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EXISTENCE AND STABILITY OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH HADAMARD DERIVATIVE

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Abstract. In this paper, we study nonlinear fractional differential equations with Hadamard derivative and Ulam stability in the weighted space of continuous functions. Firstly, some new nonlinear integral inequalities with Hadamard type singular kernel are established, which can be used in the theory of certain classes of fractional differential equations. Secondly, some sufficient conditions for existence of solutions are given by using fixed point theorems via a prior estimation in the weighted space of the continuous functions. Meanwhile, a sufficient condition for nonexistence of blowing-up solutions is derived. Thirdly, four types of Ulam–Hyers stability definitions for fractional differential equations with Hadamard derivative are introduced and Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability results are presented. Finally, some examples and counterexamples on Ulam–Hyers stability are given.

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1. Introduction

Fractional differential equations have recently proved to be strong tools in the modelling of many physical phenomena. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details on fractional calculus theory, one can see the monographs of Diethelm [12], Kilbas et al. [16], Lakshmikantham et al. [17], Miller and Ross [24], Podlubny [27] and Tarasov [30].

Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions (see for example [1], [2], [4], [5], [8], [25], [33]–[39]), however, there are few works on fractional differential equations involving the Hadamard fractional derivative, even if it has been presented many years ago. Especially, the classical Cauchy problems and blowing-up solutions for fractional differential equations with Hadamard fractional derivative have not been studied extensively.

On the other hand, the stability problem of functional equations originated from a question of Ulam, posed in 1940, concerning the stability of group homomorphisms. In the next year, Hyers gave a partial affirmative answer to the question of Ulam in the context of Banach spaces, that was the first significant breakthrough and a step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. Particular, numerous monographs have appeared devoted to the data dependence in the theory of ordinary differential equations (see for example [3], [9], [10], [13], [26], [28]). We also remark that there are some special kinds of data dependence: Ulam–Hyers, Ulam–Hyers–Rassias, Ulam–Hyers–Bourgin, Aoki–Rassias in the theory of functional equations (see [7], [14], [15]). Although, there are some works on the stability of solutions for fractional differential equations (see for example [11], [18], [19]), there are a few works on Ulam–Hyers stability for fractional differential equations. It is worth remark that Wang et al. [31], [32] discuss four types Ulam stability of fractional differential equations with Caputo derivative and obtain some new and interesting stability results. Unfortunately, Ulam stability of fractional differential equations with Hadamard derivative is still not studied until now.

Motivated by [1], [16], [23], [29], [31], [32], we will study the Cauchy problems, blowing-up solutions and Ulam–Hyers stability for fractional differential with Hadamard derivative. By generalizing some new generalized nonlinear integral inequalities with Hadamard type singular kernel, some existence results of solutions will be given by utilizing fixed point methods and a prior estimation in the weighted space of the continuous functions via the nonlinear integral inequalities with Hadamard type singular kernel. Meanwhile, a sufficient condition for the nonexistence of blowing-up solutions will be presented. Further, we will give four types of Ulam stability definitions for a certain fractional differential equations with Hadamard derivative: Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability. We present the generalized Ulam–Hyers–Rassias stability results for a fractional differential equation with Hadamard derivative.

2. Preliminaries

Let $+\infty \geq T > a \geq -\infty$ and $C[a,T]$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ with the norm $\|f\|_C = \max\{|f(x)| : x \in [a,T]\}$. When $[a,T]$ is a finite interval and for $n-1 < \alpha \leq n$, $0 \leq \gamma < 1$, we denote the space $C^{\alpha}_{n-\alpha,\gamma}[a,T]$ by

$$C^{\alpha}_{n-\alpha,\gamma}[a,T] := \{f(x) \in C_{n-\alpha,\ln}[a,T] : {}_HD^{\alpha}_{a,x}f(x) \in C_{\gamma,\ln}[a,T]\},$$

where $C_{\gamma,\ln}[a,T]$ is the weighted space of the continuous functions f on the finite interval $[a,T]$, which is given by

$$C_{\gamma,\ln}[a,T] := \left\{ f(x) : \left(\ln\frac{x}{a}\right)^{\gamma} f(x) \in C[a,T] \right\}.$$

Obviously, $C_{\gamma,\ln}[a,T]$ is the Banach space with the norm

$$\|f\|_{C_{\gamma,\ln}} = \left\| \left(\ln\frac{x}{a}\right)^{\gamma} f(x) \right\|_C$$

and $C^n_{n-\alpha,\gamma}[a,T]$ is the Banach space with the norm

$$\|f\|_{C^n_{\gamma,\ln}} = \sum_{i=0}^{n-1} \left\| \left(x\frac{d}{dx}\right)^i f \right\|_C + \left\| \left(x\frac{d}{dx}\right)^n f \right\|_{C_{\gamma,\ln}}.$$

Moreover, $C_{0,\ln}[a,T] := C[a,T]$.

For integrable functions $h\colon [a,T] \to \mathbb{R}$, define the norm

$$\|h\|_{L^p([a,T])} = \left(\int_a^T |h(t)|^p \, dt \right)^{1/p}, \quad 1 < p < \infty.$$

We denote $L^p([a,T],\mathbb{R})$ the Banach space of all the p-th power integrable functions $h : [a,T] \to \mathbb{R}$ with $\|h\|_{L^p([a,T])} < \infty$.

We need the basic definitions of fractional Hadamard derivative, which are widely used in the sequel.

Definition 2.1 ([16]). The *Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(x)$*, for all $x > 1$, is defined by

$${}_HD^{-\alpha}_{1,x}f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln\frac{x}{t})^{\alpha-1} f(t) \, \frac{dt}{t},$$

where $\Gamma(\,\cdot\,)$ is the Euler Gamma function.

Definition 2.2 ([16]). The *Hadamard derivative of order* $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$ *of function* $f(x)$ is given as follows

$$_H D_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx} \right)^n \int_1^x \left(\ln \frac{x}{t} \right)^{n-\alpha-1} f(t) \, \frac{dt}{t}.$$

Let G be an open set in $\mathbb{R}$ and $f\colon (a,T] \times G \to \mathbb{R}$ be a function such that $f(x,y) \in C_{\gamma,\ln}[a,T]$ for any $y \in G$. Consider the following fractional Cauchy problem:

$$\begin{cases} _H D_{a,x}^{\alpha} y(x) = f(x, y(x)), & n-1 < \alpha \leq n, \ x \in (a,T], \\ _H D_{a,x}^{\alpha-k} y(a+) = b_k, & b_k \in \mathbb{R}, \ k = 1, \ldots, n, \ n = -[-\alpha], \end{cases} \tag{2.1}$$

where $_H D_{a,x}^{\alpha-k}(a+)$ means that the limit is taken at all points of the rightsided neighbourhood $(a, a+\varepsilon)$ $(\varepsilon > 0)$ of a.

Let us define what we mean by a solution of the fractional Cauchy problem (2.1).

Definition 2.3. A function $y \in C_{n-\alpha,\gamma}^{\alpha}[a,T]$ is said to be a *solution of the fractional Cauchy problem* (2.1) if y satisfies the equation $_H D_{a,x}^{\alpha} y(x) = f(x, y(x))$ for each $x \in (a,T]$, and the conditions $_H D_{a,x}^{\alpha-k} y(a+) = b_k$, $k = 1, \ldots, n$, $n = -[-\alpha]$.

Lemma 2.4 ([16, Theorem 3.28]). *Let* $\alpha > 0, n = -[-\alpha]$ *and* $0 \leq \gamma < 1$. *Let* G *be an open set in* $\mathbb{R}$ *and let* $f\colon (a,T] \times G \to \mathbb{R}$ *be a function such that* $f(x,y) \in C_{\gamma,\ln}[a,T]$ *for any* $y \in G$. *A function* $y \in C_{n-\alpha,\ln}[a,T]$ *is a solution of the fractional integral equation*

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{x}{a} \right)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t, y(t)) \, \frac{dt}{t},$$

$(x > a > 0)$, *if and only if* y *is a solution of the following fractional Cauchy problem* (2.1). *In particular, a function* $y \in C_{1-\alpha,\gamma}^{\alpha}[a,T]$ *is a solution of fractional Cauchy problem*:

$$\begin{cases} _H D_{a,t}^{\alpha} y(x) = f(x, y(x)), & 0 < \alpha < 1, \ x \in (a,T], \\ _H D_{a,t}^{\alpha-1}(a+) = b_1, & b_1 \in \mathbb{R}, \end{cases}$$

if and only if y *is a solution of the following equation*:

$$y(x) = \frac{b_1}{\Gamma(\alpha)} \left(\ln \frac{x}{a} \right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t, y(t)) \, \frac{dt}{t}, \quad x > a > 0.$$

3. Nonlinear integral inequality with Hadamard type singular kernel

In this section, we give an important nonlinear integral inequality with Hadamard type singular kernel which can be used to deal with fractional differential equations with Hadamard derivative.

LEMMA 3.1. *If* $0 < \alpha < 1$, $1 < p < 1/(1-\alpha)$ *then*

$$\int_1^t \left(\ln \frac{t}{s}\right)^{p(\alpha-1)} s^{-p}\, ds \le \frac{(\ln t)^{p(\alpha-1)+1}}{p(\alpha-1)+1}.$$

PROOF. Let $\tau = \ln t - \ln s$. Then $s = te^{-\tau}$, $ds = -te^{-\tau}\, d\tau$, $s^{-p} = t^{-p}e^{p\tau}$ and we have

$$t\int_0^{\ln t} \tau^{p(\alpha-1)}(t^{-p}e^{p\tau}e^{-\tau})\, d\tau = t^{1-p}\int_0^{\ln t} \tau^{p(\alpha-1)}e^{(p-1)\tau}\, d\tau \le \frac{(\ln t)^{p(\alpha-1)+1}}{p(\alpha-1)+1}.$$

The proof is completed. □

The following lemma is proved in [6]. For more other nonlinear singular integral inequalities, one can see [20]–[22].

LEMMA 3.2. *Let* $a(t)$, $b(t)$, $k(t)$, $\psi(t)$ *be nonnegative, continuous functions on the interval* $J = (a, T)$ $(a < T \le \infty)$, $\omega\colon (0,\infty) \to \mathbb{R}$ *be a continuous, nonnegative and nondecreasing function,* $\omega(0) = 0$, $\omega(u) > 0$ *for* $u > 0$ *and let* $A(t) = \max_{0\le s\le t}\{a(s)\}$, $B(t) = \max_{0\le s\le t}\{b(s)\}$. *Assume that*

$$\psi(t) \le a(t) + b(t)\int_a^t k(s)\omega(\psi(s))\, ds, \quad t \in J.$$

Then

$$\psi(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_a^t k(s)\, ds\right), \quad t \in (a, T_1),$$

where

$$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}, \quad v \ge v_0,$$

and Ω^{-1} *is the inverse of* Ω *and* $T_1 > a$ *is such that*

$$\Omega(A(t)) + B(t)\int_a^t k(s)\, ds \in D(\Omega^{-1}) \quad \text{for all } t \in (a, T_1).$$

Now, we can give the following new nonlinear integral inequality with Hadamard type singular kernel.

LEMMA 3.3. *Let* $0 < \alpha < 1$, $1 < p < 1/(1-\alpha)$ *and* $a(t)$, $b(t)$, $F(t)$, $u(t)$ *be nonnegative, continuous functions on the interval* $J = (1, T)$ $(1 < T \le \infty)$, $H\colon (0,\infty) \to \mathbb{R}$ *be a continuous, nonnegative and nondecreasing function and* $H(0) = 0$, $H(u) > 0$ *for* $u > 0$. *Assume that*

$$u(t) \le a(t) + b(t)\int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} s^{-1}F(s)H(u(s))\, ds, \quad t \in J. \tag{3.1}$$

Then

$$u(t) \leq \bigg(\Lambda^{-1}\bigg[\Lambda(A(t)+B(t)\int_1^t [F(s)]^q\,ds\bigg]\bigg)^{1/q}, \quad t \in (1,T_1),$$

where $q=(p-1)/p$,

(3.2)

$$A(t)=2^{q-1}\max_{0\leq s\leq t}\{[a(s)]^q\}, \quad B(t)=2^{q-1}\frac{\max_{0\leq s\leq t}\{[b(s)]^q\}}{p(\alpha-1)+1}(\ln t)^{[p(\alpha-1)+1]q/p},$$

$$\Lambda(v)=\int_{v_0}^v \frac{dz}{[H(z^{1/q})]^q}, \quad v_0>0,$$

and Λ^{-1} *is the inverse of* Λ *and* $T_1>1$ *is such that*

$$\Lambda(A(t))+B(t)\int_1^t [F(s)]^q\,ds \in D(\Lambda^{-1}), \quad \textit{for all } t\in(1,T_1).$$

Proof. Using the Hölder inequality, it comes from (3.1) that

$$u(t)\leq a(t)+b(t)\bigg(\int_1^t (\ln t-\ln s)^{p(\alpha-1)}s^{-p}\,ds\bigg)^{1/p}\bigg(\int_1^t F(s)^q[H(u(s)^q)]\,ds\bigg)^{1/q}.$$

Using the elementary inequality $(a+b)^q\leq 2^{q-1}(a^q+b^q)$, $a,b\geq 0$ and Lemma 3.1 we obtain the inequality

$$\begin{aligned}(3.3)\qquad u(t)^q &\leq \bigg[a(t)+b(t)\bigg(\int_1^t (\ln t-\ln s)^{p(\alpha-1)}s^{-p}\,ds\bigg)^{1/p}\\ &\qquad\cdot\bigg(\int_1^t [F(s)]^q[H(u(s))]^q\,ds\bigg)^{1/q}\bigg]^q\\ &\leq 2^{q-1}\bigg[[a(t)]^q+[b(t)]^q\bigg(\int_1^t (\ln t-\ln s)^{p(\alpha-1)}s^{-p}\,ds\bigg)^{q/p}\\ &\qquad\cdot\bigg(\int_1^t [F(s)]^q[H(u(s))]^q\,ds\bigg)\bigg]\\ &\leq A(t)+B(t)\int_1^t [F(s)]^q[H(u(s))]^q\,ds,\end{aligned}$$

where $A(t)$ and $B(t)$ are as in (3.2).

Denote $v(t)=(u(t))^q$, then $u(t)=(v(t))^{1/q}$. So the inequality (3.3) can be rewritten as

$$v(t)\leq A(t)+B(t)\int_1^t [F(s)]^q[H(v(s))^{1/q}]^q\,ds.$$

By Lemma 3.2, we have

$$\Lambda(v(t))\leq \Lambda(A(t))+B(t)\int_1^t [F(s)]^q\,ds$$

and this yields the assertion of the lemma. □

Corollary 3.4. *Suppose $1 > \alpha > 0$, $\overline{a} > 0$ and $\overline{b} > 0$, and suppose $u(t)$ is nonnegative and locally integrable on $[1, +\infty)$ with*

$$u(t) \leq \overline{a} + \overline{b} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} u(s) \, \frac{ds}{s}, \quad t \in [1, +\infty).$$

Then

$$u(t) \leq \overline{a} + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\overline{b}\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\ln \frac{t}{s} \right)^{n\alpha - 1} \overline{a} \right] \frac{ds}{s}, \quad t \in [1, +\infty).$$

Remark 3.5. Under the assumptions of Lemma 3.3, we restrict $a(\,\cdot\,) = \overline{a}$, $b(\,\cdot\,) = \overline{b}$, $F(\,\cdot\,) = 1$ and $H(u) = u$. Then by Corollary 3.4, one can obtain

$$u(t) \leq \overline{a} E_{\alpha,1}(\overline{b}\Gamma(\alpha)(\ln t)^{\alpha}),$$

where $E_{\alpha,1}$ is the Mittag–Leffler function defined by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}.$$

4. Existence of solutions

In this section, we consider the following Cauchy problem for fractional differential equations with Hadamard derivative

$$\begin{cases} {}_H D^{\alpha}_{1,x} y(x) = f(x, y(x)), & 0 < \alpha < 1, \ x \in (1, b], \ b < +\infty, \\ {}_H D^{\alpha - 1}_{1,x} y(1+) = b_1, & b_1 \in \mathbb{R}. \end{cases} \tag{4.1}$$

Before stating and proving the main results in this section, we introduce the following hypotheses:

(H$_1$) $f\colon [1, b] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(x, y) \in C_{\gamma,\ln}[1, b]$ with $\gamma < \alpha$ for any $x \in \mathbb{R}$;

(H$_2$) There exists a positive constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \quad \text{for each } x \in (1, b], \text{ and all } y_1, y_2 \in \mathbb{R};$$

(H$_3$) There exists a function $h(\,\cdot\,) \in L^q([1, b], \mathbb{R})$ where $q = p/(p - 1)$ and $1 < p < 1/(1 - \alpha)$ such that

$$|f(x, y)| \leq h(x), \quad \text{for each } x \in (1, b], \text{ and all } y \in \mathbb{R}.$$

Our first result is based on Banach contraction principle.

Theorem 4.1. *Assume that* (H_1)–(H_3) *hold. If*

$$\Phi_x = \frac{L(\ln x)^{\alpha-\gamma}}{(\alpha-\gamma)\Gamma(\alpha)} < \frac{1}{(\ln b)^{\gamma}}, \quad \text{for all } x \in (1,b], \tag{4.2}$$

then the Cauchy problem (4.1) *has a unique solution on* $[1,b]$.

Proof. From Lemma 2.4, the Cauchy problem (4.1) is equivalent to the following fractional integral equation

$$y(x) = \frac{b_1}{\Gamma(\alpha)}\left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t}, \quad x > 1 > 0.$$

Let

$$r \geq \frac{(\ln b)^{\gamma}}{\Gamma(\alpha)}\left[|b_1|(\ln b)^{\alpha-1} + \frac{(\ln b)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\,\|h\|_{L^q([1,b])}\right]. \tag{4.3}$$

Now we define an operator F on $B_r := \{y \in C_{\gamma,\ln}[1,b] : \|y\|_{C_{\gamma,\ln}} \leq r\}$ as follows:

$$(Fy)(x) = \frac{b_1}{\Gamma(\alpha)}\left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t}, \tag{4.4}$$

for $x > 1 > 0$. It is obvious that F is well defined due to (H_1). Therefore, the existence of a solution of the Cauchy problem (4.1) is equivalent to that the operator F has a fixed point on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy \in B_r$ for every $y \in B_r$.

For every $y \in B_r$ and any $\delta > 0$, by (H_3), Hölder inequality and Lemma 3.1, we get

$$\begin{aligned}
|(Fy)(x+\delta) - (Fy)(x)| &= \frac{|b_1|}{\Gamma(\alpha)}\left|\left(\ln\frac{x+\delta}{1}\right)^{\alpha-1} - \left(\ln\frac{x}{1}\right)^{\alpha-1}\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_1^x \left(\ln\frac{x+\delta}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t} - \int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t}\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_x^{x+\delta} \left(\ln\frac{x+\delta}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t}\right| \\
&\leq \frac{|b_1|}{\Gamma(\alpha)}\left(\frac{1}{\ln x} - \frac{1}{\ln(x+\delta)}\right)^{1-\alpha} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^x \left[\left(\ln\frac{x}{t}\right)^{\alpha-1} - \ln\left(\frac{x+\delta}{t}\right)^{\alpha-1}\right] h(t)\,\frac{dt}{t} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_x^{x+\delta} \left(\ln\frac{x+\delta}{t}\right)^{\alpha-1} h(t)\,\frac{dt}{t} \\
&\leq \frac{|b_1|}{\Gamma(\alpha)}\left(\frac{1}{\ln x} - \frac{1}{\ln(x+\delta)}\right)^{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\|h\|_{L^q([1,b])}}{\Gamma(\alpha)} \frac{(\ln(x+\delta))^{p(\alpha-1)+1} - (\ln x)^{p(\alpha-1)+1}}{p(\alpha-1)+1} \\
&+ \frac{\|h\|_{L^q([1,b])}}{\Gamma(\alpha)} \frac{\left(\ln \frac{x+\delta}{x}\right)^{p(\alpha-1)+1}}{p(\alpha-1)+1} \\
\le{}& \frac{|b_1|}{\Gamma(\alpha)} \left(\frac{1}{\ln x} - \frac{1}{\ln(x+\delta)} \right)^{1-\alpha} + \frac{2\|h\|_{L^q([1,b])}}{\Gamma(\alpha)} \frac{\left(\ln \frac{x+\delta}{x}\right)^{p(\alpha-1)+1}}{p(\alpha-1)+1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(\ln b)^\gamma |(Fy)(x+\delta) - (Fy)(x)| \\
&\le (\ln b)^\gamma \left[\frac{|b_1|}{\Gamma(\alpha)} \left(\frac{1}{\ln x} - \frac{1}{\ln(x+\delta)} \right)^{1-\alpha} + \frac{2\|h\|_{L^q([1,b])}}{\Gamma(\alpha)} \frac{\left(\ln \frac{x+\delta}{x}\right)^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right].
\end{aligned}
$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore, F is continuous on $[1,b]$. Further, $Fy \in C_{\gamma,\ln}[1,b]$.

Moreover, for $y \in B_r$ and all $t \in [1,b]$, by (H_3), Hölder inequality and Lemma 3.1 again, we obtain

$$
\begin{aligned}
|(Fy)(x)| &\le \frac{|b_1|}{\Gamma(\alpha)} \left(\ln \frac{x}{1} \right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t} \right)^{\alpha-1} h(t) \, \frac{dt}{t} \\
&\le \frac{1}{\Gamma(\alpha)} \left(|b_1| (\ln b)^{\alpha-1} + \frac{(\ln b)^{p(\alpha-1)+1}}{p(\alpha-1)+1} \|h\|_{L^q([1,b])} \right),
\end{aligned}
$$

which implies that $\|Fy\|_{C_{\gamma,\ln}} \le r$ due to (4.3).

Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$, i.e. $F\colon B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $z, y \in B_r$ and any $t \in J$, using (H_2) and Hölder inequality, we get

$$
\begin{aligned}
|(Fz)(x) - (Fy)(x)| &\le \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t} \right)^{\alpha-1} |f(t,z(t)) - f(t,y(t))| \, \frac{dt}{t} \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t} \right)^{\alpha-\gamma-1} \left(\ln \frac{x}{t} \right)^{\gamma} |f(t,z(t)) - f(t,y(t))| \, \frac{dt}{t} \\
&\le \frac{L}{\Gamma(\alpha)} \left(\int_1^x \left(\ln \frac{x}{t} \right)^{\alpha-\gamma-1} \frac{dt}{t} \right) \|z-y\|_{C_{\gamma,\ln}} \\
&\le \frac{L(\ln x)^{\alpha-\gamma}}{(\alpha-\gamma)\Gamma(\alpha)} \|z-y\|_{C_{\gamma,\ln}}.
\end{aligned}
$$

So we obtain

$$
\|Fz - Fy\|_{C_{\gamma,\ln}} \le (\ln b)^\gamma \Phi_x \|z-y\|_{C_{\gamma,\ln}},
$$

where Φ_x is as in (4.2). Thus, F is a contraction due to the condition (4.2). By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the Cauchy problem (4.1). $\square$

Our second result is based on the well known Schaefer's fixed point theorem.

We use the following linear growth condition to replace (H_3):

(H_3') There exists a constant $N > 0$ such that

$$|f(x,y)| \le N|y|, \quad \text{for each } x \in [1,b] \text{ and all } y \in \mathbb{R}.$$

THEOREM 4.2. *Assume that* (H_1) *and* (H_3') *hold. Then the Cauchy problem* (4.1) *has at least one solution on* $[1,b]$.

PROOF. Transform the Cauchy problem (4.1) into a fixed point problem. Consider the operator $F\colon C_{\gamma,\ln}[1,b] \to C_{\gamma,\ln}[1,b]$ defined as (4.4).

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C_{\gamma,\ln}[1,b]$. Then for each $x \in [1,b]$, we have

$$\begin{aligned}
|(Fy_n)(x) - (Fy)(x)| &\le \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} |f(t,y_n(t)) - f(t,y(t))| \,\frac{dt}{t} \\
&\le \frac{\|f(\cdot\,,y_n(\cdot)) - f(\cdot\,,y(\cdot))\|_{C_{\gamma,\ln}}}{\Gamma(\alpha)} \left(\int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-\gamma-1} \frac{dt}{t}\right) \\
&\le \frac{(\ln b)^{\alpha-\gamma}}{(\alpha-\gamma)\Gamma(\alpha)} \|f(\cdot\,,y_n(\cdot)) - f(\cdot\,,y(\cdot))\|_{C_{\gamma,\ln}}.
\end{aligned}$$

Since $f \in C_{\gamma,\ln}[1,b]$, we have

$$\|Fy_n - Fy\|_{C_{\gamma,\ln}} \le \frac{(\ln b)^{\alpha}}{(\alpha-\gamma)\Gamma(\alpha)} \|f(\cdot\,,y_n(\cdot)) - f(\cdot\,,y(\cdot))\|_{C_{\gamma,\ln}} \to 0 \quad \text{as } n \to \infty.$$

Step 2. F maps bounded sets into bounded sets in $C_{\gamma,\ln}[1,b]$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C_{\gamma,\ln}[1,b] : \|y\|_{C_{\gamma,\ln}} \le \eta^*\}$, we have $\|Fy\|_{C_{\gamma,\ln}} \le \ell$.

For each $t \in [1,b]$, we get

$$\begin{aligned}
|(Fy)(x)| &\le \frac{|b_1|}{\Gamma(\alpha)} \left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{N}{\Gamma(\alpha)} \int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} |y(t)| \,\frac{dt}{t} \\
&\le \frac{|b_1|(\ln b)^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\eta^*(\ln b)^{\alpha-\gamma}}{(\alpha-\gamma)\Gamma(\alpha)},
\end{aligned}$$

which implies that

$$\|Fy\|_{C_{\gamma,\ln}} \le (\ln b)^{\gamma} \left[\frac{|b_1|(\ln b)^{\alpha-1}}{\Gamma(\alpha)} + \frac{N\eta^*(\ln b)^{\alpha-\gamma}}{(\alpha-\gamma)\Gamma(\alpha)}\right] := \ell.$$

Step 3. F is equicontinuous of $C_{\gamma,\ln}[1,b]$.

Let $0 \le x_1 < x_2 \le b$, $y \in B_{\eta^*}$. Using (H_3') and noting that $|y(t)| \le \eta^*$, we have:

$$\begin{aligned}
|(Fy)(x_2) - (Fy)(x_1)| &\le \frac{|b_1|}{\Gamma(\alpha)}\left(\frac{1}{\ln x_1} - \frac{1}{\ln x_2}\right)^{1-\alpha} \\
&\quad + \frac{N}{\Gamma(\alpha)}\int_1^{x_1}\left[\left(\ln\frac{x_1}{t}\right)^{\alpha-1} - \left(\ln\frac{x_2}{t}\right)^{\alpha-1}\right]|y(t)|\,\frac{dt}{t} \\
&\quad + \frac{N}{\Gamma(\alpha)}\int_x^{x+\delta}\left(\ln\frac{x+\delta}{t}\right)^{\alpha-1}|y(t)|\,\frac{dt}{t} \\
&\le \frac{|b_1|}{\Gamma(\alpha)}\left(\frac{1}{\ln x_1} - \frac{1}{\ln x_2}\right)^{1-\alpha} \\
&\quad + \frac{N\eta^*(\ln b)^{-\gamma}}{\Gamma(\alpha)}\int_1^{x_1}\left[\left(\ln\frac{x_1}{t}\right)^{\alpha-1} - \left(\ln\frac{x_2}{t}\right)^{\alpha-1}\right]\frac{dt}{t} \\
&\quad + \frac{N\eta^*(\ln b)^{-\gamma}}{\Gamma(\alpha)}\int_{x_1}^{x_2}(\ln\frac{x_2}{t})^{\alpha-1}\frac{dt}{t} \\
&\le \frac{|b_1|}{\Gamma(\alpha)}\left(\frac{1}{\ln x_1} - \frac{1}{\ln x_2}\right)^{1-\alpha} + \frac{2N\eta^*(\ln b)^{-\gamma}}{\Gamma(\alpha+1)}\left(\ln\frac{x_2}{x_1}\right)^{\alpha}.
\end{aligned}$$

As $x_2 \to x_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous.

As a consequence of Steps 1–3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{y \in C_{\gamma,\ln}[1,b] : y = \lambda F y, \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $y \in E(F)$, then $y = \lambda F y$ for some $\lambda \in (0,1)$. Thus, for each $t \in [1,b]$, we have

$$y(x) = \lambda\left(\frac{b_1}{\Gamma(\alpha)}\left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^x\left(\ln\frac{x}{t}\right)^{\alpha-1} f(t,y(t))\,\frac{dt}{t}\right).$$

For each $t \in [1,b]$, we have

$$|y(x)| \le \frac{|b_1|}{\Gamma(\alpha)}\left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{N}{\Gamma(\alpha)}\int_1^x\left(\ln\frac{x}{t}\right)^{\alpha-1}|y(t)|\,\frac{dt}{t}.$$

By Corollary 3.4, we have

$$|y(x)| \le \frac{|b_1|}{\Gamma(\alpha)}(\ln x)^{\alpha-1}E_{\alpha,1}(N(\ln x)^{\alpha}), \quad x \in [1,b].$$

Thus for every $t \in [0,b]$, we have

$$\|y\|_{C_{\gamma,\ln}} \le \frac{(\ln b)^{\gamma}|b_1|(\ln x)^{\alpha-1}E_{\alpha,1}(N(\ln x)^{\alpha})}{\Gamma(\alpha)}, \quad x \in [1,b].$$

This shows that the set $E(F)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the Cauchy problem (4.1). $\square$

In the following theorem we apply the nonlinear alternative of Leray–Schauder type in which the condition (H_3') is weakened to the nonlinear growth condition.

(H_3'') There exist a nonnegative, continuous functions ϕ_f and a continuous, nonnegative and nondecreasing function ψ with $\psi(0) = 0$, $\psi(z) > 0$ for $z > 0$ such that

$$|f(x, y)| \le \phi_f(x)\psi(|y|), \quad \text{for each } x \in [1, b] \text{ and all } y \in \mathbb{R}.$$

THEOREM 4.3. *Assume that* (H_1) *and* (H_3'') *hold. Then the Cauchy problem* (4.1) *has at least one solution on* $[1, b]$.

PROOF. Consider the operator F defined in Theorem 4.2. It can be easily shown that F is continuous and completely continuous. Repeating the same process in Step 4 in Theorem 4.2, using (H_3'') we have

$$|y(x)| \le \frac{|b_1|}{\Gamma(\alpha)}\left(\ln\frac{x}{1}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} \phi_f(t)\psi(|y(t)|)\,\frac{dt}{t}.$$

By Lemma 3.3, we have for each $x \in [1, b]$, there exists a $M^* > 0$ such that $\|y\|_C \le M^*$, which implies that $\|y\|_{C_{\gamma,\ln}} \le (\ln b)^\gamma M^*$. Let

$$U = \{y \in C_{\gamma,\ln}[1, b] : \|y\|_{C_{\gamma,\ln}} < (\ln b)^\gamma M^* + 1\}.$$

The operator $F\colon \overline{U} \to C_{\gamma,\ln}[1, b]$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda F(y)$, $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that F has a fixed point $y \in \overline{U}$, which implies that the Cauchy problem (4.1) has at least one solution $y \in C_{\gamma,\ln}[1, b]$. $\square$

5. Nonexistence of blowing-up solutions

In this section, a sufficient for the nonexistence of blowing-up solutions of the Cauchy problem (4.1) will be proved, where by a blowing-up solution of this equation we mean a solution $y(x)$ for which there is a point $1 < \tau < +\infty$ such that it is defined on the interval $[1, \tau)$ and $\lim_{x\to\tau^-} |y(x)| = +\infty$.

THEOREM 5.1. *Let* $0 < \alpha < 1$, $1 < p < 1/(1 - \alpha)$ *and* $q = (p - 1)/p$. *Assume that* $f\colon [1, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is continuous function and there are a continuous* *nonnegative function* $\mathcal{R}\colon [1,+\infty) \times \mathbb{R} \to \mathbb{R}$ *and a continuous, nonnegative, nondecreasing function* $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\omega(u) > 0$ *for* $u > 0$ *such that the following two conditions hold*:

$$\text{(K}_1\text{)} \qquad |f(x,u)| \le \mathcal{R}(x)\omega(u), \quad (x,u) \in [1,+\infty) \times \mathbb{R};$$

$$\text{(K}_2\text{)} \qquad \int_{v_0}^{\infty} \frac{\sigma^{q-1}}{[\omega(\sigma)]^q}\, d\sigma = \infty, \quad v_0 > 0.$$

Then the Cauchy problem (4.1) *has no blowing-up solution.*

Proof. Suppose that $y(x)$ is a solution of the Cauchy problem (4.1) defined on the interval $[1,\tau)$, where $1 < \tau < +\infty$ and $\lim\limits_{x\to\tau^-} |y(x)| = +\infty$.

From Lemma 2.4 and the condition (K_1) it follows that for $x \in [1,\tau)$ we have:

$$u(x) \le \frac{b_1}{\Gamma(\alpha)} \left(\ln \frac{x}{1} \right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t} \right)^{\alpha-1} \frac{1}{t}\, \mathcal{R}(t)\omega(u(t))\, dt,$$

where $u(x) := |y(x)|$. This inequality has the form (3.1) with

$$a(x) = \frac{b_1}{\Gamma(\alpha)} \left(\ln \frac{x}{1} \right)^{\alpha-1}, \quad F(x) = \mathcal{R}(x) \quad \text{and} \quad H(u) = \omega(u).$$

Therefore from Lemma 3.3 we obtain the inequality

$$\Lambda([u(x)]^q) \le \Lambda(A(x)) + B(x) \int_1^x [F(s)]^q\, ds, \quad x \in (1,\tau), \tag{5.1}$$

where $q = (p-1)/p$,

$$A(t) = 2^{q-1} \max_{0\le s\le t} \{[a(s)]^q\}, \quad B(t) = 2^{q-1}\, \frac{\max\limits_{0\le s\le t} \{[b(s)]^q\}}{p(\alpha-1)+1}\, (\ln t)^{[p(\alpha-1)+1]q/p}, \tag{5.2}$$

and

$$\Lambda(v) = \int_{a_0}^{v} \frac{1}{[\omega(z^{1/q})]^q}\, dz, \quad v_0 = a_0^q.$$

Obviously, the limit of the right-hand side of the inequality (5.1) as $x \to \tau^-$ is finite, however, the condition (K_2) yields that

$$\lim_{x\to\tau^-} \Lambda([u(x)]^q) = q \int_{v_0}^{\infty} \frac{\sigma^{q-1}}{[\omega(\sigma)]^q}\, d\sigma = +\infty$$

and this is the contradiction. □

6. Ulam–Hyers stability results

Let $0 < \alpha < 1$, $1 < b \leq +\infty$, ε is a positive real number, $f\colon [1,b) \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(t,x) \in C_{\gamma,\ln}[1,b)$ with $\gamma < \alpha$ for any $x \in \mathbb{R}$ and $\varphi\colon [1,b) \to \mathbb{R}^+$ be a continuous function. We consider the following fractional differential equation:

$$ {}_H D^{\alpha}_{1,t} x(t) = f(t, x(t)), \quad t \in (1,b), \tag{6.1} $$

and the following fractional differential inequations:

$$ |{}_H D^{\alpha}_{1,t} y(t) - f(t, y(t))| \leq \varepsilon, \qquad t \in (1,b), \tag{6.2} $$

$$ |{}_H D^{\alpha}_{1,t} y(t) - f(t, y(t))| \leq \varphi(t), \quad t \in (1,b), \tag{6.3} $$

$$ |{}_H D^{\alpha}_{1,t} y(t) - f(t, y(t))| \leq \varepsilon\varphi(t), \quad t \in (1,b). \tag{6.4} $$

Definition 6.1. The equation (6.1) is *Ulam–Hyers stable* if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the inequation (6.2) there exists a solution $x \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the equation (6.1) with

$$ |y(t) - x(t)| \leq c_f \varepsilon, \quad t \in (1,b). $$

Definition 6.2. The equation (6.1) is *generalized Ulam–Hyers stable* if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each solution $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the inequation (6.2) there exists a solution $x \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the equation (6.1) with

$$ |y(t) - x(t)| \leq \theta_f(\varepsilon), \quad t \in (1,b). $$

Definition 6.3. The equation (6.1) is *Ulam–Hyers–Rassias stable with respect to* φ if there exists $c_{f,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the inequation (6.2) there exists a solution $x \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the equation (6.1) with

$$ |y(t) - x(t)| \leq c_{f,\varphi} \varepsilon \varphi(t), \quad t \in (1,b). $$

Definition 6.4. The equation (6.1) is *generalized Ulam-Hyers-Rassias stable with respect to* φ if there exists $c_{f,\varphi} > 0$ such that for each solution $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the inequation (6.3) there exists a solution $x \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ of the equation (6.1) with

$$ |y(t) - x(t)| \leq c_{f,\varphi} \varphi(t), \quad t \in (1,b). $$

Remark 6.5. A function $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ is a solution of (6.2) if and only if there exists a function $g \in C_{1-\alpha,\ln}[1,b)$ (which depend on y) such that

(a) $|g(t)| \leq \varepsilon$, $t \in (1,b)$;
(b) ${}_H D^{\alpha}_{1,t} y(t) = f(t, y(t)) + g(t)$, $t \in (1,b)$.

One can have similar remarks for the inequations (6.3) and (6.4).

So, the Ulam stabilities of fractional differential equations with Hadamard derivative are some special types of data dependence of the solutions of fractional differential equations with Hadamard derivative.

Remark 6.6. Let $0 < \alpha < 1$, if $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ is a solution of the inequation (6.2) then y is a solution of the following integral inequation

$$\left| y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s} \right| \le \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\varepsilon,$$

for $t \in (1,b)$, where ${}_H D^{\alpha-1}_{1,t} y(1+) = y_1$.

Indeed, by Remark 6.5 we have that

$${}_H D^{\alpha}_{1,t} y(t) = f(t,y(t)) + g(t), \quad t \in (1,b).$$

Then

$$y(t) - y(1+) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)\,\frac{ds}{s},$$

for $t \in (1,b)$. By ${}_H D^{\alpha-1}_{1,t} y(1+) = y_1$, we have

$$\begin{aligned} y(1+) &= \frac{1}{\Gamma(\alpha-1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-2} y_1\,\frac{ds}{s} \\ &= \frac{-y_1}{\Gamma(\alpha-1)} \int_1^t (\ln t - \ln s)^{\alpha-2} d(\ln t - \ln s) = \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}. \end{aligned}$$

This implies that

$$\begin{aligned} y(t) = \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)\,\frac{ds}{s}, \quad t \in (1,b). \end{aligned}$$

From this it follows that

$$\begin{aligned} &\left| y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s} \right| \\ &\quad = \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)\,\frac{ds}{s} \right| \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |g(s)|\,\frac{ds}{s} \\ &\quad \le \frac{\varepsilon}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \le \frac{(\ln t)^{\alpha}\varepsilon}{\Gamma(\alpha+1)}. \end{aligned}$$

Meanwhile, we have the following remarks for the solutions of the fractional inequations (6.3) and (6.4).

Remark 6.7. Let $0 < \alpha < 1$, if $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ is a solution of the inequation (6.3) then y is a solution of the following integral inequation:

$$\begin{aligned}\bigg|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s}\bigg| \\ \leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\varphi(s)\,\frac{ds}{s}, \quad t \in (1,b).\end{aligned}$$

Remark 6.8. Let $0 < \alpha < 1$, if $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b)$ is a solution of the inequation (6.4) then y is a solution of the following integral inequation:

$$\begin{aligned}\bigg|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s}\bigg| \\ \leq \frac{\varepsilon}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\varphi(s)\,\frac{ds}{s}, \quad t \in (1,b).\end{aligned}$$

Consider (6.1) and (6.2) in the case $b < +\infty$. We have the following generalized Ulam–Hyers stability results.

Theorem 6.9. *In the conditions* (H_1) *and* (H_2), *the equation* (6.1) *is Ulam–Hyers stable.*

Proof. Let $y \in C^{\alpha}_{1-\alpha,\gamma}[1,b]$ be a solution of the inequation (6.2). Denote by x the unique solution of the Cauchy problem

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t,x(t)), & \text{for all } t \in (1,b], \\ {}_{H}D^{\alpha-1}_{1,t}x(t)|_{t=1+} = {}_{H}D^{\alpha-1}_{1,t}y(t)|_{t=1+} = y_1. \end{cases} \tag{6.5}$$

We have that

$$x(t) = \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\,\frac{ds}{s}, \quad t \in (1,b].$$

By differential inequation (6.2), we have:

$$\begin{aligned}\bigg|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s}\bigg| \\ \leq \frac{\varepsilon}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \leq \frac{(\ln t)^{\alpha}\varepsilon}{\Gamma(\alpha+1)},\end{aligned}$$

for all $t \in (1,b]$. From above it follows:

$$\begin{aligned}|y(t) - x(t)| &= \bigg|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\,\frac{ds}{s}\bigg| \\ &= \bigg|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,y(s))\,\frac{ds}{s}\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \, \frac{ds}{s} \bigg| \\
&\leq \left| y(t) - \frac{y_1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} f(s, y(s)) \, \frac{ds}{s} \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} |f(s, y(s)) - f(s, x(s))| \, \frac{ds}{s} \\
&\leq \frac{(\ln t)^\alpha \varepsilon}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} |y(s) - x(s)| \, \frac{ds}{s}.
\end{aligned}
$$

By Lemma 3.3 and Remark 3.5, for all $t \in (1, b]$, we have that:

$$
|y(t) - x(t)| \leq \frac{(\ln t)^\alpha \varepsilon}{\Gamma(\alpha+1)} \, E_{\alpha,1}(L(\ln t)^\alpha) \leq \frac{(\ln b)^\alpha E_{\alpha,1}(L(\ln b)^\alpha)\varepsilon}{\Gamma(\alpha+1)}.
$$

Thus, the equation (6.1) is Ulam–Hyers stable. □

Next, we consider the equation (6.1) and the inequation (6.3) in the case $b = +\infty$. We suppose that:

- (H_1') $f\colon [1, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(t, x) \in C_{\gamma,\ln}[1, +\infty)$ for any $x \in \mathbb{R}$;
- (H_2') There exists a $L > 0$ such that

  $$
  |f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|,
  $$

  for each $t \in [1, +\infty)$ and all $u_1, u_2 \in \mathbb{R}$;
- (H_3) $\varphi \in C([1, +\infty), \mathbb{R}_+)$ is continuous, nondecreasing, and there exists $\lambda_\varphi > 0$ such that

  $$
  \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \varphi(s) \, \frac{ds}{s} \leq \lambda_\varphi \varphi(t), \quad \text{for each } t \in [1, +\infty).
  $$

We have the following generalized Ulam–Hyers–Rassias stability results.

Theorem 6.10. *In the conditions* (H_1')*,* (H_2') *and* (H_3)*, the equation* (6.1) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $y \in C^\alpha_{1-\alpha,\gamma}[1, +\infty)$ be a solution of the inequation (6.3). Denote by x the unique solution of the Cauchy problem (6.5). We have that

$$
x(t) = \frac{y_1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \quad \text{for all } t \in (1, +\infty).
$$

By differential inequation (6.3), we have

$$
\begin{aligned}
&\left| y(t) - \frac{y_1}{\Gamma(\alpha)} (\ln t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} f(s, y(s)) \, \frac{ds}{s} \right| \\
&\qquad \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} \varphi(s) \, \frac{ds}{s} \leq \lambda_\varphi \varphi(t), \quad t \in [1, +\infty).
\end{aligned}
$$

From above it follows

$$|y(t)-x(t)| \le \lambda_\varphi \varphi(t) + \frac{L}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} |y(s)-x(s)| \,\frac{ds}{s}.$$

By Lemma 3.3 and Remark 3.5, we have that

$$|y(t)-x(t)| \le \lambda_\varphi \varphi(t) E_{\alpha,1}(L(\ln t)^\alpha), \quad t \in (1,+\infty).$$

Thus, the equation (6.1) is generalized Ulam–Hyers–Rassias stable. □

7. Example

We consider the following fractional differential equation with Hadamard derivatives

$$(7.1) \qquad {}_H D^\alpha_{1,t} x(t) = 0, \quad 0<\alpha<1,\ t\in(1,b),$$

and the inequation

$$(7.2) \qquad |{}_H D^\alpha_{1,t} y(t)| \le \varepsilon, \quad t \in [1,b).$$

Let $y \in C^\alpha_{1-\alpha,\gamma}[1,b]$ be a solution of (7.2). Then there exists a $g \in C_{\gamma,\ln}[1,b]$ such that:

$$(7.3) \qquad \begin{aligned} |g(t)| &\le \varepsilon, & t\in(1,b), \\ {}_H D^\alpha_{1,t} y(t) &= g(t), & t\in(1,b). \end{aligned}$$

By (7.3) we have:

$$y(t) = \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\,\frac{ds}{s}, \quad t\in(1,b).$$

We have, for all $x \in C^\alpha_{1-\alpha,\gamma}[1,b]$:

$$\begin{aligned} |y(t)-x(t)| &= \left| \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - x(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\,\frac{ds}{s} \right| \\ &\le \left| \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - x(t) \right| + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} |g(s)|\,\frac{ds}{s} \\ &\le | \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - x(t) | + \frac{\varepsilon}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\le \left| \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} - x(t) \right| + \frac{(\ln t)^\alpha \varepsilon}{\Gamma(\alpha+1)}, \end{aligned}$$

for $t\in(1,b)$. If we take $x(t) := \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}$, then

$$|y(t)-x(t)| \le \frac{(\ln t)^\alpha \varepsilon}{\Gamma(\alpha+1)}, \quad t\in(1,b).$$

If $b<+\infty$, then

$$|y(t)-x(t)| \le \frac{(\ln b)^\alpha \varepsilon}{\Gamma(\alpha+1)}, \quad t\in(1,b).$$

So, the equation (7.1) is Ulam–Hyers stable.

Let $b = +\infty$. The function

$$y(t) = \frac{(\ln t)^{\alpha}\varepsilon}{\Gamma(\alpha+1)}$$

is a solution of the inequation (7.2) and

$$\left|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\right| = \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{\varepsilon \ln t}{\alpha} - y_1\right| \to +\infty \quad \text{as } t \to +\infty.$$

So, the equation (7.1) is not Ulam-Hyers stable on the interval $[1,+\infty)$.

Let us consider the inequation:

$$(7.4) \qquad |_H D^{\alpha}_{1,t} y(t)| \leq \varphi(t), \quad t \in (1,+\infty).$$

Let y be a solution of (7.4) and

$$x(t) = \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}, \quad t \in (1,+\infty)$$

be a solution of (7.1). We have that

$$|y(t) - x(t)| = \left|y(t) - \frac{y_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\right| \leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\varphi(s)\,\frac{ds}{s},$$

for $t \in (1,+\infty)$. If there exists $c_{\varphi} > 0$ such that

$$\frac{1}{\Gamma(\alpha)}\int_1^t (\ln\frac{t}{s})^{\alpha-1}\varphi(s)\frac{ds}{s} \leq c_{\varphi}\varphi(t), \quad t \in (1,+\infty),$$

then (7.1) is generalized Ulam–Hyers–Rassias stable on $[1,+\infty)$ with respect to φ.

References

[1] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math. **109** (2010), 973–1033.

[2] B. Ahmad and J.J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray–Schauder degree theory*, Topol. Methods Nonlinear Anal. **35** (2010), 295–304.

[3] H. Amann, *Ordinary differential equations* (1990), Walter de Gruyter, Berlin.

[4] Z.B. Bai, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Anal. **72** (2010), 916–924.

[5] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338** (2008), 1340–1350.

[6] G. Butler and T. Rogers, *A generalization of a lemma of Bihari and applications to pointwise estimates for integral equations*, J. Math. Anal. Appl. **33** (1971), 77–81.

[7] L. Cădariu, *Stabilitatea Ulam–Hyers–Bourgin Pentru Ecuatii Functionale*, Ed. Univ. Vest Timişoara, Timişara, 2007.

[8] Y.-K. Chang, V. Kavitha and M. Mallika Arjunanb, *Existence and uniqueness of mild solutions to a semilinear integrodifferential equation of fractional order*, Nonlinear Anal. **71** (2009), 5551–5559.

[9] C. Chicone, *Ordinary differential equations with applications* (2006), Springer, New York.

[10] C. Corduneanu, *Principles of Differential and Integral Equations*, Chelsea Publ. Company, New York, 1971.

[11] W. Deng, *Smoothness and stability of the solutions for nonlinear fractional differential equations*, Nonlinear Anal. **72** (2010), 1768–1777.

[12] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, 2010.

[13] S.-B. Hsu, *Ordinary Differential Equations with Applications*, World Scientific, New Jersey, 2006.

[14] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, 1998.

[15] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Mathematics Studies, vol. 204, North-Holland, Elsevier Science B.V., Amsterdam, 2006.

[17] V. Lakshmikantham, S. Leela and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[18] Y. Li, Y. Chen and I. Podlubny, *Mittag–Leffler stability of fractional order nonlinear dynamic systems*, Automatica **45** (2009), 1965–1969.

[19] ———, *Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability*, Comput. Math. Appl. **59** (2010), 1810–1821.

[20] M. Medveď, *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, J. Math. Anal. Appl. **214** (1997), 349–366.

[21] ———, *Integral inequalities and global solutions of semilinear evolution equations*, J. Math. Anal. Appl. **267** (2002), 643–650.

[22] ———, *Singular integral inequalities with several nonlinearities and integral equations with singular kernels*, Nonlinear Oscil. **11** (2007), 70–79.

[23] M. Medveď, M. Pospíšil and L. Škripková, *Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices*, Nonlinear Anal. **74** (2011), 3903–3911.

[24] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[25] G. M. Mophou, G.M. N'Guérékata, *Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay*, Appl. Math. Comput. **216** (2010), 61–69.

[26] L.C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary Differential Equations in $\mathbb{R}^n$*, Springer, Berlin, 1984.

[27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[28] I.A. Rus, *Ecuaţii Diferenţiale, Ecuaţii Integrale şi Sisteme Dinamice*, Transilvania Press, Cluj-Napoca, 1996.

[29] ———, *Ulam stability of ordinary differential equations,*, Studia Univ. Babeş-Bolyai Math. **54** (2009), 125–133.

[30] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, HEP, 2010.

[31] J. WANG AND L. LV AND Y. ZHOU, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, E.J. Qualitative Theory of Differential Equations **2011**, no. 63, e1–e10.

[32] ______, *New concepts and results in stability of fractional differential equations,*, Commun. Nonlinear Sci. Numer. Simulat. (2011), doi:10.1016/j.cnsns.2011.09.030.

[33] J. WANG AND Y. ZHOU, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. **12** (2011), 262–272.

[34] ______, *Analysis of nonlinear fractional control systems in Banach spaces*, Nonlinear Anal. **74** (2011), 5929–5942.

[35] ______, *Existence and controllability results for fractional semilinear differential inclusions*, Nonlinear Anal. **12** (2011), 3642–3653.

[36] S. ZHANG, *Existence of positive solution for some class of nonlinear fractional differential equations*, J. Math. Anal. Appl. **278** (2003), 136–148.

[37] Y. ZHOU AND F. JIAO, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. **11** (2010), 4465–4475.

[38] Y. ZHOU, F. JIAO AND J. LI, *Existence and uniqueness for p-type fractional neutral differential equations*, Nonlinear Anal. **71** (2009), 2724–2733.

[39] ______, *Existence and uniqueness for fractional neutral differential equations with infinite delay*, Nonlinear Anal. **71** (2009), 3249–3256.

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