

Existence and Stability of Compressible Current-Vortex Sheets in Three-Dimensional Magnetohydrodynamics

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Abstract

Compressible vortex sheets are fundamental waves, along with shocks and rarefaction waves, in entropy solutions to multidimensional hyperbolic systems of conservation laws. Understanding the behavior of compressible vortex sheets is an important step towards our full understanding of fluid motions and the behavior of entropy solutions. For the Euler equations in two-dimensional gas dynamics, the classical linearized stability analysis on compressible vortex sheets predicts stability when the Mach number $M > \sqrt{2}$ and instability when $M < \sqrt{2}$; and Artola and Majda's analysis reveals that the nonlinear instability may occur if planar vortex sheets are perturbed by highly oscillatory waves even when $M > \sqrt{2}$. For the Euler equations in three dimensions, every compressible vortex sheet is violently unstable and this instability is the analogue of the Kelvin–Helmholtz instability for incompressible fluids. The purpose of this paper is to understand whether compressible vortex sheets in three dimensions, which are unstable in the regime of pure gas dynamics, become stable under the magnetic effect in three-dimensional magnetohydrodynamics (MHD). One of the main features is that the stability problem is equivalent to a free-boundary problem whose free boundary is a characteristic surface, which is more delicate than noncharacteristic free-boundary problems. Another feature is that the linearized problem for current-vortex sheets in MHD does not meet the uniform Kreiss–Lopatinskii condition. These features cause additional analytical difficulties and especially prevent a direct use of the standard Picard iteration to the nonlinear problem. In this paper, we develop a nonlinear approach to deal with these difficulties in three-dimensional MHD. We first carefully formulate the linearized problem for the current-vortex sheets to show rigorously that the magnetic effect makes the problem weakly stable and establish energy estimates, especially high-order energy estimates, in terms of the nonhomogeneous terms and variable coefficients. Then we exploit these results to develop a suitable iteration scheme of the Nash–Moser–Hörmander type to deal with the loss of the order of derivative in the nonlinear level and establish its convergence,

which leads to the existence and stability of compressible current-vortex sheets, locally in time, in three-dimensional MHD.

1. Introduction

We are concerned with the existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics (MHD). The motion of inviscid MHD fluids is governed by the following system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t (\rho v) + \nabla \cdot (\rho v \otimes v - H \otimes H) + \nabla (p + \frac{1}{2}|H|^2) = 0, \\ \partial_t H - \nabla \times (v \times H) = 0, \\ \partial_t \left(\rho \left(\frac{1}{2}(|v|^2 + \frac{|H|^2}{\rho}) + e \right) \right) + \nabla \cdot \left(\rho v \left(\frac{1}{2}|v|^2 + e + \frac{p}{\rho} \right) + H \times (v \times H) \right) = 0, \end{cases} \quad (1.1)$$

and

$$\nabla \cdot H = 0, \quad (1.2)$$

where $\rho, v = (v_1, v_2, v_3), H = (H_1, H_2, H_3)$, and p are the density, velocity, magnetic field, and pressure, respectively; $e = e(\rho, S)$ is the internal energy; and S is the entropy. The Gibbs relation

$$\theta dS = de - \frac{p}{\rho^2} d\rho$$

implies the constitutive relations:

$$p = \rho^2 e_\rho(\rho, S), \quad \theta = e_S(\rho, S),$$

where θ is the temperature. The quantity $c = \sqrt{p_\rho(\rho, S)}$ is the sound speed of the fluid.

For smooth solutions, the equations in (1.1) are equivalent to

$$\begin{cases} (\partial_t + v \cdot \nabla)p + \rho c^2 \nabla \cdot v = 0, \\ \rho(\partial_t + v \cdot \nabla)v + \nabla p - (\nabla \times H) \times H = 0, \\ (\partial_t + v \cdot \nabla)H - (H \cdot \nabla)v + H \nabla \cdot v = 0, \\ (\partial_t + v \cdot \nabla)S = 0, \end{cases} \quad (1.3)$$

which can be written as an 8×8 symmetric hyperbolic system for $U = (p, v, H, S)$ of the form:

$$B_0(U) \partial_t U + \sum_{j=1}^3 B_j(U) \partial_{x_j} U = 0. \quad (1.4)$$

Compressible vortex sheets occur ubiquitously in nature (cf. [2–5, 11, 14, 24, 26, 27]) and are fundamental waves in entropy solutions to multidimensional hyperbolic systems of conservation laws. For example, the vortex sheets are one of the core waves in the Mach reflection configurations when a planar shock hits a wedge

and, more generally, in the two-dimensional Riemann solutions, which are building blocks of general entropy solutions to the Euler equations in gas dynamics. Therefore, understanding the existence and stability of compressible vortex sheets is an important step towards our full understanding of fluid motions and the behavior of entropy solutions to multidimensional hyperbolic systems of conservation laws, along with the existence and stability of shocks and rarefaction waves (cf. [1, 14, 20, 21]; also see [12, 16]).

For the Euler equations in two-dimensional gas dynamics, the classical linearized stability analysis on compressible vortex sheets predicts stability when the Mach number $M > \sqrt{2}$ since there are no growing modes in this case, and instability when $M < \sqrt{2}$ since there are exponentially exploding modes of instability (see [22, 23] for the definitive treatment). The local nonlinear stability for the two-dimensional case with $M > \sqrt{2}$ was recently established by COULOMBEL-SECCHI [9, 10] when the initial data is a small perturbation of a planar vortex sheet. In a series of papers, ARTOLA and MAJDA [2–4] studied the stability of vortex sheets by using the argument of nonlinear geometric optics to analyze the interaction between vortex sheets and highly oscillatory waves. They first observed the generation of three distinct families of kink modes traveling along the slipstream bracketed by shocks and rarefaction waves and provided a detailed explanation of the instability of supersonic vortex sheets even when $M > \sqrt{2}$. Therefore, one cannot expect nonlinear stability globally for two-dimensional compressible vortex sheets.

For the Euler equations in three-dimensional gas dynamics, the situation is even more complicated. In fact, it is well known that every compressible vortex sheet is violently unstable and this instability is the analogue of the Kelvin–Helmholtz instability for incompressible fluids.

The purpose of this paper is to understand whether compressible vortex sheets in three dimensions, which are unstable in the regime of pure gas dynamics, become stable under the magnetic effect, that is, the nonlinear stability of current-vortex sheets in three-dimensional MHD. One of the main features is that the stability problem is equivalent to a free-boundary problem whose free boundary is a characteristic surface, which is more delicate than the noncharacteristic free-boundary problems derived from the multidimensional shock problems that are endowed with a strict jump of the normal velocity on the free boundary (cf. [12, 20, 21]; also [7, 11]). Another feature is that the linearized problem for current-vortex sheets in MHD does not meet the uniform Kreiss–Lopatinskii condition (cf. [18, 20]); see the analysis in BLOKHIN and TRAKHININ [6] and TRAKHININ [25]. These features cause additional analytical difficulties and, in particular, they prevent a direct use of the standard Picard iteration to prove the existence of solutions to the nonlinear problem.

In this paper, we develop a nonlinear approach to deal with these difficulties in three-dimensional MHD. We first carefully formulate the linearized problem for the current-vortex sheet to show rigorously that, although any compressible vortex sheet may be violently unstable in the regime of pure gas dynamics and does not meet the uniform Kreiss–Lopatinskii condition, the magnetic effect makes the linearized problem weakly stable as observed in [6, 25]. In particular, we successfully establish high-order energy estimates of the solutions for the linearized problem in

terms of the nonhomogeneous terms and variable coefficients. Then we exploit these results to develop a suitable iteration scheme of the Nash–Moser–Hörmander type and establish its convergence, which leads to the existence and stability of compressible current-vortex sheets, locally in time, in the three-dimensional MHD. We observe that, even though the energy estimate given in Theorem 4.2 for the linearized problem (3.1)–(3.3) keeps the same order regularity for the solutions, nonhomogeneous terms, and coefficients formally, in system (3.1), the nonhomogeneous term F^\pm depends on the first derivative of U^\pm and the derivatives of Ψ^\pm up to order two, and the coefficient matrices depend on the first derivative of Ψ^\pm , which leads to the loss of regularity in estimate (4.14). This observation motivates us to develop the iteration scheme of the Nash–Moser–Hörmander type to deal with the loss of regularity in the nonlinear problem.

We remark that, in order to establish the energy estimates, especially *high-order energy estimates*, of solutions to the linearized problem derived from the current-vortex sheet, we have identified a well-structured decoupled formulation so that the linear problem (3.1)–(3.3) is decoupled into one standard initial-boundary value problem (3.10) for a symmetric hyperbolic system and another problem (3.11) for an ordinary differential equation for the front. This decoupled formulation is much more convenient and simpler than that in (58)–(60) given in [25] and is essential for us to establish the desired high-order energy estimates of solutions, which is one of the key ingredients for developing the suitable iteration scheme of the Nash–Moser–Hörmander type that converges.

This paper is organized as follows. In Section 2, we first set up the current-vortex sheet problem as a free-boundary problem, and then we reformulate this problem into a fixed initial-boundary value problem and state the main theorem of this paper. In Sections 3–4, we first carefully formulate the linearized problem for compressible current-vortex sheets and identify the decoupled formulation; and then we establish energy estimates, especially high-order energy estimates, of the solutions in terms of the nonhomogeneous terms and variable coefficients. Then, in Section 5, we analyze the compatibility conditions and construct the zeroth-order approximate solutions for the nonlinear problem, which is a basis for our iteration scheme. In Section 6, we develop a suitable iteration scheme of the Nash–Moser–Hörmander type. In Section 7, we establish the convergence of the iteration scheme towards a compressible current-vortex sheet. Finally, in Section 8, we complete several necessary estimates of the iteration scheme used in Section 7.

2. Current-vortex sheets and main theorem

In this section, we first set up the current-vortex sheet problem as a free-boundary problem and then reformulate this problem into a fixed initial-boundary value problem, and finally we state the main theorem of this paper.

Let a piecewise smooth function $U(t, x)$ be an entropy solution to (1.1) with the form:

$$U(t, x) = \begin{cases} U^+(t, x) & \text{for } x_1 > \psi(t, x_2, x_3), \\ U^-(t, x) & \text{for } x_1 < \psi(t, x_2, x_3). \end{cases} \quad (2.1)$$

Then, on the discontinuity front $\Gamma := \{x_1 = \psi(t, x_2, x_3)\}$, $U(t, x)$ must satisfy the Rankine–Hugoniot conditions:

$$\begin{cases} [m_N] = 0, \\ [H_N] = 0, \\ m_N[v_N] + (1 + \psi_{x_2}^2 + \psi_{x_3}^2)[q] = 0, \\ m_N[v_\tau] = H_N[H_\tau], \\ m_N[\frac{H_\tau}{\rho}] = H_N[v_\tau], \\ m_N[e + \frac{1}{2}(|v|^2 + \frac{|H|^2}{\rho})] + [qv_N - H_N(H \cdot v)] = 0, \end{cases} \quad (2.2)$$

where $[\alpha]$ denotes the jump of the function α on the front Γ , (v_N, v_τ) (resp. (H_N, H_τ)) are the normal and tangential components of v (resp. H) on Γ , that is,

$$\begin{cases} v_N := v_1 - \psi_{x_2}v_2 - \psi_{x_3}v_3, \\ v_\tau = (v_{\tau_1}, v_{\tau_2})^\top := (\psi_{x_2}v_1 + v_2, \psi_{x_3}v_1 + v_3)^\top, \\ H_N := H_1 - \psi_{x_2}H_2 - \psi_{x_3}H_3, \\ H_\tau = (H_{\tau_1}, H_{\tau_2})^\top := (\psi_{x_2}H_1 + H_2, \psi_{x_3}H_1 + H_3)^\top, \end{cases}$$

where $m_N = \rho(v_N - \psi_t)$ is the mass transfer flux, and $q = p + \frac{|H|^2}{2}$ is the total pressure.

If $m_N = 0$ on Γ , then (U^\pm, Γ) is a contact discontinuity for (1.1). In this paper, we focus on the case:

$$H_N^+ = H_N^- = 0, \quad (2.3)$$

$$H_\tau^+ \not\parallel H_\tau^-, \quad (2.4)$$

for which $U(t, x)$ in (2.1) is called a compressible current-vortex sheet. Then the current-vortex sheet is determined by (2.3) and

$$\psi_t = v_N^+ = v_N^-, \quad \left[p + \frac{|H|^2}{2} \right] = 0, \quad (2.5)$$

on which generically

$$([\rho], [v_\tau], [S]) \neq 0. \quad (2.6)$$

When $H \equiv 0$, $U(t, x)$ becomes a classical compressible vortex sheet in fluid mechanics without magnetic effect.

Since $H_N^+ = H_N^-$ on $\Gamma = \{x_1 = \psi(t, x_2, x_3)\}$, it is easy to see that *condition (2.4) is equivalent to the condition:*

$$\begin{pmatrix} H_2^+ \\ H_3^+ \end{pmatrix} \not\parallel \begin{pmatrix} H_2^- \\ H_3^- \end{pmatrix} \quad \text{on } \Gamma. \quad (2.7)$$

Indeed, when $H_N^+ = H_N^-$ on Γ , we have

$$H_\tau^\pm = ((1 + \psi_{x_2}^2)H_2^\pm + \psi_{x_2}\psi_{x_3}H_3^\pm, \psi_{x_2}\psi_{x_3}H_2^\pm + (1 + \psi_{x_3}^2)H_3^\pm)^\top,$$

which implies that $\alpha H_\tau^+ + \beta H_\tau^- = 0$ if and only if $\alpha H_k^+ + \beta H_k^- = 0$ hold for $k = 2, 3$.

Furthermore, system (1.1) has the following property: *Let (U^\pm, ψ) be the current-vortex sheet to (1.1) for $t \in [0, T)$ for some $T > 0$. Then, if $\nabla \cdot H^\pm(0, x) = 0$, we have*

$$\nabla \cdot H^\pm(t, x) = 0 \quad \text{for all } t \in [0, T). \tag{2.8}$$

This can be seen as follows: From (1.3), we find that, on both sides of Γ ,

$$(\partial_t + v^\pm \cdot \nabla)(\nabla \cdot H^\pm) + (\nabla \cdot v^\pm)(\nabla \cdot H^\pm) = 0. \tag{2.9}$$

On Γ , we use $\psi_t = v_N^\pm$ to obtain

$$\partial_t + v^\pm \cdot \nabla = \partial_t + (v_N^\pm + \psi_{x_2} v_2^\pm + \psi_{x_3} v_3^\pm) \partial_{x_1} + v_2^\pm \partial_{x_2} + v_3^\pm \partial_{x_3} = \tau^\pm \cdot (\partial_t, \nabla),$$

where the vectors $\tau^\pm = (1, v_N^\pm + \psi_{x_2} v_2^\pm + \psi_{x_3} v_3^\pm, v_2^\pm, v_3^\pm)^\top$ are orthogonal to the space-time normal vector $n = (\psi_t, -1, \psi_{x_2}, \psi_{x_3})^\top$ on Γ . Thus, $\partial_t + v^\pm \cdot \nabla$ are tangential derivative operators to Γ , and the equations in (2.9) are transport equations for $\nabla \cdot H$ on both sides of Γ , which yields (2.8).

Set

$$D(\lambda, U) := \begin{pmatrix} \tilde{D}(\lambda, U) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \tilde{D}(\lambda, U) := \begin{pmatrix} 1 & \frac{\lambda}{\rho c^2} H^\top & O_{1 \times 3} \\ \lambda \rho H & I_3 & -\rho \lambda I_3 \\ O_{3 \times 1} & -\lambda I_3 & I_3 \end{pmatrix}.$$

As in [25], property (2.8) implies that system (1.1)–(1.2) is equivalent to the following system on both sides of Γ :

$$D(\lambda^\pm, U^\pm) \left(B_0(U^\pm) \partial_t U^\pm + \sum_{j=1}^3 B_j(U^\pm) \partial_{x_j} U^\pm \right) + \lambda^\pm G^\pm \nabla \cdot H^\pm = 0, \tag{2.10}$$

provided $\nabla \cdot H^\pm(0, x) = 0$, where

$$G^\pm = -(1, 0, 0, 0, H^\pm, 0)^\top,$$

and $\lambda^\pm = \lambda^\pm(v_2^\pm, v_3^\pm, H_2^\pm, H_3^\pm)$ will be determined later so that

$$(\lambda^\pm)^2 < \frac{1}{\rho^\pm + |H^\pm|^2 / (c^\pm)^2} \tag{2.11}$$

are satisfied for the sound speeds c^\pm .

We rewrite system (2.10) as

$$A_0(U^\pm) \partial_t U^\pm + \sum_{j=1}^3 A_j(U^\pm) \partial_{x_j} U^\pm = 0, \tag{2.12}$$

which is a symmetric hyperbolic system. Then the problem of existence and stability of current-vortex sheets of form (2.1) can be formulated as the following free-boundary problem.

Free-boundary problem: Determine $U^\pm(t, x)$ and a free boundary $\Gamma = \{x_1 = \psi(t, x_2, x_3)\}$ for $t > 0$ such that

$$\begin{cases} A_0(U^\pm)\partial_t U^\pm + \sum_{j=1}^3 A_j(U^\pm)\partial_{x_j} U^\pm = 0 & \text{for } \pm(x_1 - \psi(t, x_2, x_3)) > 0, \\ U|_{t=0} = \begin{cases} U_0^+(x) & \text{for } x_1 > \psi_0(x_2, x_3), \\ U_0^-(x) & \text{for } x_1 < \psi_0(x_2, x_3) \end{cases} \end{cases} \quad (2.13)$$

satisfying the jump conditions on Γ :

$$\psi_t = v_N^+ = v_N^-, \quad H_N^+ = H_N^- = 0, \quad [p + \frac{|H|^2}{2}] = 0, \quad H_\tau^+ \nparallel H_\tau^-, \quad (2.14)$$

where $\psi_0(x_2, x_3) = \psi(0, x_1, x_2)$, $\nabla \cdot H_0^\pm(x) = 0$, and condition (2.11) is satisfied for λ^\pm given by (3.9) later.

For a piecewise smooth solution $U(t, x)$ of problem (2.13)–(2.14) with smooth (U^\pm, ψ) for $t \in [0, T)$, it can be similarly verified that $\nabla \cdot H^\pm(t, x) = 0$ for all $t \in [0, T)$ since $\nabla \cdot H_0^\pm(x) = 0$. Hence the equations in (2.13) and (1.1)–(1.2) are equivalent for such a solution in the domain $\Omega^\pm := \{\pm(x_1 - \psi(t, x_2, x_3)) > 0\}$.

Furthermore, unlike the shock case, the new difficulty here is that the free boundary Γ in problem (2.13)–(2.14) is now a characteristic surface of the equations in (2.13), rather than the noncharacteristic surface endowed with a strict jump of the normal velocity as in the shock case (see [7, 11, 12, 20, 21]).

To deal with such a free-boundary problem, it is convenient to employ the standard partial hodograph transformation:

$$\begin{cases} t = \tilde{t}, & x_2 = \tilde{x}_2, & x_3 = \tilde{x}_3, \\ x_1 = \Psi^\pm(\tilde{t}, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \end{cases} \quad (2.15)$$

with Ψ^\pm satisfying

$$\begin{cases} \pm(\Psi^\pm)_{\tilde{x}_1} \geq \kappa > 0, \\ \Psi^+|_{\tilde{x}_1=0} = \Psi^-|_{\tilde{x}_1=0} = \psi(\tilde{t}, \tilde{x}_2, \tilde{x}_3) \end{cases} \quad (2.16)$$

for some constant $\kappa > 0$. Under (2.15), the domains Ω^\pm are transformed into $\{\tilde{x}_1 > 0\}$ and the free boundary Γ into the fixed boundary $\{\tilde{x}_1 = 0\}$.

Then we define $\tilde{U}^\pm(\tilde{t}, \tilde{x}) := U^\pm(t, x)$. From the first jump condition in (2.14), the natural candidates for Ψ^\pm are those that satisfy the following eikonal equations:

$$\partial_t \Psi^\pm - v_1^\pm + v_2^\pm \partial_{\tilde{x}_2} \Psi^\pm + v_3^\pm \partial_{\tilde{x}_3} \Psi^\pm = 0 \quad \text{in } \{x_1 > 0\}, \quad (2.17)$$

where we drop the tildes in the formula here and from now on for simplicity of notation.

It is easy to check that, under (2.15), $\tilde{U}^\pm(\tilde{t}, \tilde{x}) = U^\pm(t, x)$ satisfy

$$L(U^\pm, \Psi^\pm) U^\pm = 0 \quad \text{in } \{x_1 > 0\}, \quad (2.18)$$

the boundary conditions on $\{x_1 = 0\}$:

$$\begin{cases} \Psi^+|_{x_1=0} = \Psi^-|_{x_1=0} = \psi, \\ B(U^+, U^-, \psi)|_{x_1=0} = 0, \end{cases} \tag{2.19}$$

and the initial condition at $\{t = 0\}$:

$$(U^\pm, \psi)|_{t=0} = (U_0^\pm(x), \psi_0(x_2, x_3)), \tag{2.20}$$

where the tildes have been also dropped for simplicity of notation,

$$L(U, \Psi)V = A_0(U)\partial_t V + \bar{A}_1(U, \Psi)\partial_{x_1} V + \sum_{j=2}^3 A_j(U)\partial_{x_j} V$$

with $\bar{A}_1(U, \Psi) = \frac{1}{\Psi_{x_1}}(A_1(U) - \Psi_t A_0(U) - \sum_{j=2}^3 \Psi_{x_j} A_j(U))$,

$$B(U^+, U^-, \psi) = (\psi_t - U_{v,N}^\pm, U_{H,N}^\pm, q^+ - q^-)^\top$$

with $U_{v,N}^\pm = U_2^\pm - \psi_{x_2} U_3^\pm - \psi_{x_3} U_4^\pm$, $U_{H,N}^\pm = U_5^\pm - \psi_{x_2} U_6^\pm - \psi_{x_3} U_7^\pm$, and

$$q = U_1 + \frac{1}{2}|U_H|^2 \text{ for } U_H = (U_5, U_6, U_7)^\top.$$

With these, the free-boundary problem has been reduced into the **initial-boundary value problem** (2.17)–(2.20) with a fixed boundary $\{x_1 = 0\}$.

To solve (2.17)–(2.20), as in [1, 8, 15], it is natural to introduce the tangential vector $M = (M_1, M_2, M_3)$ of $\{x_1 = 0\}$:

$$M_1 = \sigma(x_1)\partial_{x_1} \quad M_2 = \partial_{x_2}, \quad M_3 = \partial_{x_3},$$

with

$$\sigma(x_1) := \begin{cases} x_1 & \text{for } 0 \leq x_1 \leq 1, \\ 2 & \text{for } x_1 \geq 2, \end{cases}$$

and the weighted Sobolev spaces defined on $\Omega_T := \{(t, x) \in [0, T] \times \mathbb{R}^3 : x_1 > 0\}$:

$$B_\mu^s(\Omega_T) := \{u \in L^2(\Omega_T) : e^{-\mu t} M^\alpha \partial_{x_1}^k u \in L^2(\Omega_T) \text{ for } |\alpha| + 2k \leq s\}$$

for all $s \in \mathbb{N}$ and $\mu > 0$, imposed with the norms

$$\|u\|_{s,\mu,T} := \left(\int_0^T \|u(t, \cdot)\|_{s,\mu}^2 dt \right)^{1/2},$$

where

$$\|u(t, \cdot)\|_{s,\mu}^2 := \sum_{|\alpha|+2k \leq s} \mu^{2(s-|\alpha|-2k)} \|e^{-\mu t} M^\alpha \partial_{x_1}^k u(t, \cdot)\|_{L^2}^2.$$

We will also use similar notation as above for the spaces with $\mu = 0$, $B^s(\Omega_T)$, whose norm is defined by

$$\|u\|_{s,T} := \left(\sum_{|\alpha|+2k \leq s} \|M^\alpha \partial_{x_1}^k u\|_{L^2(\Omega_T)}^2 \right)^{1/2}.$$

Also denote by $b\Omega_T := \{(t, x_2, x_3) : t \in [0, T], (x_2, x_3) \in \mathbb{R}^2\}$.

Then, when the initial data (U_0^\pm, ψ_0) is a small perturbation of a planar current-vortex sheet $(\bar{U}^\pm, \bar{\psi})$ for constant states \bar{U}^\pm and $\bar{\psi} = 0$ satisfying (2.3)–(2.5), we have the following main theorem of this paper.

Theorem 2.1. (Main theorem). *Assume that, for any fixed $\alpha \geq 14$ and $s \in [\alpha + 5, 2\alpha - 9]$, the initial data functions $\psi_0 \in H^{2s+3}(\mathbb{R}^2)$ and $U_0^\pm - \bar{U}^\pm \in B^{2(s+2)}(\mathbb{R}_+^3)$ satisfy the compatibility conditions of problem (2.17)–(2.20) up to order $s + 2$ (cf. Section 5). Then there exists a solution (U^\pm, Ψ^\pm) of the initial-boundary value problem (2.17)–(2.20) such that*

$$(U^\pm - U_a^\pm, \Psi^\pm - \Psi_a^\pm) \in B^\alpha(\Omega_T) \quad \text{and} \quad \psi = \Psi^\pm|_{x_1=0} \in H^{\alpha-1}(b\Omega_T)$$

for some functions U_a^\pm and Ψ_a^\pm satisfying

$$(U_a^\pm - \bar{U}^\pm, \Psi_a^\pm \mp x_1) \in B^{s+3}(\mathbb{R}_+ \times \mathbb{R}_+^3).$$

We remark that, since $\pm(\Psi^\pm)_{x_1} \geq \kappa > 0$, the corresponding vector function of the solution (U^\pm, Ψ^\pm) in Theorem 2.1 under the inverse of the partial hodograph transform is a solution of the free-boundary problem that is the current-vortex sheet problem, which implies the existence and stability of compressible current-vortex sheets to (2.1)–(2.2) under the initial perturbation. To achieve this, we start with the linear stability in Sections 3–4 and then establish the nonlinear stability by developing nonlinear techniques in Sections 5–7. Some estimates used in Sections 5–7 for the iteration scheme of the Nash–Moser–Hörmander type are given in Section 8.

3. Linear stability I: linearized problems

To study the linear stability of current-vortex sheets, we first derive a linearized problem from the nonlinear problem (2.17)–(2.20). By a direct calculation, we have

$$\begin{aligned} & \frac{d}{ds} (L(U + sV, \Psi + s\Phi)(U + sV))|_{s=0} \\ &= L(U, \Psi)W + E(U, \Psi)W + \frac{\Phi}{\Psi_{x_1}}(L(U, \Psi)U)_{x_1}, \end{aligned}$$

where

$$W = V - \frac{\Phi}{\Psi_{x_1}}U_{x_1}$$

is the good unknown as introduced in [1] (see also [12, 21]) and

$$E(U, \Psi)W = W \cdot \nabla_U (\bar{A}_1(U, \Psi)) U_{x_1} + \sum_{j=2}^3 W \cdot \nabla_U A_j(U) U_{x_j}.$$

Then we obtain the following linearized problem of (2.17)–(2.20):

$$L(U^\pm, \Psi^\pm)W^\pm + E(U^\pm, \Psi^\pm)W^\pm = F^\pm \quad \text{in } \{x_1 > 0\} \tag{3.1}$$

with the boundary conditions on $\{x_1 = 0\}$:

$$\begin{cases} \phi_t - (W_2^\pm - \psi_{x_2} W_3^\pm - \psi_{x_3} W_4^\pm) + U_3^\pm \phi_{x_2} + U_4^\pm \phi_{x_3} = h_1^\pm, \\ W_5^\pm - \psi_{x_2} W_6^\pm - \psi_{x_3} W_7^\pm - U_6^\pm \phi_{x_2} - U_7^\pm \phi_{x_3} = h_2^\pm, \\ W_1^+ - W_1^- + \sum_{j=5}^7 (U_j^+ W_j^+ - U_j^- W_j^-) = h_3, \end{cases} \tag{3.2}$$

and the initial condition at $\{t = 0\}$:

$$(W^\pm, \phi)|_{t=0} = 0, \tag{3.3}$$

for some functions F^\pm and $h := (h_1^\pm, h_2^\pm, h_3)^\top$, where $(\psi, \phi) = (\Psi, \Phi)|_{x_1=0}$.

To separate the characteristic and noncharacteristic components of the unknown W^\pm , we introduce $J^\pm = J(U^\pm, \Psi^\pm)$ as an 8×8 regular matrix such that

$$X^\pm = (J^\pm)^{-1} W^\pm \tag{3.4}$$

satisfies

$$\begin{cases} X_1^\pm = W_1^\pm + \sum_{j=5}^7 U_j^\pm W_j^\pm, \\ X_2^\pm = W_2^\pm - (\Psi^\pm)_{x_2} W_3^\pm - (\Psi^\pm)_{x_3} W_4^\pm, \\ X_5^\pm = W_5^\pm - (\Psi^\pm)_{x_2} W_6^\pm - (\Psi^\pm)_{x_3} W_7^\pm, \\ (X_3^\pm, X_4^\pm, X_6^\pm, X_7^\pm, X_8^\pm) = (W_3^\pm, W_4^\pm, W_6^\pm, W_7^\pm, W_8^\pm). \end{cases}$$

Then, under transformation (3.4), problem (3.1)–(3.3) for (W^\pm, ϕ) is equivalent to the following problem for (X^\pm, ϕ) :

$$\tilde{L}(U^\pm, \Psi^\pm) X^\pm + \tilde{E}(U^\pm, \Psi^\pm) X^\pm = \tilde{F}^\pm \quad \text{in } \{x_1 > 0\} \tag{3.5}$$

with the boundary conditions on $\{x_1 = 0\}$:

$$\begin{cases} \phi_t - X_2^\pm + U_3^\pm \phi_{x_2} + U_4^\pm \phi_{x_3} = h_1^\pm, \\ X_5^\pm - U_6^\pm \phi_{x_2} - U_7^\pm \phi_{x_3} = h_2^\pm, \\ X_1^+ - X_1^- = h_3, \end{cases} \tag{3.6}$$

and the initial condition at $\{t = 0\}$:

$$(X^\pm, \phi)|_{t=0} = 0, \tag{3.7}$$

where $\tilde{F}^\pm = (J^\pm)^\top F^\pm$,

$$\tilde{L}(U^\pm, \Psi^\pm) = \tilde{A}_0(U^\pm, \Psi^\pm)\partial_t + \sum_{j=1}^3 \tilde{A}_j(U^\pm, \Psi^\pm)\partial_{x_j}$$

with $\tilde{A}_1(U^\pm, \Psi^\pm) = (J^\pm)^\top \tilde{A}_1(U^\pm, \Psi^\pm) J^\pm$, $\tilde{A}_j(U^\pm, \Psi^\pm) = (J^\pm)^\top A_j(U^\pm) J^\pm$, $j \neq 1$, and

$$\tilde{E}(U^\pm, \Psi^\pm) X^\pm = ((J^\pm)^\top E(U^\pm, \Psi^\pm) J^\pm) X^\pm + ((J^\pm)^\top L(U^\pm, \Psi^\pm) J^\pm) X^\pm.$$

For simplicity of notation, we will drop the tildes in (3.5). By a direct calculation, we see that $A_1(U^\pm, \Psi^\pm)$ can be decomposed into three parts:

$$A_1(U^\pm, \Psi^\pm) = A_1^{\pm,0} + A_1^{\pm,1} + A_1^{\pm,2}$$

with

$$A_1^{\pm,0} = \frac{1}{(\Psi^\pm)_{x_1}} \begin{bmatrix} 0 & a \\ a^\top & O_{7 \times 7} \end{bmatrix}, \quad A_1^{\pm,1} = \frac{\Psi_t^\pm - U_{v,N}^\pm}{(\Psi^\pm)_{x_1}} \tilde{A}_1^{\pm,1}, \quad A_1^{\pm,2} = \frac{U_{H,N}^\pm}{(\Psi^\pm)_{x_1}} \tilde{A}_1^{\pm,2},$$

where $a = (1, 0, 0, -\lambda^\pm, 0, 0, 0)$, $U_{v,N}^\pm = U_2^\pm - (\Psi^\pm)_{x_2} U_3^\pm - (\Psi^\pm)_{x_3} U_4^\pm$, and $U_{H,N}^\pm = U_5^\pm - (\Psi^\pm)_{x_2} U_6^\pm - (\Psi^\pm)_{x_3} U_7^\pm$.

When the state (U^\pm, Ψ^\pm) satisfies the boundary conditions given in (2.19), we know from (3.5) that, on $\{x_1 = 0\}$,

$$\left\langle \begin{pmatrix} A_1(U^+, \Psi^+) & 0 \\ 0 & A_1(U^-, \Psi^-) \end{pmatrix} \begin{pmatrix} X^+ \\ X^- \end{pmatrix}, \begin{pmatrix} X^+ \\ X^- \end{pmatrix} \right\rangle = 2X_1^\pm [X_2 - \lambda X_5], \quad (3.8)$$

when $[X_1] = 0$ on $\{x_1 = 0\}$.

Furthermore, from (3.6), we have

$$[X_2 - \lambda X_5] = \phi_{x_2}[U_3 - \lambda U_6] + \phi_{x_3}[U_4 - \lambda U_7] - [h_1 + \lambda h_2].$$

Thus, from assumption (2.4) that is equivalent to (2.7), there exists a unique $\lambda^\pm = \lambda^\pm(v_2^\pm, v_3^\pm, H_2^\pm, H_3^\pm)$ such that

$$\begin{pmatrix} v_2^+ \\ v_3^+ \end{pmatrix} - \begin{pmatrix} v_2^- \\ v_3^- \end{pmatrix} = \lambda^+ \begin{pmatrix} H_2^+ \\ H_3^+ \end{pmatrix} - \lambda^- \begin{pmatrix} H_2^- \\ H_3^- \end{pmatrix}, \quad (3.9)$$

that is, $[U_3 - \lambda U_6] = [U_4 - \lambda U_7] = 0$, which implies

$$[X_2 - \lambda X_5] = -[h_1 + \lambda h_2].$$

Therefore, with the choice of λ^\pm in (3.9), problem (3.5)–(3.7) is decomposed into

$$\begin{cases} L(U^\pm, \Psi^\pm) X^\pm + E(U^\pm, \Psi^\pm) X^\pm = F^\pm & \text{in } \{x_1 > 0\}, \\ [X_2 - \lambda X_5] = -[h_1 + \lambda h_2] & \text{on } \{x_1 = 0\}, \\ X_1^+ - X_1^- = h_3 & \text{on } \{x_1 = 0\}, \\ X^\pm|_{t \leq 0} = 0, & \end{cases} \quad (3.10)$$

which is maximally dissipative in the sense of LAX and FRIEDRICHS [13], and on $\{x_1 = 0\}$,

$$\begin{cases} \phi_t = X_2^\pm - (U_3^\pm, U_4^\pm) \begin{pmatrix} U_6^+ & U_7^+ \\ U_6^- & U_7^- \end{pmatrix}^{-1} \begin{pmatrix} X_5^+ - h_2^+ \\ X_5^- - h_2^- \end{pmatrix} + h_1^\pm, \\ \phi|_{t=0} = 0. \end{cases} \tag{3.11}$$

Remark 3.1. To have the identities in (3.11) requires compatibility, that is, that the right-hand side of (3.11) for the plus sign is equal to the term for the minus sign. This is also guaranteed by the choice of λ^\pm in (3.9).

4. Linear stability II: energy estimates

In this section, we establish the energy estimates for the linear problem (3.1)–(3.3), which implies the existence and uniqueness of solutions. We know from Section 3 that it suffices to study the *linear problems* (3.10)–(3.11).

As noted as above, when the state (U^\pm, Ψ^\pm) satisfies the eikonal equation (2.17) in $\{x_1 > 0\}$ and the boundary conditions on $\{x_1 = 0\}$ in (2.19), we know that the boundary $\{x_1 = 0\}$ is the uniform characteristic plane of multiplicities 12 for the equations in (3.10).

First, we have the following elementary properties in the space $B^s(\Omega_T)$, whose proof can be found in [1].

- Lemma 4.1.** (i) *For any fixed $s > 5/2$, the identity mapping is a bounded embedding from $B^s(\Omega_T)$ to $L^\infty(\Omega_T)$;*
 (ii) *If $s > 1$ and $u \in B^s(\Omega_T)$, then $u|_{x_1=0} \in H^{s-1}(b\Omega_T)$, the classical Sobolev space. Conversely, if $v \in H^\sigma(b\Omega_T)$ for a fixed $\sigma > 0$, then there exists $u \in B^{\sigma+1}(\Omega_T)$ such that $u|_{x_1=0} = v$.*

Denote by $coef(t, x)$ all the coefficient functions appeared in the equations in (3.10), and $coef(t, x) = coef(t, x) - coef(0)$. With the decoupled formulation (3.10)–(3.11) for the linearized problem, we can now employ the LAX–FRIEDRICHS theory [13] to establish the desired energy estimates, especially the high-order energy estimates, of solutions for the linear problem.

Theorem 4.1. *For any fixed $s_0 > 17/2$, there exist constants C_0 and μ_0 depending only on $\|coef\|_{s_0, T}$ for the coefficient functions in (3.10) such that, for any $s \geq s_0$ and $\mu \geq \mu_0$, the estimate*

$$\begin{aligned} & \max_{0 \leq t \leq T} \|X^\pm(t)\|_{s, \mu}^2 + \mu \|X^\pm\|_{s, \mu, T}^2 \\ & \leq \frac{C_0}{\mu} \left(\|F^\pm\|_{s, \mu, T}^2 + \|h\|_{H_\mu^{s+1}(b\Omega_T)}^2 \right. \\ & \quad \left. + \|coef\|_{s, \mu, T}^2 \left(\|F^\pm\|_{s_0, T}^2 + \|h\|_{H^{s_0+1}(b\Omega_T)}^2 \right) \right) \end{aligned} \tag{4.1}$$

holds, provided that the eikonal equations (2.17) and

$$U_{H,N}^{\pm} := U_5^{\pm} - (\Psi^{\pm})_{x_2} U_6^{\pm} - (\Psi^{\pm})_{x_3} U_7^{\pm} = 0$$

are valid for (U^{\pm}, Ψ^{\pm}) on $\{x_1 = 0\}$, where $h = (h_1^{\pm}, h_2^{\pm}, h_3)^{\top}$ and the norm in $H_{\mu}^s(b\Omega_T)$ are defined as that of $B_{\mu}^s(\Omega_T)$ with functions independent of x_1 .

Proof. The proof is divided into two steps.

Step 1. We first study problem (3.10) with homogeneous boundary conditions:

$$h_1^{\pm} = h_2^{\pm} = h_3 = 0 \quad \text{on } \{x_1 = 0\}. \quad (4.2)$$

Define

$$P^{\pm} = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} P_2^{\pm} & 0 \\ 0 & I_5 \end{pmatrix}$$

with

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda^{\pm} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\mathcal{A}_1^{\pm} = (P^{\pm})^{\top} \mathcal{A}_1^{\pm} P^{\pm} = \begin{pmatrix} A_{11}^{\pm,1} & A_{12}^{\pm,1} \\ A_{21}^{\pm,1} & A_{22}^{\pm,1} \end{pmatrix}$$

with

$$A_{11}^{\pm,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{12}^{\pm,1} = A_{21}^{\pm,1} = A_{22}^{\pm,1} = 0 \quad (4.3)$$

on $\{x_1 = 0\}$.

Thus, from (3.10) and (4.2), we find that the vector function

$$Y^{\pm} = (P^{\pm})^{-1} X^{\pm}$$

satisfies the following problem:

$$\begin{cases} \mathcal{L}(U^{\pm}, \Psi^{\pm})Y^{\pm} + \mathcal{E}^{\pm}(U^{\pm}, \Psi^{\pm})Y^{\pm} = \mathcal{F}^{\pm} & \text{in } \{x_1 > 0\}, \\ [Y_1] = [Y_2] = 0 & \text{on } \{x_1 = 0\}, \\ Y^{\pm}|_{t \leq 0} = 0, \end{cases} \quad (4.4)$$

where $\mathcal{F}^{\pm} = (P^{\pm})^{\top} F^{\pm}$,

$$\mathcal{L}(U^{\pm}, \Psi^{\pm}) = \mathcal{A}_0^{\pm} \partial_t + \sum_{j=1}^3 \mathcal{A}_j^{\pm} \partial_{x_j}$$

with $\mathcal{A}_j^\pm = (P^\pm)^\top A_j^\pm P^\pm, 0 \leq j \leq 3$, and

$$\mathcal{E}^\pm = (P^\pm)^\top E P^\pm + (P^\pm)^\top \sum_{j=0}^3 A_j^\pm (P^\pm)_{x_j}.$$

L^2 -estimate: With setup (4.4), we can apply the Lax–Friedrichs theory to obtain the following L^2 -estimate:

$$\max_{0 \leq t \leq T} \|Y^\pm(t)\|_{0,\mu}^2 + \mu \|Y^\pm\|_{0,\mu,T}^2 \leq \frac{C_0}{\mu} \|\mathcal{F}^\pm\|_{0,\mu,T}^2 \tag{4.5}$$

when $\mu \geq \mu_0 > 0$, where

$$C_0 = C_0(\|coef\|_{L^\infty}, \|\nabla \cdot coef\|_{L^\infty}).$$

Estimates on the tangential derivatives. From (4.4), we find that $M^\alpha Y^\pm, |\alpha| \leq s$, satisfy

$$\begin{cases} \mathcal{L} M^\alpha Y^\pm + \mathcal{E}^\pm M^\alpha Y^\pm = M^\alpha \mathcal{F}^\pm + [\mathcal{L} + \mathcal{E}^\pm, M^\alpha] Y^\pm & \text{in } \{x_1 > 0\}, \\ [M^\alpha Y_1] = [M^\alpha Y_2] = 0 & \text{on } \{x_1 = 0\}, \\ M^\alpha Y^\pm|_{t \leq 0} = 0, \end{cases} \tag{4.6}$$

which, by the Lax–Friedrichs theory again, yields

$$\begin{aligned} & \max_{0 \leq t \leq T} |Y^\pm(t)|_{s,\mu}^2 + \mu |Y^\pm|_{s,\mu,T}^2 \\ & \leq \frac{C_0}{\mu} \left(|\mathcal{F}^\pm|_{s,\mu,T}^2 + (\|coef\|_T^*)^2 |Y^\pm|_{s,\mu,T}^2 \right. \\ & \quad \left. + \|\nabla Y^\pm\|_{L^\infty}^2 |coef|_{s,\mu,T}^2 + |\partial_{x_1} Y^\pm|_{s-1,\mu,T}^2 \right), \end{aligned} \tag{4.7}$$

where $Y_I^\pm = (Y_1^\pm, Y_2^\pm)$, the norms $|\cdot|_{s,\mu}$ and $|\cdot|_{s,\mu,T}$ are the special cases of the norms $\|\cdot\|_{s,\mu}$ and $\|\cdot\|_{s,\mu,T}$, respectively, without the normal derivatives $\partial_{x_1}^k$, and

$$\|u\|_T^* = \sum_{|\alpha| \leq 2} \|M^\alpha u\|_{L^\infty(\Omega_T)} + \sum_{|\alpha| \leq 1} \|M^\alpha \partial_{x_1} u\|_{L^\infty(\Omega_T)}.$$

Estimates on the normal derivatives. Set $\mathcal{A}_j = \text{diag}[\mathcal{A}_j^+, \mathcal{A}_j^-], 0 \leq j \leq 3$, with

$$\mathcal{A}_j^\pm = \begin{pmatrix} A_{11}^{\pm,j} & A_{12}^{\pm,j} \\ A_{21}^{\pm,j} & A_{22}^{\pm,j} \end{pmatrix}.$$

From (4.3)–(4.4), we find that $Y_I^\pm = (Y_1^\pm, Y_2^\pm)$ and $Y_{II}^\pm = (Y_3^\pm, \dots, Y_8^\pm)$ satisfy the following equations:

$$A_{11}^{\pm,1} \partial_{x_1} Y_I^\pm = coef \cdot M Y^\pm + coef \cdot Y^\pm + \mathcal{F}_I^\pm - A_{12}^{\pm,1} \partial_{x_1} Y_{II}^\pm, \tag{4.8}$$

and

$$\begin{cases} \sum_{j=0}^3 A_{22}^{\pm,j} \partial_{x_j} Y_{II}^{\pm} + C_{22}^{\pm} Y_{II}^{\pm} = \mathcal{F}_{II}^{\pm} - \sum_{j=0}^3 A_{21}^{\pm,j} \partial_{x_j} Y_I^{\pm} - C_{21}^{\pm} Y_I^{\pm}, \\ Y_{II}^{\pm}|_{t \leq 0} = 0, \end{cases} \quad (4.9)$$

where the linear operator $\sum_{j=0}^3 A_{22}^{\pm,j} \partial_{x_j} + C_{22}^{\pm}$ is symmetric hyperbolic and tangential to $\{x_1 = 0\}$.

Thus, from (4.8), we obtain

$$\begin{aligned} |\partial_{x_1} Y_I^{\pm}(t)|_{s-1,\mu} &\leq |\mathcal{F}_I^{\pm}(t)|_{s-1,\mu} + \|coef\|_{Lip(T)} |Y^{\pm}(t)|_{s,\mu} \\ &\quad + \|Y^{\pm}\|_{Lip(T)} |coef(t)|_{s,\mu}, \end{aligned} \quad (4.10)$$

with $\|\cdot\|_{Lip(T)}$ denoting the usual norm in the space of Lipschitz continuous functions in Ω_T and, from (4.9), we deduce the following estimate on $\partial_{x_1} Y_{II}^{\pm}$ by using the classical hyperbolic theory:

$$\begin{aligned} &\max_{0 \leq t \leq T} |\partial_{x_1} Y_{II}^{\pm}(t)|_{s-2,\mu}^2 + \mu |\partial_{x_1} Y_{II}^{\pm}|_{s-2,\mu,T}^2 \\ &\leq \frac{C_0}{\mu} \left(|\partial_{x_1} \mathcal{F}_{II}^{\pm}|_{s-2,\mu,T}^2 + \|coef\|_{Lip(T)}^2 (|Y_I^{\pm}|_{s,\mu,T}^2 + |\partial_{x_1} Y_I^{\pm}|_{s-1,\mu,T}^2) \right. \\ &\quad \left. + (\|coef\|_T^* |Y_{II}^{\pm}|_{s,\mu,T}^N)^2 + (\|Y^{\pm}\|_T^* |coef|_{s,\mu,T}^N)^2 \right), \end{aligned} \quad (4.11)$$

where

$$|u|_{s,\mu,T}^N = |u|_{s,\mu,T} + |u_{x_1}|_{s-2,\mu,T}.$$

Combining estimates (4.7) with (4.10)–(4.11), applying an inductive argument on the order of normal derivatives $\partial_{x_1}^k$, and noting that all the coefficient functions of \mathcal{L} and \mathcal{E}^{\pm} in problem (4.4) depending on $(U^{\pm}, \nabla_{t,x} \Psi^{\pm})$ and $(U^{\pm}, \nabla_{t,x} U^{\pm}, \{\partial_{x_1}^{\alpha_1} \nabla_{t,x_2,x_3}^{\alpha_2} \Psi^{\pm}\}_{|\alpha_1| \leq 1, |\alpha_2| \leq 2, |\alpha_1 + |\alpha_2| \leq 2})$, respectively, we can conclude estimate (4.1) for the case of homogeneous boundary conditions by choosing $s_0 > 17/2$.

Step 2. With Step 1, for the case of nonhomogeneous boundary conditions in (3.10) with $(h_1^{\pm}, h_2^{\pm}, h_3) \in H^{s+1}([0, T] \times \mathbb{R}^2)$, it suffices to study (3.10) when $F^{\pm} = 0$.

By Lemma 4.1, there exists $X_0^{\pm} \in B^{s+2}(\Omega_T)$ satisfying

$$[X_{0,2} - \lambda X_{0,5}] = -[h_1 + \lambda h_2], \quad [X_{0,1}] = h_3 \quad \text{on } \{x_1 = 0\}. \quad (4.12)$$

Thus, from (3.10), we know that $Y^{\pm} := X^{\pm} - X_0^{\pm}$ satisfies the following problem:

$$\begin{cases} L(U^{\pm}, \Psi^{\pm})Y^{\pm} + E(U^{\pm}, \Psi^{\pm})Y^{\pm} = f^{\pm} & \text{in } \{x_1 > 0\}, \\ [Y_1] = [Y_2 - \lambda Y_5] = 0 & \text{on } \{x_1 = 0\}, \\ Y^{\pm}|_{t \leq 0} = 0, \end{cases} \quad (4.13)$$

with $f^{\pm} = -L(U^{\pm}, \Psi^{\pm})X_0^{\pm} - E(U^{\pm}, \Psi^{\pm})X_0^{\pm} \in B^s(\Omega_T)$.

Employing estimate (4.1) in the case $h = 0$ for problem (4.13) established in Step 1 yields an estimate on Y^{\pm} , which implies an estimate of $X^{\pm} = Y^{\pm} + X_0^{\pm}$. Combining this estimate with that for the case of homogeneous boundary conditions yields (4.1) for the general case.

By choosing first $\mu \gg 1$ and then T so that $\mu T \leq 1$ in (4.1), we can directly conclude:

Theorem 4.2. *For any fixed $s_0 > 17/2$, there exists a constant $C_0 > 0$ depending only on $\|\dot{c}o\dot{e}f\|_{s_0, T}$ such that, for any $s \geq s_0$, the following estimate holds:*

$$\begin{aligned} \|X^\pm\|_{s, T}^2 &\leq C_0 \left(\|F^\pm\|_{s, T}^2 + \|h\|_{H^{s+1}(b\Omega_T)}^2 \right. \\ &\quad \left. + \|\dot{c}o\dot{e}f\|_{s, T}^2 (\|F^\pm\|_{s_0, T}^2 + \|h\|_{H^{s_0+1}(b\Omega_T)}^2) \right). \end{aligned} \tag{4.14}$$

5. Nonlinear stability I: construction of the zeroth-order approximate solutions

With the linear estimates in Section 4, we now establish the local existence of current-vortex sheets of problem (2.17)–(2.20) under the compatibility conditions on the initial data. From now on, we focus on the initial data (U_0^\pm, ψ_0) that is a small perturbation of a planar current-vortex sheet $(\bar{U}^\pm, \bar{\psi})$ with constant states \bar{U}^\pm and $\bar{\psi} = 0$ satisfying (2.3)–(2.5).

We start with the compatibility conditions on $\{t = x_1 = 0\}$. For fixed $k \in \mathbb{N}$, it is easy to formulate the j^{th} -order compatibility conditions, $0 \leq j \leq k$, for the initial data (U_0^\pm, ψ_0) of problem (2.17)–(2.20), from which we determine the data:

$$\{\partial_t^j U^\pm|_{t=x_1=0}\}_{0 \leq j \leq k} \quad \text{and} \quad \{\partial_t^j \psi|_{t=0}\}_{0 \leq j \leq k+1}. \tag{5.1}$$

For any fixed integer $s > 9/2$ and given data $\psi_0 \in H^{s-1}(\mathbb{R}^2)$ and U_0^\pm with

$$\dot{U}_0^\pm = U_0^\pm - \bar{U}^\pm \in B^s(\mathbb{R}_+^3),$$

by using Lemma 4.1 for $T = 0$, we can extend ψ_0 to be $\dot{\psi}_0^\pm \in B^s(\mathbb{R}_+^3)$ satisfying

$$U_{0,5}^\pm - (\dot{\psi}_0^\pm)_{x_2} U_{0,6}^\pm - (\dot{\psi}_0^\pm)_{x_3} U_{0,7}^\pm = 0 \quad \text{in } \{x_1 > 0\}, \tag{5.2}$$

which is possible by using assumption (2.4). Let $\Psi_0^\pm = \dot{\psi}_0^\pm \pm x_1$ be the initial data for solving Ψ^\pm from the eikonal equations (2.17).

Let $U_j^\pm(x) = \partial_t^j U^\pm|_{t=0}$ and $\Psi_j^\pm = \partial_t^j \Psi^\pm|_{t=0}$ for any $j \geq 0$.

For all $k \leq [s/2]$, from problem (2.17)–(2.20), we determine

$$\{U_j^\pm\}_{1 \leq j \leq k} \quad \text{and} \quad \{\Psi_j^\pm\}_{0 \leq j \leq k+1} \tag{5.3}$$

in terms of (U_0^\pm, Ψ_0^\pm) and

$$\begin{cases} U_j^\pm \in B^{s-2j}(\mathbb{R}_+^3), & 1 \leq j \leq k, \\ \Psi_1^\pm \in B^{s-1}(\mathbb{R}_+^3), \quad \Psi_j^\pm \in B^{s-2(j-1)}(\mathbb{R}_+^3), & 2 \leq j \leq k+1. \end{cases} \tag{5.4}$$

Furthermore, we have the estimate

$$\begin{aligned} & \sum_{j=1}^{[s/2]} \|U_j^\pm\|_{B^{s-2j}(\mathbb{R}_+^3)} + \sum_{j=2}^{[s/2]+1} \|\Psi_j^\pm\|_{B^{s-2j+2}(\mathbb{R}_+^3)} + \|\Psi_1^\pm\|_{B^{s-1}(\mathbb{R}_+^3)} \\ & \leq C_0(\|\dot{U}_0^\pm\|_{B^s(\mathbb{R}_+^3)} + \|\psi_0\|_{H^{s-1}(\mathbb{R}^2)}) \end{aligned} \quad (5.5)$$

with C_0 depending only on s , $\|\dot{U}_0^\pm\|_{W^{1,\infty}(\mathbb{R}_+^3)}$, and $\|\psi_0\|_{W^{1,\infty}(\mathbb{R}^2)}$.

Set $s_1 = [s/2] + 1$. We now construct the *zeroth-order approximate solutions* (U_a^\pm, Ψ_a^\pm) . First, by using a classical extension argument (cf. [19]), we construct

$$\dot{\Psi}_a^\pm \in B^{s_1+1}(\mathbb{R}_+ \times \mathbb{R}_+^3) \quad (5.6)$$

satisfying $\partial_t^j \dot{\Psi}_a^\pm|_{t=0} = \dot{\Psi}_j^\pm$, $0 \leq j \leq s_1$, and $\dot{\Psi}_a^+|_{x_1=0} = \dot{\Psi}_a^-|_{x_1=0}$, and

$$\dot{U}_{a,l}^\pm \in B^{s_1}(\mathbb{R}_+ \times \mathbb{R}_+^3), \quad l \neq 2, \quad (5.7)$$

satisfying $\partial_t^j \dot{U}_{a,l}^\pm|_{t=0} = \dot{U}_{j,l}^\pm$, $0 \leq j \leq s_1 - 1$.

Moreover, by using the compatibility conditions of problem (2.17)–(2.20), we have

$$\begin{cases} U_{a,5}^\pm - (\Psi_a^\pm)_{x_2} U_{a,6}^\pm - (\Psi_a^\pm)_{x_3} U_{a,7}^\pm = 0 & \text{on } \{x_1 = 0\}, \\ [U_{a,1} + \frac{1}{2}|U_{a,H}|^2] = 0 & \text{on } \{x_1 = 0\}, \end{cases} \quad (5.8)$$

where $U_a^\pm = \bar{U}^\pm + \dot{U}_a^\pm$, $\Psi_a^\pm = \dot{\Psi}_a^\pm \pm x_1$, and $U_{a,H} = (U_{a,5}, U_{a,6}, U_{a,7})^\top$.

Finally, we construct $U_{a,2}^\pm \in B^{s_1}(\mathbb{R}_+ \times \mathbb{R}_+^3)$ by requiring

$$(\Psi_a^\pm)_t - U_{a,2}^\pm + (\Psi_a^\pm)_{x_2} U_{a,3}^\pm + (\Psi_a^\pm)_{x_3} U_{a,4}^\pm = 0 \quad \text{in } \{x_1 > 0\}. \quad (5.9)$$

Then we conclude

Lemma 5.1. *For any fixed integer $s > 9/2$, assume that the initial data $(\dot{U}_0^\pm, \psi_0) = (U_0^\pm - \bar{U}^\pm, \psi_0) \in B^s(\mathbb{R}_+^3) \times H^{s-1}(\mathbb{R}^2)$ is bounded in the norms. Then there exists (U_a^\pm, Ψ_a^\pm) such that $\dot{U}_a^\pm = U_a^\pm - \bar{U}^\pm \in B^{[s/2]+1}(\mathbb{R}_+ \times \mathbb{R}_+^3)$, $\dot{\Psi}_a^\pm = \Psi_a^\pm \mp x_1 \in B^{[s/2]+2}(\mathbb{R}_+ \times \mathbb{R}_+^3)$,*

$$\partial_t^j (L(U_a^\pm, \Psi_a^\pm)U_a^\pm)|_{t=0} = 0 \quad \text{for } 0 \leq j \leq [s/2] - 1, \quad (5.10)$$

and (5.8)–(5.9) hold.

With this, we set $\psi_a = \Psi_a^\pm|_{x_1=0}$. From the compatibility conditions of problem (2.17)–(2.20) and Lemma 5.1, we have

$$\partial_t^j (B(U_a^+, U_a^-, \psi_a))|_{t=0} = 0 \quad \text{for } 0 \leq j \leq [s/2]. \quad (5.11)$$

Set

$$V^\pm = U^\pm - U_a^\pm, \quad \Phi^\pm = \Psi^\pm - \Psi_a^\pm. \quad (5.12)$$

Then it follows from Lemma 5.1 and (5.11) that problem (2.17)–(2.20) is equivalent to the following fixed initial-boundary value problem for (V^\pm, Φ^\pm) :

$$\begin{cases} \mathcal{L}(V^\pm, \Phi^\pm)V^\pm = f_a^\pm & \text{in } \{t > 0, x_1 > 0\}, \\ \mathcal{E}(V^\pm, \Phi^\pm) = 0 & \text{in } \{t > 0, x_1 > 0\}, \\ \Phi^+|_{x_1=0} = \Phi^-|_{x_1=0} = \phi, \\ \mathcal{B}(V^+, V^-, \phi) = 0 & \text{on } \{x_1 = 0\}, \\ V^\pm|_{t \leq 0} = 0, \Phi^\pm|_{t \leq 0} = 0, \end{cases} \tag{5.13}$$

where $f_a^\pm = -L(U_a^\pm, \Psi_a^\pm)U_a^\pm$,

$$\begin{aligned} \mathcal{L}(V^\pm, \Phi^\pm)V^\pm &= L(U_a^\pm + V^\pm, \Psi_a^\pm + \Phi^\pm)(U_a^\pm + V^\pm) - L(U_a^\pm, \Psi_a^\pm)U_a^\pm, \\ \mathcal{E}(V^\pm, \Phi^\pm) &= \partial_t \Phi^\pm - V_2^\pm + (V_3^\pm + U_{a,3}^\pm)\partial_{x_2} \Phi^\pm + (V_4^\pm + U_{a,4}^\pm)\partial_{x_3} \Phi^\pm \\ &\quad + (\Psi_a^\pm)_{x_2} V_3^\pm + (\Psi_a^\pm)_{x_3} V_4^\pm, \end{aligned}$$

and

$$\mathcal{B}(V^+, V^-, \phi) = B(U_a^+ + V^+, U_a^- + V^-, \psi_a + \phi).$$

6. Nonlinear stability II: iteration scheme

From Theorem 4.1, we observe that the high-order energy estimates of solutions for the linearized problem (3.10) in terms of the nonhomogeneous terms F^\pm and the variable coefficients *cœf* keep the same order in the linear level. Based on these estimates and the structure of the nonlinear system, we now develop a suitable iteration scheme of the Nash–Moser–Hörmander type (cf. [17]) to deal with the loss of regularity for the *nonlinear problem* (5.13).

To do this, we first recall a standard family of smoothing operators (cf. [1, 10]):

$$\{S_\theta\}_{\theta>0} : B_\mu^0(\Omega_T) \longrightarrow \cap_{s \geq 0} B_\mu^s(\Omega_T) \tag{6.1}$$

satisfying

$$\begin{cases} \|S_\theta u\|_{s,T} \leq C\theta^{(s-\alpha)_+} \|u\|_{\alpha,T} & \text{for all } s, \alpha \geq 0, \\ \|S_\theta u - u\|_{s,T} \leq C\theta^{s-\alpha} \|u\|_{\alpha,T} & \text{for all } s \in [0, \alpha], \\ \|\frac{d}{d\theta} S_\theta u\|_{s,T} \leq C\theta^{s-\alpha-1} \|u\|_{\alpha,T} & \text{for all } s, \alpha \geq 0, \end{cases} \tag{6.2}$$

and

$$\|(S_\theta u_+ - S_\theta u_-)|_{x_1=0}\|_{H^s(b\Omega_T)} \leq C\theta^{(s+1-\alpha)_+} \|(u_+ - u_-)|_{x_1=0}\|_{\alpha,T} \tag{6.3}$$

for all $s, \alpha \geq 0$.

Similarly, one has a family of smoothing operators (still denoted by) $\{S_\theta\}_{\theta>0}$ acting on $H^s(b\Omega_T)$, satisfying (6.2) as well for the norms of $H^s(b\Omega_T)$ (cf. [1, 10]).

Now we construct the iteration scheme for solving the nonlinear problem (5.13) in $\mathbb{R}_+ \times \mathbb{R}^3$.

ITERATION SCHEME: Let $V^{\pm,0} = \Phi^{\pm,0} = 0$. Assume that $(V^{\pm,k}, \Phi^{\pm,k})$ have been known for $k = 0, \dots, n$, and satisfy

$$\begin{cases} (V^{\pm,k}, \Phi^{\pm,k}) = 0 & \text{in } \{t \leq 0\}, \\ \Phi^{+,k} = \Phi^{-,k} = \phi^k & \text{on } \{x_1 = 0\}. \end{cases} \quad (6.4)$$

Denote the $(n+1)^{\text{th}}$ approximate solutions to (5.13) in $\mathbb{R}_+ \times \mathbb{R}^3$ by

$$V^{\pm,n+1} = V^{\pm,n} + \delta V^{\pm,n}, \quad \Phi^{\pm,n+1} = \Phi^{\pm,n} + \delta \Phi^{\pm,n}, \quad \phi^{n+1} = \phi^n + \delta \phi^n. \quad (6.5)$$

Let $\theta_0 \geq 1$ and $\theta_n = \sqrt{\theta_0^2 + n}$ for any $n \geq 1$. Let S_{θ_n} be the associated smoothing operator defined as above. We now determine the problem of the increments $(\delta V^{\pm,n}, \delta \Phi^{\pm,n})$ as follows.

Construction of $\delta V^{\pm,n}$: First, it is easy to see

$$\begin{aligned} & \mathcal{L}(V^{\pm,n+1}, \Phi^{\pm,n+1})V^{\pm,n+1} - \mathcal{L}(V^{\pm,n}, \Phi^{\pm,n})V^{\pm,n} \\ &= L'_{e, (U_a^\pm + V^{\pm,n+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})} \delta \dot{V}^{\pm,n} + e_{\pm,n}, \end{aligned} \quad (6.6)$$

where $V^{\pm,n+\frac{1}{2}}$ is the modified state of $S_{\theta_n} V^{\pm,n}$ defined in (6.24)–(6.25) later, which guarantees that the boundary $\{x_1 = 0\}$ is the uniform characteristic plane of constant multiplicity at each iteration step,

$$\begin{aligned} & L'_{e, (U_a^\pm + V^{\pm,n+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})} \delta \dot{V}^{\pm,n} \\ &= L(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n}) \delta \dot{V}^{\pm,n} \\ &+ E(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n}) \delta \dot{V}^{\pm,n} \end{aligned} \quad (6.7)$$

is the effective linear operator,

$$\delta \dot{V}^{\pm,n} = \delta V^{\pm,n} - \delta \Phi^{\pm,n} \frac{(U_a^\pm + V^{\pm,n+\frac{1}{2}})_{x_1}}{(\Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})_{x_1}} \quad (6.8)$$

is the good unknown, and

$$e_{\pm,n} = \sum_{j=1}^4 e_{\pm,n}^{(j)} \quad (6.9)$$

is the total error with the first error resulting from the Newton iteration scheme:

$$\begin{aligned} e_{\pm,n}^{(1)} &= L(U_a^\pm + V^{\pm,n+1}, \Psi_a^\pm + \Phi^{\pm,n+1})(U_a^\pm + V^{\pm,n+1}) \\ &- L(U_a^\pm + V^{\pm,n}, \Psi_a^\pm + \Phi^{\pm,n})(U_a^\pm + V^{\pm,n}) \\ &- L'_{(U_a^\pm + V^{\pm,n}, \Psi_a^\pm + \Phi^{\pm,n})}(\delta V^{\pm,n}, \delta \Phi^{\pm,n}), \end{aligned} \quad (6.10)$$

the second error resulting from the substitution:

$$\begin{aligned} e_{\pm,n}^{(2)} &= L'_{(U_a^\pm + V^{\pm,n}, \Psi_a^\pm + \Phi^{\pm,n})}(\delta V^{\pm,n}, \delta \Phi^{\pm,n}) \\ &- L'_{(U_a^\pm + S_{\theta_n} V^{\pm,n}, \Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})}(\delta V^{\pm,n}, \delta \Phi^{\pm,n}), \end{aligned} \quad (6.11)$$

the third error resulting from the second substitution:

$$\begin{aligned}
 e_{\pm,n}^{(3)} = & L'_{(U_a^\pm + S_{\theta_n} V^{\pm,n}, \psi_a^\pm + S_{\theta_n} \phi^{\pm,n})} (\delta V^{\pm,n}, \delta \Phi^{\pm,n}) \\
 & - L'_{(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a^\pm + S_{\theta_n} \phi^{\pm,n})} (\delta V^{\pm,n}, \delta \Phi^{\pm,n}),
 \end{aligned} \tag{6.12}$$

and the remaining error:

$$e_{\pm,n}^{(4)} = \frac{\delta \Phi^{\pm,n}}{(\psi_a^\pm + S_{\theta_n} \phi^{\pm,n})_{x_1}} \left(L(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a^\pm + S_{\theta_n} \phi^{\pm,n})(U_a^\pm + V^{\pm,n+\frac{1}{2}}) \right)_{x_1}. \tag{6.13}$$

Similarly, we have

$$\begin{aligned}
 & \mathcal{B}(V^{+,n+1}, V^{-,n+1}, \phi^{n+1}) - \mathcal{B}(V^{+,n}, V^{-,n}, \phi^n) \\
 & = B'_{(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a + S_{\theta_n} \phi^n)} (\delta \dot{V}^{+,n}, \delta \dot{V}^{-,n}, \delta \phi^n) + \tilde{e}_n,
 \end{aligned} \tag{6.14}$$

where

$$\tilde{e}_n = \sum_{j=1}^4 \tilde{e}_n^{(j)} \tag{6.15}$$

with the first error resulting from the Newton iteration scheme:

$$\begin{aligned}
 \tilde{e}_n^{(1)} = & B(U_a^+ + V^{+,n+1}, U_a^- + V^{-,n+1}, \psi_a + \phi^{n+1}) \\
 & - B(U_a^+ + V^{+,n}, U_a^- + V^{-,n}, \psi_a + \phi^n) \\
 & - B'_{(U_a^\pm + V^{\pm,n}, \psi_a + \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n),
 \end{aligned} \tag{6.16}$$

the second error resulting from the substitution:

$$\begin{aligned}
 \tilde{e}_n^{(2)} = & B'_{(U_a^\pm + V^{\pm,n}, \psi_a + \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n) \\
 & - B'_{(U_a^\pm + S_{\theta_n} V^{\pm,n}, \psi_a + S_{\theta_n} \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n),
 \end{aligned} \tag{6.17}$$

the third error resulting from the second substitution:

$$\begin{aligned}
 \tilde{e}_n^{(3)} = & B'_{(U_a^\pm + S_{\theta_n} V^{\pm,n}, \psi_a + S_{\theta_n} \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n) \\
 & - B'_{(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a + S_{\theta_n} \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n),
 \end{aligned} \tag{6.18}$$

and the remaining error:

$$\begin{aligned}
 \tilde{e}_n^{(4)} = & B'_{(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a + S_{\theta_n} \phi^n)} (\delta V^{+,n}, \delta V^{-,n}, \delta \phi^n) \\
 & - B'_{(U_a^\pm + V^{\pm,n+\frac{1}{2}}, \psi_a + S_{\theta_n} \phi^n)} (\delta \dot{V}^{+,n}, \delta \dot{V}^{-,n}, \delta \phi^n).
 \end{aligned} \tag{6.19}$$

Using (6.6) and (6.14) and noting that

$$\mathcal{L}(V^{\pm,0}, \Phi^{\pm,0}) V^{\pm,0} = 0, \quad \mathcal{B}(V^{+,0}, V^{-,0}, \phi^0) = 0,$$

we obtain

$$\left\{ \begin{aligned} & \mathcal{L}(V^{\pm, n+1}, \Phi^{\pm, n+1}) V^{\pm, n+1} \\ &= \sum_{j=0}^n (L'_{e, (U_a^{\pm} + V^{\pm, j+1/2}, \Psi_a^{\pm} + S_{\theta_j} \Phi^{\pm, j})} \delta \dot{V}^{\pm, j} + e_{\pm, j}), \\ & \mathcal{B}(V^{+, n+1}, V^{-, n+1}, \phi^{n+1}) \\ &= \sum_{j=0}^n (B'_{(U_a^{\pm} + V^{\pm, j+1/2}, \Psi_a + S_{\theta_j} \phi^j)} (\delta \dot{V}^{+, j}, \delta \dot{V}^{-, j}, \delta \phi^j) + \tilde{e}_j). \end{aligned} \right. \quad (6.20)$$

Observe that, if the limit of $(V^{\pm, n}, \Phi^{\pm, n})$ exists which is expected to be a solution to problem (5.13), then the left-hand side in the first equation of (6.20) should converge to f_a^{\pm} , and the left-hand side in the second one of (6.20) should tend to zero when $n \rightarrow \infty$. Thus, from (6.20), it suffices to study the following problem:

$$\left\{ \begin{aligned} & L'_{e, (U_a^{\pm} + V^{\pm, n+1/2}, \Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})} \delta \dot{V}^{\pm, n} = f_n^{\pm} && \text{in } \Omega_T, \\ & B'_{(U_a^{\pm} + V^{\pm, n+1/2}, \Psi_a + S_{\theta_n} \phi^n)} (\delta \dot{V}^{+, n}, \delta \dot{V}^{-, n}, \delta \phi^n) = g_n && \text{on } b\Omega_T, \\ & \delta \dot{V}^{\pm, n}|_{t \leq 0} = 0, \end{aligned} \right. \quad (6.21)$$

where f_n^{\pm} and g_n are defined by

$$\sum_{j=0}^n f_j^{\pm} + S_{\theta_n} \left(\sum_{j=0}^{n-1} e_{\pm, j} \right) = S_{\theta_n} f_a^{\pm}, \quad \sum_{j=0}^n g_j + S_{\theta_n} \left(\sum_{j=0}^{n-1} \tilde{e}_j \right) = 0, \quad (6.22)$$

by induction on n , with $f_0^{\pm} = S_{\theta_0} f_a^{\pm}$ and $g_0 = 0$.

We now define the modified state $V^{\pm, n+1/2}$ to guarantee that the boundary $\{x_1 = 0\}$ is the uniform characteristic plane of constant multiplicity at each iteration step (6.21). To achieve this, we require

$$(\mathcal{B}(V^{+, n+1/2}, V^{-, n+1/2}, S_{\theta_n} \phi^n))_i^{\pm}|_{x_1=0} = 0 \quad \text{for } i = 1, 2, \text{ all } n \in \mathbb{N}, \quad (6.23)$$

which leads one to define

$$V_j^{+, n+1/2} = S_{\theta_n} V_j^{\pm, n} \quad \text{for } j \in \{3, 4, 6, 7\}, \quad (6.24)$$

and

$$\left\{ \begin{aligned} & V_2^{\pm, n+1/2} = (\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2} V_3^{\pm, n+1/2} + (\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_3} V_4^{\pm, n+1/2} \\ & \quad + (S_{\theta_n} \Phi^{\pm, n})_t + U_{a,3}^{\pm} (S_{\theta_n} \Phi^{\pm, n})_{x_2} + U_{a,4}^{\pm} (S_{\theta_n} \Phi^{\pm, n})_{x_3}, \\ & V_5^{\pm, n+1/2} = (\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2} V_6^{\pm, n+1/2} + (\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_3} V_7^{\pm, n+1/2} \\ & \quad + U_{a,6}^{\pm} (S_{\theta_n} \Phi^{\pm, n})_{x_2} + U_{a,7}^{\pm} (S_{\theta_n} \Phi^{\pm, n})_{x_3}. \end{aligned} \right. \quad (6.25)$$

Construction of $\delta \Phi^{\pm, n}$ satisfying $\delta \Phi^{\pm, n}|_{x_1=0} = \delta \phi^n$. Clearly, we have

$$\mathcal{E}(V^{\pm, n+1}, \Phi^{\pm, n+1}) - \mathcal{E}(V^{\pm, n}, \Phi^{\pm, n}) = \mathcal{E}'_{(V^{\pm, n+1/2}, S_{\theta_n} \Phi^{\pm, n})} (\delta \dot{V}^{\pm, n}, \delta \Phi^{\pm, n}) + \bar{e}_{\pm, n}, \quad (6.26)$$

where

$$\begin{aligned} \mathcal{E}'_{(V^\pm, \Phi^\pm)}(W^\pm, \Theta^\pm) = & \partial_t \Theta^\pm - W_2^\pm + (U_{a,3}^\pm + V_3^\pm) \partial_{x_2} \Theta^\pm + (U_{a,4}^\pm + V_4^\pm) \partial_{x_3} \Theta^\pm \\ & + (\Psi_a^\pm + \Phi^\pm)_{x_2} W_3^\pm + (\Psi_a^\pm + \Phi^\pm)_{x_3} W_4^\pm \end{aligned} \quad (6.27)$$

is the linearized operator of \mathcal{E} and

$$\bar{e}_{\pm, n} = \sum_{j=1}^4 \bar{e}_{\pm, n}^{(j)} \quad (6.28)$$

with the first error resulting from the Newton iteration scheme:

$$\begin{aligned} \bar{e}_{\pm, n}^{(1)} = & \mathcal{E}(V^{\pm, n+1}, \Phi^{\pm, n+1}) - \mathcal{E}(V^{\pm, n}, \Phi^{\pm, n}) - \mathcal{E}'_{(V^{\pm, n}, \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}) \\ = & (\delta \Phi^{\pm, n})_{x_2} \delta V_3^{\pm, n} + (\delta \Phi^{\pm, n})_{x_3} \delta V_4^{\pm, n}, \end{aligned} \quad (6.29)$$

the second error resulting from the substitution:

$$\begin{aligned} \bar{e}_{\pm, n}^{(2)} = & \mathcal{E}'_{(V^{\pm, n}, \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}) - \mathcal{E}'_{(S_{\theta_n} V^{\pm, n}, S_{\theta_n} \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}) \\ = & ((I - S_{\theta_n}) \Phi^{\pm, n})_{x_2} \delta V_3^{\pm, n} + ((I - S_{\theta_n}) \Phi^{\pm, n})_{x_3} \delta V_4^{\pm, n} \\ & + (I - S_{\theta_n}) V_3^{\pm, n} (\delta \Phi^{\pm, n})_{x_2} + (I - S_{\theta_n}) V_4^{\pm, n} (\delta \Phi^{\pm, n})_{x_3}, \end{aligned} \quad (6.30)$$

the third error resulting from the second substitution:

$$\bar{e}_{\pm, n}^{(3)} = \mathcal{E}'_{(S_{\theta_n} V^{\pm, n}, S_{\theta_n} \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}) - \mathcal{E}'_{(V^{\pm, n+\frac{1}{2}}, S_{\theta_n} \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}), \quad (6.31)$$

which vanishes due to (6.24) and that all the coefficients of the linearized operator $\mathcal{E}'_{(V^\pm, \Phi^\pm)}$ are independent of (V_2^\pm, V_5^\pm) , and the remaining error:

$$\bar{e}_{\pm, n}^{(4)} = \mathcal{E}'_{(V^{\pm, n+\frac{1}{2}}, S_{\theta_n} \Phi^{\pm, n})}(\delta V^{\pm, n}, \delta \Phi^{\pm, n}) - \mathcal{E}'_{(V^{\pm, n+\frac{1}{2}}, S_{\theta_n} \Phi^{\pm, n})}(\delta \dot{V}^{\pm, n}, \delta \Phi^{\pm, n}). \quad (6.32)$$

Thus, from (6.26), we have

$$\mathcal{E}(V^{\pm, n+1}, \Phi^{\pm, n+1}) = \sum_{j=0}^n (\mathcal{E}'_{(V^{\pm, j+\frac{1}{2}}, S_{\theta_j} \Phi^{\pm, j})}(\delta V^{\pm, j}, \delta \Phi^{\pm, j}) + \bar{e}_{\pm, j}), \quad (6.33)$$

which leads to define $\delta \Phi^{\pm, n}$ to be governed by the following problem:

$$\begin{cases} \mathcal{E}'_{(V^{\pm, n+\frac{1}{2}}, S_{\theta_n} \Phi^{\pm, n})}(\delta \dot{V}^{\pm, n}, \delta \Phi^{\pm, n}) = h_n^\pm & \text{on } \Omega_T, \\ \delta \Phi^{\pm, n}|_{t \leq 0} = 0, \end{cases} \quad (6.34)$$

where h_n^\pm is defined by

$$\sum_{k=0}^n h_k^\pm + S_{\theta_n} \left(\sum_{k=0}^{n-1} \bar{e}_{\pm, k} \right) = 0 \quad (6.35)$$

by induction on n starting with $h_0^\pm = 0$.

By comparing (6.34) with (6.21), it is easy to verify

$$\delta\Phi^{\pm,n}|_{x_1=0} = \delta\phi^n. \quad (6.36)$$

From (5.10)–(5.11) and (6.34), we know that the compatibility conditions hold for the initial-boundary value problem (6.21) for all $n \geq 0$.

The steps for determining $(\delta V^{\pm,n}, \delta\Phi^{\pm,n})$ are to solve first $\delta\dot{V}^{\pm,n}$ from (6.21) and then $\delta\Phi^{\pm,n}$ from (6.34), and to obtain finally $\delta V^{\pm,n}$ from (6.8).

7. Nonlinear stability III: convergence of the iteration scheme and existence of the current-vortex sheet

Fix any $s_0 \geq 9$, $\alpha \geq s_0 + 5$, and $s_1 \in [\alpha + 5, 2\alpha - s_0]$. Let the zeroth-order approximate solutions for the initial data (U_0^\pm, ψ_0) constructed in Section 5 satisfy

$$\|\dot{U}_a^\pm\|_{s_1+3,T} + \|\dot{\psi}_a^\pm\|_{s_1+3,T} + \|f_a^\pm\|_{s_1-4,T} \leq \varepsilon, \quad \|f_a^\pm\|_{\alpha+1,T}/\varepsilon \text{ is small}, \quad (7.1)$$

for some small constant $\varepsilon > 0$.

Before we prove the convergence of the iteration scheme (6.21) and (6.34), we first introduce the following lemmas, under the inductive assumption that (7.10) hold for all $0 \leq k \leq n - 1$.

Lemma 7.1. *For any $k = 0, 1, \dots, n$, we have*

$$\begin{cases} \|(V^{\pm,k}, \Phi^{\pm,k})\|_{\alpha,T} + \|\phi^k\|_{H^{\alpha-1}(b\Omega_T)} \leq C\varepsilon \log \theta_k, \\ \|(V^{\pm,k}, \Phi^{\pm,k})\|_{s,T} + \|\phi^k\|_{H^{s-1}(b\Omega_T)} \leq C\varepsilon \theta_k^{(s-\alpha)_+}, \end{cases} \quad (7.2)$$

for $s \in [s_0, s_1]$ with $s \neq \alpha$,

$$\begin{cases} \|(S_{\theta_k} V^{\pm,k}, S_{\theta_k} \Phi^{\pm,k})\|_{\alpha,T} + \|S_{\theta_k} \phi^k\|_{H^{\alpha-1}(b\Omega_T)} \leq C\varepsilon \log \theta_k, \\ \|(S_{\theta_k} V^{\pm,k}, S_{\theta_k} \Phi^{\pm,k})\|_{s,T} + \|S_{\theta_k} \phi^k\|_{H^{s-1}(b\Omega_T)} \leq C\varepsilon \theta_k^{(s-\alpha)_+}, \end{cases} \quad (7.3)$$

for $s \geq s_0$ with $s \neq \alpha$, and

$$\|(I - S_{\theta_k})V^{\pm,k}, (I - S_{\theta_k})\Phi^{\pm,k}\|_{s,T} + \|(I - S_{\theta_k})\phi^k\|_{H^{s-1}(b\Omega_T)} \leq C\varepsilon \theta_k^{s-\alpha} \quad (7.4)$$

for $s \in [s_0, s_1]$.

These results can be easily obtained by using the triangle inequality, the classical comparison between series and integrals, and the properties in (6.2) of the smoothing operators S_θ .

Lemma 7.2. *If $4 \leq s_0 \leq \alpha$ and $\alpha \geq 5$, then, for the modified state $V^{\pm,n+\frac{1}{2}}$, we have*

$$\|V^{\pm,n+\frac{1}{2}} - S_{\theta_n} V^{\pm,n}\|_{s,T} \leq C\varepsilon \theta_n^{s+1-\alpha} \quad \text{for any } s \in [4, s_1 + 2]. \quad (7.5)$$

Lemma 7.3. *Let $s_0 \geq 5$ and $\alpha \geq s_0 + 3$. For all $k \leq n - 1$, we have*

$$\begin{cases} \|e_{\pm,k}\|_{s,T} \leq C\varepsilon^2\theta_k^{L^1(s)}\Delta_k & \text{for } s \in [s_0, s_1 - 4], \\ \|\bar{e}_{\pm,k}\|_{s,T} \leq C\varepsilon^2\theta_k^{L^2(s)}\Delta_k & \text{for } s \in [s_0, s_1 - 3], \\ \|\tilde{e}_k\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_k^{L^3(s)}\Delta_k & \text{for } s \in [s_0 - 1, s_1 - 4], \end{cases} \quad (7.6)$$

where $\Delta_k = \theta_{k+1} - \theta_k$ and

$$L^1(s) = \begin{cases} \max((s + 2 - \alpha)_+ + s_0 - \alpha - 1, s + s_0 + 4 - 2\alpha) & \text{if } \alpha \neq s + 2, s + 4, \\ \max(s_0 - \alpha, 2(s_0 - \alpha) + 4) & \text{if } \alpha = s + 4, \\ s_0 + 2 - \alpha & \text{if } \alpha = s + 2, \end{cases}$$

$$L^2(s) = \begin{cases} \max((s + 2 - \alpha)_+ + 2(s_0 - \alpha), s + s_0 + 2 - 2\alpha) & \text{if } \alpha \neq s + 2, s + 3, \\ s_0 - \alpha - 1 & \text{if } \alpha = s + 3, \\ s_0 - \alpha & \text{if } \alpha = s + 2, \end{cases}$$

and

$$L^3(s) = \begin{cases} \max((s + 3 - \alpha)_+ + 2(s_0 - \alpha), s + s_0 + 3 - 2\alpha) & \text{if } \alpha \neq s + 3, s + 4, \\ s_0 - \alpha - 1 & \text{if } \alpha = s + 4, \\ s_0 - \alpha & \text{if } \alpha = s + 3. \end{cases}$$

Denote the accumulated errors by

$$E_{\pm,n} = \sum_{k=0}^{n-1} e_{\pm,k}, \quad \bar{E}_n = \sum_{k=0}^{n-1} \bar{e}_k, \quad \tilde{E}_{\pm,n} = \sum_{k=0}^{n-1} \tilde{e}_{\pm,k}. \quad (7.7)$$

Then, as a corollary of Lemma 7.3, we have

Lemma 7.4. *Let $s_0 \geq 5$, $\alpha \geq s_0 + 3$, and $s_1 \leq 2\alpha - s_0 - 1$. Then*

$$\begin{cases} \|E_{\pm,n}\|_{s,T} \leq C\varepsilon^2\theta_n & \text{for } s \in [s_0, s_1 - 4], \\ \|\bar{E}_{\pm,n}\|_{s,T} \leq \varepsilon^2 & \text{for } s \in [s_0, s_1 - 3], \\ \|\tilde{E}_n\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_n & \text{for } s \in [s_0 - 1, s_1 - 4]. \end{cases} \quad (7.8)$$

Lemma 7.5. *For any $s_0 \geq 5$, $\alpha \geq s_0 + 5$, and $s_1 \in [\alpha + 5, 2\alpha - s_0 - 1]$, we have*

$$\begin{cases} \|f_n^\pm\|_{s,T} \leq C\Delta_n\theta_n^{s-\alpha-1}(\|f_a^\pm\|_{\alpha,T} + \varepsilon^2), \\ \|g_n\|_{H^{s+1}(b\Omega_T)} \leq C\varepsilon^2\Delta_n\theta_n^{s-\alpha-1}, \\ \|h_n^\pm\|_{s,T} \leq C\varepsilon^2\Delta_n\theta_n^{s-\alpha-1} \end{cases} \quad (7.9)$$

for all $s \geq s_0$.

The proofs of Lemmas 7.2– 7.3 and 7.5 will be given in Section 8. With these lemmas, we can now prove the following key result for the convergence of the iteration scheme.

Proposition 7.1. For the solution sequence $(\delta V^{\pm,k}, \delta \Phi^{\pm,k}, \delta \phi^k)$ given by (6.21) and (6.34), we have

$$\begin{cases} \|(\delta V^{\pm,k}, \delta \Phi^{\pm,k})\|_{s,T} + \|\delta \phi^k\|_{H^{s-1}(b\Omega_T)} \leq \varepsilon \theta_k^{s-\alpha-1} \Delta_k & \text{for } s \in [s_0, s_1], \\ \|\mathcal{L}(V^{\pm,k}, \Phi^{\pm,k})V^{\pm,k} - f_a^{\pm}\|_{s,T} \leq 2\varepsilon \theta_k^{s-\alpha-1} & \text{for } s \in [s_0, s_1 - 4], \\ \|\mathcal{B}(V^{+,k}, V^{-,k}, \phi^k)\|_{H^{s-1}(b\Omega_T)} \leq \varepsilon \theta_k^{s-\alpha-1} & \text{for } s \in [s_0, s_1 - 2] \end{cases} \quad (7.10)$$

for any $k \geq 0$, where $\Delta_k = \theta_{k+1} - \theta_k$.

Proof. Estimate (7.10) is proved by induction on $k \geq 0$.

Step 1. Verification of (7.10) for $k = 0$. We first notice from (5.8)–(5.9) that $(U_a^{\pm}, \Psi_a^{\pm})$ satisfies the Rankine–Hugoniot conditions, and $V^{\pm,0} = \Phi^{\pm,0} = 0$ imply that $V^{\pm, \frac{1}{2}} = 0$. Thus, $\delta \dot{V}^{\pm,0}$ satisfies the following problem:

$$\begin{cases} L'_{e,(U_a^{\pm}, \Psi_a^{\pm})} \delta \dot{V}^{\pm,0} = S_{\theta_0} f_a^{\pm} & \text{in } \Omega_T, \\ B'_{(U_a^{\pm}, \Psi_a^{\pm})}(\delta \dot{V}^{+,0}, \delta \dot{V}^{-,0}, \delta \phi^0) = 0 & \text{on } b\Omega_T, \\ \delta \dot{V}^{\pm,0}|_{t \leq 0} = 0. \end{cases} \quad (7.11)$$

Applying Theorem 4.2 to problem (7.11), we have

$$\|\delta \dot{V}^{\pm,0}\|_{s,T} \leq C \theta_0^{(s-\alpha-1)_+} \|f_a^{\pm}\|_{\alpha+1,T} \quad \text{for all } s \in [s_0, s_1]. \quad (7.12)$$

Similarly, from (6.34), we obtain

$$\|\delta \Phi^{\pm,0}\|_{s,T} \leq C \left(\|\tilde{h}_0^{\pm}\|_{s,T} + \|\dot{U}_a^{\pm}\|_{s,T} \|\tilde{h}_0^{\pm}\|_{s_0,T} \right), \quad (7.13)$$

where $\tilde{h}_0^{\pm} = \delta \dot{V}_2^{\pm,0} - (\Psi_a^{\pm})_{x_2} \delta \dot{V}_3^{\pm,0} - (\Psi_a^{\pm})_{x_3} \delta \dot{V}_4^{\pm,0}$ satisfies

$$\|\tilde{h}_0^{\pm}\|_{s,T} \leq C \|\delta \dot{V}^{\pm,0}\|_{s,T} \leq C_0 \theta_0^{(s-\alpha-1)_+} \|f_a^{\pm}\|_{\alpha+1,T}. \quad (7.14)$$

Thus, we have

$$\|\delta \Phi^{\pm,0}\|_{s,T} \leq C_1 \theta_0^{(s-\alpha-1)_+} \|f_a^{\pm}\|_{\alpha+1,T}. \quad (7.15)$$

Since

$$\delta \dot{V}^{\pm,0} = \delta V^{\pm,0} - \delta \Phi^{\pm,0} \frac{(\dot{U}_a^{\pm})_{x_1}}{(\Psi_a^{\pm})_{x_1}},$$

we use (7.12) and (7.15) to find

$$\|\delta V^{\pm,0}\|_{s,T} \leq C_2 \theta_0^{(s-\alpha-1)_+} \|f_a^{\pm}\|_{\alpha+1,T}. \quad (7.16)$$

From (7.15)–(7.16), we deduce

$$\|(\delta V^{\pm,0}, \delta \Phi^{\pm,0})\|_{s,T} \leq \varepsilon \theta_0^{s-\alpha-1} \Delta_0 \quad \text{for any } s \in [s_0, s_1], \quad (7.17)$$

provided that $\|f_a^{\pm}\|_{\alpha+1,T}/\varepsilon$ is small.

The other two inequalities in (7.10) hold obviously when $k = 0$.

Step 2. Suppose that (7.10) holds for all $k \leq n - 1$, we now verify (7.10) for $k = n$. First, we note that, when $\alpha \geq s_0 + 3$,

$$\begin{aligned} & \|(\dot{U}_a^\pm + V^{\pm, n+\frac{1}{2}}, \dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})\|_{s_0+2, T} \\ & \leq \|(\dot{U}_a^\pm, \dot{\Psi}_a^\pm)\|_{s_0+2, T} + \|V^{\pm, n+\frac{1}{2}} - S_{\theta_n} V^{\pm, n}\|_{s_0+2, T} \\ & \quad + \|(S_{\theta_n} V^{\pm, n}, S_{\theta_n} \Phi^{\pm, n})\|_{s_0+2, T} \\ & \leq C\varepsilon(1 + \theta_n^{s_0+3-\alpha} + \theta_n^{(s_0+2-\alpha)_+}) \leq C\varepsilon, \end{aligned} \tag{7.18}$$

by using assumption (7.1) and Lemmas 7.1–7.2. Applying Theorem 4.2 to problem (6.21), we have

$$\begin{aligned} & \|\delta \dot{V}^{\pm, n}\|_{s, T} \\ & \leq C\left(\|f_n^\pm\|_{s, T} + \|g_n\|_{H^{s+1}(b\Omega_T)} + \|c\dot{o}e\dot{f}\|_{s, T}(\|f_n^\pm\|_{s_0, T} + \|g_n\|_{H^{s_0+1}(b\Omega_T)})\right) \\ & \leq C\left(\|f_n^\pm\|_{s, T} + \|g_n\|_{H^{s+1}(b\Omega_T)}\right. \\ & \quad \left. + \|(\dot{U}_a^\pm + V^{\pm, n+\frac{1}{2}}, \dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})\|_{s+2, T}(\|f_n^\pm\|_{s_0, T} + \|g_n\|_{H^{s_0+1}(b\Omega_T)})\right). \end{aligned} \tag{7.19}$$

On the other hand, similar to (7.18), from assumption (7.1) and Lemmas 7.1–7.2, we have

$$\|(\dot{U}_a^\pm + V^{\pm, n+\frac{1}{2}}, \dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})\|_{s+2, T} \leq C\varepsilon\left(1 + \theta_n^{s+3-\alpha} + \theta_n^{(s+2-\alpha)_+}\right), \tag{7.20}$$

which implies

$$\begin{aligned} & \|(\dot{U}_a^\pm + V^{\pm, n+\frac{1}{2}}, \dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})\|_{s+2, T}(\|f_n^\pm\|_{s_0, T} + \|g_n\|_{H^{s_0+1}(b\Omega_T)}) \\ & \leq C\theta_n^{s-\alpha-1} \Delta_n(\|f_a^\pm\|_{\alpha, T} + \varepsilon^2) \end{aligned} \tag{7.21}$$

for all $s \in [s_0, s_1]$ by using $\alpha \geq s_0 + 3$ and (7.9).

Thus, from (7.19), we conclude

$$\|\delta \dot{V}^{\pm, n}\|_{s, T} \leq C\theta_n^{s-\alpha-1} \Delta_n(\|f_a^\pm\|_{\alpha, T} + \varepsilon^2) \quad \text{for } s \in [s_0, s_1]. \tag{7.22}$$

For problem (6.34), we can easily obtain the following estimate:

$$\|\delta \Phi^{\pm, n}\|_{s, T} \leq C\left(\|\tilde{h}_n^\pm\|_{s, T} + \|\dot{U}_a^\pm + S_{\theta_n} V^{\pm, n}\|_{s, T} \|\tilde{h}_n^\pm\|_{s_0, T}\right) \quad \text{for any } s \geq s_0, \tag{7.23}$$

where $C > 0$ depends only on $\|\dot{U}_a^\pm + S_{\theta_n} V^{\pm, n}\|_{s_0, T}$, and

$$\tilde{h}_n^\pm = h_n^\pm + \delta \dot{V}_2^{\pm, n} - (\dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})_{x_2} \delta \dot{V}_3^{\pm, n} - (\dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n})_{x_3} \delta \dot{V}_4^{\pm, n}. \tag{7.24}$$

Obviously, we have

$$\begin{aligned} \|\tilde{h}_n^\pm\|_{s, T} & \leq \|h_n^\pm\|_{s, T} + (1 + \|\dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n}\|_{s_0, T}) \|\delta \dot{V}^{\pm, n}\|_{s, T} \\ & \quad + \|\dot{\Psi}_a^\pm + S_{\theta_n} \Phi^{\pm, n}\|_{s+1, T} \|\delta \dot{V}^{\pm, n}\|_{s_0, T}, \end{aligned}$$

which implies

$$\|\tilde{h}_n^\pm\|_{s,T} \leq C \Delta_n \theta_n^{s-\alpha-1} (\|f_a^\pm\|_{\alpha,T} + \varepsilon^2) \quad (7.25)$$

by using (7.22) and Lemmas 7.1 and 7.5.

Substituting (7.25) into (7.23), we find

$$\|\delta \Phi^{\pm,n}\|_{s,T} \leq C \Delta_n \theta_n^{s-\alpha-1} (\|f_a^\pm\|_{\alpha,T} + \varepsilon^2) \quad \text{for all } s \in [s_0, s_1]. \quad (7.26)$$

Together (7.22) with (7.26), we obtain (7.10) for $(\delta V^{\pm,n}, \delta \Phi^{\pm,n})$ by using

$$\delta V^{\pm,n} = \delta \dot{V}^{\pm,n} + \frac{(U_a^\pm + V^{\pm,n+\frac{1}{2}})_{x_1}}{(\Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})_{x_1}} \delta \Phi^{\pm,n}$$

and choosing $\|f_a^\pm\|_{\alpha,T}/\varepsilon$ and $\varepsilon > 0$ to be small.

Finally, we verify the other inequalities in (7.10). First, from (6.21)–(6.22), we have

$$\mathcal{L}(V^{\pm,n}, \Phi^{\pm,n}) V^{\pm,n} - f_a^\pm = (S_{\theta_{n-1}} - I) f_a^\pm + (I - S_{\theta_{n-1}}) E_{\pm,n-1} + e_{\pm,n-1}. \quad (7.27)$$

From Lemma 7.4, we have

$$\|(I - S_{\theta_{n-1}}) E_{\pm,n-1}\|_{s,T} \leq C \theta_n^{s-\tilde{s}} \|E_{\pm,n-1}\|_{\tilde{s},T} \leq C \theta_n^{s-\tilde{s}+1} \varepsilon^2 \leq C \theta_n^{s-\alpha-1} \varepsilon^2 \quad (7.28)$$

by choosing $\tilde{s} = \alpha + 2$.

From Lemma 7.3, we obtain

$$\|e_{\pm,n-1}\|_{s,T} \leq C \theta_n^{L^1(s)} \varepsilon^2 \leq C \theta_n^{s-\alpha-1} \varepsilon^2 \quad \text{for all } s \in [s_0, s_1 - 4], \quad (7.29)$$

by using $\alpha \geq s_0 + 5$.

When $s \leq \alpha + 1$, we have

$$\|(S_{\theta_{n-1}} - I) f_a^\pm\|_{s,T} \leq C \theta_n^{s-\alpha-1} \|f_a^\pm\|_{\alpha+1,T}; \quad (7.30)$$

while $s \in (\alpha + 1, s_1 - 4]$, we have

$$\|(S_{\theta_{n-1}} - I) f_a^\pm\|_{s,T} \leq \|S_{\theta_{n-1}} f_a^\pm\|_{s,T} + \|f_a^\pm\|_{s,T} \leq C \theta_n^{s-\alpha-1} \|f_a^\pm\|_{\alpha+1,T} + \varepsilon \quad (7.31)$$

by using (6.2) and (7.1).

Substitution of (7.28)–(7.31) into (7.27) yields

$$\|\mathcal{L}(V^{\pm,n}, \Phi^{\pm,n}) V^{\pm,n} - f_a^\pm\|_{s,T} \leq 2\varepsilon \theta_n^{s-\alpha-1} \quad \text{for } s \in [s_0, s_1 - 4],$$

provided that $\|f_a^\pm\|_{s_1-4,T} \leq \varepsilon$ and $\|f_a^\pm\|_{\alpha+1,T}/\varepsilon$ is small.

Similarly, we have

$$\|\mathcal{B}(V^{+,n}, V^{-,n}, \phi^n)\|_{H^{s-1}(b\Omega_T)} \leq \varepsilon \theta_n^{s-\alpha-1}. \quad (7.32)$$

Thus, we obtain (7.10) for $k = n$.

CONVERGENCE OF THE ITERATION SCHEME: From Proposition 7.1, we have

$$\sum_{n \geq 0} \left(\|(\delta V^{\pm, n}, \delta \Phi^{\pm, n})\|_{\alpha, T} + \|\delta \phi^n\|_{H^{\alpha-1}(b\Omega_T)} \right) < \infty, \tag{7.33}$$

which implies that there exists $(V^\pm, \Phi^\pm) \in B^\alpha(\Omega_T)$ with $\phi \in H^{\alpha-1}(b\Omega_T)$ such that

$$\begin{cases} (V^{\pm, n}, \Phi^{\pm, n}) \longrightarrow (V^\pm, \Phi^\pm) & \text{in } B^\alpha(\Omega_T) \times B^\alpha(\Omega_T), \\ \phi^n \longrightarrow \phi & \text{in } H^{\alpha-1}(b\Omega_T). \end{cases} \tag{7.34}$$

Thus, we conclude

Theorem 7.1. *Let $\alpha \geq 14$ and $s_1 \in [\alpha + 5, 2\alpha - 9]$. Let $\psi_0 \in H^{2s_1+3}(\mathbb{R}^2)$ and $U_0^\pm - \bar{U}^\pm \in B^{2(s_1+2)}(\mathbb{R}_+^3)$ satisfy the compatibility conditions of problem (2.17)–(2.20) up to order $s_1 + 2$, and let conditions (2.3)–(2.4) and (7.1) be satisfied. Then there exists a solution $(V^\pm, \Phi^\pm) \in B^\alpha(\Omega_T)$ with $\phi \in H^{\alpha-1}(b\Omega_T)$ to problem (5.13).*

Then Theorem 2.1 (main theorem) in Section 2 directly follows from Theorem 7.1.

8. Error estimates: proofs of Lemmas 7.2–7.3 and 7.5

In this section, we study the error estimates for the iteration scheme (6.21)–(6.22) and (6.34)–(6.35) to provide the proofs for Lemmas 7.2–7.3 and 7.5 under the assumption that (7.10) holds for all $0 \leq k \leq n - 1$. We start with the proof of Lemma 7.2.

PROOF OF LEMMA 7.2. Denote by

$$\mathcal{E}_1^{\pm, n} := \mathcal{E}(V^{\pm, n}, \Phi^{\pm, n}), \tag{8.1}$$

and

$$\begin{aligned} \mathcal{E}_2^{\pm, n} := & V_5^{\pm, n} - (\Psi_a^\pm + \Phi^{\pm, n})_{x_2} V_6^{\pm, n} - (\Psi_a^\pm + \Phi^{\pm, n})_{x_3} V_7^{\pm, n} \\ & - U_{a,6}^\pm(\Phi^{\pm, n})_{x_2} - U_{a,7}^\pm(\Phi^{\pm, n})_{x_3} \end{aligned} \tag{8.2}$$

as the extension of $(\mathcal{B}(V^{\pm, n}, \phi^n))_2^\pm$ in Ω_T .

By the definition in (6.25) for $V^{\pm, n+\frac{1}{2}}$, we have

$$\left\{ \begin{array}{l} V_2^{\pm, n+\frac{1}{2}} - S_{\theta_n} V_2^{\pm, n} \\ = S_{\theta_n} \mathcal{E}_1^{\pm, n} + [\partial_t, S_{\theta_n}] \Phi^{\pm, n} \\ \quad + ((\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2} S_{\theta_n} V_3^{\pm, n} - S_{\theta_n} ((\Psi_a^{\pm} + \Phi^{\pm, n})_{x_2} V_3^{\pm, n})) \\ \quad + ((\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_3} S_{\theta_n} V_4^{\pm, n} - S_{\theta_n} ((\Psi_a^{\pm} + \Phi^{\pm, n})_{x_3} V_4^{\pm, n})) \\ \quad + ((S_{\theta_n} \Phi^{\pm, n})_{x_2} U_{a,3}^{\pm} - S_{\theta_n} ((\Phi^{\pm, n})_{x_2} U_{a,3}^{\pm})) \\ \quad + ((S_{\theta_n} \Phi^{\pm, n})_{x_3} U_{a,4}^{\pm} - S_{\theta_n} ((\Phi^{\pm, n})_{x_3} U_{a,4}^{\pm})), \\ V_5^{\pm, n+\frac{1}{2}} - S_{\theta_n} V_5^{\pm, n} \\ = ((\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2} S_{\theta_n} V_6^{\pm, n} - S_{\theta_n} ((\Psi_a^{\pm} + \Phi^{\pm, n})_{x_2} V_6^{\pm, n})) \\ \quad + ((\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_3} S_{\theta_n} V_7^{\pm, n} - S_{\theta_n} ((\Psi_a^{\pm} + \Phi^{\pm, n})_{x_3} V_7^{\pm, n})) \\ \quad + ((S_{\theta_n} \Phi^{\pm, n})_{x_2} U_{a,6}^{\pm} - S_{\theta_n} ((\Phi^{\pm, n})_{x_2} U_{a,6}^{\pm})) \\ \quad + ((S_{\theta_n} \Phi^{\pm, n})_{x_3} U_{a,7}^{\pm} - S_{\theta_n} ((\Phi^{\pm, n})_{x_3} U_{a,7}^{\pm})) - S_{\theta_n} \mathcal{E}_2^{\pm, n}. \end{array} \right. \quad (8.3)$$

On the other hand, we have

$$\left\{ \begin{array}{l} \mathcal{E}_1^{\pm, n} = \mathcal{E}_1^{\pm, n-1} + (\delta \Phi^{\pm, n-1})_t + (\Psi_a^{\pm} + \Phi^{\pm, n})_{x_2} \delta V_3^{\pm, n-1} \\ \quad + (\Psi_a^{\pm} + \Phi^{\pm, n})_{x_3} \delta V_4^{\pm, n-1} - \delta V_2^{\pm, n-1} \\ \quad + (\delta \Phi^{\pm, n-1})_{x_2} (U_{a,3}^{\pm} + V_3^{\pm, n-1}) + (\delta \Phi^{\pm, n-1})_{x_3} (U_{a,4}^{\pm} + V_4^{\pm, n-1}), \\ \mathcal{E}_2^{\pm, n} = \mathcal{E}_2^{\pm, n-1} - (\Psi_a^{\pm} + \Phi^{\pm, n})_{x_2} \delta V_6^{\pm, n-1} \\ \quad - (\Psi_a^{\pm} + \Phi^{\pm, n})_{x_3} \delta V_7^{\pm, n-1} + \delta V_5^{\pm, n-1} \\ \quad - (\delta \Phi^{\pm, n-1})_{x_2} (U_{a,6}^{\pm} + V_6^{\pm, n-1}) + (\delta \Phi^{\pm, n-1})_{x_3} (U_{a,7}^{\pm} + V_7^{\pm, n-1}), \end{array} \right. \quad (8.4)$$

which implies

$$\|(\mathcal{E}_1^{\pm, n}, \mathcal{E}_2^{\pm, n})\|_{s_0, T} \leq C \varepsilon \theta_n^{s_0-1-\alpha} \quad (8.5)$$

by using $\Delta_n = O(\theta_n^{-1})$, the inductive assumption for (7.10), and Lemma 7.1. Thus we deduce

$$\|S_{\theta_n}(\mathcal{E}_1^{\pm, n}, \mathcal{E}_2^{\pm, n})\|_{s, T} \leq C \varepsilon \theta_n^{s-\alpha-1} \quad \text{for any } s \geq s_0. \quad (8.6)$$

The discussion for the commutators in (8.3) follows an argument from COULOMBEL and SECCHI [10]. We now analyze the third term of $V_2^{\pm, n+\frac{1}{2}} - S_{\theta_n} V_2^{\pm, n}$ given by (8.3) in detail.

When $s \in [\alpha + 1, s_1 + 2]$, we have

$$\begin{aligned} & \|(\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2} S_{\theta_n} V_3^{\pm, n}\|_{s, T} \\ & \leq C \left(\|(\Psi_a^{\pm} + S_{\theta_n} \Phi^{\pm, n})_{x_2}\|_{L^\infty} \|S_{\theta_n} V_3^{\pm, n}\|_{s, T} \right. \\ & \quad \left. + \|\dot{\Psi}_a^{\pm} + S_{\theta_n} \dot{\Phi}^{\pm, n}\|_{s+1, T} \|S_{\theta_n} V_3^{\pm, n}\|_{L^\infty} \right) \\ & \leq C \varepsilon^2 \theta_n^{s+1-\alpha}, \end{aligned}$$

and

$$\begin{aligned} & \|S_{\theta_n}((\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n})\|_{s,T} \\ & \leq C\theta_n^{s-\alpha} \left(\|(\Psi_a^\pm + \Phi^{\pm,n})_{x_2}\|_{L^\infty} \|V_3^{\pm,n}\|_{\alpha,T} + \|\dot{\Psi}_a^\pm + \Phi^{\pm,n}\|_{\alpha+1,T} \|V_3^{\pm,n}\|_{L^\infty} \right) \\ & \leq C\varepsilon^2\theta_n^{s+1-\alpha} \end{aligned}$$

by using (7.1), the induction assumption for (7.10), and Lemma 7.1, which implies

$$\|(\Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})_{x_2} S_{\theta_n} V_3^{\pm,n} - S_{\theta_n}((\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n})\|_{s,T} \leq C\varepsilon^2\theta_n^{s-\alpha+1}. \tag{8.7}$$

When $s \in [s_0, \alpha]$, we have

$$\begin{aligned} & \|(\Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})_{x_2} S_{\theta_n} V_3^{\pm,n} - S_{\theta_n}((\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n})\|_{s,T} \\ & \leq \|(I - S_{\theta_n})((\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n})\|_{s,T} \\ & \quad + \|(\Psi_a^\pm + \Phi^{\pm,n})_{x_2} (S_{\theta_n} - I) V_3^{\pm,n}\|_{s,T} \\ & \quad + \|((S_{\theta_n} - I)\Phi^{\pm,n})_{x_2} S_{\theta_n} V_3^{\pm,n}\|_{s,T} \\ & \leq C\theta_n^{s-\alpha} (\|(\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n}\|_{\alpha,T} + \|(\Psi_a^\pm + \Phi^{\pm,n})_{x_2}\|_{L^\infty} \|V_3^{\pm,n}\|_{\alpha,T}) \\ & \quad + C\theta_n^{3-\alpha} \|V_3^{\pm,n}\|_{\alpha,T} \|\Psi_a^\pm + \Phi^{\pm,n}\|_{s+1,T} \\ & \quad + \|((S_{\theta_n} - I)\Phi^{\pm,n})_{x_2} S_{\theta_n} V_3^{\pm,n}\|_{s,T} \\ & \leq C\varepsilon^2\theta_n^{s+1-\alpha}. \end{aligned} \tag{8.8}$$

Together (8.7) with (8.8), it follows that

$$\|(\Psi_a^\pm + S_{\theta_n} \Phi^{\pm,n})_{x_2} S_{\theta_n} V_3^{\pm,n} - S_{\theta_n}((\Psi_a^\pm + \Phi^{\pm,n})_{x_2} V_3^{\pm,n})\|_{s,T} \leq C\varepsilon^2\theta_n^{s-\alpha+1} \tag{8.9}$$

for all $s \in [s_0, s_1 + 2]$.

The other commutators appearing in (8.3) satisfy estimates similar to the above. Thus, we complete the proof.

To show the error estimates given in Lemma 7.3, we first show several lemmas dealing with different types of errors.

Lemma 8.1. *Let $\alpha \geq s_0 + 1 \geq 6$. For the quadratic errors, we have*

$$\begin{cases} \|e_{\pm,k}^{(1)}\|_{s,T} \leq C\varepsilon^2\theta_k^{L_1(s)} \Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2], \\ \|e_{\pm,k}^{(1)}\|_{s,T} \leq C\varepsilon^2\theta_k^{s-2\alpha+s_0-2} \Delta_k & \text{for } s \in [s_0 - 1, s_1 - 1], \\ \|e_k^{(1)}\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_k^{s-2\alpha+s_0-1} \Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2] \end{cases} \tag{8.10}$$

for all $k \leq n - 1$, where

$$L_1(s) = \max((s+2-\alpha)_+ + 2(s_0-\alpha) - 3, s+s_0-1-2\alpha, s+2s_0-3-3\alpha, s_0-\alpha-3). \tag{8.11}$$

Proof. The proof is divided into three steps.

Step 1. From the definition of $e_{\pm,k}^{(1)}$, we first have

$$e_{\pm,k}^{(1)} = \int_0^1 (1 - \tau) L'' (U_a^\pm + V^{\pm,k} + \tau \delta V^{\pm,k}; \psi_a^\pm + \phi^{\pm,k} + \tau \delta \phi^{\pm,k}) \times ((\delta V^{\pm,k}, \delta \phi^{\pm,k}), (\delta V^{\pm,k}, \delta \phi^{\pm,k})) \, d\tau. \tag{8.12}$$

From (7.1) and Lemma 7.1, we find

$$\sup_{0 \leq \tau \leq 1} \left(\|\dot{U}_a^\pm + V^{\pm,k} + \tau \delta V^{\pm,k}\|_{W^{1,\infty}(\Omega_T)} + \|\dot{\psi}_a^\pm + \phi^{\pm,k} + \tau \delta \phi^{\pm,k}\|_{W^{1,\infty}(\Omega_T)} \right) \leq C\varepsilon. \tag{8.13}$$

On the other hand, we have

$$\begin{aligned} & \|L''_{(U^\pm, \psi^\pm)}((V^{\pm,1}, \phi^{\pm,1}), (V^{\pm,2}, \phi^{\pm,2}))\|_{s,T} \\ & \leq C \left(\|\dot{U}^\pm, \dot{\psi}^\pm\|_{s+2,T} \| (V^{\pm,1}, \phi^{\pm,1}) \|_{W^{1,\infty}} \| (V^{\pm,2}, \phi^{\pm,2}) \|_{W^{1,\infty}} \right. \\ & \quad + \| (V^{\pm,1}, \phi^{\pm,1}) \|_{s+2,T} \| (V^{\pm,2}, \phi^{\pm,2}) \|_{W^{1,\infty}} \\ & \quad \left. + \| (V^{\pm,1}, \phi^{\pm,1}) \|_{W^{1,\infty}} \| (V^{\pm,2}, \phi^{\pm,2}) \|_{s+2,T} \right). \end{aligned} \tag{8.14}$$

Therefore, we obtain

$$\begin{aligned} & \|e_{\pm,k}^{(1)}\|_{s,T} \\ & \leq C \left(\varepsilon^2 \theta_k^{2(s_0-1-\alpha)} \Delta_k^2 (\varepsilon + \| (V^{\pm,k}, \phi^{\pm,k}) \|_{s+2,T} + \| (\delta V^{\pm,k}, \delta \phi^{\pm,k}) \|_{s+2,T}) \right. \\ & \quad \left. + \varepsilon \theta_k^{s_0-1-\alpha} \Delta_k \| (\delta V^{\pm,k}, \delta \phi^{\pm,k}) \|_{s+2,T} \right), \end{aligned}$$

which implies

$$\begin{cases} \|e_{\pm,k}^{(1)}\|_{s,T} \leq C \varepsilon^2 \theta_k^{L(s)} \Delta_k & \text{when } s + 2 \neq \alpha, \, s \leq s_1 - 2, \\ \|e_{\pm,k}^{(1)}\|_{s,T} \leq C \varepsilon^2 \theta_k^{\max(s_0-\alpha-3, 2s_0-2-2\alpha)} \Delta_k & \text{when } s + 2 = \alpha, \end{cases} \tag{8.15}$$

by Lemma 7.1, the inductive assumption for (7.10), and $\Delta_k = O(\theta_k^{-1})$, where

$$L(s) = \max((s + 2 - \alpha)_+ + 2(s_0 - \alpha) - 3, s + s_0 - 1 - 2\alpha, s + 2s_0 - 3 - 3\alpha).$$

Thus, we conclude the first result in (8.10).

Step 2. Obviously, from the definition of $\bar{e}_{\pm,k}^{(1)}$ in (6.29), we have

$$\|\bar{e}_{\pm,k}^{(1)}\|_{s,T} \leq C \left(\|\delta V^{\pm,k}\|_{s,T} \|\delta \phi^{\pm,k}\|_{W^{1,\infty}} + \|\delta V^{\pm,k}\|_{L^\infty} \|\delta \phi^{\pm,k}\|_{s+1,T} \right), \tag{8.16}$$

which implies the second result in (8.10).

Step 3. Since $\bar{e}_{\pm,k}^{(1)}|_{x_1=0} = (\tilde{e}_k)_1^\pm$, we obtain that $(\tilde{e}_k)_1^\pm$ satisfies the estimate given in (8.10). Moreover, from the definition of $\tilde{e}_k^{(1)}$, we have

$$\begin{cases} (\tilde{e}_k^{(1)})_2^\pm = -(\delta\phi^k)_{x_2}\delta V_6^{\pm,k} - (\delta\phi^k)_{x_3}\delta V_7^{\pm,k}, \\ (\tilde{e}_k^{(1)})_3 = \frac{1}{2}(|\delta V_H^{+,k}|^2 - |\delta V_H^{-,k}|^2), \end{cases} \quad (8.17)$$

which implies

$$\begin{cases} \|(\tilde{e}_k^{(1)})_2^\pm\|_{H^s(b\Omega_T)} \leq C(\|\delta\phi^k\|_{H^{s+1}(b\Omega_T)}\|\delta V^{\pm,k}\|_{L^\infty} \\ \quad + \|\delta\phi^k\|_{W^{1,\infty}(b\Omega_T)}\|\delta V^{\pm,k}\|_{H^s(b\Omega_T)}), \\ \|(\tilde{e}_k^{(1)})_3\|_{H^s(b\Omega_T)} \leq C\|\delta V^{\pm,k}\|_{L^\infty}\|\delta V^{\pm,k}\|_{H^s(b\Omega_T)}, \end{cases} \quad (8.18)$$

yielding the last estimate of (8.10) by the inductive assumption for (7.10).

For the first substitution errors, we have

Lemma 8.2. *Let $\alpha \geq s_0 + 1 \geq 6$. Then, for all $k \leq n - 1$, we have*

$$\begin{cases} \|e_{\pm,k}^{(2)}\|_{s,T} \leq C\varepsilon^2\theta_k^{L_2(s)}\Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2], \\ \|e_{\pm,k}^{(2)}\|_{s,T} \leq C\varepsilon^2\theta_k^{s-2\alpha+s_0}\Delta_k & \text{for } s \in [s_0 - 1, s_1 - 1], \\ \|e_{\pm,k}^{(2)}\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_k^{s-2\alpha+s_0+1}\Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2], \end{cases} \quad (8.19)$$

where

$$L_2(s) = \begin{cases} (s + 2 - \alpha)_+ + s_0 - \alpha - 1 & \text{for } \alpha \neq s + 2, \\ \max(5 - \alpha, s_0 - 1 - \alpha) & \text{for } \alpha = s + 2. \end{cases} \quad (8.20)$$

Proof. The proof is divided into three steps.

Step 1. From the definition of $e_{\pm,k}^{(2)}$, we have

$$\begin{aligned} e_{\pm,k}^{(2)} &= \int_0^T L''(U_a^\pm + S_{\theta_k}V^{\pm,k} + \tau(1 - S_{\theta_k})V^{\pm,k}; \Psi_a^\pm + S_{\theta_k}\Phi^{\pm,k} + \tau(1 - S_{\theta_k})\Phi^{\pm,k}) \\ &\quad \times ((\delta V^{\pm,k}, \delta\Phi^{\pm,k}), ((1 - S_{\theta_k})V^{\pm,k}, (1 - S_{\theta_k})\Phi^{\pm,k})) \, d\tau. \end{aligned} \quad (8.21)$$

As in (8.13), from the inductive assumption for (7.10), we find

$$\sup_{0 \leq \tau \leq 1} \|\dot{U}_a^\pm + S_{\theta_k}V^{\pm,k} + \tau(1 - S_{\theta_k})V^{\pm,k}\|_{W^{1,\infty}(\Omega_T)} \leq C\varepsilon, \quad (8.22)$$

and

$$\sup_{0 \leq \tau \leq 1} \|\dot{\Psi}_a^\pm + S_{\theta_k}\Phi^{\pm,k} + \tau(1 - S_{\theta_k})\Phi^{\pm,k}\|_{W^{1,\infty}(\Omega_T)} \leq C\varepsilon. \quad (8.23)$$

Then we use (8.14) in (8.21) to obtain

$$\begin{aligned} \|e_{\pm,k}^{(2)}\|_{s,T} \leq & C \left(\|\delta V^{\pm,k}, \delta \Phi^{\pm,k}\|_{W^{1,\infty}} \|(1 - S_{\theta_k})(V^{\pm,k}, \Phi^{\pm,k})\|_{W^{1,\infty}} \right. \\ & \times (\|\dot{U}_a^\pm, \dot{\Psi}_a^\pm\|_{s+2,T} + \|S_{\theta_k}(V^{\pm,k}, \Phi^{\pm,k})\|_{s+2,T} \\ & \quad \left. + \|(1 - S_{\theta_k})(V^{\pm,k}, \Phi^{\pm,k})\|_{s+2,T}) \right. \\ & + \|\delta V^{\pm,k}, \delta \Phi^{\pm,k}\|_{s+2,T} \|(1 - S_{\theta_k})(V^{\pm,k}, \Phi^{\pm,k})\|_{W^{1,\infty}} \\ & \left. + \|\delta V^{\pm,k}, \delta \Phi^{\pm,k}\|_{W^{1,\infty}} \|(1 - S_{\theta_k})(V^{\pm,k}, \Phi^{\pm,k})\|_{s+2,T} \right). \end{aligned} \quad (8.24)$$

Using the properties of the smoothing operators, the inductive assumption for (7.10), and Lemma 7.1 in (8.24), we conclude the first estimate in (8.19) when $s \leq s_1 - 2$.

Step 2. From the definition of $\bar{e}_{\pm,k}^{(2)}$ in (6.30), we have

$$\begin{aligned} \|\bar{e}_{\pm,k}^{(2)}\|_{s,T} \leq & C \left(\|\delta V^{\pm,k}\|_{s,T} \|\nabla_{(x_2,x_3)}(1 - S_{\theta_k})\Phi^{\pm,k}\|_{L^\infty} \right. \\ & + \|(1 - S_{\theta_k})\Phi^{\pm,k}\|_{s+1,T} \|\delta V^{\pm,k}\|_{L^\infty} \\ & + \|(1 - S_{\theta_k})V^{\pm,k}\|_{s,T} \|\nabla_{(x_2,x_3)}\delta\Phi^{\pm,k}\|_{L^\infty} \\ & \left. + \|\delta\Phi^{\pm,k}\|_{s+1,T} \|(1 - S_{\theta_k})V^{\pm,k}\|_{L^\infty} \right), \end{aligned} \quad (8.25)$$

which implies the second result in (8.19) when $s \leq s_1 - 1$ by the inductive assumption for (7.10) and Lemma 7.1 in (8.25).

Step 3. Noting that

$$\left(\tilde{e}_k^{(2)}\right)_1^\pm = \bar{e}_{\pm,k}^{(2)}|_{x_1=0}$$

from the above discussion, we conclude that $(\tilde{e}_k^{(2)})_1^\pm$ satisfy the estimate given in (8.19) when $s \leq s_1 - 2$.

From the definition of $\tilde{e}_k^{(2)}$, we have

$$\begin{cases} (\tilde{e}_k^{(2)})_2^\pm = ((S_{\theta_k} - I)\phi^k)_{x_2} \delta V_6^{\pm,k} + ((S_{\theta_k} - I)\phi^k)_{x_3} \delta V_7^{\pm,k} \\ \quad + (S_{\theta_k} - I)V_6^{\pm,k}(\delta\phi^k)_{x_2} + (S_{\theta_k} - I)V_7^{\pm,k}(\delta\phi^k)_{x_3}, \\ (\tilde{e}_k^{(2)})_3 = (I - S_{\theta_k})V_H^{+,k} \delta V_H^{+,k} - (I - S_{\theta_k})V_H^{-,k} \delta V_H^{-,k}, \end{cases} \quad (8.26)$$

which implies

$$\begin{cases} \|(\tilde{e}_k^{(2)})_2^\pm\|_{H^s(b\Omega_T)} \leq C \left(\|\delta V^{\pm,k}\|_{H^s(b\Omega_T)} \|(1 - S_{\theta_k})\phi^k\|_{W^{1,\infty}} \right. \\ \quad + \|(1 - S_{\theta_k})\phi^k\|_{H^{s+1}(b\Omega_T)} \|\delta V^{\pm,k}\|_{L^\infty} \\ \quad + \|(1 - S_{\theta_k})V^{\pm,k}\|_{H^s(b\Omega_T)} \|\delta\phi^k\|_{W^{1,\infty}} \\ \quad \left. + \|\delta\phi^k\|_{H^{s+1}(b\Omega_T)} \|(1 - S_{\theta_k})V^{\pm,k}\|_{L^\infty} \right), \\ \|(\tilde{e}_k^{(2)})_3\|_{H^s(b\Omega_T)} \leq C \left(\|\delta V^{\pm,k}\|_{H^s(b\Omega_T)} \|(1 - S_{\theta_k})V^{\pm,k}\|_{L^\infty} \right. \\ \quad \left. + \|(1 - S_{\theta_k})V^{\pm,k}\|_{H^s(b\Omega_T)} \|\delta V^{\pm,k}\|_{L^\infty} \right). \end{cases} \quad (8.27)$$

Using the inductive assumption for (7.10) and Lemma 7.1 in (8.27), we conclude the last result in (8.19) when $s \leq s_1 - 2$.

The representations of the second substitution errors $(e_{\pm,k}^{(3)}, \tilde{e}_k^{(3)})$ are similar to those of $(e_{\pm,k}^{(2)}, \tilde{e}_k^{(2)})$. Thus, using Lemma 7.2 and the same argument as the proof of Lemma 8.2, we conclude

Lemma 8.3. *Let $\alpha \geq s_0 + 1 \geq 6$. For the second substitution errors, we have*

$$\begin{cases} \|e_{\pm,k}^{(3)}\|_{s,T} \leq C\varepsilon^2\theta_k^{L_3(s)}\Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2], \\ \|\tilde{e}_k^{(3)}\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_k^{s+s_0+2-2\alpha}\Delta_k & \text{for } s \in [s_0 - 2, s_1 - 2] \end{cases} \quad (8.28)$$

for all $k \leq n - 1$, where

$$L_3(s) = \max(s + s_0 + 2 - 2\alpha, s + 2s_0 + 3 - 3\alpha, (s + 2 - \alpha)_+ + 2(s_0 - \alpha) + 1).$$

Lemma 8.4. *Let $s_0 \geq 5$ and $\alpha \geq s_0 + 3$. For the last errors, we have*

$$\begin{cases} \|e_{\pm,k}^{(4)}\|_{s,T} \leq C\varepsilon^2\theta_k^{L_4(s)}\Delta_k & \text{for } s \in [s_0, s_1 - 4], \\ \|\tilde{e}_{\pm,k}^{(4)}\|_{s,T} \leq C\varepsilon^2\theta_k^{L_5(s-1)}\Delta_k & \text{for } s \in [s_0, s_1 - 3], \\ \|\tilde{e}_k^{(4)}\|_{H^s(b\Omega_T)} \leq C\varepsilon^2\theta_k^{L_5(s)}\Delta_k & \text{for } s \in [s_0 - 1, s_1 - 4] \end{cases} \quad (8.29)$$

for all $k \leq n - 1$, where

$$L_4(s) = \begin{cases} \max((s + 2 - \alpha)_+ + 2 + 2(s_0 - \alpha), s + 4 + s_0 - 2\alpha) & \text{if } \alpha \neq s + 2, s + 4, \\ \max(s_0 - \alpha, 2(s_0 - \alpha) + 4) & \text{if } \alpha = s + 4, \\ s_0 + 2 - \alpha & \text{if } \alpha = s + 2, \end{cases}$$

and

$$L_5(s) = \begin{cases} \max((s + 3 - \alpha)_+ + 2(s_0 - \alpha), s + 3 + s_0 - 2\alpha) & \text{if } \alpha \neq s + 3, s + 4, \\ s_0 - \alpha & \text{if } \alpha = s + 3, \\ s_0 - 1 - \alpha & \text{if } \alpha = s + 4. \end{cases}$$

Proof. The proof is divided into three steps.

Step 1. Set

$$R_k^\pm = (L(U_a^\pm + V^{\pm,k+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_k}\Phi^{\pm,k})(U_a^\pm + V^{\pm,k+\frac{1}{2}}))_{x_1}. \quad (8.30)$$

Then we have

$$\begin{aligned} \|R_k^\pm\|_{s,T} &\leq \|L(U_a^\pm + V^{\pm,k+\frac{1}{2}}, \Psi_a^\pm + S_{\theta_k}\Phi^{\pm,k})(U_a^\pm + V^{\pm,k+\frac{1}{2}}) \\ &\quad - L(U_a^\pm + V^{\pm,k}, \Psi_a^\pm + \Phi^{\pm,k})(U_a^\pm + V^{\pm,k})\|_{s+2,T} \\ &\quad + \|L(V^{\pm,k}, \Phi^{\pm,k})V^{\pm,k} - f_a^\pm\|_{s+2,T}, \end{aligned}$$

which implies that, for all $s \in [s_0 - 2, s_1 - 4]$,

$$\begin{aligned}
 & \|R_k^\pm\|_{s,T} \\
 & \leq C \left(\|V^{\pm,k+\frac{1}{2}} - V^{\pm,k}\|_{W^{1,\infty}} (\|\dot{U}_a^\pm + V^{\pm,k+\frac{1}{2}}\|_{s+2,T} + \|\dot{\Psi}_a^\pm + S_{\theta_k} \Phi^{\pm,k}\|_{s+4,T}) \right. \\
 & \quad + \|\dot{U}_a^\pm + V^{\pm,k}\|_{s+4,T} (\|V^{\pm,k+\frac{1}{2}} - V^{\pm,k}\|_{L^\infty} + \|(I - S_{\theta_k}) \Phi^{\pm,k}\|_{W^{1,\infty}}) \\
 & \quad + \|\dot{U}_a^\pm + V^{\pm,k}\|_{W^{1,\infty}} (\|V^{\pm,k+\frac{1}{2}} - V^{\pm,k}\|_{s+2,T} + \|(I - S_{\theta_k}) \Phi^{\pm,k}\|_{s+4,T}) \\
 & \quad \left. + \|V^{\pm,k+\frac{1}{2}} - V^{\pm,k}\|_{s+4,T} + \|\mathcal{L}(V^{\pm,k}, \Phi^{\pm,k}) V^{\pm,k} - f_a^\pm\|_{s+2,T} \right) \\
 & \leq \begin{cases} C \varepsilon^2 \theta_k^{(s+4-\alpha)_+ + s_0 + 1 - \alpha} + C \varepsilon \theta_k^{s+5-\alpha} & \text{for } s + 4 \neq \alpha, \\ C \varepsilon^2 \theta_k^{s_0 + 2 - \alpha} + C \varepsilon \theta_k & \text{for } s + 4 = \alpha. \end{cases} \tag{8.31}
 \end{aligned}$$

Thus we find that

$$e_{\pm,k}^{(4)} = \frac{R_k^\pm \delta \Phi^{\pm,k}}{(\Psi_a^\pm + S_{\theta_k} \Phi^{\pm,k})_{x_1}}$$

satisfy

$$\begin{aligned}
 \|e_{\pm,k}^{(4)}\|_{s,T} & \leq C \varepsilon \left(\|R_k^\pm\|_{s_0-2,T} (\theta_k^{s-1-\alpha} \Delta_k + \theta_k^{s_0-1-\alpha} \Delta_k (\varepsilon + \varepsilon \theta_k^{(s+2-\alpha)_+})) \right. \\
 & \quad \left. + \theta_k^{s_0-1-\alpha} \Delta_k \|R_k^\pm\|_{s,T} \right), \tag{8.32}
 \end{aligned}$$

when $s + 2 \neq \alpha$, and

$$\begin{aligned}
 \|e_{\pm,k}^{(4)}\|_{s,T} & \leq C \varepsilon \left(\|R_k^\pm\|_{s_0-2,T} (\theta_k^{s-1-\alpha} \Delta_k + \theta_k^{s_0-1-\alpha} \Delta_k (\varepsilon + \varepsilon \log \theta_k)) \right. \\
 & \quad \left. + \theta_k^{s_0-1-\alpha} \Delta_k \|R_k^\pm\|_{s,T} \right), \tag{8.33}
 \end{aligned}$$

when $s + 2 = \alpha$, which yields the first estimate in (8.29) for any $s \in [s_0, s_1 - 4]$, provided $\alpha \geq s_0 + 3$ by using (8.31).

Step 2. Set

$$R_k^b = B(U_a^\pm + V^{\pm,k+\frac{1}{2}}, \psi_a + S_{\theta_k} \phi^k). \tag{8.34}$$

Then we have

$$\begin{aligned}
 & \|R_k^b\|_{H^s(b\Omega_T)} \\
 & \leq \|B(U_a^\pm + V^{\pm,k+\frac{1}{2}}, \psi_a + S_{\theta_k} \phi^k) - B(U_a^\pm + V^{\pm,k}, \psi_a + \phi^k)\|_{H^s(b\Omega_T)} \\
 & \quad + \|B(V^{\pm,k}, \phi^k)\|_{H^s(b\Omega_T)},
 \end{aligned}$$

which implies

$$\begin{aligned}
 & \| (R_k^b)_1^\pm \|_{H^s(b\Omega_T)} \\
 & \leq C \left(\| (S_{\theta_k} - 1)\phi^k \|_{H^{s+1}(b\Omega_T)} + \| V_2^{\pm, k+\frac{1}{2}} - V_2^{\pm, k} \|_{H^s(b\Omega_T)} \right. \\
 & \quad + \| (S_{\theta_k} - I)V^{\pm, k} \|_{L^\infty} \| \dot{\psi}_a + S_{\theta_k}\phi^k \|_{H^{s+1}(b\Omega_T)} \\
 & \quad + \| (S_{\theta_k} - I)V^{\pm, k} \|_{H^s(b\Omega_T)} \| \dot{\psi}_a + S_{\theta_k}\phi^k \|_{W^{1,\infty}} \\
 & \quad + \| (S_{\theta_k} - 1)\phi^k \|_{H^{s+1}(b\Omega_T)} \| \dot{U}_a^\pm + V^{\pm, k} \|_{L^\infty} \\
 & \quad + \| (S_{\theta_k} - 1)\phi^k \|_{W^{1,\infty}(b\Omega_T)} \| \dot{U}_a^\pm + V^{\pm, k} \|_{H^s(b\Omega_T)} \\
 & \quad \left. + \| \mathcal{B}(V^{\pm, k}, \phi^k) \|_{H^s(b\Omega_T)} \right). \tag{8.35}
 \end{aligned}$$

Therefore, we have the estimate

$$\| (R_k^b)_1^\pm \|_{H^s(b\Omega_T)} \leq \begin{cases} C\varepsilon\theta_k^{\max((s+2-\alpha)_+ + s_0 - \alpha, s+2-\alpha)} & \text{if } \alpha \neq s+1, s+2, \\ C\varepsilon\theta_k & \text{if } \alpha = s+1, \\ C\varepsilon & \text{if } \alpha = s+2 \end{cases} \tag{8.36}$$

for all $s \in [s_0 - 1, s_1 - 2]$. Thus we find that

$$(\tilde{e}_k^{(4)})_1^\pm = - \frac{((R_k^b)_1^\pm)_{x_1}}{(\Psi_a^\pm + S_{\theta_k}\Phi^{\pm, k})_{x_1}|_{x_1=0}} \delta\phi^k$$

satisfy

$$\begin{aligned}
 \| (\tilde{e}_k^{(4)})_1^\pm \|_{H^s(b\Omega_T)} & \leq C \left(\| (R_k^b)_1^\pm \|_{H^4(b\Omega_T)} (\varepsilon\theta_k^{s-\alpha} \Delta_k + \varepsilon^2\theta_k^{s_0-1-\alpha} \Delta_k \theta_k^{(s+3-\alpha)_+}) \right. \\
 & \quad \left. + \varepsilon^2\theta_k^{s_0-1-\alpha} \Delta_k \| (R_k^b)_1^\pm \|_{H^{s+2}(b\Omega_T)} \right), \tag{8.37}
 \end{aligned}$$

when $s+3 \neq \alpha$, and

$$\begin{aligned}
 \| (\tilde{e}_k^{(4)})_1^\pm \|_{H^s(b\Omega_T)} & \leq C \left(\| (R_k^b)_1^\pm \|_{H^4(b\Omega_T)} (\varepsilon\theta_k^{s-\alpha} \Delta_k + \varepsilon^2\theta_k^{s_0-1-\alpha} \Delta_k \log \theta_k) \right. \\
 & \quad \left. + \varepsilon^2\theta_k^{s_0-1-\alpha} \Delta_k \| (R_k^b)_1^\pm \|_{H^{s+2}(b\Omega_T)} \right), \tag{8.38}
 \end{aligned}$$

when $s+3 = \alpha$, which yields the estimate of $(\tilde{e}_k^{(4)})_1^\pm$ given in (8.29) for any $s \in [s_0 - 1, s_1 - 4]$ by using (8.36).

Estimate (8.29) for the other components of $\tilde{e}_k^{(4)}$ can be proved by the same argument as above.

Step 3. Noting that the restriction of $\tilde{e}_{\pm, k}^{(4)}$ on $\{x_1 = 0\}$ is the same as $(\tilde{e}_k^{(4)})_1^\pm$, we conclude the estimate of $\tilde{e}_{\pm, k}^{(4)}$ in (8.29) for any $s \in [s_0, s_1 - 3]$.

PROOF OF LEMMA 7.3. Summarizing all the results in Lemmas 8.1–8.4, we obtain the estimates given in Lemma 7.3.

PROOF OF LEMMA 7.5 . From the definitions of (f_n^\pm, g_n, h_n^\pm) given in (6.22) and (6.35), respectively, we have

$$\begin{cases} f_n^\pm = (S_{\theta_n} - S_{\theta_{n-1}})f_a^\pm - (S_{\theta_n} - S_{\theta_{n-1}})E_{\pm, n-1} - S_{\theta_n}e_{\pm, n-1}, \\ g_n = -(S_{\theta_n} - S_{\theta_{n-1}})\tilde{E}_{n-1} - S_{\theta_n}\tilde{e}_{n-1}, \\ h_n^\pm = (S_{\theta_{n-1}} - S_{\theta_n})\bar{E}_{\pm, n-1} - S_{\theta_n}\bar{e}_{\pm, n-1}. \end{cases} \quad (8.39)$$

Using the properties of the smoothing operators, we have

$$\begin{cases} \|(S_{\theta_n} - S_{\theta_{n-1}})f_a^\pm\|_{s, T} \leq C\theta_n^{s-\tilde{s}-1}\Delta_n\|f_a^\pm\|_{\tilde{s}, T} & \text{for } \tilde{s} \geq 0, \\ \|(S_{\theta_n} - S_{\theta_{n-1}})E_{\pm, n-1}\|_{s, T} \leq C\theta_n^{s-\tilde{s}-1}\Delta_n\|E_{\pm, n-1}\|_{\tilde{s}, T} \\ \leq C\varepsilon^2\theta_n^{s-\tilde{s}}\Delta_n & \text{for } \tilde{s} \in [s_0, s_1 - 4], \\ \|S_{\theta_n}e_{\pm, n-1}\|_{s, T} \leq C\theta_n^{(s-\tilde{s})_+}\|e_{\pm, n-1}\|_{\tilde{s}, T} \\ \leq C\varepsilon^2\theta_n^{(s-\tilde{s})_+ + L^1(\tilde{s})}\Delta_n & \text{for } \tilde{s} \in [s_0, s_1 - 4]. \end{cases} \quad (8.40)$$

Estimate (8.40)₃ can be represented as the following three cases.

Case 1. $s = s_0$. For this case, if we choose $\tilde{s} \in [s_0, \alpha - 5]$, then

$$L^1(\tilde{s}) = \max(s_0 - \alpha - 1, \tilde{s} + s_0 + 4 - 2\alpha) = s_0 - \alpha - 1,$$

which implies from (8.40)₃ that

$$\|S_{\theta_n}e_{\pm, n-1}\|_{s_0, T} \leq C\varepsilon^2\theta_n^{s_0-\alpha-1}\Delta_n. \quad (8.41)$$

Case 2. $s = s_0 + 1$ or $s = s_0 + 2$. For these two cases, if we choose $\tilde{s} = \alpha - 4 \geq s_0 + 1$, then

$$L^1(\tilde{s}) = \max(s_0 - \alpha, 2(s_0 - \alpha) + 4) = s_0 - \alpha,$$

which implies

$$\|S_{\theta_n}e_{\pm, n-1}\|_{s, T} \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n. \quad (8.42)$$

Case 3. $s \geq s_0 + 3$. For this case, if we choose $\tilde{s} = \alpha - 2 \geq s_0$, then

$$(s - \tilde{s})_+ + L^1(\tilde{s}) = (s + 2 - \alpha)_+ + s_0 + 2 - \alpha \leq s - \alpha - 1$$

by using $\alpha \geq s_0 + 5$, which implies from (8.40)₃ that

$$\|S_{\theta_n}e_{\pm, n-1}\|_{s, T} \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n. \quad (8.43)$$

On the other hand, we have

$$\|(S_{\theta_n} - S_{\theta_{n-1}})E_{\pm, n-1}\|_{s, T} \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n \quad (8.44)$$

by setting $\tilde{s} = \alpha + 1$ in (8.40)₂ if $s_1 \geq \alpha + 5$.

From (8.41)–(8.44) with (8.39), it follows that

$$\|f_n^\pm\|_{s, T} \leq C\theta_n^{s-\alpha-1}\Delta_n(\varepsilon^2 + \|f_a^\pm\|_{\alpha, T}) \quad \text{for all } s \geq s_0. \quad (8.45)$$

Similarly, we have

$$\left\{ \begin{aligned} \|(S_{\theta_n} - S_{\theta_{n-1}})\tilde{E}_{n-1}\|_{H^{s+1}(b\Omega_T)} &\leq C\theta_n^{s-\tilde{s}}\Delta_n\|\tilde{E}_{n-1}\|_{H^{\tilde{s}}(b\Omega_T)} \\ &\leq C\varepsilon^2\theta_n^{s+1-\tilde{s}}\Delta_n, \\ \|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(b\Omega_T)} &\leq C\theta_n^{(s+1-\tilde{s})_+}\|\tilde{e}_{n-1}\|_{H^{\tilde{s}}(b\Omega_T)} \\ &\leq C\theta_n^{(s+1-\tilde{s})_++L^3(\tilde{s})}\varepsilon^2\Delta_n \end{aligned} \right. \quad (8.46)$$

for $\tilde{s} \in [s_0 - 1, s_1 - 4]$.

Estimate (8.46)₂ can be represented as the following three cases.

Case 1. $s = s_0$. For this case, if we choose $\tilde{s} = \alpha - 4$, then $L^3(\alpha - 4) = s_0 - \alpha - 1$, it follows from (8.46)₃ that

$$\|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s_0+1}(b\Omega_T)} \leq C\varepsilon^2\theta_n^{s_0-\alpha-1}\Delta_n;$$

Case 2. $s_0 + 1 \leq s \leq \alpha - 3$. For this case, if we choose $\tilde{s} = \alpha - 3$, then $L^3(\alpha - 3) = s_0 - \alpha$, which implies

$$\|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(b\Omega_T)} \leq C\varepsilon^2\theta_n^{s_0-\alpha}\Delta_n \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n;$$

Case 3. $s \geq \alpha - 2$. For this case, if we choose $\tilde{s} = \alpha - 5 \geq s_0$, then $L^3(\tilde{s}) = s_0 - 2 - \alpha$, which implies

$$\|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(b\Omega_T)} \leq C\varepsilon^2\theta_n^{(s+1-\tilde{s})_++L^3(\tilde{s})}\Delta_n \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n.$$

In summary, we obtain

$$\|S_{\theta_n}\tilde{e}_{n-1}\|_{H^{s+1}(b\Omega_T)} \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n \quad \text{for all } s \geq s_0. \quad (8.47)$$

From the assumption $s_1 \geq \alpha + 6$, it is possible to let $\tilde{s} = \alpha + 2$ in (8.46)₁, which yields

$$\|g_n\|_{H^{s+1}(b\Omega_T)} \leq C\varepsilon^2\theta_n^{s-\alpha-1}\Delta_n \quad \text{for all } s \geq s_0, \quad (8.48)$$

by using (8.46)₁ and (8.47) in (8.39)₂.

Furthermore, we have

$$\left\{ \begin{aligned} \|(S_{\theta_n} - S_{\theta_{n-1}})\bar{E}_{\pm,n-1}\|_{s,T} &\leq C\theta_n^{s-\tilde{s}-1}\Delta_n\|\bar{E}_{\pm,n-1}\|_{\tilde{s},T} \leq C\varepsilon^2\theta_n^{s-\tilde{s}-1}\Delta_n, \\ \|S_{\theta_n}\bar{e}_{\pm,n-1}\|_{s,T} &\leq C\theta_n^{(s-\tilde{s})_+}\|\bar{e}_{\pm,n-1}\|_{\tilde{s},T} \leq C\theta_n^{(s-\tilde{s})_++L^2(\tilde{s})}\varepsilon^2\Delta_n \end{aligned} \right. \quad (8.49)$$

for $\tilde{s} \in [s_0, s_1 - 3]$.

Estimate (8.49)₂ can be represented as the following three cases.

Case 1. $s = s_0$. For this case, if we choose $\tilde{s} = \alpha - 3$, then $L^2(\tilde{s}) = s_0 - \alpha - 1$, which implies from (8.49)₂ that

$$\|S_{\theta_n}\bar{e}_{\pm,n-1}\|_{s_0,T} \leq C\varepsilon^2\theta_n^{s_0-\alpha-1}\Delta_n;$$

Case 2. $s = s_0 + 1$. For this case, if we choose $\tilde{s} = \alpha - 2$, then $L^2(\tilde{s}) = s_0 - \alpha$, which implies

$$\|S_{\theta_n}\bar{e}_{\pm,n-1}\|_{s_0+1,T} \leq C\varepsilon^2\theta_n^{s_0-\alpha}\Delta_n;$$

Case 3. $s \geq s_0 + 2$. For this case, if we choose $\tilde{s} = \alpha - 4 \geq s_0$, then $L^2(\tilde{s}) = s_0 - 2 - \alpha$, which implies

$$\|S_{\theta_n} \bar{e}_{\pm, n-1}\|_{s, T} \leq C \varepsilon^2 \theta_n^{s-\alpha-1} \Delta_n.$$

In summary, we obtain

$$\|S_{\theta_n} \bar{e}_{\pm, n-1}\|_{s, T} \leq C \varepsilon^2 \theta_n^{s-\alpha-1} \Delta_n \quad \text{for all } s \geq s_0. \quad (8.50)$$

Letting $\tilde{s} = \alpha$ in (8.49)₁, we find

$$\|h_n^{\pm}\|_{s, T} \leq C \varepsilon^2 \theta_n^{s-\alpha-1} \Delta_n \quad \text{for all } s \geq s_0, \quad (8.51)$$

by using (8.50) in (8.39)₃.

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References

1. ALINHAC, S.: Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Commun. Partial Differ. Eqs.* **14**, 173–230 (1989)
2. ARTOLA, M., MAJDA, A.: Nonlinear development of instability in supersonic vortex sheets, I: The basic kink modes. *Phys. D.* **28**, 253–281 (1987)
3. ARTOLA, M., MAJDA, A.: Nonlinear development of instability in supersonic vortex sheets. II *SIAM J. Appl. Math.* **49**, 1310–1349 (1989)
4. ARTOLA, M., MAJDA, A.: Nonlinear kind modes for supersonic vortex sheets. *Phys. Fluids* **1A**, 583–596 (1989)
5. BEGELMAN, M.C., BLANDFORD, R.D., REES, M.J.: Theory of exagalactic radio sources. *Rev. Mod. Phys.* **56**, 255–351 (1984)
6. BLOKHIN, A., TRAKHININ, Y.: Stability of strong discontinuities in fluids and MHD, In: *Handbook of Mathematical Fluid Dynamics*. Vol. I, pp. 545–652, North-Holland, Amsterdam, 2002
7. CHEN, G.-Q., FELDMAN, M.: Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type. *J. Am. Math. Soc.* **3**, 461–494 (2003)
8. CHEN, S.: On the initial-boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary (in Chinese). *Chinese Anal. Math.* **3**, 222–232 (1982)
9. COULOMBEL, J.F., SECCHI, P.: Stability of compressible vortex sheet in two space dimensions. *Indiana Univ. Math. J.* **53**, 941–1012 (2004)
10. COULOMBEL, J.F., SECCHI, P.: Nonlinear compressible vortex sheets in two space dimensions. Preprint, 2006
11. COURANT, R., FRIEDRICHS, K.: *Supersonic Flow and Shock Waves*. Springer, New York 1948
12. FRANCHETEAU, J., MÉTIVIER, G.: Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Astérisque* **268**, 1–198 (2000)

13. FRIEDRICHS, K.O., LAX, P.D.: Boundary value problems for first order operators. *Comm. Pure Appl. Math.* **18**, 355–388 (1965)
14. GLIMM, J., MAJDA, A.: *Multidimensional Hyperbolic Problems and Computations*, The IMA Volumes in Mathematics and its Applications, Vol. 29. Springer, New York, 1991
15. GUÈS, O.: Problème mixte hyperbolique quasi-linéaire caractéristique. *Commun. Partial Differ. Eqs.* **15**, 595–645 (1990)
16. GUÈS, O., MÉTIVIER, G., WILLIAMS, M., ZUMBRUN, K.: Existence and stability of multidimensional shock fronts in the vanishing viscosity limit. *Arch. Ration. Mech. Anal.* **175**, 151–244 (2005)
17. HÖRMANDER, L.: The boundary problems of physical geodesy, *Arch. Ration. Mech. Anal.* **62**, 1–52 (1982)
18. KREISS, H.-O.: Initial boundary value problems of hyperbolic systems. *Comm. Pure Appl. Math.* **23**, 277–296 (1970)
19. LIONS, J.L., MAGENES, E.: *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1–3, Springer, New York, 1972
20. MAJDA, A.: *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*. Springer, New York, 1984
21. MÉTIVIER, G.: Stability of multidimensional shocks. In: *Advances in The Theory of Shock Waves*, Vol. 47, pp. 25–103. PNDEA, Birkhäuser, Boston, 2001
22. MILES, J.W.: On the reflection of sound at an interface of relative motion. *J. Acoust. Soc. Am.* **29**, 226–228 (1957)
23. MILES, J.W.: On the disturbed motion of a plane vortex sheet. *J. Fluid Mech.* **4**, 538–552 (1958)
24. SMART, L.L., NORMAN, M.L., WINKLER, K.A.: Shocks, interfaces, and patterns in supersonic jets. *Physica* **12D**, 83–106 (1984)
25. TRAKHININ, Y.: Existence of compressible current-vortex sheets: variable coefficients linear analysis. *Arch. Ration. Mech. Anal.* **177**, 331–366 (2005)
26. VAN DYKE, M.: *An Album of Fluid Motion*. Parabolic, Stanford, 1982
27. VON NEUMANN, J.: *Collected Works*, Vol. 5. Pergamon, New York, 1963

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