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EXISTENCE AND STABILITY OF STRONG POTENTIAL DOUBLE LAYERS*

By

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ABSTRACT

An elementary integral equation technique is used to construct strong and weak stationary shock solutions from the one-dimensional Vlasov equation. It is shown that the plasma is Penrose stable in all points in space under certain conditions.

I. INTRODUCTION

Potential double layers, or sheaths, are regions in a plasma where charge-neutrality is not valid. They have been observed in gaseous discharges and could possibly be responsible for the acceleration of auroral particles in the ionosphere (Gurnett, 1972; Block, 1972; Alfén, 1958).

Using the two fluid equations, Block (1972) derives a number of self-consistency conditions for stationary potential double layers. The model of the double layer used by him has the following characteristics:

(1) The electrostatic potential φ is a function of x only and varies monotonically from $\varphi = 0$ at $x \geq x_1$ to a certain value φ_0 at $x \leq 0$. The electric field $E = -(d/dx)\varphi = 0$ for $x \geq x_1$ and $x \leq 0$.

(2) Cold electrons move into the layer from $x = +\infty$ with a velocity $u_{e\infty}$; cold ions move in from $x = -\infty$ with a velocity $u_{i\infty}$. Both particles are accelerated through the layer. The total current density $j = e(n_{i\infty} u_{i\infty} - n_{e\infty} u_{e\infty}) > 0$ is necessarily constant.

(3) There are also warm ions and electrons with temperatures T_i and T_e, respectively. Some of these particles are reflected inside the layer.

For strong layers (potential difference much larger than $\kappa T_e/e$) warm particles are trapped, i.e., reflected inside the layer. Block derives certain necessary conditions for the existence of a stationary double layer of this kind.

It is, however, not clear from his work whether the plasma at $x \geq x_1$ and $x \leq 0$ is stable or not. It appears that the particular velocity distributions assumed by Block, namely δ-functions for the cold electrons and ions, should give rise to an electrostatic two-stream instability. It is also not clear whether a physical distribution function for the trapped particles exist at all under these conditions, particularly for strong shocks. This question can only be answered within the framework of the Vlasov equation.

It is the purpose of this paper to derive necessary conditions for the existence of a double layer and to investigate the stability of the plasma in the regions where the potential is almost constant. The model we adopt for the double layer is similar to that of Block, which was chosen to represent realistically the conditions of potential double layers in the ionosphere. We will allow a thermal spread in the distribution function of the electrons and ions moving in from $x = -\infty$ and $x = +\infty$, respectively. In Figure 1 we have plotted the potential and the phase space of ions and electrons.

Our technique of constructing the trapped particle distribution by an integral equation, which appears to be particularly convenient for our purpose, has been described by Bernstein, Green, and Kruskal (1958). Another method of solving for the potential has been applied in a related case by Montgomery and Joyce (1969). They showed for the first time that shock-like solutions do indeed exist. The asymptotic stability of their solutions was first discussed by Biskamp (1969).

II. THE METHOD

We consider stationary one-dimensional solutions of the Vlasov-Poisson system which yield a monotonically increasing potential $\varphi(x)$ with $\lim_{x\to\pm\infty} \varphi(x) = \pm\varphi_o$. This implies that the number densities of electrons and (singly charged) positive ions are equal at $x = \pm\infty$. The distribution functions f_j of the electrons and ions can be written as functions of the total energy

$$W = \frac{1}{2} m_j v^2 + q_j \varphi(x) \quad . \tag{1}$$

Depending on the value of W, trapped and free particles can be described by the following parameter ranges of W:

$$\begin{array}{ll} \text{Trapped ions} & e\varphi(x) < W < e\varphi_o \\ \text{Free ions} & e\varphi_o < W < \infty \\ \text{Trapped electrons} & -e\varphi(x) < W < e\varphi_o \\ \text{Free electrons} & e\varphi_o < W < \infty \end{array} \tag{2}$$

The density of any free or trapped species is given by

$$n_j(x) = \int \frac{f_j(W)\, dW}{\sqrt{2m_j}\,\sqrt{W - q_j\varphi(x)}} \quad . \tag{3}$$

The limits of the integral are obvious from the regions (2). Note that distributions for trapped particles must be symmetric in v in order to preserve stationarity. Poisson's equation can be written as

$$\frac{1}{4\pi}\,\varphi(x)'' + \sum_j q_j n_{jt}[\varphi(x)] + \sum_j q_j n_{jf}[\varphi(x)] = 0 \tag{4}$$

where the indices t and f refer to trapped and free particles. We require a potential with the following properties:

(1) $\varphi(x)' \geq 0$ in $-\infty < x < +\infty$.

(2) $\varphi(x)''$ can be expressed as a simple function of φ. Equation (4) can then be considered as an integral equation for f_{et}, the trapped electron distribution, with φ as variable rather than x.

(3) $\lim_{x \to \pm\infty} \varphi(x) = \pm\varphi_o$.

(4) $\varphi(x)$ is analytic.

(5) The transition region is characterized by the scale length ξ.

A simple example satisfying the above requirements is

$$\varphi(x)'' = -\left(\frac{2}{\xi^2}\right)\varphi\left[1 - \left(\frac{\varphi}{\varphi_o}\right)^2\right] \tag{5}$$

which has the solution

$$\varphi(x) = \varphi_o \ \mathrm{tgh}\left(\frac{x}{\xi}\right) \qquad (6)$$

(see Figure 1a). For brevity we define the energy

$$C = e\varphi_o \quad .$$

III. THE INTEGRAL EQUATION

With Eq. (3) Poisson's equation (4) can be written as

$$\frac{1}{\sqrt{2m_e}} \int_{-e\varphi(x)}^{C} [W' + e\varphi(x)]^{-1/2}\, 2f_{et}(W')\, dW'$$

$$= \frac{e\varphi(x)''}{4\pi e^2} - n_{ef}^{(1)} - n_{ef}^{(2)} + n_{if}^{(3)} + n_{it}^{(4)}$$

$$\equiv g[e\varphi(x)] \quad . \tag{7}$$

The factor 2 in front of f_{et} accounts for the symmetry of f_{et} for $v \gtrless 0$. We have labeled all different kinds of particle groups by a running upper index: Two species of free electrons with $v > 0$ and $v < 0$, free and trapped ions. Arbitrary groups, which may be partly free or trapped, can be added as necessity arises.

Consistency at $-e\varphi(x) = e\varphi_o$ at $x = -\infty$ requires

$$g(-e\varphi_o) = 0 = - N_{ef}^{(1)} - N_{ef}^{(2)} + N_{if}^{(3)} + N_{it}^{(4)} = 0 \tag{8}$$

where $n_{\nu\mu}^{(i)}(-e\varphi_o) = N_{\nu\mu}^{(i)}$. Physically this characterizes the charge neutrality at $x = -\infty$. The right side of Eq. (7) is a function of the potential $\varphi(x)$ only and is an Abel integral equation for $f_{et}[e\varphi(x)]$. In solving, we introduce the variable $s = -e\varphi(x)$, multiply Eq. (7) with $(s - W)^{-1/2}$ and integrate over s from W to C. The left side of Eq. (7) can then be written as

$$\frac{1}{\sqrt{2m_e}} \int_W^C ds \int_s^C dW' \, 2f_{et}(W')(s - W)^{-1/2} (W' - s)^{-1/2}$$

$$= \frac{\pi}{\sqrt{2m_e}} \int_W^C 2f_{et}(W') \, dW' \quad . \tag{9}$$

This result is obtained by introducing the variable t by $s = W + (W' - W)t$. The integral over t can be performed and results in the factor π. By differentiating Eq. (9) with respect to W, one obtains the solution of Eq. (7) as

$$\frac{\pi}{\sqrt{2m_e}} 2f_{et}(W) = \int_W^C h(s) \, (s - W)^{-1/2} \, ds \tag{10}$$

where $h(s) = (d/ds) \, g(-s)$ and use has been made of Eq. (8). Each of the density contributions $n^{(i)}$, $i = 1...4$, gives a contribution $f_{et}^{(i)}$ to the trapped electrons. For consistency we label the contribution due to $e\varphi(x)''$ as $f_{et}^{(0)}$.

We can think of Eq. (7) as establishing a linear transformation between the function $f_{\nu\mu}^{(i)}$ and $f_{et}^{(i)}$. It establishes a one-to-one correspondence in function space. We will study the properties of this transformation by choosing simple base functions for which the integrals can easily be performed. On the other hand the base functions can be used to assemble more general, physically meaningful distributions by linear superposition. Examples of this will be given in sections VI and VIII.

IV. THE PENROSE CRITERION

It is advantageous to combine this program with the consideration of stability. We specify for example the distribution of free electrons in terms of W [compare Eq. (1) and (2)]. At $x = -\infty$ we have to consider the free electrons and the trapped electrons via Eq. (7). Whereas it is comparatively easy to construct a stable distribution at $x = -\infty$ it is much more complicated at $x = +\infty$ because two electron distributions, which depend on each other, have to be taken into account. A necessary and sufficient criterion for stability has been given by Penrose (1960)

$$P = \int_{-\infty}^{+\infty} \frac{F(u) - F(u_o)}{(u - u_o)^2} = \int_{-\infty}^{+\infty} \frac{1}{u - u_o} \frac{dF(u)}{du} < 0 \qquad (11)$$

for stability. u_o is the location of a minimum of the distribution, and

$$F(u) = f_e(u) + \frac{m_e}{m_i} f_i(u) \qquad (12)$$

is a weighted sum of electron (f_e) and ion (f_i) distributions.

To keep the mathematics simple we confine ourselves to the case where v = 0 is the only nontrivial minimum of F(v), if there is any. Then Eq. (11) can be written for arbitrary x for the considered case of free and trapped electrons

$$P_e = P^e_{et} + P^e_{ef} = \sqrt{\frac{m_e}{2}} \int_{-e\varphi(x)}^{C} 2\, \frac{df^e_{et}(W)}{dW} \frac{dW}{\sqrt{W + C}}$$

$$+ \sqrt{\frac{m_e}{2}} \int_{C}^{\infty} \frac{df^e_{ef}(W)}{dW} \frac{dW}{\sqrt{W + C}} \quad . \tag{13}$$

If we deal with free ions which create a trapped electron distribution Eq. (11) can be written as

$$P^i = P^i_{et} + P^i_{if} = \sqrt{\frac{m_e}{2}} \int_{-e\varphi(x)}^{C} 2\, \frac{df^i_{et}(W)}{dW} \frac{dW}{\sqrt{W + C}}$$

$$+ \frac{m_e}{m_i}\sqrt{\frac{m_i}{2}} \int_{o}^{\infty} \frac{df^i}{dW} \frac{dW}{\sqrt{W - C}} \quad . \tag{14}$$

The lower index characterizes the species (electron, ion) and the mode (trapped, free), the upper index the primary species (electron, ion). Note in the second term the factor m_e/m_i and the different sign of C. A similar equation holds for the trapped ions. There is also a contribution from the potential term

$$P_e^o = \sqrt{\frac{m_e}{2}} \int 2 \frac{df_{et}^{(o)}(W)}{dW} \frac{dW}{\sqrt{W + C}} \quad .$$

An application of the Eqs. (13) and (14) to the base functions discussed in section V appears to be meaningless because the base functions may violate the condition that the only minimum be at v = 0. Nevertheless the results for base functions indicate which parameters are favorable for stability. We will thus be able to construct distributions by linear superposition which satisfy all requirements of stability.

The Penrose criterion applies strictly for homogeneous plasmas only. If the gradients in the transition regions are not too large we can consider the plasma as locally homogeneous and then the Penrose criterion will indicate stability with respect to certain modes whose wavelengths are small compared with the scale length, even in the inhomogeneous region. There may be other modes associated with the density gradient, however small, which are not covered by this analysis. Their treatment is much more complicated and beyond the scope of this paper.

V. THE BASE FUNCTIONS

A. Free Electrons

For free electrons we assume a waterbag, i.e., a distribution which is constant in a certain energy range.

$$f_{ef} = \text{const} = \frac{\sqrt{2m_e}}{2} \frac{N_e}{\sqrt{\theta_e}} \quad , \qquad C < W < C + \theta_e \quad . \qquad (15)$$

C is the boundary between free and trapped particles. θ_e characterizes a "temperature" of the electrons. Their velocity is either greater than or less than zero. The density is normalized to N_e for $x = -\infty$ and is given by

$$n_{ef} = \frac{N_e}{\sqrt{\theta_e}} \left[\sqrt{C + e\varphi(x) + \theta_e} - \sqrt{C + e\varphi(x)}\right] \quad . \qquad (16)$$

Any distribution which decreases with W can be obtained by superposition of base functions with positive weight. Arbitrary distributions (the less interesting ones) are obtained using negative weight functions. The function h(s) of Eq. (10) is given by

$$- \frac{N_e}{2\sqrt{\theta_e}} [(\theta_e + C - s)^{-1} - (C - s)^{-1}] \tag{17}$$

and the trapped electrons are

$$2f^e_{et} = \frac{\sqrt{2m_e}}{\pi} \frac{N_e}{\sqrt{\theta_e}} \left(\frac{\pi}{2} - \text{arc sin} \sqrt{\frac{C - W}{\theta_e + C - W}} \right) \quad . \tag{18}$$

For W = C we have

$$2f^e_{et}(C) = \frac{\sqrt{2m_e}}{2} \frac{N_e}{\sqrt{\theta}} = f_{ef}(C)$$

because of the symmetry of trapped particles we conclude immediately that

$$f^e_{et}(C) = \frac{1}{2}[f_{ef}(C, v > 0) + f_{ef}(C, v < 0)]$$

for general distributions. Hence the electron distribution is continuous, if and only if $f_{ef}(C, v > 0) = f_{ef}(C, v < 0)$. Further we find from Eq. (18)

$$\frac{d}{dW} 2f^e_{et} = \frac{\sqrt{2m_e}}{2\pi} N_e(C - W)^{-1/2} (\theta_e + C - W)^{-1} > 0 \quad . \tag{19}$$

The trapped distribution increases with W if the free distribution decreases (compare Figure 2). Such distributions are always double humped, and an application of the Penrose criterion to this base function is physically meaningful. We find

$$P^e_{ef} = -\frac{m_e}{2}\frac{N_e}{\sqrt{\theta_e}}\frac{1}{\sqrt{2C+\theta_e}}$$

$$P^e_{et} = +\frac{m_e}{2}\frac{N_e}{\sqrt{\theta_e}}\frac{1}{\sqrt{2C+\theta_e}}\frac{2}{\pi}\,\mathrm{arc\ tg}\left(\sqrt{\frac{2C+\theta_e}{\theta_e}}\sqrt{\frac{C+e\varphi}{C-e\varphi}}\right) \quad . \tag{20}$$

This gives for $e\varphi = C$ at $x = +\infty$

$$P_e = P^e_{ef} + P^e_{et} = 0 \quad .$$

Thus the electrons are marginally stable at $x = +\infty$ for any free electron distribution which decreases with energy! In the opposite case the theorem obviously does not hold any more. For example, let

$$f_e = \text{const in} \qquad C + \theta_1 < W < C + \theta_2 \quad ,$$
$$0 < \theta_1 < \theta_2 \quad .$$

It is well known that this distribution is unstable because it contains a gap with no particles at all (see, e.g., Krall and Trivelpiece, 1973). A special case of our general result has already been obtained by Biskamp. At $x < +\infty$, P_e is negative and the electrons are stable.

B. Free Ions

We choose as base function a δ-function

$$f_i = \sqrt{2m_i}\, N_i \sqrt{W_o + C}\, \delta(W - W_o) \quad , \quad C < W_o \quad . \tag{21}$$

The density is given by

$$n_i = N_i \frac{\sqrt{W_o + C}}{\sqrt{W_o - e\varphi(x)}} \tag{22}$$

and is normalized to N_i at $x = -\infty$. The trapped electrons are given by

$$2f^i_{ef} = \frac{\sqrt{2m_e}}{\pi} N_i \frac{\sqrt{C - W}}{W_o + W} \tag{23}$$

which goes to zero at the trapped-untrapped boundary W = C with infinite slope. The derivative

$$\frac{d}{dW}\, 2f^{i}_{et} = - \frac{\sqrt{2m_e}}{2\pi}\, N_i\, \frac{W_o + W + 2(C - W)}{(W_o + W)^2 \sqrt{C - W}} \tag{24}$$

is always negative, in contrast to Eq. (19). Finally we find

$$P^{i}_{f} = \frac{m_e N_i}{2} (W_o + C)^{1/2} (W_o - C)^{-3/2}$$

$$P^{i}_{et} = - \frac{m_e N_i}{2} (W_o - C)^{-1} \left\{ \sqrt{\frac{W_o + C}{W_o - C}} \left[\frac{1}{2} + \frac{1}{\pi} \arc\sin \frac{W_o e\varphi - C^2}{(W_o - e\varphi)C} \right] \right.$$

$$\left. - \frac{2}{\pi} \frac{\sqrt{C^2 - (e\varphi)^2}}{W_o - e\varphi} \right\} \quad . \tag{25}$$

For $e\varphi = C$ at $x = +\infty$ we obtain the surprising result

$$P^{i} = P^{i}_{f} + P^{i}_{et} = 0 \quad . \tag{26}$$

Thus a combined ion-trapped electron distribution with minimum at $v = 0$ is also marginally stable provided that the restrictions discussed above are met. We reiterate that the application of our Penrose criterion is only meaningful if the total plasma distribution $F(v)$ has a single minimum at $v = 0$.

C. Trapped Ions

A convenient base for trapped ions is given by

$$f_{it} = \begin{cases} \dfrac{\sqrt{2m_i}}{2} N_{it} \,(W_o + C)^{-1/2} , & e\varphi(x) \le W < W_o \le C \\ 0 , & e\varphi(x) > W_o \end{cases} \tag{27}$$

Again, distributions decreasing with W can be assembled with positive weight and general distributions with positive and negative weight functions. The density is

$$n_{it}[e\varphi(x)] = \begin{cases} N_{it} \sqrt{\dfrac{W_o - e\varphi(x)}{W_o + C}} , & e\varphi(x) < W_o \\ 0 , & e\varphi(x) > W_o \end{cases} \tag{28}$$

with $n_{it}(-C) = N_{it}$ at $x = -\infty$. The trapped electron contribution is

$$2f_{et} = - \frac{\sqrt{2m_e}}{\pi} N_{it}(W_o + C)^{-1/2} \ln \frac{\sqrt{C - W} + \sqrt{W_o + C}}{\sqrt{|W_o + W|}} \tag{29}$$

with an (integrable) singularity at $W = -W_o$. It is always negative.

At $x = +\infty$ one finds

$$P_{it} = - \sqrt{2m_e}\, N_{it}\, (W_o + C)^{-1} \quad ,$$

$$P^i_{et} \le \frac{m_e}{\pi} \frac{N_{it}}{W_o + C} \ln \frac{\sqrt{2C} + \sqrt{W_o + C}}{\sqrt{|W_o - C|}} \quad .$$

D. The Potential Term

The potential term $e\varphi''/4\pi e^2$ also creates a trapped particle distribution which is partly negative. Equation (6) may serve as a convenient example. We find

$$f^{(o)}_{ef} = \frac{\sqrt{2m_e}}{\pi} 4 \frac{N_e \lambda^2_{D,e}}{\theta_e \xi^2} \sqrt{C - W} \left[1 - 2\left(1 - \frac{W}{C}\right) + \frac{4}{5}\left(1 - \frac{W}{C}\right)^2\right] \quad . \tag{30}$$

We have written $(4\pi e^2)^{-1} = 2N_e\lambda^2_{D,e}/\theta_e$ where $\lambda_{D,e}$ is the Debye length of the electrons in the system. Expression (30) is negative in $-(\sqrt{5} + 1)/4 < W/C < (\sqrt{5} - 1)/4$.

The potential contribution to the Penrose function is

$$P^o_{et} = \frac{\sqrt{2m_e}}{\pi} 4 \frac{N_e \lambda^2}{\theta_e \xi^2} \left[- \arcsin \frac{e\varphi}{C} + \frac{2}{C^2} (C + e\varphi) \sqrt{C^2 - (e\varphi)^2} - \frac{\pi}{2}\right] .$$

A schematic sketch of the contributions to $f_{et}(W)$ is shown in Figure 2. Only terms from the potential and the trapped ions are partially negative. The first expression can be made arbitrarily small by choosing a large characteristic length ξ, the second term can be made to vanish by putting $N_{it} = 0$. Thus we can always easily construct physically meaningful trapped particle distribution functions $f_{et} > 0$.

VI. THE STRONG SHOCK

We define a strong shock by the condition

$$\frac{C}{\theta_e} \gg 1 , \tag{31}$$

where θ_e is an effective electron temperature of the model. In order to apply our expressions (20) and (25) for the Penrose criterion, we want to construct an $F(v)$ which has only one minimum at $v = 0$. We impose this condition only for mathematical convenience. Distribution functions with minima elsewhere may be equally stable, but it is more complicated to analyse them. For the electrons we take two equal base functions with $\theta_e \ll C$, one for $v > 0$ and the other for $v < 0$. For the ions we take a "ramp" defined by the weight function

$$a(W_o) = \alpha\left(1 - \frac{W_o - W_1}{W_2 - W_1}\right) = \rho - \sigma W_o , \quad \text{in } W_1 < W_o < W_2 . \tag{32}$$

The weight function has to be normalized to one in order to keep Eq. (8) unchanged. This gives

$$\alpha = 2(W_2 - W_1)^{-1} \quad ; \quad \rho = 2W_2(W_2 - W_1)^{-2} \quad ; \quad \sigma = 2(W_2 - W_1)^{-2} . \tag{33}$$

The ion distribution is, according to (32) and (21)

$$f_i = \sqrt{2m_i}\, N_i \sqrt{W + C}\, (\rho - \sigma W) \quad , \qquad C < W_1 < W < W_2$$

and

$$\frac{d}{dW} f_i = \sqrt{2m_i}\, N_i \left[- \sigma\sqrt{W + C} + \frac{1}{2} (\rho - \sigma W)(W + C)^{-1/2} \right] \quad . \quad (34)$$

This expression can evidently be minimized by neglecting the second term on the right.

From Eq. (8) we have without trapped ions

$$N_{ef}^{(1)} + N_{ef}^{(2)} = N_{if} \quad . \qquad (35)$$

The functions are plotted at $x = -\infty$ in Figure 3. It is evident that whenever

$$W_1 + C < \frac{m_i}{m_e} \theta_e \qquad (36)$$

the ions and electrons overlap in such a way that the distribution $F(v)$ is single humped and stable. Figure 4 shows the situation at $x = +\infty$. The ion ramp has moved to smaller energies, the free electrons have moved away from the origin. The trapped electrons exhibit a minimum. In Figure 4c F(v) is shown to have only one minimum at v = 0. We have to make sure that

$$\frac{\partial}{\partial v} F(v) < 0 \qquad \text{in } -2\sqrt{\frac{C}{m_e}} < v < 0 \quad . \tag{37}$$

As is seen from Figure 4 this is equivalent to

$$\frac{\partial}{\partial v}\left(f^e_{et} + f^i_{et}\right) < 0 \;, \quad \text{for } v < 0 \text{ in } 0 < \frac{1}{2} m_e v^2 < 2C \tag{38}$$

and

$$\frac{\partial}{\partial v}\left(f^e_{et} + f^i_{et} + \frac{m_e}{m_i} f^i_{if}\right) < 0 \qquad \text{for } v < 0$$

$$\text{in} \qquad \frac{m_e}{m_i}(W_1 - C) < \frac{1}{2} m_e v^2 < \frac{m_e}{m_i}(W_2 - C) \quad . \tag{39}$$

With $\epsilon = 1/2\, m_e v^2$ one can write $W = \epsilon - C$ for the electrons and $W = m_i/m_e\, \epsilon + C$ for the ions.

From Eq. (24) one sees that $(d/dW)f^i_{et}$ is a decreasing function of W_o. We maximize expressions (38) and (39) if we take $df^{(i)}_{et}/dW$

from Eq. (24) with $W_o = W_1$, rather than the exact form due to the ramp distribution. Using Eqs. (19) and (24) we see that Eqs. (38) and (39) are satisfied if

$$(\theta + 2C - \epsilon)^{-1} - [(W_1 - C + \epsilon) + 2(2C - \epsilon)](W_1 - C + \epsilon)^{-2} > 0$$

$$\text{in } 0 < \epsilon < 2C \tag{40}$$

and

$$[(\theta + 2C - \epsilon)^{-1} - (W_1 - C + \epsilon)^{-1} - 2(C - \epsilon)(W_1 - C + \epsilon)^{-2}]$$

$$- \sqrt{2C - \epsilon}\left\{\sigma\sqrt{\frac{m_i}{m_e}\epsilon + 2C} - \frac{1}{2}\left[\rho - \sigma\left(\frac{m_i}{m_e}\epsilon + C\right)\right]\right.$$

$$\left. \times \left(\frac{m_i}{m_e}\epsilon + 2C\right)^{-1/2}\right\} > 0 \qquad \text{for } W_1 - C < \frac{m_i}{m_e}\epsilon < W_2 - C \quad . \tag{41}$$

Equation (40) can be satisfied by making W_1 large enough. In Eq. (41) all ϵ without the factor m_i/m_e may be replaced by zero. Thus one finds that Eqs. (39) and (40) are satisfied for example by $W_1/C = 11$,

$W_2/C = 111$, and $\theta/C \ll 1$. This completes the proof that strong shocks can be constructed which are stable at $x = \pm\infty$. Many more examples could be given.

It appears probable that asymptotically stable distributions $F(v)$ with more than two humps can be constructed but the proof is probably cumbersome as already mentioned. Also it can be discussed how small the shock thickness ξ can be made without violating stability or making f_{et} negative.

VII. STABILITY FOR FINITE x

The marginal stability at $x = \infty$ should be supplemented by Penrose stability for finite x. If the sum of expressions (20) and (25), each multiplied by appropriate positive weight functions, can be kept negative the plasma is Penrose stable everywhere if it has at most one minimum at $v = 0$.

We show that if the energy of the ions is large enough the first term of the Taylor series of $F(v)$ in powers of $\epsilon = 1 - e\varphi/C$ can indeed be made negative. We expand the sum of expressions (20) in ϵ and obtain

$$P^e = - \frac{m_e N_e}{\sqrt{2}\,\pi(2C + \theta)} \epsilon^{1/2} \quad .$$

A corresponding expansion for the sum of expressions (25) gives

$$P^i = \frac{m_e N_i \sqrt{W_o + C}}{\sqrt{2}\,\pi (W_o + C)^{3/2}} \left[\left(\frac{W_o + C}{W_o - C} \right)^{1/2} + \frac{2C}{W_o - C} \right] \sqrt{\epsilon} \quad .$$

Inspections show that $P^e + P^i < 0$ for $\theta < 0.1\,C$; $W_o \geq 4C$. Thus we expect that plasmas can be constructed which are Penrose stable everywhere for a fast enough ion distribution.

VIII. THE WEAK SHOCK

We have only considered so far an electron distribution with two humps at $x = +\infty$. It is also possible to construct an electron distribution which decreases with W with its peak at $v = 0$. We have to require that

$$\frac{d}{dW}\left(f^{e}_{et} + f^{i}_{et}\right) < 0 \quad . \tag{42}$$

Let us assume for simplicity that f^{e}_{et} is produced by two symmetric free electron base functions and f^{i}_{et} by a free ion δ-function. With Eqs. (19) and (24) inserted into Eq. (42) we obtain as necessary and sufficient condition

$$-\frac{W_o + 2C - W}{(W_o + W)^2} + \frac{1}{\theta_e + C - W} < 0 \quad .$$

The inequality is satisfied for $-C < W < C$ if it is satisfied for $W = C$ and we obtain

$$\theta_e > W_o + C \qquad \text{or} \qquad \frac{\theta_e}{2C} > 1 \quad . \tag{43}$$

A waterbag ion distribution such that

$$f_i = \text{const} \qquad \text{for } W_1 \leq W \leq W_2$$

leads to the analogous condition

$$\frac{\theta_e}{C} > \sqrt{\left(\frac{W_1}{C} + 1\right)\left(\frac{W_1}{C} + 1\right)} \quad . \tag{44}$$

We conclude that single humped electron distributions allow only weak shocks ($\theta_e/C < 1$) if trapped ions are neglected. The example above can easily be extended to a completely stable mode for a weak shock if the ion distribution is supplemented such that it goes continuously to zero at low energies. This is accomplished, e.g., by adding a distribution

$$f_{ii} = f_i \left(\frac{W - C}{W_1 - C}\right)^2 \quad , \qquad C < W < W_1 \quad .$$

$F(v)$ is single humped at $x = -\infty$. At $x = +\infty$ it has a hardly noticable depression between the electron and ion peak, which however is absolutely Penrose stable.

IX. CONCLUSION

The integral equation method first suggested by Bernstein, Green, and Kruskal (1957) is used to give an elementary discussion of the relation between free electrons and ions and its trapped electron counterpart. Certain simple "base functions" are defined for the different particle species. Arbitrary distributions can be represented by a linear superposition of such base functions. The Penrose criterion is evaluated for the base functions assuming that the only nontrivial minimum of the distribution is at $v = 0$. This method allows to construct explicitly distributions which are Penrose stable at $x = \pm\infty$ and which represent strong or weak shock solutions. Single humped electron distributions at $x = +\infty$ allow only weak shocks. A strong shock always has a double humped electron distribution at $x = +\infty$ if trapped ions are neglected.

The Penrose criterion is also applied to regions where the plasma is no longer homogeneous. In case of stability this tells us that certain modes with wave length smaller than the scale length of the shock are stable. The stability analysis is not complete because it does not cover modes due to the inhomogeneity. Nevertheless it helps to understand better observations of double layers in space and in laboratory experiments. Double layers are also simulated in computer experiments (Joyce, 1974). Preliminary results appear to be consistent with our findings.

ACKNOWLEDGMENTS

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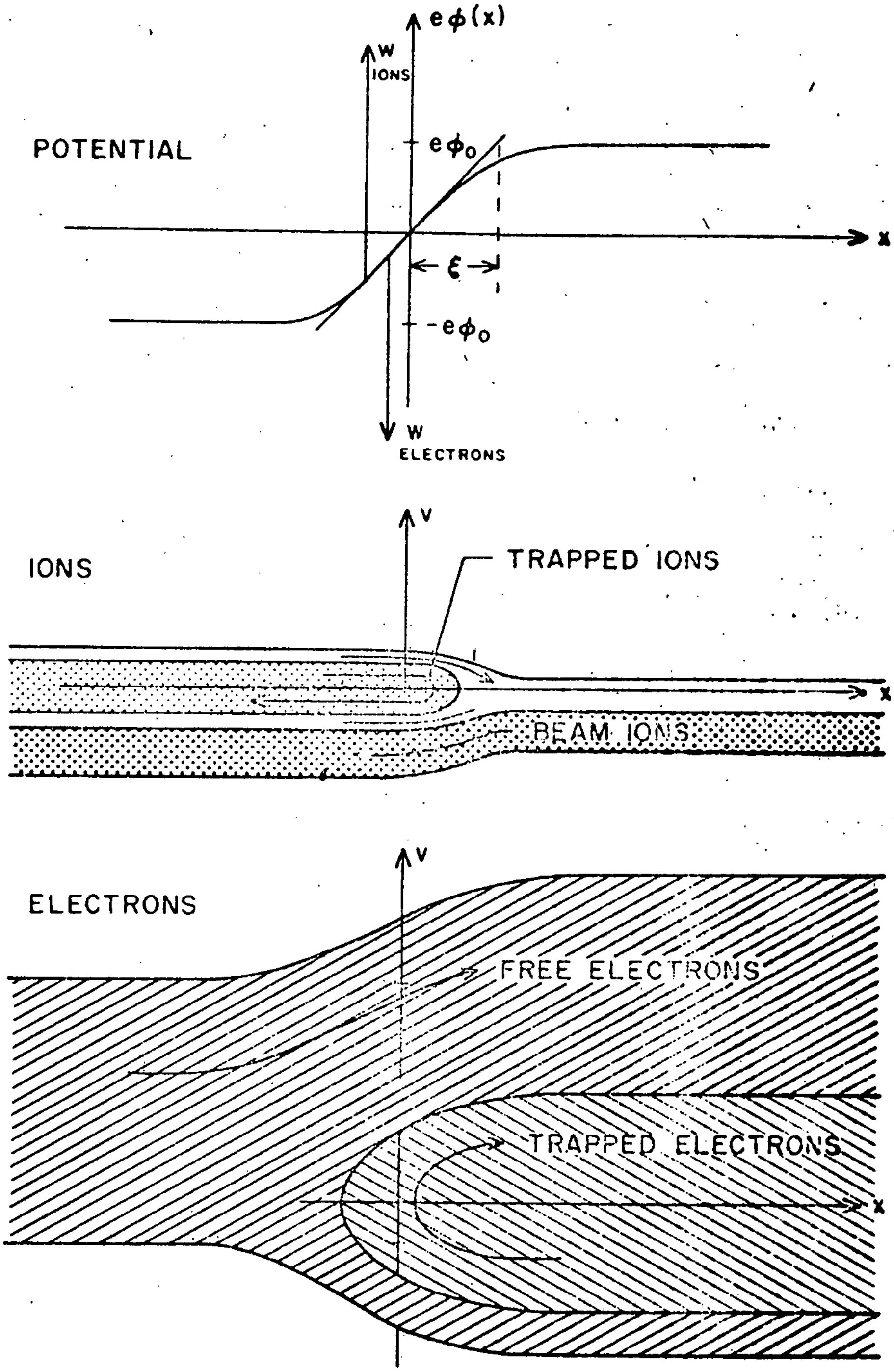
LIST OF FIGURES

B-G74-207

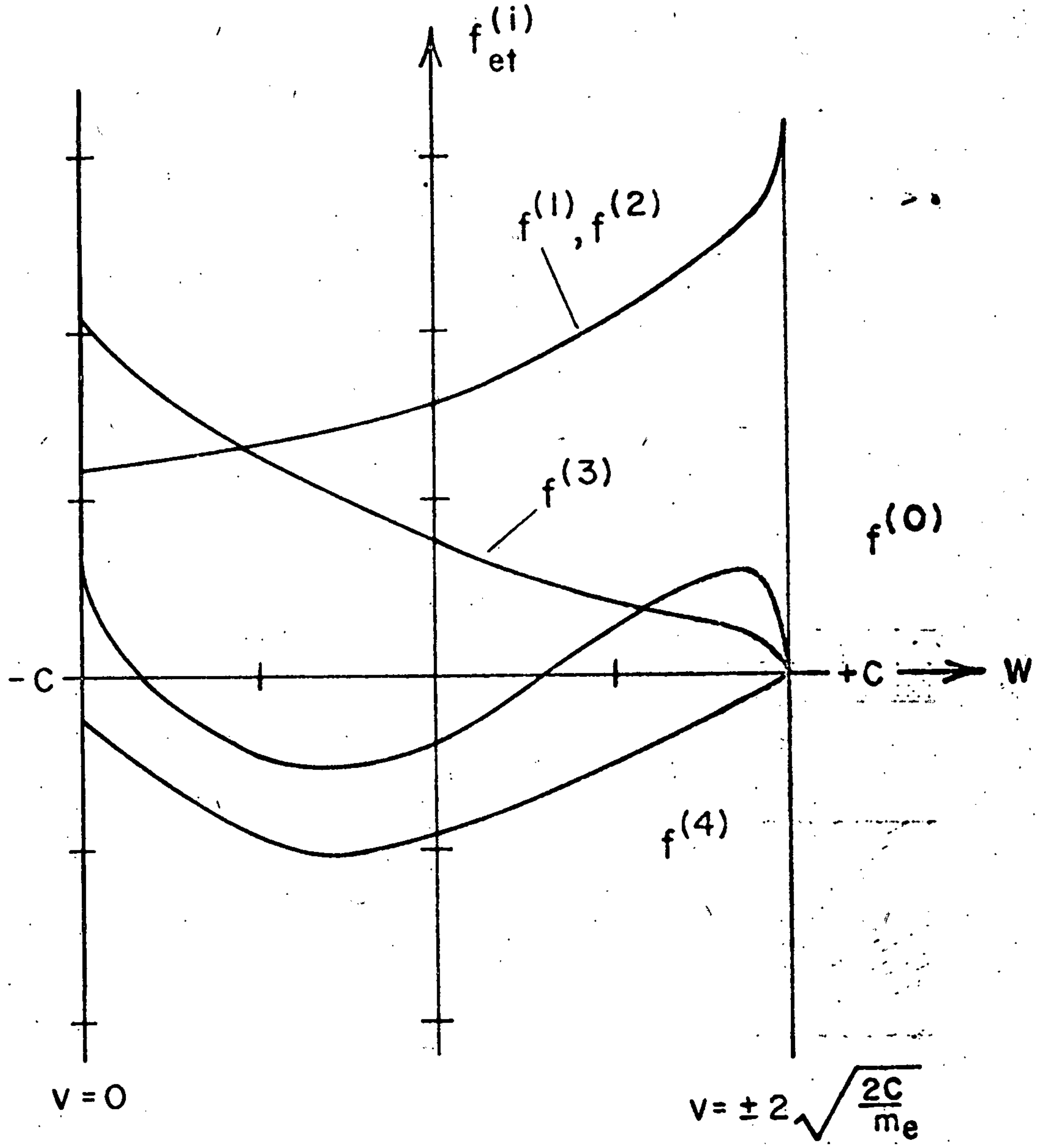
(a)

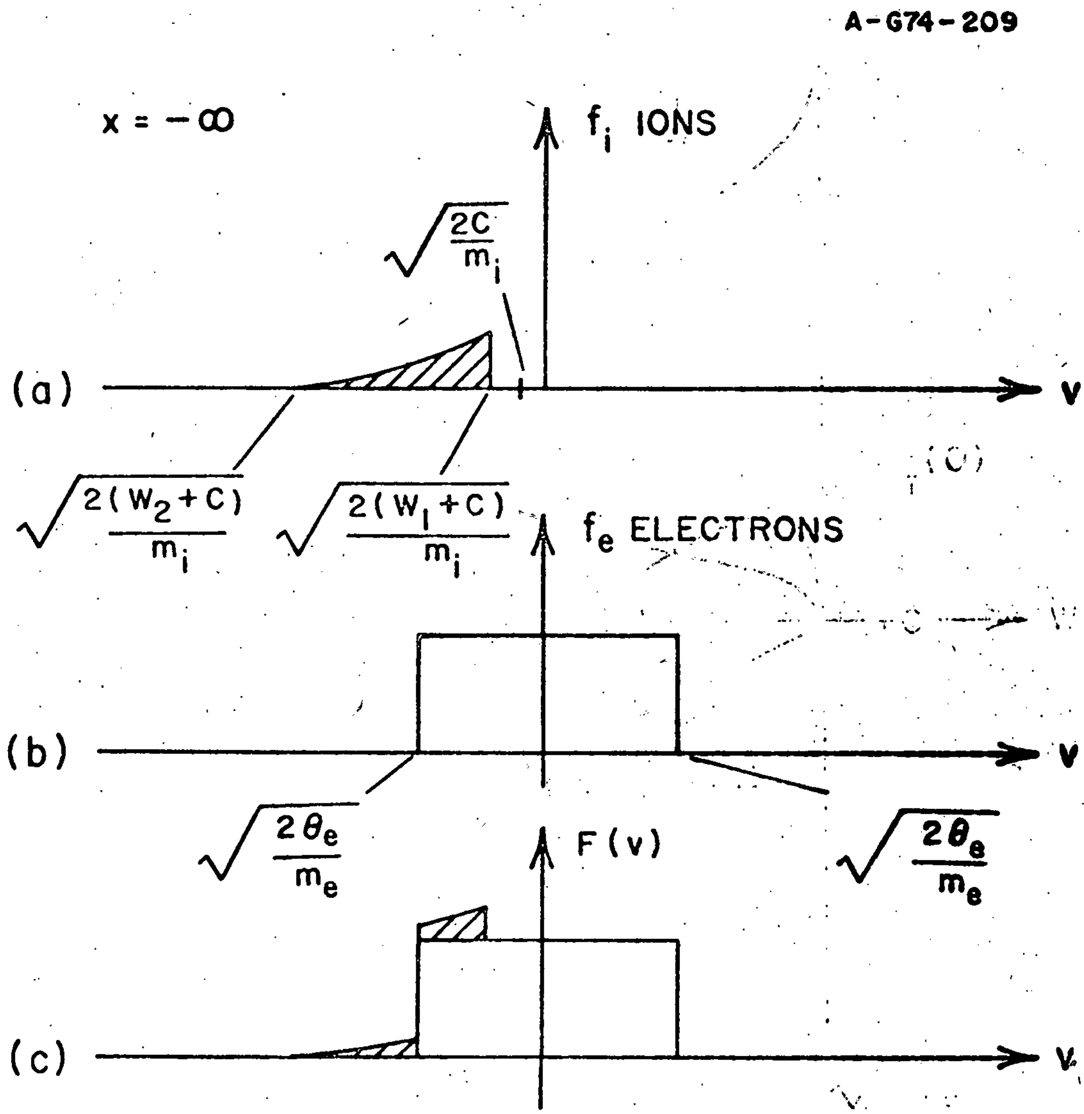

eφ(x)
W IONS
POTENTIAL
eφ₀
X
ξ
-eφ₀
W ELECTRONS
V
IONS
TRAPPED IONS
BEAM IONS
X
V
ELECTRONS
FREE ELECTRONS
TRAPPED ELECTRONS
X

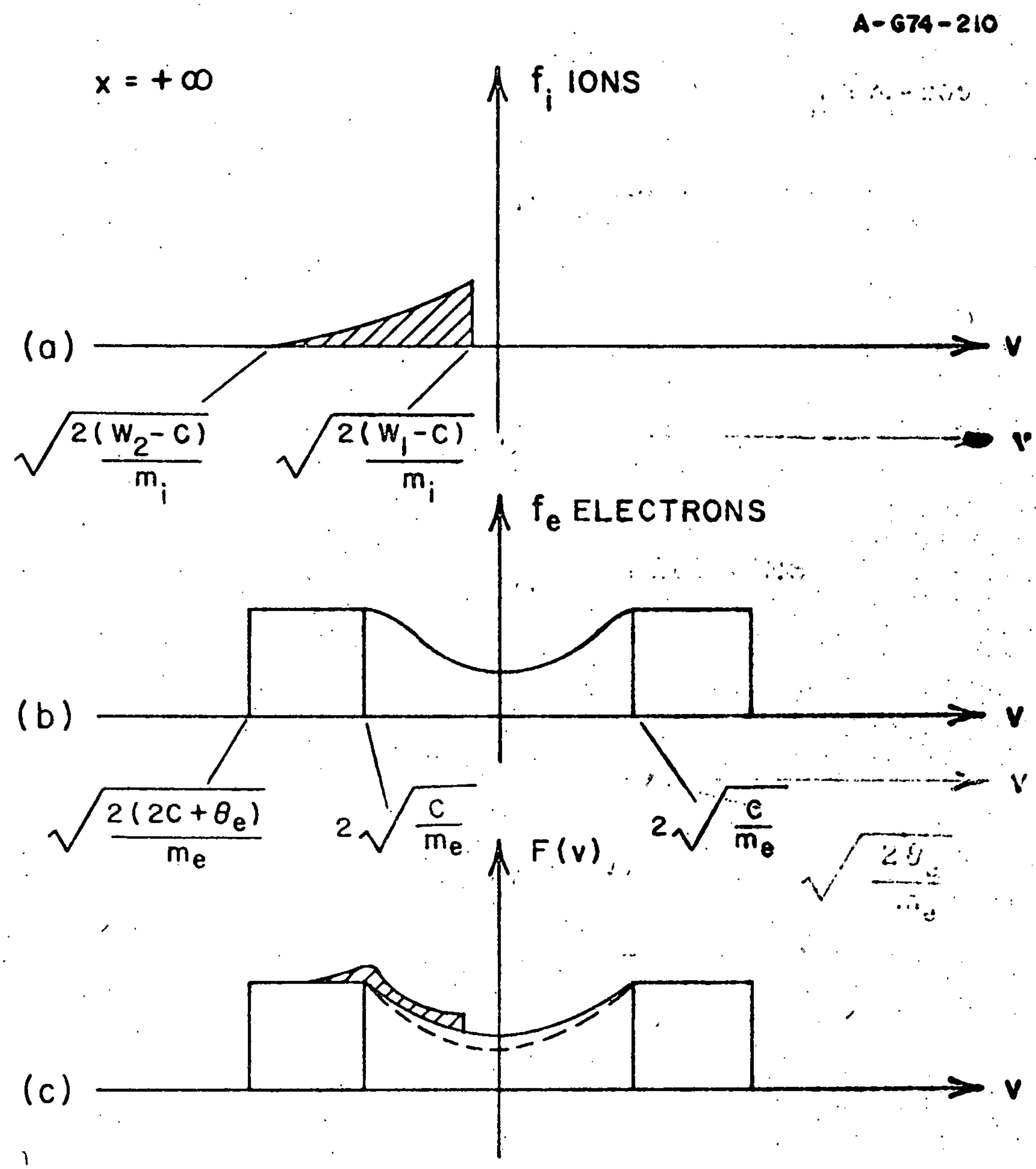
(b)

(c)

A-G74-208




A-G74-209
x = −∞
f_i IONS
$\sqrt{\frac{2C}{m_i}}$
(a)
v
$\sqrt{\frac{2(W_2+C)}{m_i}}$
$\sqrt{\frac{2(W_1+C)}{m_i}}$
f_e ELECTRONS
(b)
v
$\sqrt{\frac{2\theta_e}{m_e}}$
$\sqrt{\frac{2\theta_e}{m_e}}$
F(v)
(c)
v


A-G74-210
x = +∞
f_i IONS
(a)
v
$\sqrt{\frac{2(W_2-C)}{m_i}}$
$\sqrt{\frac{2(W_1-C)}{m_i}}$
f_e ELECTRONS
(b)
v
$\sqrt{\frac{2(2C+\theta_e)}{m_e}}$
$2\sqrt{\frac{C}{m_e}}$
$2\sqrt{\frac{C}{m_e}}$
F(v)
(c)
v