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Research Article

Existence and Stability of Solutions for Hadamard-Stieltjes Fractional Integral Equations

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We give some existence results and Ulam stability results for a class of Hadamard-Stieltjes integral equations. We present two results: the first one is an existence result based on Schauder's fixed point theorem and the second one is about the generalized Ulam-Hyers-Rassias stability.

1. Introduction

Fractional differential and integral equations have recently been applied in various areas of engineering, mathematics, physics, bioengineering, and other applied sciences [1, 2]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the excellent classical monograph of Kilbas et al. [3] or the recent monograph of Abbas et al. [4].

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [5]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [6]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [7] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [8, 9]. Bota-Boriceanu and Petrusel [10], Petru et al. [11], and Rus [12, 13] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [14], and Jung [15] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. More details from historical point of view and recent developments of such stabilities are reported in [12, 16].

In [17], Butzer et al. investigate properties of the Hadamard fractional integral and the derivative. In [18], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and in [19], Pooseh et al. obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators are summarized in [20] and the references therein.

This paper deals with the existence of the Ulam stability of solutions to the following Hadamard-Stieltjes fractional integral equation:

$$u(x,y)=\mu(x,y)+\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\cdot\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}d_tg_2(y,t)\,d_sg_1(x,s);\quad \text{if } (x,y)\in J, \tag{1}$$

where $J:=[1,a]\times[1,b]$, $a,b>1$, $r_1,r_2>0$ and $\mu:J\to\mathbb{R}$, $f:J\times\mathbb{R}\to\mathbb{R}$, $g_1:[1,a]^2\to\mathbb{R}$, $g_2:[1,b]^2\to\mathbb{R}$ are given continuous functions, and $\Gamma(\cdot)$ is the Euler gamma function.

Our investigations are conducted with an application of Schauder's fixed point theorem for the existence of solutions of the integral equation (1). Also, we obtain some results about the generalized Ulam-Hyers-Rassias stability of solutions of (1). Finally, we present an example illustrating the applicability of the imposed conditions.

This paper initiates the study of the existence and the Ulam stability of such class of integral equations.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote by $L^1(J,\mathbb{R})$ the Banach space of functions $u:J\to\mathbb{R}$ that are Lebesgue integrable with norm

$$\|u\|_{L^1}=\int_1^a\int_1^b|u(x,y)|\,dy\,dx. \tag{2}$$

Let $C:=C(J,\mathbb{R})$ be the Banach space of all continuous functions $u:J\to\mathbb{R}$ with the norm

$$\|u\|_C=\sup_{(x,y)\in J}|u(x,y)|. \tag{3}$$

Definition 1 (see [3, 21]). The Hadamard fractional integral of order $q>0$ for a function $g\in L^1([1,a],\mathbb{R})$ is defined as

$$\left({}^HI_1^rg\right)(x)=\frac{1}{\Gamma(q)}\int_1^x\left(\log\frac{x}{s}\right)^{q-1}\frac{g(s)}{s}ds. \tag{4}$$

Definition 2. Let $r_1,r_2\geq 0$, $\sigma=(1,1)$, and $r=(r_1,r_2)$. For $w\in L^1(J,\mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$\left({}^HI_\sigma^rw\right)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\cdot\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{w(s,t)}{st}dt\,ds. \tag{5}$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w:[a,b]\times[c,b]\to\mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t\to w(t,s)$ on the interval $[p,q]\subset[a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [22].

If u and φ are two real functions defined on the interval $[a,b]$, then under some conditions (see [22]) we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b u(t)\,d\varphi(t) \tag{6}$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a,b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [22]. One of the most frequently used requirements are that u is continuous and φ is of bounded variation on $[a,b]$.

In what follows we use the following properties of the Stieltjes integral ([23], section 8.13).

If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then

$$\left|\int_a^b u(t)\,d\varphi(t)\right|\leq\int_a^b|u(t)|\,d\left(\bigvee_a^t\varphi\right). \tag{7}$$

If u and v are Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ such that $u(t)\leq v(t)$ for $t\in[a,b]$, then

$$\int_a^b u(t)\,d\varphi(t)\leq\int_a^b v(t)\,d\varphi(t). \tag{8}$$

In the sequel we consider Stieltjes integrals of the form

$$\int_a^b u(t)\,d_sg(t,s) \tag{9}$$

and Hadamard-Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}u(s)\,d_sg(t,s), \tag{10}$$

where $g:[1,\infty)\times[1,\infty)\to\mathbb{R}$, $q\in(0,\infty)$, and the symbol d_s indicates the integration with respect to s.

Definition 3. Let $r_1,r_2\geq 0$, $\sigma=(1,1)$, and $r=(r_1,r_2)$. For $w\in L^1(J,\mathbb{R})$, define the Hadamard-Stieltjes partial fractional integral of order r by the expression

$$\left({}^{HS}I_\sigma^rw\right)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\cdot\left(\log\frac{y}{t}\right)^{r_2-1}\frac{w(s,t)}{st}d_tg_2(y,t)\,d_sg_1(x,s), \tag{11}$$

where $g_1,g_2:[1,\infty)\times[1,\infty)\to\mathbb{R}$.

Now, we consider the Ulam stability for the integral equation (1). Consider the operator $N:C\to C$ defined by

$$(Nu)(x,y)=\mu(x,y)+\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\cdot\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}d_tg_2(y,t)\,d_sg_1(x,s). \tag{12}$$

Clearly, the fixed points of the operator N are solution of the integral equation (1). Let $\epsilon > 0$ and $\Phi : J \to [0, \infty)$ be a continuous function. We consider the following inequalities:

$$|u(x, y) - (Nu)(x, y)| \le \epsilon; \quad (x, y) \in J, \tag{13}$$

$$|u(x, y) - (Nu)(x, y)| \le \Phi(x, y); \quad (x, y) \in J, \tag{14}$$

$$|u(x, y) - (Nu)(x, y)| \le \epsilon\Phi(x, y); \quad (x, y) \in J. \tag{15}$$

Definition 4 (see [12, 24]). Equation (1) is Ulam-Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C$ of the inequality (13) there exists a solution $v \in C$ of (1) with

$$|u(x, y) - v(x, y)| \le \epsilon c_N; \quad (x, y) \in J. \tag{16}$$

Definition 5 (see [12, 24]). Equation (1) is generalized Ulam-Hyers stable if there exists $c_N : C([0, \infty), [0, \infty))$ with $c_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathscr{C}$ of the inequality (13) there exists a solution $v \in C$ of (1) with

$$|u(x, y) - v(x, y)| \le c_N(\epsilon); \quad (x, y) \in J. \tag{17}$$

Definition 6 (see [12, 24]). Equation (1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C$ of the inequality (15) there exists a solution $v \in C$ of (1) with

$$|u(x, y) - v(x, y)| \le \epsilon c_{N,\Phi}\Phi(x, y); \quad (x, y) \in J. \tag{18}$$

Definition 7 (see [12, 24]). Equation (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in C$ of the inequality (14) there exists a solution $v \in \mathscr{C}$ of (1) with $|u(x, y) - v(x, y)| \le c_{N,\Phi}\Phi(x, y)$; $(x, y) \in J$.

Remark 8. It is clear that (i) Definition 4 $\Rightarrow$ Definition 5, (ii) Definition 6 $\Rightarrow$ Definition 7, and (iii) Definition 6 for $\Phi(\cdot, \cdot) = 1$ $\Rightarrow$ Definition 4.

One can have similar remarks for the inequalities (13) and (15).

3. Existence and Ulam Stabilities Results

In this section, we discuss the existence of solutions and we present conditions for the Ulam stability for the Hadamard integral equation (1).

The following hypotheses will be used in the sequel.

(H_1) There exist functions $p_1, p_2 \in C(J, \mathbb{R}_+)$ such that, for any $u \in \mathbb{R}$ and $(x, y) \in J$,

$$|f(x, y, u)| \le p_1(x, y) + \frac{p_2(x, y)}{1 + |u(x, y)|}|u(x, y)|, \tag{19}$$

with

$$p_i^* = \sup_{(x,y)\in J} \sup_{(s,t)\in[1,x]\times[1,y]} \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \cdot \frac{p_i(s, t)}{st\Gamma(r_1)\Gamma(r_2)}; \quad i = 1, 2. \tag{20}$$

(H_2) For all $x_1, x_2 \in [1, a]$ such that $x_1 < x_2$, the function $s \mapsto g(x_2, s) - g(x_1, s)$ is nondecreasing on $[1, a]$. Also, for all $y_1, y_2 \in [1, b]$ such that $y_1 < y_2$, the function $s \mapsto g(y_2, t) - g(y_1, t)$ is nondecreasing on $[1, b]$.

(H_3) The functions $s \mapsto g_1(0, s)$ and $t \mapsto g_2(0, t)$ are nondecreasing on $[1, a]$ or $[1, b]$, respectively.

(H_4) The functions $s \mapsto g_1(x, s)$ and $x \mapsto g_1(x, s)$ are continuous on $[1, a]$ for each fixed $x \in [1, a]$ or $s \in [1, a]$, respectively. Also, the functions $t \mapsto g_2(y, t)$ and $y \mapsto g_2(y, t)$ are continuous on $[1, b]$ for each fixed $y \in [1, b]$ or $t \in [1, b]$, respectively.

(H_5) There exists $\lambda_\Phi > 0$ such that, for each $(x, y) \in J$, we have

$$\left({}^{HS}I_\sigma^r\Phi\right)(x, y) \le \lambda_\Phi\Phi(x, y). \tag{21}$$

Set

$$g^* = \sup_{(x,y)\in J} \bigvee_{k_2=1}^{y} g_2(y, k_2) \bigvee_{k_1=1}^{x} g_1(x, k_1). \tag{22}$$

Theorem 9. *Assume that the hypotheses (H_1)–(H_4) hold. Then the integral equation (1) has a solution defined on J.*

Proof. Let $\rho > 0$ be a constant such that

$$\rho > \|\mu\|_\infty + g^*(p_1^* + p_2^*). \tag{23}$$

We will use Schauder's theorem [25], to prove that the operator N defined in (12) has a fixed point. The proof will be given in four steps.

Step 1 (N transforms the ball $B_\rho := \{u \in \mathscr{C} : \|u\|_C \le \rho\}$ into itself). For any $u \in B_\rho$ and each $(x, y) \in J$, we have

$$\begin{aligned}
|(Nu)(x, y)| &\le |\mu(x, y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \cdot \frac{p_1(s, t)}{st}|d_t g_2(y, t)\, d_s g_1(x, s)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \cdot \frac{p_2(s, t)|u(s, t)|}{st(1 + |u(s, t)|)}|d_t g_2(y, t)\, d_s g_1(x, s)| \\
&\le \|\mu\|_C + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \cdot \frac{p_1(s, t) + p_2(s, t)\rho}{st} d_t \bigvee_{k_2=1}^{t} g_2(y, k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x, k_1) \\
&\le \|\mu\|_C + (p_1^* + p_2^*) \cdot \int_1^x \int_1^y d_t \bigvee_{k_2=1}^{t} g_2(y, k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x, k_1) \\
&\le \|\mu\|_C + g^*(p_1^* + p_2^*) \le \rho.
\end{aligned} \tag{24}$$

Thus, $\|(Nu)\|_C \leq \rho$. This implies that N transforms the ball B_ρ into itself.

Step 2 ($N : B_\rho \rightarrow B_\rho$ is continuous). Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \rightarrow u$ in B_ρ. Then

$$\begin{aligned}
&\left|(Nu_n)(x,y) - (Nu)(x,y)\right| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \\
&\quad \cdot \frac{\left|f(s,t,u_n(s,t)) - f(s,t,u(s,t))\right|}{st} d_t g_2(y, \\
&\quad t)\, d_s g_1(x,s) \\
&\quad \leq \frac{\sup_{(s,t)\in J}\left|f(s,t,u_n(s,t)) - f(s,t,u(s,t))\right|}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} d_t \bigvee_{k_2=1}^{t} g_2(y,k_2)\, d_s \\
&\quad \cdot \bigvee_{k_1=1}^{s} g_1(x,k_1) \leq g^* \left\|f(\cdot,\cdot,u_n(\cdot,\cdot)) - f(\cdot,\cdot,u(\cdot,\cdot))\right\|_C.
\end{aligned} \tag{25}$$

From Lebesgue's dominated convergence theorem and the continuity of the function f, we get

$$\left|(Nu_n)(x,y) - (Nu)(x,y)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{26}$$

Step 3 ($N(B_\rho)$ is bounded). This is clear since $N(B_\rho) \subset B_\rho$ and B_ρ is bounded.

Step 4 ($N(B_\rho)$ is equicontinuous). Let $(x_1,y_1),(x_2,y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$. Then

$$\begin{aligned}
&\left|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)\right| \leq \left|\mu(x_1,y_1) - \mu(x_2,y_2)\right| \\
&\quad + \left|\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1}\int_1^{y_1} \left|\log\frac{x_2}{s}\right|^{r_1-1} \left|\log\frac{y_2}{t}\right|^{r_2-1}\right. \\
&\quad \cdot \frac{f(s,t,u(s,t))}{st} d_t g_2(y_2,t)\, d_s g_1(x_2,s) \\
&\quad - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1}\int_1^{y_1} \left|\log\frac{x_1}{s}\right|^{r_1-1} \left|\log\frac{y_1}{t}\right|^{r_2-1} \\
&\quad \left.\cdot \frac{f(s,t,u(s,t))}{st} d_t g_2(y_1,t)\, d_s g_1(x_1,s)\right| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left|\log\frac{x_2}{s}\right|^{r_1-1} \left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\quad \cdot \frac{\left|f(s,t,u(s,t))\right|}{st} \left|d_t g_2(y_2,t)\right| d_s g_1(x_2,s) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1}\int_{y_1}^{y_2} \left|\log\frac{x_2}{s}\right|^{r_1-1} \left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\quad \cdot \frac{\left|f(s,t,u(s,t))\right|}{st} \left|d_t g_2(y_2,t)\, d_s g_1(x_2,s)\right| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_1^{y_1} \left|\log\frac{x_2}{s}\right|^{r_1-1} \left|\log\frac{y_2}{t}\right|^{r_2-1} \\
&\quad \cdot \frac{\left|f(s,t,u(s,t))\right|}{st} \left|d_t g_2(y_2,t)\, d_s g_1(x_2,s)\right|.
\end{aligned} \tag{27}$$

Thus, we obtain

$$\begin{aligned}
&\left|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)\right| \leq \left|\mu(x_1,y_1) - \mu(x_2,\right. \\
&\quad \left. y_2)\right| + (p_1^* + p_2^*) \\
&\quad \cdot \int_1^{x_1}\int_1^{y_1} \left| d_t \bigvee_{k_2=1}^{t} g_2(y_2,k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x_2,k_1) \right. \\
&\quad \left. - d_t \bigvee_{k_2=1}^{t} g_2(y_1,k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x_1,k_1)\right| + (p_1^* + p_2^*) \\
&\quad \cdot \int_{x_1}^{x_2}\int_{y_1}^{y_2} d_t \bigvee_{k_2=1}^{t} g_2(y_2,k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x_2,k_1) \\
&\quad + (p_1^* + p_2^*) \\
&\quad \cdot \int_1^{x_1}\int_{y_1}^{y_2} d_t \bigvee_{k_2=1}^{t} g_2(y_2,k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x_2,k_1) \\
&\quad + (p_1^* + p_2^*) \\
&\quad \cdot \int_{x_1}^{x_2}\int_1^{y_1} d_t \bigvee_{k_2=1}^{t} g_2(y_2,k_2)\, d_s \bigvee_{k_1=1}^{s} g_1(x_2,k_1).
\end{aligned} \tag{28}$$

Hence, we get

$$\begin{aligned}
&\left|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)\right| \leq \left|\mu(x_1,y_1)\right. \\
&\quad \left. - \mu(x_2,y_2)\right| + (p_1^* + p_2^*) \\
&\quad \cdot \left| \bigvee_{k_2=1}^{y_1} g_2(y_2,k_2) \bigvee_{k_1=1}^{x_1} g_1(x_2,k_1) \right. \\
&\quad \left. - \bigvee_{k_2=1}^{y_1} g_2(y_1,k_2) \bigvee_{k_1=1}^{x_1} g_1(x_1,k_1)\right| + (p_1^* + p_2^*) \\
&\quad \cdot \bigvee_{k_2=y_1}^{y_2} g_2(y_2,k_2) \bigvee_{k_1=x_1}^{x_2} g_1(x_2,k_1) + (p_1^* + p_2^*) \\
&\quad \cdot \bigvee_{k_2=y_1}^{y_2} g_2(y_2,k_2) \bigvee_{k_1=1}^{x_2} g_1(x_2,k_1) + (p_1^* + p_2^*) \\
&\quad \cdot \bigvee_{k_2=1}^{y_2} g_2(y_2,k_2) \bigvee_{k_1=x_1}^{x_2} g_1(x_2,k_1).
\end{aligned} \tag{29}$$

As $x_1 \rightarrow x_2$ and $y_1 \rightarrow y_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 4 together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Schauder's theorem [25], we deduce that N has a fixed point u which is a solution of the integral equation (1). □

Now, we are concerned with the stability of solutions for the integral equation (1).

Theorem 10. *Assume that (H_1)–(H_5) hold. Furthermore, suppose that there exists $q_i \in C(J, \mathbb{R}_+)$, $i = 1, 2$, such that, for each $(x, y) \in J$, we have*

$$p_i(x, y) \le q_i(x, y)\Phi(x, y). \tag{30}$$

Then the integral equation (1) is generalized Ulam-Hyers-Rassias stable.

Proof. Let u be a solution of the inequality (14). By Theorem 9, there exists v which is a solution of the integral equation (1). Hence

$$\begin{aligned} v(x, y) = {} & \mu(x, y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \\ & \cdot \frac{f(s, t, v(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t)\, d_s g_1(x, s). \end{aligned} \tag{31}$$

By the inequality (14) for each $(x, y) \in J$, we have

$$\begin{aligned} & \left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \right. \\ & \left. \cdot \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t)\, d_s g_1(x, s) \right| \\ & \le \Phi(x, y). \end{aligned} \tag{32}$$

Set

$$q_i^* = \sup_{(x,y)\in J} q_i(x, y); \quad i = 1, 2. \tag{33}$$

For each $(x, y) \in J$, we have

$$\begin{aligned} |u(x, y) - v(x, y)| \le {} & \left| u(x, y) - \mu(x, y) \right. \\ & - \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \\ & \left. \cdot \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t)\, d_s g_1(x, s) \right| \\ & + \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \\ & \cdot \frac{|f(s, t, u(s, t)) - f(s, t, v(s, t))|}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, \\ & t)\, d_s g_1(x, s) \le \Phi(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\ & \cdot \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \left(2q_1^* + \frac{q_2^* |u(s, t)|}{1 + |u|}\right. \\ & \left. + \frac{q_2^* |v(s, t)|}{1 + |v|}\right) \frac{\Phi(s, t)}{st} d_t g_2(y, t)\, d_s g_1(x, s) \\ & \le \Phi(x, y) + 2(q_1^* + q_2^*)\left({}^{HS}I_\sigma^r \Phi\right)(x, y) \le \left[1\right. \\ & \left. + 2(q_1^* + q_2^*)\lambda_\phi\right] \Phi(x, y) := c_{N,\Phi}\Phi(x, y). \end{aligned} \tag{34}$$

Hence the integral equation (1) is generalized Ulam-Hyers-Rassias stable. □

4. An Example

As an application of our results we consider the following Hadamard-Stieltjes integral equation

$$\begin{aligned} u(x, y) = {} & \mu(x, y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \\ & \cdot \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t)\, d_s g_1(x, s); \\ & (x, y) \in [1, e] \times [1, e], \end{aligned} \tag{35}$$

where

$$\begin{aligned} & r_1, r_2 > 0, \\ & \mu(x, y) = x + y^2; \quad (x, y) \in [1, e] \times [1, e], \\ & g_1(x, s) = s, \\ & g_2(y, t) = t; \\ & \qquad s, t \in [1, e], \\ & f(x, y, u(x, y)) = xy^2\left(e^{-7} + \frac{u(x, y)}{e^{x+y+5}}\right); \\ & \qquad (x, y) \in [1, e] \times [1, e]. \end{aligned} \tag{36}$$

The condition (H_1) is satisfied with $p_1(x, y) = xy^2e^{-7}$ and $p_2^* = xy^2/e^{x+y+5}$. We can see that the functions g_1 and g_2 satisfy (H_2)–(H_4). Consequently Theorem 9 implies that the Hadamard integral equation (35) has a solution defined on $[1, e] \times [1, e]$.

Also, the hypothesis (H_5) is satisfied with

$$\begin{aligned} \Phi(x, y) &= e^3, \\ \lambda_\Phi &= \frac{1}{\Gamma(1 + r_1)\Gamma(1 + r_2)}. \end{aligned} \tag{37}$$

Indeed, for each $(x, y) \in [1, e] \times [1, e]$ we get

$$\left({}^{HS}I_\sigma^r \Phi\right)(x, y) \le \frac{e^3}{\Gamma(1 + r_1)\Gamma(1 + r_2)} = \lambda_\Phi \Phi(x, y). \tag{38}$$

Consequently, Theorem 10 implies that (35) is generalized Ulam-Hyers-Rassias stable.

Conflict of Interests

The authors declare no conflict of interests.

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