dot{\varphi}^-(0) = A_T\varphi(0) + F(a, \varphi)$, $u(t)$ *is continuously differentiable and satisfies*

$$(2.6)\quad \begin{aligned} &d/dt\, u(t) = A_T u(t) + F(t, u_t), \quad a \leq t \leq b, \\ &u_a = \varphi. \end{aligned}$$

PROOF. By virtue of Proposition 2.1 we can solve

$$\begin{aligned} v(t) &= T(t-a)(A_T\varphi(0) + F(a, \varphi)) \\ &\quad + \int_a^t T(t-s)(F_1(s, u_s) + F_2(s, u_s)v_s)\,ds, \qquad a \leqslant t \leqslant b, \\ v_a &= \dot{\varphi}. \end{aligned} \tag{2.7}$$

Define $w(t) = \varphi(t-a)$ for $a - r \leqslant t \leqslant a$, $w(t) = \varphi(0) + \int_a^t v(s)\,ds$ for $a \leqslant t \leqslant b$. We will show that $w(t) = u(t)$, which will establish that $u(t)$ is continuously differentiable. First, by taking the limit of the difference quotient, one obtains that for $a \leqslant t \leqslant b$,

$$\begin{aligned} \frac{d}{dt}\int_a^t T(t-s)F(s, w_s)\,ds &= \int_a^t T(t-s)(F_1(s, w_s) + F_2(s, w_s)v_s)\,ds \\ &\quad + T(t-a)F(a, \varphi). \end{aligned} \tag{2.8}$$

Then (2.8) yields

$$\begin{aligned} \int_a^t T(t-a)F(a, \varphi)\,ds &= \int_a^t T(t-s)F(s, w_s)\,ds \\ &\quad - \int_a^t\int_a^s T(s-\tau)(F_1(\tau, w_\tau) + F_2(\tau, w_\tau)v_\tau)\,d\tau\,ds. \end{aligned} \tag{2.9}$$

Using the fact that for $z \in D(A_T)$, $\int_a^t T(t-s)A_T z\,ds = T(t-a)z - z$, (2.7) and (2.9) imply

$$\begin{aligned} w(t) &= T(t-a)\varphi(0) + \int_a^t T(t-s)F(s, w_s)\,ds \\ &\quad + \int_a^t\int_a^s T(s-\tau)(F_1(\tau, u_\tau) - F_1(\tau, w_\tau) \\ &\qquad + (F_2(\tau, u_\tau) - F_2(\tau, w_\tau))v_\tau)\,d\tau\,ds. \end{aligned} \tag{2.10}$$

Then (2.10), (2.4), and (2.5) yield $\|w(t) - u(t)\|_X \leqslant \text{const} \int_a^t \|w_\tau - u_\tau\|_C\,d\tau$, which implies

$$\|w_t - u_t\|_C \leqslant \text{const} \int_a^t \|w_\tau - u_\tau\|_C\,d\tau.$$

By Gronwall's lemma $w(t) = u(t)$. Therefore, $\int_a^t T(t-s)F(s, u_s)\,ds$ is of the form $\int_0^d T(s)g(s)\,ds$ where $g(s)$: $[0, d] \longrightarrow X$ is continuously differentiable. By Theorem 1.9, p. 486 of [8], $u(t)$ is a solution of (2.6) and the proof is finished.

We remark that the equation (2.2) is more general than (2.6). In fact, in the case of Proposition 2.1 $u(t)$, $t > 0$, may not belong to $D(A_T)$ and may not be differentiable (even if $\varphi(0) \in D(A_T)$). The following proposition will be of fundamental importance in §4.

PROPOSITION 2.4. *Suppose the hypothesis of Proposition* 2.1 *and in addition suppose that* $T(t)$ *is compact for each* $t > 0$. *Then,* $(t, \varphi) \longrightarrow u_t \overset{\mathrm{def}}{=} u_t(\varphi)$, *the mapping defined by solutions, is compact in* φ *for each fixed* $t > r$.

Before proving Proposition 2.4 we require two lemmas.

LEMMA 2.5. *Let* $T(t)$, $t \geq 0$, *and* ω *be as in* (1.2) *and in addition let* $T(t)$ *be compact for each* $t > 0$. *Let* B *be a bounded subset of* X *and let* $\{f_\gamma: \gamma \in \Gamma\}$ *be a set of continuous functions from the finite interval* $[c, d] \subset (0, \infty)$ *to* B. *Then* $K = \{\int_c^d T(s) f_\gamma(s)\, ds: \gamma \in \Gamma\}$ *is a precompact subset of* X.

PROOF. Let $H = \{T(t)x: t \in [c, d], x \in B\}$. We will use the fact that $T(t)$ is uniformly continuous from $[c, d]$ to $B(X, X)$ (see [7, Theorem 10.22, p. 304]) to show that H is totally bounded (see [14, p. 13]). Let $\epsilon > 0$ and let M be a bound for B. There exists $c = t_1 < t_2 < \cdots < t_n = d$ such that

$$|T(t_i) - T(t)| \leq \epsilon/2M \quad \text{for } t_{i-1} \leq t \leq t_i. \tag{2.11}$$

Since for each t_i, $T(t_i)B$ is totally bounded, there exists $\{x_1^i, x_2^i, \cdots, x_{k(i)}^i\} \subset B$ such that if $x \in B$, then

$$\| T(t_i)x_j^i - T(t_i)x \|_X \leq \epsilon/2 \quad \text{for some } x_j^i. \tag{2.12}$$

One uses (2.11) and (2.12) to demonstrate the total boundedness of H. Then H is precompact and therefore so is the convex hull of H (see [12, Exercise 4, p. 134]). The conclusion follows since K is contained in the closed convex hull of $(d - c)H$.

LEMMA 2.6. *Let* $\{f_\gamma: \gamma \in \Gamma\} \subset C$ *be an equicontinuous family such that for each* $\theta \in [-r, 0]$, $\{f_\gamma(\theta): \gamma \in \Gamma\}$ *is precompact in* X. *Then* $\{f_\gamma: \gamma \in \Gamma\}$ *is precompact in* C.

PROOF. A proof may be found in [10, Theorem 33, p. 179].

PROOF OF PROPOSITION 2.4. Let $\{\varphi_\gamma: \gamma \in \Gamma\}$ be a bounded subset of C and let $t > r$. For each $\gamma \in \Gamma$ define $f_\gamma \in C$ by $f_\gamma = u_t(\varphi_\gamma)$. Then, for $\theta \in [-r, 0]$, $t + \theta > 0$, and so

$$f_\gamma(\theta) = u_t(\varphi_\gamma)(\theta) = u(\varphi_\gamma)(t + \theta) = T(t + \theta)\varphi_\gamma(0)$$

$$+ \int_0^{t+\theta} T(t + \theta - s) F(s, u_s(\varphi_\gamma))\, ds.$$

We shall apply Lemma 2.6 to the family $\{f_\gamma: \gamma \in \Gamma\}$.

First, we show this family is equicontinuous. Recall (2.1) and (2.3) to argue that $\{F(s, u_s(\varphi_\gamma)): s \in [0, t], \gamma \in \Gamma\}$ is bounded by a constant, say M.

Let $\gamma \in \Gamma$, let $0 < c < t - r$, let $-r \leqslant \hat{\theta} < \theta \leqslant 0$, and observe that

$$\begin{aligned}
\|f_\gamma(\theta) - f_\gamma(\hat{\theta})\|_X &\leqslant \|T(t+\theta)\varphi_\gamma(0) - T(t+\hat{\theta})\varphi_\gamma(0)\|_X \\
&\quad + \left\| \int_0^{t+\theta} T(t+\theta-s)F(s, u_s(\varphi_\gamma))\,ds - \int_0^{t+\hat{\theta}} T(t+\theta-s)F(s, u_s(\varphi_\gamma))\,ds \right\|_X \\
&\quad + \left\| \int_{t+\hat{\theta}-c}^{t+\hat{\theta}} (T(t+\theta-s) - T(t+\hat{\theta}-s))F(s, u_s(\varphi_\gamma))\,ds \right\|_X \\
&\quad + \left\| \int_0^{t+\hat{\theta}-c} (T(t+\theta-s) - T(t+\hat{\theta}-s))F(s, u_s(\varphi_\gamma))\,ds \right\|_X \\
&\leqslant |T(t+\theta) - T(t+\hat{\theta})| \, \|\varphi_\gamma(0)\|_X + |\theta - \hat{\theta}| e^{\omega t} M + c\, 2 e^{\omega t} M \\
&\quad + t \sup_{s \in [0, t+\hat{\theta}-c]} |T(t+\theta-s) - T(t+\hat{\theta}-s)| M.
\end{aligned}$$

One now uses the uniform continuity of $T(s)$, $s \in [c, t]$, in $B(X, X)$ to demonstrate the claimed equicontinuity.

Next, we show that for fixed $\theta \in [-r, 0]$, $\{f_\gamma(\theta): \gamma \in \Gamma\}$ is precompact in X. Obviously, $\{T(t+\theta)\varphi_\gamma(0): \gamma \in \Gamma\}$ is precompact, since $t + \theta > 0$ and $\|\varphi_\gamma(0)\|_X$ is bounded independent of γ. We will show that

$$K = \left\{ \int_0^{t+\theta} T(t+\theta-s)F(s, u_s(\varphi_\gamma))\,ds: \ \gamma \in \Gamma \right\}$$

is totally bounded. Observe that if $0 < c < t + \theta$, then

$$\left\| \int_{t+\theta-c}^{t+\theta} T(t+\theta-s)F(s, u_s(\varphi_\gamma))\,ds \right\| \leqslant c e^{\omega t} M \tag{2.13}$$

for all $\gamma \in \Gamma$. By Lemma 2.5, if $0 < c < t + \theta$, then

$$K_c = \left\{ \int_0^{t+\theta-c} T(t+\theta-s)F(s, u_s(\varphi_\gamma))\,ds: \gamma \in \Gamma \right\}$$

is precompact in X. This fact together with (2.13) yield the precompactness of K. Thus the hypothesis of Lemma 2.6 is verified and the proof is complete.

3. The semigroup and infinitesimal generator in the autonomous case. Throughout this section we will suppose the hypothesis of Proposition 2.1 except that we require F to be autonomous, that is, $F: C \longrightarrow X$. By virtue of Proposition 2.1 there exists for each $\varphi \in C$ a unique continuous function $u(\varphi)(t)$: $[-r, \infty) \longrightarrow X$ satisfying

$$\begin{aligned}
u(\varphi)(t) &= T(t)\varphi(0) + \int_0^t T(t-s)F(u_s(\varphi))\,ds, \qquad t \geqslant 0, \\
u_0(\varphi) &= \varphi.
\end{aligned} \tag{3.1}$$

For each $t \geq 0$ define $U(t)\colon C \longrightarrow C$ by $U(t)\varphi = u_t(\varphi)$.

PROPOSITION 3.1. *$U(t)$, $t \geq 0$, is a strongly continuous semigroup of (possibly nonlinear) operators on C satisfying for $\varphi, \hat{\varphi} \in C$, $t \geq 0$,*

$$\begin{aligned} &\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leq \|\varphi - \hat{\varphi}\|_C e^{(\omega + L)t} && \text{if } \omega \geq 0, \\ &\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leq e^{-\omega r}\|\varphi - \hat{\varphi}\|_C e^{(\omega + Le^{-\omega r})t} && \text{if } \omega < 0. \end{aligned} \tag{3.2}$$

PROOF. The strong continuity follows from the fact that solutions of (3.1) are continuous. The semigroup property follows from the fact that for $t, \hat{t} \geq 0$, $\varphi \in C$,

$$\begin{aligned} u(\varphi)(t + \hat{t}) &= T(t + \hat{t})\varphi(0) + \int_0^t T(t + \hat{t} - s)F(u_s(\varphi))\,ds \\ &\quad + \int_t^{t+\hat{t}} T(t + \hat{t} - s)F(u_s(\varphi))\,ds \\ &= T(\hat{t})\left(T(t)\varphi(0) + \int_0^t T(t - s)F(u_s(\varphi))\,ds\right) + \int_0^{\hat{t}} T(\hat{t} - s)F(u_{t+s}(\varphi))\,ds \\ &= T(\hat{t})u(\varphi)(t) + \int_0^{\hat{t}} T(\hat{t} - s)F(u_{t+s}(\varphi))\,ds. \end{aligned}$$

By the uniqueness of solutions to (3.1) this implies that $u_{t+\hat{t}}(\varphi) = u_{\hat{t}}(u_t(\varphi))$. Lastly, (3.2) follows from Corollary 2.2.

We next investigate the infinitesimal generator of $U(t)$, $t \geq 0$. Define A_U: $C \longrightarrow C$ as follows:

$$\begin{aligned} &(A_U\varphi)(\theta) = \dot{\varphi}(\theta), \quad -r \leq \theta \leq 0, \\ &D(A_U) = \{\varphi \in C\colon \dot{\varphi} \in C, \varphi(0) \in D(A_T), \dot{\varphi}^-(0) = A_T\varphi(0) + F(\varphi)\}. \end{aligned} \tag{3.3}$$

PROPOSITION 3.2. *A_U is the infinitesimal generator of $U(t)$, $t \geq 0$.*

PROOF. First, let φ belong to the domain of the infinitesimal generator of $U(t)$, $t \geq 0$. We will show that $\varphi \in D(A_U)$ and $A_U\varphi = \psi$, where

$$\psi(\theta) = \lim_{t \to 0^+} t^{-1}(U(t)\varphi(\theta) - \varphi(\theta)), \quad \theta \in [-r, 0].$$

Obviously, $\psi(\theta) = \dot{\varphi}^+(\theta)$ for $-r \leq \theta < 0$. Since $\psi \in C$, $\lim_{\theta \to 0^-} \dot{\varphi}^+(\theta)$ exists and must be $\psi(0)$. But this means that $\dot{\varphi}$ exists on $(-r, 0)$, φ^- exists at 0, and $\dot{\varphi}^-(0) = \psi(0)$ (see [14, p. 239]). It remains to show that $\varphi(0) \in D(A_T)$ and $\psi(0) = A_T\varphi(0) + F(\varphi)$. Observe that

$$\lim_{t \to 0^+} \frac{1}{t}\int_0^t T(t - s)F(u_s(\varphi))\,ds = F(\varphi), \tag{3.4}$$

since

$$\left\|\frac{1}{t}\int_0^t T(t-s)F(u_s(\varphi))\,ds - F(\varphi)\right\|_X \leqslant \frac{1}{t}\int_0^t \|T(t-s)F(u_s(\varphi)) - F(\varphi)\|_X\,ds$$

$$\leqslant \max_{s\in[0,t]} \left(e^{\omega t}\|F(u_s(\varphi)) - F(\varphi)\|_X + \|T(t-s)F(\varphi) - F(\varphi)\|_X\right).$$

By virtue of (3.4) the limit as $t \longrightarrow 0^+$ on the left side of

$$\frac{1}{t}(u(\varphi)(t) - u(\varphi)(0)) - \frac{1}{t}\int_0^t T(t-s)F(u_s(\varphi))\,ds = \frac{1}{t}\,(T(t)\varphi(0) - \varphi(0)) \tag{3.5}$$

exists and is $\psi(0) - F(\varphi)$. But this implies that the limit as $t \longrightarrow 0^+$ on the right side of (3.5) exists, and is $A_T\varphi(0)$ by definition of A_T.

Now let $\varphi \in D(A_U)$. We must show that

$$\lim_{t\to 0^+} (1/t)(U(t)\varphi - \varphi) \text{ exists in } C \text{ and } = A_U\varphi. \tag{3.6}$$

Recall that $\|t^{-1}(U(t)\varphi(\theta) - \varphi(\theta)) - \dot{\varphi}(\theta)\|_X =$

$$\|(t^{-1}(\varphi(t+\theta) - \varphi(\theta)) - \dot{\varphi}(\theta)\|_X \quad \text{if } -r \leqslant t+\theta \leqslant 0, \tag{3.7}$$

and

$$\left\|\frac{1}{t}\left(T(t+\theta)\varphi(0) + \int_0^{t+\theta} T(t+\theta-s)F(u_s(\varphi))\,ds - \varphi(\theta)\right) - \dot{\varphi}(\theta)\right\|_X \tag{3.8}$$

$$\text{if } 0 < t+\theta.$$

Suppose $t+\theta > 0$. Then (3.8) $\leqslant$

$$\left\|\frac{1}{t}\left(T(t+\theta)\varphi(0) - \varphi(0) + \int_0^{t+\theta} T(t+\theta-s)F(u_s(\varphi))\,ds\right) - (t+\theta)t^{-1}\dot{\varphi}^-(0)\right\|_X \tag{3.9}$$

$$+ \|t^{-1}(\varphi(0) - \varphi(\theta)) - \dot{\varphi}(\theta) + (t+\theta)t^{-1}\dot{\varphi}^-(0)\|_X. \tag{3.10}$$

Then (3.9) $\leqslant$

$$\frac{1}{t}\int_0^{t+\theta} (\|T(s)A_T\varphi(0) - A_T\varphi(0)\|_X + \|T(t+\theta-s)F(u_s(\varphi)) - F(\varphi)\|_X)\,ds \tag{3.11}$$

and (3.10) $\leqslant$

$$\frac{1}{t}\int_\theta^0 \|\dot{\varphi}(s) - \dot{\varphi}^-(0)\|_X\,ds + \|\dot{\varphi}(\theta) - \dot{\varphi}^-(0)\|_X. \tag{3.12}$$

If $\epsilon > 0$ choose δ sufficiently small such that if $0 < t < \delta$, $-r \leqslant \theta \leqslant 0$, and $t+\theta > 0$, then (3.11) and (3.12) $< \epsilon/3$, and if $t+\theta \leqslant 0$, then (3.7) $< \epsilon/3$.

This establishes (3.6) and thus the proof of the proposition.

PROPOSITION 3.3. *If* $-L \leqslant \omega$ *and* $\operatorname{Re}\lambda > L + \omega$, *then* $(A_U - \lambda I)^{-1}$ *exists and has domain all of* C.

PROOF. Given $\psi \in C$ we must solve

$$(3.13)\quad (A_U - \lambda I)\varphi = \dot{\varphi} - \lambda\varphi = \psi, \qquad \dot{\varphi}(0) = \lambda\varphi(0) + \psi(0) = A_T\varphi(0) + F(\varphi).$$

This means that

$$(3.14)\quad \begin{aligned} \varphi(\theta) &= e^{\lambda\theta}\varphi(0) + \int_0^\theta e^{\lambda(\theta-s)}\psi(s)\,ds, \\ \varphi(0) &= (A_T - \lambda I)^{-1}(\psi(0) - F(\varphi)). \end{aligned}$$

The mapping

$$x \longrightarrow (A_T - \lambda I)^{-1}\left(\psi(0) - F\left(e^{\lambda\theta}x + \int_0^\theta e^{\lambda(\theta-s)}\psi(s)\,ds\right)\right)$$

is a strict contraction from X to X, since by (1.3)

$$\|(A_T - \lambda I)^{-1}(F(e^{\lambda\theta}x) - F(e^{\lambda\theta}\hat{x}))\|_X \leqslant (L/(\operatorname{Re}\lambda - \omega))\|x - \hat{x}\|_X.$$

Then (3.14) and hence (3.13) has a unique solution. But this means that $(A_U - \lambda I)$ is onto and injective and the proof is complete.

PROPOSITION 3.4. *If* $-L \leqslant \omega$ *and* $\operatorname{Re}\lambda > L + \omega$, *then* $(A_U - \lambda I)^{-1}$ *is Lipschitz continuous with Lipschitz constant* $\leqslant 1/(\operatorname{Re}\lambda - (L + \omega))$.

PROOF. Let $\varphi = (A_U - \lambda I)^{-1}\psi, \hat{\varphi} = (A_U - \lambda I)^{-1}\hat{\psi}$ for $\psi, \hat{\psi} \in C$. Let $\epsilon > 0$ and let $\theta \in [-r, 0]$ have the property that $\|\varphi(\theta) - \hat{\varphi}(\theta)\|_X > \|\varphi - \hat{\varphi}\|_C - \epsilon$. Using (3.14) and (1.3) we have that

$$\begin{aligned} \|\varphi(\theta) - \hat{\varphi}(\theta)\|_X &\leqslant \|e^{\lambda\theta}(A_T - \lambda I)^{-1}(\psi(0) - F(\varphi) - \hat{\psi}(0) + F(\hat{\varphi}))\|_X \\ &\quad + \left\|\int_0^\theta e^{\lambda(\theta-s)}(\psi(s) - \hat{\psi}(s))ds\right\|_X \\ &\leqslant \frac{e^{\operatorname{Re}\lambda\theta}}{\operatorname{Re}\lambda - \omega}(\|\psi - \hat{\psi}\|_C + L\|\varphi - \hat{\varphi}\|_C) + \frac{(1 - e^{\operatorname{Re}\lambda\theta})}{\operatorname{Re}\lambda}\|\psi - \hat{\psi}\|_C \\ &= \frac{Le^{\operatorname{Re}\lambda\theta}}{\operatorname{Re}\lambda - \omega}\|\varphi - \hat{\varphi}\|_C + \frac{1 - (\omega/\operatorname{Re}\lambda)(1 - e^{\operatorname{Re}\lambda\theta})}{\operatorname{Re}\lambda - \omega}\|\psi - \hat{\psi}\|_C. \end{aligned}$$

But this implies

$$\frac{\operatorname{Re}\lambda - \omega - Le^{\operatorname{Re}\lambda\theta}}{\operatorname{Re}\lambda - \omega}\|\varphi - \hat{\varphi}\|_C \leqslant \epsilon + \frac{1 - (\omega/\operatorname{Re}\lambda)(1 - e^{\operatorname{Re}\lambda\theta})}{\operatorname{Re}\lambda - \omega}\|\psi - \hat{\psi}\|_C.$$

Since

$$\frac{1-(\omega/\mathrm{Re}\,\lambda)(1-e^{\mathrm{Re}\,\lambda\theta})}{\mathrm{Re}\,\lambda-\omega-e^{\mathrm{Re}\,\lambda\theta}L}\leqslant\frac{1}{\mathrm{Re}\,\lambda-\omega-L},$$

the assertion follows.

PROPOSITION 3.5. *For each* $\psi\in C$, $\lim_{\lambda\to 0^+}(I-\lambda A_U)^{-1}\psi=\psi$ *and, consequently,* $D(A_U)$ *is dense in* C.

PROOF. By virtue of Propositions 3.3 and 3.4 and the fact that $(I-\lambda A_U)^{-1}=(A_U-(1/\lambda)I)^{-1}(-1/\lambda)$, we have (choosing $\omega\geqslant -L$ if necessary) that

(3.15) $(I-\lambda A_U)^{-1}$ exists with domain C and is Lipschitz continuous with Lipschitz constant $\leqslant 1/(1-\lambda(L+\omega))$ for all real λ such that $0<\lambda<1/(L+\omega)$.

For $\lambda>0$, define $B_\lambda\colon C\longrightarrow C$ by

$$(B_\lambda\psi)(\theta)=\frac{e^{\theta/\lambda}}{\lambda}\int_\theta^0 e^{-s/\lambda}\psi(s)\,ds,\qquad \psi\in C,\ \theta\in[-r,0].$$

We will use the fact that $\lim_{\lambda\to 0^+}B_\lambda\psi=\psi$ provided that $\psi(0)=0$ (see [13, §2]) and alsc that $\lim_{\lambda\to 0^+}\|(I-\lambda A_T)^{-1}x-x\|_X=0$ for all $x\in X$ (see [14, p. 241]). Let $\psi\in C$ and let λ be real such that $0<\lambda<1/(L+\omega)$. Then

$$\begin{aligned}
&\|\psi-(I-\lambda A_U)^{-1}\psi\|_C\\
&\quad=\|\psi-(e^{\theta/\lambda}(I-\lambda A_T)^{-1}(\psi(0)+\lambda F((I-\lambda A_U)^{-1}\psi)))+B_\lambda\psi\|_C\\
&\quad\leqslant(\lambda/(1-\lambda\omega))(L\|(I-\lambda A_U)^{-1}\psi\|_C+\|F(0)\|_X)\\
&\qquad+\|\psi-\psi(0)-B_\lambda(\psi-\psi(0))\|_C+\|e^{\theta/\lambda}(I-\lambda A_T)^{-1}\psi(0)-e^{\theta/\lambda}\psi(0)\|_C.
\end{aligned}$$

Also, $\|(I-\lambda A_U)^{-1}\psi\|_C\leqslant\|(I-\lambda A_U)^{-1}\psi-\psi\|_C+\|\psi\|_C$. Thus,

$$\begin{aligned}
&(1-\lambda L/(1-\lambda\omega))\|\psi-(I-\lambda A_U)^{-1}\psi\|_C\\
&\quad\leqslant(\lambda L/(1-\lambda\omega))\|\psi\|_C+(\lambda/(1-\lambda\omega))\|F(0)\|_X\\
&\qquad+\|\psi-\psi(0)-B_\lambda(\psi-\psi(0))\|_C+\|(I-\lambda A_T)^{-1}\psi(0)-\psi(0)\|_X
\end{aligned}$$

and the assertion follows.

PROPOSITION 3.6. *For each* $\psi\in C$, $t\geqslant 0$, $\lim_{n\to\infty}(I-(t/n)A_U)^{-n}\psi=U(t)\psi$.

PROOF. To establish this exponential formula for $U(t)$, $t \geqslant 0$, we will use results of M. Crandall and T. Liggett in [3] and H. Brezis and A. Pazy in [1]. From Theorem 1 of [3] we have by virtue of Proposition 3.5 and (3.15) that

$$\lim_{n\to\infty} (I - (t/n)A_U)^{-n}\psi \overset{\text{def}}{=} V(t)\psi$$

exists for all $\psi \in C$, $t \geqslant 0$, and $V(t)$, $t \geqslant 0$, is a strongly continuous semigroup of nonlinear operators on C. From Corollary 4.3 of [1] we have by virtue of Propositions 3.2 and 3.5 and also (3.2) that

$$\lim_{n\to\infty} U(t/n)^n\psi = U(t)\psi = V(t)\psi$$

for all $\psi \in C$, $t \geqslant 0$, and the proof is complete.

COROLLARY 3.7. *If* $-L = \omega$, *then for* $\varphi, \hat{\varphi} \in C$, $t \geqslant 0$,

$$\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leqslant \|\varphi - \hat{\varphi}\|_C. \tag{3.16}$$

That is, if $-L = \omega$, *then* $U(t)$ *is nonexpansive for all* $t \geqslant 0$.

PROOF. (3.16) follows directly from (3.15) and Proposition 3.6.

COROLLARY 3.8. *If* $-L > \omega$, *then for* $\varphi, \hat{\varphi} \in C$, $t \geqslant 0$, *and each positive integer* n,

(3.17)

$$\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leqslant \left[(-L/\omega)^n + \left(\sum_{k=0}^{n-1} L^k(1 - (-L/\omega)^{n-k})e^{\omega(t-(k+1)r)}t^k/k!\right)\right]\|\varphi - \hat{\varphi}\|_C.$$

Furthermore, there exists a unique $\varphi_0 \in C$ *such that* $U(t)\varphi_0 = \varphi_0$ *for* $t > r$ *and* $\lim_{t\to\infty} U(t)\varphi = \varphi_0$ *for all* $\varphi \in C$.

PROOF. Since $-L > \omega$, (3.16) holds by virtue of Corollary 3.7 (ω can always be chosen larger than any given ω). Then, for $t \geqslant 0$

$$\begin{aligned} \|u(\varphi)(t) - u(\hat{\varphi})(t)\|_X & \\ \leqslant e^{\omega t}\|\varphi(0) - \hat{\varphi}(0)\|_X &+ L\int_0^t e^{\omega(t-s)}\|u_s(\varphi) - u_s(\hat{\varphi})\|_C\, ds \\ \leqslant [(-L/\omega) &+ (1 - (-L/\omega))e^{\omega t}]\|\varphi - \hat{\varphi}\|_C. \end{aligned} \tag{3.18}$$

Then, for $t \geqslant r$,

$$\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leqslant [(-L/\omega) + (1 - (-L/\omega))e^{\omega(t-r)}]\,\|\varphi - \hat{\varphi}\|_C. \tag{3.19}$$

But since $e^{\omega(t-r)} \geqslant 1$ for $0 \leqslant t \leqslant r$, (3.19) holds for all $t \geqslant 0$. In a similar manner one substitutes inequality (3.19) into (3.18) and integrates to obtain

$$\|U(t)\varphi - U(t)\hat{\varphi}\|_C \leqslant [(-L/\omega)^2 + (1 - (-L/\omega)^2)e^{\omega(t-r)} + L(1 - (-L/\omega))e^{\omega(t-2r)}t]\,\|\varphi - \hat{\varphi}\|_C$$

for $t \geqslant 0$. An induction argument yields (3.17). By virtue of (3.19), $U(t)$, $t > r$, is a commutative family of strict contractions on C and therefore has a unique common fixed point. That is, for $s, t > r$, $U(t)\varphi_t = \varphi_t$ implies $U(s)\varphi_t = U(s)U(t)\varphi_t = U(t)U(s)\varphi_t$ implies $\varphi_t = U(s)\varphi_t$ implies $\varphi_s = \varphi_t$. The last statement now follows from (3.17).

4. **The spectral properties of A_U in the linear case.** Throughout this section F will be as in §3 except that we require F to be linear with norm $|F| = L$. $T(t)$, $t \geqslant 0$, and A_T will be as before except that we require $T(t)$ to be compact for each $t > 0$. $U(t)$, $t \geqslant 0$, and A_U, which are now linear, will be as in §3. For each scalar λ define the linear operator $\Delta(\lambda)\colon D(A_T) \longrightarrow X$ by

$$\Delta(\lambda)x = A_T x = \lambda x + F(e^{\lambda\theta}x), \qquad x \in D(A_T). \tag{4.1}$$

We will say that λ satisfies the "characteristic equation" of (3.1) provided $\Delta(\lambda)x = 0$ for some $x \neq 0$.

PROPOSITION 4.1. *Suppose β is real such that if λ satisfies the characteristic equation of* (3.1), *then* $\operatorname{Re}\lambda \leqslant \beta$. *For each $\gamma > 0$ there exists a constant $K(\gamma) \geqslant 1$ such that for all $t \geqslant 0$,*

$$\|U(t)\varphi\|_C \leqslant K(\gamma)e^{(\beta+\gamma)t}\|\varphi\|_C. \tag{4.2}$$

We shall require three lemmas for the proof.

LEMMA 4.2. *For $t > r$, $\sigma(U(t))$ is a countable set and is compact with only possible accumulation point* 0, *and if $\mu \neq 0 \in \sigma(U(t))$, then $\mu \in P\sigma(U(t))$.*

PROOF. The lemma follows immediately from Proposition 2.4 and Theorem 6.26, p. 185, of [8].

LEMMA 4.3. *For $t > r$, $P\sigma(U(t)) = e^{tP\sigma(A_U)}$ plus possibly* $\{0\}$. *More specifically, if $\mu = \mu(t) \in P\sigma(U(t))$ for some $t > r$ and $\mu \neq 0$, then there exists $\lambda \in P\sigma(A_U)$ such that $e^{\lambda t} = \mu$. Furthermore, if $\{\lambda_n\}$ consists of all distinct points in $P\sigma(A_U)$ such that $e^{\lambda_n t} = \mu$, then for arbitrary k, $N(U(t) - \mu I)^k$ is the linear extension of the linear independent manifolds $N(A_U - \lambda_n I)^k$, where n ranges over $e^{\lambda_n t} = \mu$.*

PROOF. See Lemma 22.1 and the exercise which follows it in [6, p. 112].

LEMMA 4.4. *If $S(t)$, $t \geqslant 0$, is a strongly continuous semigroup of linear operators on X and for some $s > 0$ the spectral radius ρ of $S(s)$ is $\neq 0$ and $\tau = (1/s)\log\rho$, then for all $\gamma > 0$ there exists a constant $K(\gamma) \geqslant 1$ such that*

that

$$\|S(t)x\| \leqslant K(\gamma)e^{(\tau+\gamma)t}\|x\| \quad \textit{for all} \quad t \geqslant 0, \ x \in X.$$

PROOF. See Lemma 22.2, p. 112 of [6].

PROOF OF PROPOSITION 4.1. Suppose $\mu \neq 0 \in \sigma(U(t))$ where t is some fixed number $> r$. By Lemma 4.2, $\mu \in P\sigma(U(t))$. By Lemma 4.3, $\mu = e^{t\lambda}$ where $\lambda \in P\sigma(A_U)$. Then there exists

$$\varphi \neq 0 \in D(A_U), \qquad \dot{\varphi} - \lambda\varphi = 0. \tag{4.3}$$

But this is equivalent to

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \qquad \varphi(0) \neq 0, \qquad \dot{\varphi}^-(0) = A_T\varphi(0) + F(\varphi). \tag{4.4}$$

Then $\Delta(\lambda)\varphi(0) = 0$, and by hypothesis $\operatorname{Re}\lambda \leqslant \beta$. Thus the spectral radius of $U(t) \leqslant e^{t\beta}$ and (4.2) follows immediately by application of Lemma 4.4.

PROPOSITION 4.5. *If* $\lambda \in P\sigma(A_U)$ *then* $M_\lambda(A_U)$ *is finite dimensional.*

PROOF. The proof follows immediately from Lemma 4.3 and the following lemma.

LEMMA 4.6. *For* $t > r$, *if* $\mu \in P\sigma(U(t)), \mu \neq 0$, *then* $N(U(t) - \mu I)^k$ *is of finite dimension for all* k, *and there exists a positive integer* n *such that* $M_\mu(U(t)) = N(U(t) - \mu I)^n$. *Moreover,* $U(t)M_\mu(U(t)) \subset M_\mu(U(t))$.

PROOF. A proof is given in [7, Theorem 5.7.3, p. 182]. The last statement is a consequence of the fact that $U(t)$ and $(U(t) - \mu I)^k$ commute.

PROPOSITION 4.7. *There exists a real number* β *such that* $\operatorname{Re}\lambda \leqslant \beta$ *for all* $\lambda \in \sigma(A_U)$ *and if* γ *is any real number there exists only a finite number of* $\lambda \in P\sigma(A_U)$ *such that* $\gamma \leqslant \operatorname{Re}\lambda$.

PROOF. The existence of the constant β follows immediately from Propositions 3.3 and 3.4 (in fact, one can choose $\beta = \max\{0, L + \omega\}$). Assume that $\{\lambda_k\}$ is an infinite sequence of distinct points in $P\sigma(A_U)$ such that $\operatorname{Re}\lambda_k > \gamma$ for all k, where γ is a given real number. By Lemma 4.3 $e^{\lambda_k t} \in P\sigma(U(t))$ for a fixed $t > r$. If $\{e^{\lambda_k t}\}$ is infinite, then $P\sigma(U(t))$ has an accumulation point different from 0, which contradicts Lemma 4.2. If $\{e^{\lambda_k t}\}$ is finite, then

$$e^{\lambda_{n_k} t} = \mu = \text{constant for some infinite subsequence } \{\lambda_{n_k}\}.$$

Then $N(U(t) - \mu I)$ is infinite dimensional, since it contains the linearly independent manifolds $N(A_U - \lambda_{n_k} I)$ by Lemma 4.3. But this contradicts Lemma 4.6. Thus the assumption is false and the proof is finished.

COROLLARY 4.8. *Let* β *be the smallest real number such that if* λ

satisfies the characteristic equation of (3.1), *then* $\operatorname{Re}\lambda \leqslant \beta$. *If* $\beta < 0$, *then for all* $\varphi \in C$, $\|U(t)\varphi\|_C \longrightarrow 0$ *as* $t \longrightarrow \infty$. *If* $\beta = 0$, *then there exists* $\varphi \neq 0 \in C$ *such that* $\|U(t)\varphi\|_C = \|\varphi\|_C$ *for all* $t \geqslant 0$. *If* $\beta > 0$, *then there exists* $\varphi \in C$ *such that* $\|U(t)\varphi\|_C \longrightarrow \infty$ *as* $t \longrightarrow \infty$.

PROOF. The claim for $\beta < 0$ is immediate from (4.2). If $\beta = 0$ let $x \neq 0 \in D(A_T)$ such that $\Delta(\lambda)x = 0$ where $\operatorname{Re}\lambda = 0$ (such a λ exists by Proposition 4.7). As in (4.3) and (4.4), $\varphi(\theta) = e^{\lambda\theta}\varphi(0)$, $\varphi(0) = x$ solves $(A_U - \lambda I)\varphi = 0, \varphi \neq 0$. Thus $U(t)\varphi = e^{\lambda t}\varphi$ and $\|U(t)\varphi\|_C = |e^{it\operatorname{Im}\lambda}|\|\varphi\|_C = \|\varphi\|_C$. If $\beta > 0$, let $x \neq 0 \in D(A_T)$ such that $\Delta(\lambda)x = 0$ and $\operatorname{Re}\lambda > 0$. Again $\varphi(\theta) \doteq e^{\lambda\theta}\varphi(0)$, $\varphi(0) = x$ solves $(A_U - \lambda I)\varphi = 0, \varphi \neq 0$. Thus, $U(t)\varphi = e^{\lambda t}\varphi$ and $\|U(t)\varphi\|_C = e^{\operatorname{Re}\lambda t}\|\varphi\|_C$.

Our next objective is to decompose the space C using the eigenvalues of A_U. This will be done by means of Propositions 4.10, 4.11, and 4.12, which are proved just as in [6, Chapters 20 and 22]. We first require the following proposition.

PROPOSITION 4.9. *Suppose* X *is complex and* $\lambda_0 \in P\sigma(A_U)$ *such that* λ_0 *is a pole of order* n_0 *of* $(A_U - \lambda I)^{-1}$. *Then*

$$(4.5)\quad C = N(A_U - \lambda_0 I)^{n_0} \oplus R(A_U - \lambda_0 I)^{n_0} \overset{\text{def}}{=} M_{\lambda_0}(A_U) \oplus R_{\lambda_0}(A_U).$$

Moreover, $M_{\lambda_0}(A_U) = N(A_U - \lambda_0 I)^n$ *for all* $n \geqslant n_0$ *and* n_0 *is the smallest such positive integer. Also*

$$(4.6)\qquad A_U(M_{\lambda_0}(A_U)) \subseteq M_{\lambda_0}(A_U), \quad A_U(R_{\lambda_0}(A_U)) \subseteq R_{\lambda_0}(A_U),$$

$$(4.7)\quad U(t)(M_{\lambda_0}(A_U)) \subseteq M_{\lambda_0}(A_U),\ U(t)(R_{\lambda_0}(A_U)) \subseteq R_{\lambda_0}(A_U)\ \textit{for all}\ t \geqslant 0.$$

PROOF. (4.5) and (4.6) follow directly from Theorem 5.8-A, p. 306 of [12]. (4.7) follows from the fact that $U(t)$ and A_U commute on $D(A_U)$ (see [14, Theorem 2, p. 239].

PROPOSITION 4.10. *Suppose* X *is complex,* $\lambda \in P\sigma(A_U)$ *is a pole of* $(A_U - \mu I)^{-1}, \varphi_1^\lambda, \cdots, \varphi_d^\lambda$ *is a basis for the finite dimensional subspace* $M_\lambda(A_U)$, *and* $\Phi_\lambda = (\varphi_1^\lambda, \cdots, \varphi_d^\lambda)$. *Since* $A_U(M_\lambda(A_U)) \subseteq M_\lambda(A_U)$, *there exists a* $d \times d$ *constant matrix* B_λ *such that* $A_U\Phi_\lambda = \Phi_\lambda B_\lambda$. *The only eigenvalue of* B_λ *is* λ, $\Phi_\lambda(\theta) = \Phi_\lambda(0)e^{B_\lambda\theta}$, $-r \leqslant \theta \leqslant 0$, $U(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t}$ *for* $t \geqslant 0$, $(U(t)\Phi_\lambda)(\theta) = \Phi_\lambda(0)e^{B_\lambda(t+\theta)}$ *for* $-r \leqslant \theta \leqslant 0, t \geqslant 0$. *Then* $U(t)$ *can be defined for all* $t \in (-\infty, \infty)$ *on* $M_\lambda(A_U)$.

PROPOSITION 4.11. *Suppose* X *is complex,* $\Lambda = \{\lambda_1, \cdots, \lambda_p\}$ *is a finite set in* $P\sigma(A_U)$ *such that each* λ_k *is a pole of* $(A - \lambda I)^{-1}$, $\Phi_\Lambda =$

$(\Phi_{\lambda_1}, \cdots, \Phi_{\lambda_p})$, $B_\Lambda = \operatorname{diag}(B_{\lambda_1}, \cdots, B_{\lambda_p})$ *where* Φ_{λ_k} *and* B_{λ_k} *are as in Proposition* 4.10. *Then for any vector* a *of the same dimension as* Φ_Λ, *the solution* $U(t)\Phi_\lambda a$ *with initial value* $\Phi_\lambda a$ *at* $t = 0$ *may be defined on* $(-\infty, \infty)$ *by*

$$U(t)\Phi_\Lambda a = \Phi_\Lambda e^{B_\Lambda t} a,$$

$$\Phi_\Lambda(\theta) = \Phi_\Lambda(0) e^{B_\Lambda \theta}, \qquad -r \leqslant \theta \leqslant 0.$$

Furthermore, there exists a subspace Q_Λ *of* C *such that* $U(t)Q_\Lambda \subset Q_\Lambda$ *for all* $t \geqslant 0$ *and* $C = P_\Lambda \oplus Q_\Lambda$, *where* $P_\Lambda = \{\varphi \in C: \varphi = \Phi_\Lambda a$ *for some vector* a *of the same dimension as* $\Phi_\Lambda\}$.

PROPOSITION 4.12. *Suppose* X *is complex*, $\Lambda = \Lambda(\beta) = \{\lambda \in P\sigma(A_U)$: $\operatorname{Re} \lambda \geqslant \beta\}$, *where* β *is a given real number, and let* $C = P_\Lambda \oplus Q_\Lambda$ *be decomposed as in Proposition* 4.11. *There exist positive constants* K, γ *such that for all* $\varphi \in C$

$$\|U(t)\varphi^{P_\Lambda}\|_C \leqslant K e^{(\beta-\gamma)t} \|\varphi^{P_\Lambda}\|_C, \qquad t \leqslant 0,$$

$$\|U(t)\varphi^{Q_\Lambda}\|_C \leqslant K e^{(\beta-\gamma)t} \|\varphi^{Q_\Lambda}\|_C, \qquad t \geqslant 0,$$

where $\varphi = \varphi^{P_\Lambda} + \varphi^{Q_\Lambda}$ *is the decomposition of* φ *in* $P_\Lambda \oplus Q_\Lambda$.

In the case of Proposition 4.12 we will say that $C = P_\Lambda \oplus Q_\Lambda$ is decomposed by Λ. Propositions 4.11 and 4.12 permit us to distinguish the behavior of the solutions on the two subspaces P_Λ and Q_Λ, since these subspaces are invariant under A_U and $U(t)$. In the remaining propositions of this section we shall investigate the "dual operator" of A_U relative to a certain bilinear form. In particular, we shall obtain an "alternative" theorem for the nonhomogeneous equation $(A_U - \lambda I)\varphi = \psi$.

In the propositions which follow we will suppose that F has the form

$$F(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \qquad \varphi \in C, \tag{4.8}$$

where η: $[-r, 0] \longrightarrow B(X, X)$ is of bounded variation. In addition we let $C' = C([0, r]; X')$ where X' is the dual space of X. Also let A'_T: $X' \longrightarrow X'$ be the dual operator of A_T, which exists since A_T is densely defined (see [14, p. 193]. Let $\eta'(\theta) \in B(X', X')$, $-r \leqslant \theta \leqslant 0$, be the dual of $\eta(\theta)$ and note that η': $[-r, 0] \longrightarrow B(X', X')$ is of bounded variation since η is (see [14, Theorem 2, p. 195]). Define A'_U: $C' \longrightarrow C'$ by

$$(A'_U\alpha)(s) = -\dot{\alpha}(s), \qquad 0 \leqslant s \leqslant r,$$

$$D(A'_U) = \Big\{\alpha \in C' : \dot{\alpha} \in C', \alpha(0) \in D(A'_T), \text{ and } -\dot{\alpha}(0) = A'_T\alpha(0) + \int_{-r}^{0} d\eta'(\theta)\alpha(-\theta)\Big\}. \tag{4.9}$$

Lastly, define the bilinear form $\langle\ ,\ \rangle$ from $C \times C'$ to the scalar field by

$$\langle\varphi, \alpha\rangle = (\varphi(0), \alpha(0)) - \int_{-r}^{0}\int_{0}^{\theta} (d\eta(\theta)\varphi(\xi), \alpha(\xi - \theta))d\xi, \tag{4.10}$$

where (x, x') means $x'(x)$ for $x \in X, x' \in X'$.

Proposition 4.13. *$\langle A_U\varphi, \alpha\rangle = \langle\varphi, A'_U\alpha\rangle$ for $\varphi \in D(A_U)$ and $\alpha \in D(A'_U)$. Also, if $\lambda \in P\sigma(A_U)$, $\mu \in P\sigma(A'_U)$, $\lambda \neq \mu$, $\varphi \in N(A_U - \lambda I)$, and $\alpha \in N(A'_U - \mu I)$, then $\langle\varphi, \alpha\rangle = 0$.*

Proof. The first statement follows from

$$\langle A_U\varphi, \alpha\rangle = \left(A_T\varphi(0) + \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \alpha(0) - \int_{-r}^{0}\int_{0}^{\theta} (d\eta(\theta)\dot{\varphi}(\xi), \alpha(\xi - \theta))d\xi\right)$$

$$= (A_T\varphi(0), \alpha(0)) + \left(\int_{-r}^{0} d\eta(\theta)\varphi(\theta), \alpha(0)\right)$$

$$-\int_{-r}^{0} d\eta(\theta)\varphi(\theta), \alpha(\xi - \theta)\ \Bigg|_{\xi = 0}^{\xi = \theta} + \int_{-r}^{0}\int_{0}^{\theta} (d\eta(\theta)\varphi(\theta), \dot{\alpha}(\xi - \theta))d\xi$$

$$= \left(\varphi(0), A'_T\alpha(0) + \int_{-r}^{0} d\eta'(\theta)\alpha(-\theta)\right) - \int_{-r}^{0}\int_{0}^{\theta} (d\eta(\theta)\varphi(\xi), -\dot{\alpha}(\xi - \theta))d\xi$$

$$= \langle\varphi, A'_U\alpha\rangle.$$

The second statement follows from the first.

Proposition 4.14. *Suppose $N(\Delta(\lambda)) \neq \{0\}$ iff $N(\Delta(\Lambda)') \neq \{0\}$, where $\Delta(\lambda) = A_T - \lambda I + \int_{-r}^{0} d\eta(\theta)e^{\lambda\theta}$ as in* (4.1) *and $\Delta(\lambda)' = A'_T - \lambda I + \int_{-r}^{0} d\eta'(\theta)e^{\lambda\theta}$. Then $\lambda \in P\sigma(A_U)$ iff $\lambda \in P\sigma(A'_U)$.*

Proof. As in the proof of Proposition 4.1, $\varphi \neq 0 \in N(A_U - \lambda I)$ iff $\varphi(\theta) = e^{\lambda\theta}\varphi(0)$ where $\varphi(0) \neq 0$ satisfies $\lambda\varphi(0) = A_T\varphi(0) + F(\varphi)$ iff $\varphi(0) \neq 0 \in N(\Delta(\lambda))$. Similarly, $\alpha \neq 0 \in N(A'_U - \lambda I)$ iff $\alpha(s) = e^{-\lambda s}\alpha(0)$ where $\alpha(0) \neq 0$ satisfies $\lambda\alpha(0) = A'_T\alpha(0) + \int_{-r}^{0} d\eta'(\theta)\alpha(-\theta)$ iff $\alpha(0) \neq 0 \in N(\Delta(\lambda)')$.

Proposition 4.15. *Suppose $\Delta(\lambda)$ has closed range and let $\psi \in C$. Then $(A_U - \lambda I)\varphi = \psi$ has a solution $\varphi \in C$ iff $\langle\psi, \alpha\rangle = 0$ for all $\alpha \in N(A'_U - \lambda I)$.*

Proof. We first observe that $\Delta(\lambda)$ is closed and densely defined since

A_T is, a fact which we will need in order to apply the closed range theorem (see [14, p. 205]). Suppose $\langle \psi, \alpha \rangle = 0$ for every $\alpha \in N(A'_U - \lambda I)$. To show there exists $\varphi \in C$ satisfying $(A_U - \lambda I)\varphi = \psi$ it suffices to show that there exists $\varphi(0) \in X$ satisfying

$$\Delta(\lambda)\varphi(0) = \psi(0) - \int_{-r}^{0} \int_{0}^{\theta} e^{\lambda(\theta - \xi)} d\eta(\theta) \psi(\xi) d\xi \stackrel{\text{def}}{=} \gamma. \tag{4.11}$$

That is, there exists $\varphi \in D(A_U)$ such that $\dot{\varphi} - \lambda\varphi = \psi$ iff

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) + \int_0^{\theta} e^{\lambda(\theta - \xi)} \psi(\xi) d\xi, \quad \dot{\varphi}(0) = A_T\varphi(0) + F(\varphi) \tag{4.12}$$

iff (4.11). Since $\Delta(\lambda)$ has closed range, by the closed range theorem, $\gamma \in R(\Delta(\lambda))$ iff $\gamma \in N(\Delta(\lambda)')^{\perp} = \{x \in X: (x, x') = 0$ for all $x' \in N(\Delta(\lambda)')\}$. So let $x' \in N(\Delta(\lambda)')$ and define $\alpha(s) = e^{-\lambda s}\alpha(0)$ where $\alpha(0) = x'$. As in the proof of Proposition 4.14, $\alpha \in N(A'_U - \lambda I)$. By hypothesis

$$\begin{aligned} 0 = \langle \psi, \alpha \rangle &= (\psi(0), \alpha(0)) - \int_{-r}^{0} \int_{0}^{\theta} (d\eta(\theta)\, \psi(\xi), e^{\lambda(\theta - \xi)} \alpha(0))\, d\xi \\ &= \left(\psi(0) - \int_{-r}^{0} \int_{0}^{\theta} e^{\lambda(\theta - \xi)} d\eta(\theta) \psi(\xi) d\xi, \alpha(0)\right) = (\gamma, x'). \end{aligned}$$

Hence, $\gamma \in R(\Delta(\lambda))$ and so there exists $\varphi \in C$ satisfying $(A_U - \lambda I)\varphi = \psi$. The converse follows immediately from Proposition 4.13.

In the case that X is reflexive one can consider the "dual equation"

$$\begin{aligned} &v(\alpha)(t) = T'(-t)\alpha(0) + \int_0^t T'(s - t) F'(v^s(\alpha)) ds, \quad t \leq 0, \\ &v^0(\alpha) = \alpha. \end{aligned} \tag{4.13}$$

Here $v(\alpha)(t): (-\infty, r] \rightarrow X', \alpha \in C', T'(t)$ is the dual of $T(t), T'(t), t \geq 0$, is a strongly continuous semigroup on X' with infinitesimal generator A' (see [2, Corollary 1.4.8, p. 52]), $F'(\alpha) = \int_{-r}^{0} d\eta'(\theta)\alpha(-\theta)$ where η is as in (4.8), and for $t \leq 0, v^t(\alpha) \in C'$ is given by $v^t(\alpha)(s) = v(\alpha)(t + s), 0 \leq s \leq r$. One solves (4.12) just as in Proposition 2.1. Define $U'(t)\alpha = v^{-t}(\alpha)$ for $t \geq 0$, $U'(t): C' \rightarrow C'$. One can verify that $U'(t), t \geq 0$, is a strongly continuous semigroup on C' (as in Proposition 3.1) and A'_U defined by (4.9) is its infinitesimal generator.

5. **Stability of solutions and examples.** Throughout this section we shall consider the equation (3.1), that is, $U(t), t \geq 0$, will be as in §3. We shall call the zero solution $u(0)(t)$ of (3.1) stable iff for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\varphi\|_C < \delta$, then $\|U(t)\varphi\|_C < \epsilon$ for all $t \geq 0$. We shall call the zero solution $u(0)(t)$ of (3.1) asymptotically stable iff it is stable and $\lim_{t \to \infty} \|U(t)\varphi\|_C = 0$ for all $\varphi \in C$. If $F(0) = 0$ (which implies $U(t)0 = 0$ for all $t \geq 0$ by the uniqueness of solutions), then Corollaries 3.7 and 3.8 say

that the zero solution of (3.1) is stable if $-L = \omega$ and asymptotically stable if $-L > \omega$, For the linear version of (3.1) Corollary 4.8 says that the zero solution of (3.1) is stable if $-L = \omega$ and asymptotically stable if $-L > \omega$. For the linear version of (3.1) Corollary 4.8 says that the zero solution is asymptotically stable iff $\beta < 0$ and not stable if $\beta > 0$.

EXAMPLE 5.1. Let $X = C_0[0, \pi]$, the space of continuous scalar-valued functions on $[0, \pi]$, which are 0 at 0 and π, and having supremum norm. Let $A_T: X \longrightarrow X$ be defined by

$$A_T y = \ddot{y} + \omega y, \qquad D(A_T) = \{y \in X: \ddot{y} \in X\},$$

where ω is a given real number. Then A_T is the infinitesimal generator of a semigroup $T(t), t \geqslant 0$, as in (1.2). Let f be a Lipschitz continuous scalar-valued function on the scalar field with Lipschitz constant L and such that $f(0) = 0$. Define $F: C \longrightarrow X$ by

$$F(\varphi)(x) = f(\varphi(-r)(x)), \qquad \varphi \in C, \ x \in [0, \pi].$$

Then the hypothesis of Proposition 2.1 is satisfied and if f is continuously differentiable, then the hypothesis of Proposition 2.3 is satisfied. In the latter case one can show that $u(\varphi)(t)(x) \stackrel{\text{def}}{=} w(x, t), t \geqslant 0, x \in [0, \pi]$, satisfies the equation

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) + \omega w(x, t) + f(w(x, t - r)), \qquad 0 \leqslant x \leqslant \pi, \ t \geqslant 0, \\ &w(0, t) = w(\pi, t) = 0, \qquad t \geqslant 0, \\ &w(x, t) = \varphi(t)(x), \qquad 0 \leqslant x \leqslant \pi, \ -r \leqslant t \leqslant 0, \end{aligned} \tag{5.1}$$

in the "classical sense". If $-L = \omega$ then the remarks above apply and the zero solution of (5.1) is stable if $-L = \omega$ and asymptotically stable if $-L > \omega$.

EXAMPLE 5.2. We wish to determine the exact region of stability of the linear equation

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) - aw(x, t) - bw(x, t - r), \qquad 0 \leqslant x \leqslant \pi, \ t \geqslant 0, \\ &w(0, t) = w(\pi, t) = 0, \qquad t \geqslant 0, \\ &w(x, t) = \varphi(t)(x), \qquad 0 \leqslant x \leqslant \pi, \ -r \leqslant t \leqslant 0, \end{aligned} \tag{5.2}$$

as a function of a, b, and r, where the solutions are in the sense of (3.1) for $X = L^2[0, \pi]$. Let $A_T: X \longrightarrow X$ be defined by

$$\begin{aligned} &A_T y = \ddot{y}, \\ &D(A_T) = \{y \in X: y \text{ and } \dot{y} \text{ are absolutely continuous}, \\ &\qquad\qquad \ddot{y} \in X, y(0) = y(\pi) = 0\}. \end{aligned} \tag{5.3}$$

Then A_T is the infinitesimal generator of a semigroup $T(t), t \geqslant 0$, as in (1.2) with $\omega = -1$. Let $F: C \longrightarrow X$ by $F(\varphi) = -a\varphi(0) - b\varphi(-r)$. The characteristic equation from (4.1) is

$$\Delta(\lambda)f = (A_T - (\lambda + a + be^{-\lambda r})I)f = 0, \quad f \neq 0 \in D(A_T). \tag{5.4}$$

By Corollary 4.8 the zero solution of (5.2) is asymptotically stable iff $\beta < 0$. In order to apply Corollary 4.8 we need to establish the fact that $T(t)$ is compact for $t > 0$. To this purpose we will use the following lemma due to A. Pazy (Theorem 3.4 of [9]).

LEMMA 5.3. *Let $T(t), t \geqslant 0$, be a strongly continuous semigroup of linear operators on X with infinitesimal generator A. Suppose that $R(\mu + i\tau; A)$ exists for $\mu > \mu_0$, and for some $\mu > \mu_0$*

$$\overline{\lim_{|\tau| \to \infty}} \ln |\tau| \, |R(\mu + i\tau; A)| = c < \infty. \tag{5.5}$$

Also suppose that for some λ and $t_0 \geqslant 0, R(\lambda; A)T(t_0)$ is compact. Then $T(t)$ is compact for $t > \max(t_0, c)$.

When A_T is defined by (5.3), it is well known that $R(\mu + i\tau; A_T)$, which exists for $\mu > 0$, is compact. Therefore $R(\lambda; A_T)T(0)$ is compact where $\operatorname{Re}\lambda > 0$. In the next example we will show that

$$R(\lambda; A_T)f = \sum_{n=1}^{\infty} \frac{f_n \varphi_n(x)}{-n^2 - \lambda}, \qquad \text{where } \varphi_n(x) = (\sqrt{2/\pi}) \sin nx \tag{5.6}$$

$$\text{and } f_n = \frac{\sqrt{2}}{\pi} \int_0^\pi f(x) \sin nx \, dx.$$

Thus, $|R(\lambda; A_T)| = (\sum_{n=1}^{\infty} 1/|n^2 + \lambda|^2)^{1/2}$ and it follows that (5.5) holds with $C = 0$. Hence, by Lemma 5.3, $T(t)$ is compact for each $t > 0$.

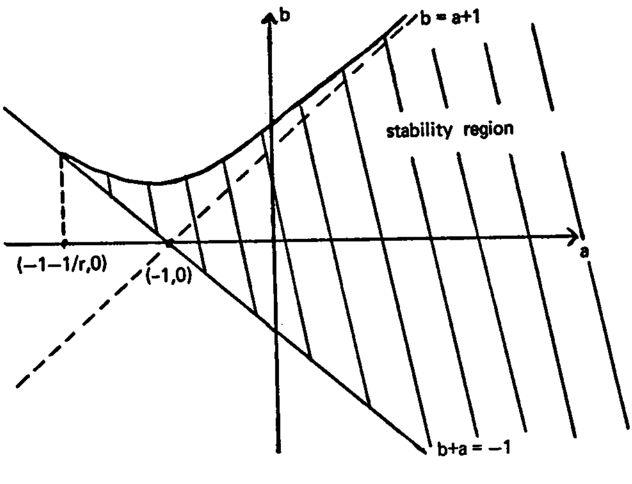

FIGURE 1

Since the eigenvalues of A_T are $-n^2, n = 1, 2, \cdots$, we have from (5.4) that the system is asymptotically stable iff all the roots of the equations

$$\lambda + a + be^{-\lambda r} = -n^2, \quad n = 1, 2, \cdots,$$

have negative real parts. The exact region of stability of (5.1) as a function of a, b, and r is indicated in Figure 1.

EXAMPLE 5.4. Consider the linear equation

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) + w(x, t) - (\pi/2)w(x, t-1), \quad 0 \leqslant x \leqslant \pi, \ t \geqslant 0, \\ &w(0, t) = w(\pi, t) = 0, \quad t \geqslant 0, \\ &w(x, t) = \varphi(t)(x), \quad 0 \leqslant x \leqslant \pi, \ -1 \leqslant t \leqslant 0. \end{aligned} \tag{5.7}$$

Let X be as in Example 5.2, let $A_T = B + I$ where B is the "A_T" defined in (5.3), and observe that A_T is self-dual. Let $F: C \longrightarrow X$ be defined by $F(\varphi) = -(\pi/2)\varphi(-1)$ (here we take $r = 1$). Note that (4.8) holds where $\eta(\theta) = 0$ if $-1 < \theta \leqslant 0$ and $\eta(\theta) = -(\pi/2)I$ if $\theta = -1$. The dual equation (4.13) of (5.7) is

$$\begin{aligned} &v_t(x, t) = v_{xx}(x, t) + v(x, t) + (\pi/2)v(x, t+1), \quad 0 \leqslant x \leqslant \pi, \ t \leqslant 0, \\ &v(0, t) = v(\pi, t) = 0, \quad t \leqslant 0, \\ &v(x, t) = \alpha(t)(x), \quad 0 \leqslant x \leqslant \pi, \ 0 \leqslant t \leqslant 1. \end{aligned} \tag{5.8}$$

The bilinear form (4.10) is

$$\langle \varphi, \alpha \rangle = (\varphi(0), \alpha(0)) - \frac{\pi}{2}\int_{-1}^{0} (\varphi(\xi), \alpha(\xi + 1))\, d\xi$$

where $(x, y) = \int_0^\pi x(s)y(s)\,ds$, $x \in X$, $y \in X'$. The operators A_U, A_U' are given by

$$(A_U)(\theta) = \dot{\varphi}(\theta) \ \text{ if } -1 \leqslant \theta < 0,$$

$$(A_U\varphi)(\theta) = A_T\varphi(0) - (\pi/2)\varphi(-1) \quad \text{if } \theta = 0,$$

and

$$(A_U'\alpha)(s) = -\dot{\alpha}(s) \quad \text{if } 0 < s \leqslant 1,$$

$$(A_U'\alpha)(s) = A_T\alpha(0) - (\pi/2)\alpha(1) \quad \text{if } s = 0.$$

Observe that $\varphi \in N(A_U - \lambda I)$ iff $\varphi(\theta) = \varphi(0)e^{\lambda\theta}$, $-1 \leqslant \theta \leqslant 0$, where λ satisfies $\Delta(\lambda)\varphi(0) = (A_T - \lambda I - (\pi/2)e^{-\lambda}I)\varphi(0) = 0$. Thus, $\varphi(0) = \sin nx$, $\lambda + (\pi/2)e^{-\lambda} = 1 - n^2$ for $n = 1, 2, \cdots$. Also, $\alpha \in N(A_U' - \lambda I)$ iff $\alpha(s) = \alpha(0)e^{-\lambda s}$, $0 \leqslant s \leqslant 1$, where λ satisfies

$$\Delta(\lambda)'\alpha(0) = (A_T' - \lambda I - (\pi/2)e^{-\lambda}I)\alpha(0) = 0.$$

Since A_T is self-dual, we conclude that $\alpha(0) = \sin nx$, $\lambda + (\pi/2)e^{-\lambda} = 1 - n^2$ for $n = 1, 2, \cdots$.

It can be shown that $\lambda + (\pi/2)e^{-\lambda} = 0$ has two simple roots $\pm i\pi/2$ and the remaining roots have negative real parts. It can also be shown that all the solutions of $\lambda + (\pi/2)e^{-\lambda} = 1 - n^2, n = 2, 3, \cdots$, have negative real parts. Let $\Lambda = \{i\pi/2, -i\pi/2\}$. Then $\Phi = \{\varphi_1, \varphi_2\}$, where

$$\varphi_1(\theta, x) = \sin(\pi\theta/2)\sin x, \quad -1 \leqslant \theta \leqslant 0,\ 0 \leqslant x \leqslant \pi,$$

$$\varphi_2(\theta, x) = \cos(\pi\theta/2)\sin x, \quad -1 \leqslant \theta \leqslant 0,\ 0 \leqslant x \leqslant \pi,$$

is a basis for the "generalized eigenspace" $P = P_\Lambda$ of (5.7) where $C = P_\Lambda \oplus Q_\Lambda$ is decomposed by Λ. Also, $\Psi = \{\psi_1, \psi_2\}$, where

$$\psi_1(s, x) = \sin(\pi s/2)\sin x, \quad 0 \leqslant s \leqslant 1,\ 0 \leqslant x \leqslant \pi,$$

$$\psi_2(s, x) = \cos(\pi s/2)\sin x, \quad 0 \leqslant s \leqslant 1,\ 0 \leqslant x \leqslant \pi,$$

is a basis for the "generalized eigenspace" P'_Λ of (5.8) where $C' = P'_\Lambda \oplus Q'_\Lambda$ is decomposed by Λ.

Before we can decompose C by Λ we must verify the compactness of $T(t)$ for $t > 0$ and the hypothesis of Proposition 4.11. Since $T(t)$ is just e^t multiplied times the "$T(t)$" of Example 5.2, it is compact for $t > 0$. It remains to show that each $\lambda_0 \in P\sigma(A_U)$ is a pole of $(A_U - \lambda I)^{-1}$ and this will be done in a series of lemmas.

LEMMA 5.5. *$(A_U - \lambda I)^{-1}$ has a pole of order n at λ_0 iff $\Delta(\lambda)^{-1}$ has a pole of order n at λ_0.*

PROOF. $\dot\varphi - \lambda\varphi = \psi \in C$ iff (4.11) and (4.12). Thus, for $\gamma = \gamma(\lambda, \psi) \overset{\text{def}}{=} \psi(0) + (\pi/2)\int_{-1}^0 e^{-\lambda(1+s)}\psi(s)\,ds$ as in (4.11),

$$(A_U - \lambda I)^{-1}\psi = e^{\lambda\theta}\Delta(\lambda)^{-1}\gamma(\lambda, \psi) + \int_0^\theta e^{\lambda(\theta - s)}\psi(s)\,ds.$$

Since γ, $\int_0^\theta e^{\lambda(\theta-s)}\psi(s)\,ds$, and $e^{\lambda\theta}$ are entire functions of λ, the lemma follows.

LEMMA 5.6. *Suppose $h(z)$ is an entire complex-valued function of the complex variable z such that $h(z) - w_0$ has a zero of order n at z_0. Suppose w_0 is a pole of order m of the Banach space-valued function $g(w)$ of the complex variable w. Then the function $f(z) = g(h(z))$ has a pole of order mn at z_0.*

PROOF. Since $h(z) - w_0$ has a zero of order n at $z_0, h(z) - w_0 = (z_0 - z_0)^n\hat h(z)$ where $\hat h(z_0) \neq 0$ and $\hat h(z)$ is analytic. Since $g(w)$ has a pole of order m at $w_0, g(w) = \hat g(w)/(w - w_0)^m$ where $\hat g(w_0) \neq 0$ and $\hat g(w)$ is analytic. Thus

$$f(z) = g(h(z)) = \frac{\hat{g}(h(z))}{(h(z) - w_0)^m} = \frac{\hat{g}(h(z))}{(z - z_0)^{mn}\hat{h}(z)^m}.$$

which implies that $f(z)$ has a pole of order mn at z_0.

LEMMA 5.7. $(A_T - \lambda I)^{-1}$ *has simple poles at* $\lambda = 1 - n^2$, $n = 1, 2, \cdots$.

PROOF. Consider

$$\ddot{u} + u - \lambda u = f, \qquad u(0) = u(\pi) = 0, \qquad f \in X. \tag{5.9}$$

The related eigenvalue problem is

$$\ddot{\varphi} + \varphi = \lambda\varphi, \qquad \varphi(0) = \varphi(\pi) = 0. \tag{5.10}$$

Nontrivial solutions of (5.10) can be obtained iff λ is one of the eigenvalues $\lambda_n = 1 - n^2$, $n = 1, 2, \cdots$. The corresponding normalized eigenfunctions are $\varphi_n(x) = \sqrt{2/\pi}\sin nx$. These eigenfunctions form a complete orthonormal set. Thus we can write

$$f(x) = \sum_{n=1}^{\infty} f_n\varphi_n(x), \qquad f_n = (f, \varphi_n) = \sqrt{2/\pi}\int_0^{\pi} f(x)\sin nx\,dx,$$

$$u(x) = \sum_{n=1}^{\infty} u_n\varphi_n(x), \qquad u_n = (u, \varphi_n) = \sqrt{2/\pi}\int_0^{\pi} u(x)\sin nx\,dx,$$

where convergence is in the mean. From (5.9) we obtain

$$(\ddot{u}, \varphi_n) + (u, \varphi_n) - \lambda(u, \varphi_n) = (f, \varphi_n),$$

which, after integrating by parts, becomes $(1 - n^2 - \lambda)(u, \varphi_n) = (f, \varphi_n)$. If $\lambda \neq 1 - n^2$, then

$$u_n = \frac{f_n}{1 - n^2 - \lambda} \quad \text{and} \quad u(x) = \sum_{n=1}^{\infty} \frac{f_n\varphi_n(x)}{1 - n^2 - \lambda}.$$

It is easily shown that the convergence is uniform on $[0, \pi]$. Thus,

$$(A_T - \lambda I)^{-1}f = \sum_{n=1}^{\infty} \frac{f_n\varphi_n(x)}{1 - n^2 - \lambda},$$

and the lemma now follows.

LEMMA 5.8. $(A_U - \lambda I)^{-1}$ *is analytic except at roots to* $\lambda + (\pi/2)e^{-\lambda} = 1 - n^2$, $n = 1, 2, \cdots$, *where it has simple poles.*

PROOF. Set $g(w) = (A_T - wI)^{-1}$ and $h(z) = z + (\pi/2)e^{-z}$. The function $h(z) - (1 - n^2) = z + (\pi/2)e^{-\lambda} - (1 - n^2)$ has simple zeroes for $n = 1, 2, \cdots$. That is, suppose z_0 is a zero which is not simple. Then

$h(z_0) - (1 - n^2) = z_0 + (\pi/2)e^{-z_0} - (1 - n^2) = 0$ and $h'(z_0) = 1 - (\pi/2)e^{-z_0} = 0$ must hold, which is easily seen to be a contradiction. The proof of Lemma 5.8 now follows from Lemmas 5.5–5.7.

It now follows from Proposition 4.11 that we can decompose C by Λ. Furthermore, by Proposition 4.15 any $\varphi \in C$ can be written as $\varphi = \varphi^P + \varphi^Q$, where P is the "generalized eigenspace" associated with Φ and Q is the orthogonal complement of the "generalized eigenspace" associated with Ψ. By Proposition 4.12 there exist positive constants K, γ such that $\|U(t)\varphi^Q\|_C \leqslant Ke^{-\gamma t}\|\varphi^Q\|_C$, $t \geqslant 0$, and, consequently, the subspace Q of C is asymptotically stable.

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