DOI 10.7603/s40956-015-0002-9

Moroccan J. Pure and Appl. Anal.(MJPAA) RESEARCH ARTICLE
Volume 1(1), 2015, Pages 22–37
ISSN: 2351-8227

# Existence and Stability Results for Nonlinear Boundary Value Problem for Implicit Differential Equations of Fractional Order

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Abstract. In this paper, we establish sufficient conditions for the existence and stability of solutions for a class of boundary value problem for implicit fractional differential equations with Caputo fractional derivative. The arguments are based upon the Banach contraction principle. Two examples are included to show the applicability of our results.

**2010 Mathematics Subject Classification.** 26A33, 34A08.

**Key words and phrases.** Boundary value conditions, Caputo's fractional derivative, implicit fractional differential equations, fractional integral, existence, stability, fixed point.

## 1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). See, for example, the books ([2, 3, 6, 9, 10, 24, 25]), the papers [4, 5, 11] and the references therein.

In recent years, fractional differential equations arise naturally in various fields such as rheology, fractals, chaotic dynamics, modeling and control theory, signal processing, bioengineering and biomedical applications, etc; Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. We refer the reader, for example, to the books [10, 21, 30] and the references therein.

Received January 8, 2015 - Accepted June 25, 2015.



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Published online: 23 July 2015

The stability problem of functional equations (of group homomorphisms) was raised by Ulam in 1940 in a talk given at Wisconsin University ([31, 32]). The question posed by Ulam was "Under what conditions does there exist an additive mapping near an approximately additive mapping?" In 1941, Hyers [15] gave the first answer to the question of Ulam (for the additive mapping) in the case Banach spaces. In 1978, Rassias established the Hyers-Ulam stability of linear and nonlinear mapping. Jung [17, 18] investigated in 1988, the Hyers-Ulam stability of more general mapping on restricted domains. Obloza [23] in 1993, is the first author who has investigated the Hyers-Ulam stability of linear differential equations. After, many articles and books on this subject have been published in order to generalize the results of Hyers in many directions. For more detailed definitions of the Hyers-Ulam stability and the generalized Hyers-Ulam stability, we refer the reader to the papers [1, 7, 8, 14, 16, 19, 20, 22, 26, 29, 34, 35, 36] and the books [13, 27, 28]. Let us notice that Ulam-Hyers stability concept is quite significant in realistic problems in numerical analysis, biology and economics.

The purpose of this paper is to establish four types of Ulam stability, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability for the following problems of implicit fractional-order differential equations

$$^cD^{\alpha}y(t) = f(t, y(t), ^cD^{\alpha}y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \alpha \leq 1 \tag{1}$$

$$ay(0) + by(T) = c \tag{2}$$

where $^cD^{\alpha}$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function, and $a, b, c$ are real constants with $a + b \neq 0$, and

$$^cD^{\alpha}y(t) = f(t, y(t), ^cD^{\alpha}y(t)), \quad \text{for every } t \in J := [0,T], T > 0, \quad 0 < \alpha \leq 1 \tag{3}$$

$$y(0) + g(y) = y_0 \tag{4}$$

where $g : C([0,T], \mathbb{R}) \longrightarrow \mathbb{R}$ a continuous function and $y_0$ a real constant. This type of non-local Cauchy problem was introduced by Byszewski [12]. The author observed that the non-local condition is more appropriate than the local condition (initial) to describe correctly some physics phenomenons [12], and proved the existence and the uniqueness of weak solutions and also classical solutions for this type of problems. We take an example of non-local conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i)$$

where $c_i, \ i = 1, \ldots, p$ are constants and $0 < t_1 < \ldots < t_p \leq T$.

The present results initiate the concept of Ulam stablity for such class of problems.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of continuous functions from $J$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

By $L^1(J)$ we denote the space of Lebesgue-integrable functions $y : J \to \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

**Definition 2.1.** *([25]) The fractional (arbitrary) order integral of the function $h \in L^1([0,T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

*where $\Gamma$ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} t^{\alpha-1}e^{-t}dt$, $\alpha > 0$.*

**Definition 2.2.** *([21]) For a function $h$ given on the interval $[0,T]$, the Caputo fractional-order $\alpha$ of $h$, is defined by*

$$({}^cD^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

*where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number $\alpha$.*

**Lemma 2.1.** *([21]) Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$I^\alpha\left({}^cD^\alpha f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

**Lemma 2.2.** *([25]) Let $\alpha > 0$, so the homogenous differential equation of fractional order:*

$${}^cD^\alpha h(t) = 0,$$

*has a solution:*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

*where $c_i$, $i = 1, \ldots, n$ are constants and $n = [\alpha] + 1$.*

We state the following generalization of Gronwall's lemma for singular kernels.

**Lemma 2.3.** *([33]) Let $\upsilon : [0,T] \longrightarrow [0,+\infty)$ be a real function and $\omega(.)$ is a nonnegative, locally integrable function on $[0,T]$. Assume that there are constants $a > 0$ and $0 < \alpha \le 1$ such that*

$$\upsilon(t) \le \omega(t) + a\int_0^t (t-s)^{-\alpha} \upsilon(s)\, ds.$$

*Then, there exists a constant $K = K(\alpha)$ such that*

$$\upsilon(t) \le \omega(t) + Ka\int_0^t (t-s)^{-\alpha} \omega(s)\, ds, \quad \textit{for every } t \in [0,T].$$

For the implicit fractional-order differential equation (1), we adopt the definition in Rus [29] of the Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

**Definition 2.3.** *The equation* (1) *is Ulam-Hyers stable if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon, \ t \in J,$$

*there exists a solution* $y \in C^1(J, \mathbb{R})$ *of equation* (1) *with*

$$|z(t) - y(t)| \leq c_f \epsilon, \ t \in J.$$

**Definition 2.4.** *The equation* (1) *is generalized Ulam-Hyers stable if there exists* $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon, \ t \in J,$$

*there exists a solution* $y \in C^1(J, \mathbb{R})$ *of the equation* (1) *with*

$$|z(t) - y(t)| \leq \psi_f(\epsilon), \ t \in J.$$

**Definition 2.5.** *The equation* (1) *is Ulam-Hyers-Rassias stable with respect to* $\varphi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon\varphi(t), \ t \in J,$$

*there exists a solution* $y \in C^1(J, \mathbb{R})$ *of equation* (1) *with*

$$|z(t) - y(t)| \leq c_f \epsilon \varphi(t), \ t \in J.$$

**Definition 2.6.** *The equation* (1) *is generalized Ulam-Hyers-Rassias stable with respect to* $\varphi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_{f,\varphi} > 0$ *such that for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \varphi(t), \ t \in J,$$

*there exists a solution* $y \in C^1(J, \mathbb{R})$ *of equation* (1) *with*

$$|z(t) - y(t)| \leq c_{f,\varphi} \varphi(t), \ t \in J.$$

**Remark 2.1.** *A function* $z \in C^1(J, \mathbb{R})$ *is a solution of the inequality*

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon, \ t \in J,$$

*if and only if there exists a function* $g \in C(J, \mathbb{R})$ *(which depends on solution* $y$*) such that*

***i*):** $|g(t)| \leq \epsilon$, $\forall t \in J$.
***ii*):** ${}^cD^\alpha z(t) = f(t, z(t), {}^cD^\alpha z(t)) + g(t)$, $t \in J$.

**Remark 2.2.** *Clearly,*

***i*):** *Definition* (2.6) $\Rightarrow$ *Definition* (2.7)
***ii*):** *Definition* (2.8) $\Rightarrow$ *Definition* (2.9).

**Remark 2.3.** *A solution of the implicit fractional differential inequality*

$$|^c D^\alpha z(t) - f(t, z(t), ^c D^\alpha z(t))| \leq \epsilon, \ t \in J,$$

*is called an fractional $\epsilon-$solution of the implicit fractional differential equation* (1).

## 3. Existence and Ulam-Hyers stability of the boundary value problem

**Lemma 3.1.** *Let $0 < \alpha \leq 1$ and $h : [0,T] \longrightarrow \mathbb{R}$ be a continuous function. Then the linear problem*

$$^c D^\alpha y(t) = h(t), \ t \in J \tag{5}$$

$$ay(0) + by(T) = c \tag{6}$$

*has a unique solution which is given by:*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &- \frac{1}{a+b} \left[ \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds - c \right]. \end{aligned} \tag{7}$$

**Proof.** By integration of formula (5) we obtain :

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \tag{8}$$

We use condition (6) to compute the constant $y_0$, so we have:

$$ay(0) = ay_0 \ \ \text{and} \ \ by(T) = by_0 + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds$$

then, $ay(0) + by(T) = c$, since

$$y_0 = \frac{-1}{(a+b)} \left[ \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds - c \right].$$

Substituting in equation (8) leads to formula (7).

**Lemma 3.2.** *Let $f(t,u,v) : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then the problem* (1)-(2) *is equivalent to the problem:*

$$y(t) = \tilde{A} + I^\alpha g(t) \tag{9}$$

*where $g \in C(J, \mathbb{R})$ satisfies the functional equation*

$$g(t) = f(t, \tilde{A} + I^\alpha g(t), g(t))$$

*and*

$$\tilde{A} = \frac{1}{a+b} \left[ c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \right].$$

**Proof.** Let $y$ be solution of (9). We shall show that $y$ is solution of (1)$-$(2). We have

$$y(t) = \tilde{A} + I^{\alpha} g(t).$$

So, $y(0) = \tilde{A}$ and $y(T) = \tilde{A} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^T (T-s)^{\alpha-1} g(s)ds.$

$$\begin{aligned} ay(0) + by(T) &= \frac{-ab}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds \\ &\quad + \frac{ac}{a+b} - \frac{b^2}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds \\ &\quad + \frac{bc}{a+b} + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds. \\ &= c. \end{aligned}$$

Then

$$y(0) + by(T) = c.$$

On the other hand, we have

$$\begin{aligned} {}^cD^{\alpha} y(t) &= {}^cD^{\alpha}(\tilde{A} + I^{\alpha} g(t)) = g(t) \\ &= f(t, y(t), {}^cD^{\alpha} y(t)). \end{aligned}$$

Thus, $y$ is solution of problem (1)-(2).

**Lemma 3.3.** *Assume assumption*

(H1) *there exist two constants $K > 0$ et $0 < L < 1$ such that*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq K\,|u - \overline{u}| + L\,|v - \overline{v}| \quad \text{for each } t \in J \text{ and } u, \overline{u}, v, \overline{v} \in \mathbb{R}.$$

*If*

$$\frac{KT^{\alpha}}{(1-L)\,\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) < 1, \tag{10}$$

*the problem (1)-(2) has a unique solution.*

**Proof.** Let the operator

$$\begin{aligned} N \quad &: \quad C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R}) \\ Ny(t) \quad &= \quad \tilde{A}_y + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds, \end{aligned}$$

where

$$g_y(t) = f(t, \tilde{A}_y + I^{\alpha} g_y(t), g_y(t)),$$

and

$$\tilde{A}_y = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g_y(s)ds\right].$$

By Lemmas 3.1 and 3.2, it is clear that the fixed points of $N$ are solutions of (1)-(2). Let $y_1, y_2 \in C(J,\mathbb{R})$, and $t \in J$, then we have

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|g_{y_1}(s) - g_{y_2}(s)\right| ds \\ &\quad + \frac{|b|}{|a+b|\,\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|g_{y_1}(s) - g_{y_2}(s)\right| ds, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \left|g_{y_1}(t) - g_{y_2}(t)\right| &= |f(t, y_1(t), {}^cD^\alpha y_1(t)) - f(t, y_2(t), {}^cD^\alpha y_2(t))| \\ &\leq K\,|y_1(t) - y_2(t)| + L\left|g_{y_1}(t) - g_{y_2}(t)\right|. \end{aligned}$$

Thus

$$\left|g_{y_1}(t) - g_{y_2}(t)\right| \leq \frac{K}{1-L}\,|y_1(t) - y_2(t)|. \tag{12}$$

By replacing (12) in the inequality (11), we obtain

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq \frac{K}{(1-L)\,\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y_1(s) - y_2(s)|\, ds \\ &\quad + \frac{|b|\,K}{(1-L)\,|a+b|\,\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |y_1(s) - y_2(s)|\, ds \\ &\leq \frac{KT^\alpha}{(1-L)\,\Gamma(\alpha+1)} \|y_1 - y_2\|_\infty \\ &\quad + \frac{|b|\,KT^\alpha}{(1-L)\,|a+b|\,\Gamma(\alpha+1)} \|y_1 - y_2\|_\infty. \end{aligned}$$

Then

$$\|Ny_1 - Ny_2\|_\infty \leq \left[\frac{KT^\alpha}{(1-L)\,\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\right] \|y_1 - y_2\|_\infty.$$

From (10), it follows that $N$ has a unique fixed point which is solution of problem (1)-(2).

**Theorem 3.1.** *Assume that* $(H1)$ *and (10) are satisfied, then the problem (1)-(2) is Ulam-Hyers stable.*

**Proof.** Let $\epsilon > 0$ and let $z \in C^1(J,\mathbb{R})$ be a function which satisfies the inequality:

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon \quad \text{for any } t \in J \tag{13}$$

and let $y \in C(J,\mathbb{R})$ be the unique solution of the following Cauchy problem

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)); \;\; t \in J; \;\; 0 < \alpha \leq 1 \\ y(0) = z(0), y(T) = z(T). \end{cases}$$

Using Lemmas 3.1 and 3.2, we obtain

$$y(t) = \tilde{A}_y + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s) ds.$$

On the other hand, if $y(T) = z(T)$ and $y(0) = z(0)$, then $\tilde{A}_y = \tilde{A}_z$. Indeed

$$\left|\tilde{A}_y - \tilde{A}_z\right| \leq \frac{|b|}{|a+b|\,\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|g_y(s) - g_z(s)\right| ds$$

and by the inequality (12), we find

$$\begin{aligned}\left|\tilde{A}_y - \tilde{A}_z\right| &\leq \frac{|b|\,K}{(1-L)\,|a+b|\,\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left|y(s) - z(s)\right| ds \\ &= \frac{|b|\,K}{(1-L)\,|a+b|} I^{\alpha} \left|y(T) - z(T)\right| = 0.\end{aligned}$$

Thus

$$\tilde{A}_y = \tilde{A}_z.$$

Then, we have

$$y(t) = \tilde{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)\, ds.$$

By integration of the inequality (13), we obtain

$$\left|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds\right| \leq \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)} \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)},$$

with

$$g_z(t) = f(t, \tilde{A}_z + I^{\alpha} g_z(t), g_z(t)).$$

We have for any $t \in J$

$$\begin{aligned}|z(t) - y(t)| &= \left|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds\right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(g_z(s) - g_y(s)\right) ds\right| \\ &\leq \left|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds\right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|g_z(s) - g_y(s)\right| ds.\end{aligned}$$

Using (12), we obtain

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{(1-L)\,\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|z(s) - y(s)\right| ds,$$

and by the Gronwall's lemma, we get

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} \left[1 + \frac{\gamma K T^{\alpha}}{(1-L)\,\Gamma(\alpha+1)}\right] := c\epsilon$$

where $\gamma = \gamma(\alpha)$ a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c\epsilon$; $\psi(0) = 0$, then the problem (1)-(2) is generalized Ulam-Hyers stable.

**Theorem 3.2.** *Assume that (H1), (10) and*

(H2) *there exists an increasing function* $\varphi \in C(J, \mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$

$$I^\alpha \varphi(t) \le \lambda_\varphi \varphi(t)$$

*are satisfied, then, the problem (1)-(2) is Ulam-Hyers-Rassias stable.*

**Proof.** Let $z \in C^1(J, \mathbb{R})$ be solution of the following inequality

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \le \epsilon\varphi(t)\ ,\ t \in J\ ,\ \epsilon > 0 \tag{14}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of Cauchy problem:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)); & t \in J;\ 0 < \alpha \le 1 \\ y(0) = z(0),\ y(T) = z(T). \end{cases}$$

By Lemmas 3.1 and 3.2, we have

$$y(t) = \tilde{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)\, ds,$$

where $g_y \in C(J, \mathbb{R})$ satisfies the equation:

$$g_y(t) = f(t, \tilde{A}_z + I^\alpha g_y(t), g_y(t)),$$

and

$$\tilde{A}_z = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g_z(s)\, ds\right].$$

By integration of (14), we obtain

$$\begin{aligned} \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right| &\le \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds \\ &\le \epsilon\lambda_\varphi \varphi(t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (g_z(s) - g_y(s))\, ds \right| \\ &\le \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_z(s) - g_y(s)|\, ds. \end{aligned}$$

Using (12), we have

$$|z(t) - y(t)| \le \epsilon\lambda_\varphi \varphi(t) + \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)|\, ds.$$

By applying Gronwall's lemma, we get that for any $t \in J$ :

$$|z(t) - y(t)| \le \epsilon\lambda_\varphi \varphi(t) + \frac{\gamma_1 \epsilon K \lambda_\varphi}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds$$

where $\gamma_1 = \gamma_1(\alpha)$ is constant, and by $(H_2)$, we have:

$$|z(t) - y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1\epsilon K\lambda_\varphi^2\varphi(t)}{(1-L)} = \left(1 + \frac{\gamma_1 K\lambda_\varphi}{(1-L)}\right)\epsilon\lambda_\varphi\varphi(t).$$

Then for any $t \in J$ :

$$|z(t) - y(t)| \leq \left[\left(1 + \frac{\gamma_1 K\lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\epsilon\varphi(t) = c\epsilon\varphi(t)$$

which completes the proof of Theorem 3.2.

**Remark 3.1.** *Our results for the boundary value problem (1)-(2) are appropriate for the following problems:*

- *Initial value problem:* $a = 1, b = 0, c = 0$.
- *Terminal value problem:* $a = 0, b = 1, c$ *arbitrary.*
- *Anti-periodic problem:* $a = 1,\ b = 1,\ c = 0$.

*However, they are not for the periodic problem, i.e. for* $a = 1, b = -1, c = 0$.

## 4. Existence and Ulam-Hyers Stability of the nonlocal boundary value problem

**Lemma 4.1.** *Let* $0 < \alpha \leq 1$ *and let* $h : [0,T] \longrightarrow \mathbb{R}$ *a continuous function. Then the linear problem*

$$\begin{aligned} {}^cD^\alpha y(t) &= h(t), \quad t \in J \\ y(0) + g(y) &= y_0 \end{aligned}$$

*has a unique solution which is given by:*

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds.$$

**Lemma 4.2.** *Let* $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function, then the problem* (3)-(4) *is equivalent to the following problem*

$$y(t) = y_0 - g(y) + I^\alpha K_y(t)$$

*where*

$$K_y(t) = f(t, y(t), K_y(t)).$$

**Theorem 4.1.** *Assume*

(P1) *there exist* $K > 0$, $0 < \overline{K} < 1$ *and* $0 < L < 1$ *such that:*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq K|u - \overline{u}| + \overline{K}|v - \overline{v}| \text{ for any} u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

*and*

$$\|g(y) - g(\overline{y})\| \leq L\|y - \overline{y}\| \text{ for any} y, \overline{y} \in C(J, \mathbb{R}).$$

*If*

$$L + \frac{KT^\alpha}{(1-\overline{K})\Gamma(\alpha+1)} < 1 \tag{15}$$

*then, the boundary value problem (3) -(4) has a unique solution on J.*

**Proof.** Let the operator

$$\begin{aligned} N &: C(J,\mathbb{R}) \to C(J,\mathbb{R}) \\ Ny(t) &= y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_y(s)\, ds \end{aligned}$$

where

$$K_y(t) = f\left(t, y_0 - g(y) + I^\alpha K_y(t), K_y(t)\right).$$

By Lemmas 4.1 and 4.2, it is easy to see that the fixed points of $N$ are the solutions of the problem (3) -(4). Let $y_1, y_2 \in C(J,\mathbb{R})$, we have for any $t \in J$

$$|Ny_1(t) - Ny_2(t)| \leq |g(y_1) - g(y_2)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_{y_1}(s) - K_{y_2}(s)|\, ds$$

then

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq L|y_1(t) - y_2(t)| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |K_{y_1}(s) - K_{y_2}(s)|\, ds. \end{aligned} \tag{16}$$

On the other hand, we have for every $t \in J$

$$\begin{aligned} |K_{y_1}(t) - K_{y_2}(t)| &= |f(t, y_1(t), K_{y_1}(t)) - f(t, y_2(t), K_{y_2}(t))| \\ &\leq K|y_1(t) - y_2(t)| + \overline{K}|K_{y_1}(t) - K_{y_2}(t)|. \end{aligned}$$

Thus

$$|K_{y_1}(t) - K_{y_2}(t)| \leq \frac{K}{1-\overline{K}} |y_1(t) - y_2(t)|. \tag{17}$$

By replacing (17) in the inequality (16), we obtain

$$\begin{aligned} |Ny_1(t) - Ny_2(t)| &\leq L|y_1(t) - y_2(t)| \\ &\quad + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y_1(s) - y_2(s)| \\ &\leq \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] \|y_1 - y_2\|_\infty. \end{aligned}$$

Thus

$$\|Ny_1 - Ny_2\|_\infty \leq \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] \|y_1 - y_2\|_\infty$$

from which it follows that $N$ is a contraction which implies that $N$ admits a unique fixed point which is solution of the problem (3) -(4).

**Theorem 4.2.** *Assume that* $(P1)$ *and the inequality (15) are satisfied, then the problem (3)-(4) is Ulam-Hyers stable.*

**Proof.** Let $\epsilon > 0$ and let $z \in C^1(J,\mathbb{R})$ satisfying the inequality:

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon \text{ for every } t \in J \tag{18}$$

and let $y \in C(J,\mathbb{R})$ the unique solution of the Cauchy problem:

$$\begin{cases} {}^cD^{\alpha}y(t) = f\left(t, y\left(t\right), {}^cD^{\alpha}y(t)\right), & t \in J, \quad 0 < \alpha \leq 1 \\ z(0) + g(y) = y_0 \end{cases}$$

so

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_y(s)\, ds,$$

where

$$K_y(t) = f\left(t, y(t), K_y(t)\right).$$

By integration of the inequality (18), we find

$$\left| z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)\, ds \right| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}$$

where $K_z(t) = f\left(t, z(t), K_z(t)\right)$. For every $t \in J$, we have

$$\begin{aligned} |z(t) - y(t)| &\leq \left| z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)\, ds \right| \\ &\quad + \left| g(y) - g(z) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(K_z(s) - K_y(s)\right) ds \right| \\ &\leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + |g(z) - g(y)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|K_z(s) - K_y(s)\right| ds. \end{aligned}$$

Using (17), we obtain

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + L\,|z(t) - y(t)| + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)|\, ds$$

thus

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{(1-L)\,\Gamma(\alpha+1)} + \frac{K}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)|\, ds.$$

Using Gronwall's Lemma, we obtain for every $t \in J$

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{(1-L)\,\Gamma(\alpha+1)} \left[ 1 + \frac{\gamma K T^{\alpha}}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha+1)} \right] := c\epsilon$$

where $\gamma = \gamma(\alpha)$ a constant, so the problem (3)-(4) is Ulam-Hyers stable. If we set $\psi(\epsilon) = c\epsilon$; $\psi(0) = 0$, then the problem $(3)-(4)$ is generalized Ulam-Hyers stable.

**Theorem 4.3.** *Assume that* $(P1)$, *inequality* (15) *and*

(P2) *there exist an increasing function* $\varphi \in C(J, \mathbb{R}_+)$ *and* $\lambda_{\varphi} > 0$ *such that*

$$I^{\alpha}\varphi(t) \leq \lambda_{\varphi}\varphi(t) \;\; \textit{for each } t \in J$$

*are satisfied, then the problem* (3)-(4) *is Ulam-Hyers-Rassias stable.*

## 5. Examples

**Example 1.** Consider the following boundary value problem

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{1}{10e^{t+2}(1+|y(t)|+|{}^cD^{\frac{1}{2}}y(t)|)}, \text{ for each } t \in [0,1] \tag{19}$$

$$ y(0)+y(1)=0. \tag{20}$$

Set

$$ f(t,u,v) = \frac{1}{10e^{t+2}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}.$$

Clearly, the function $f$ is continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{10e^2}(|u-\bar{u}|+|v-\bar{v}|).$$

Hence condition $(H1)$ is satisfied with $K = L = \frac{1}{10e^2}$.
Thus condition

$$ \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) = \frac{3}{2(10e^2-1)\Gamma(\frac{3}{2})} = \frac{3}{(10e^2-1)\sqrt{\pi}} < 1,$$

is satisfied with $a = b = T = 1$, $c = 0$, and $\alpha = \frac{1}{2}$. It follows from Lemma 3.3 that the problem (19)-(20) has a unique solution on $J$. Moreover, Theorem 3.1 implies that the problem (19)-(20) is Ulam-Hyers stable.

**Example 2.** Consider the boundary value problem:

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|} - \frac{\left|{}^cD^{\frac{1}{2}}y(t)\right|}{1+\left|{}^cD^{\frac{1}{2}}y(t)\right|}\right], \quad t \in J = [0,1] \tag{21}$$

$$ y(0) + \sum_{i=1}^{n} c_i y(t_i) = 1, \tag{22}$$

where $0 < t_1 < t_2 < \ldots < t_n < 1$ and $c_i = 1, \ldots, n$ are positive constants with

$$ \sum_{i=1}^{n} c_i \le \frac{1}{3}.$$

Set

$$ f(t,u,v) = \frac{e^{-t}}{(9+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right], \ t \in [0,1], \ u,v \in [0,+\infty).$$

Clearly, the function $f$ is continuous. For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$ :

$$ \begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\le \frac{e^{-t}}{(9+e^t)}(|u-\bar{u}|+|v-\bar{v}|) \\ &\le \frac{1}{10}|u-\bar{u}| + \frac{1}{10}|v-\bar{v}|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |g(u)-g(\bar{u})| &= \left|\sum_{i=1}^{n} c_i u - \sum_{i=1}^{n} c_i \bar{u}\right| \\ &\leq \sum_{i=1}^{n} c_i \left|u-\bar{u}\right| \\ &\leq \frac{1}{3}\left|u-\bar{u}\right|. \end{aligned}$$

Hence condition $(P1)$ is satisfied with $K=\overline{K}=\dfrac{1}{10}$ and $L=\dfrac{1}{3}$. We have

$$L+\frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma\left(\alpha+1\right)}=\frac{1}{3}+\frac{1}{9\Gamma\left(\frac{3}{2}\right)}=\frac{9\sqrt{\pi}+6}{27\sqrt{\pi}}<1.$$

It follows from Lemma 4.1 that the problem (21)- (22) has a unique solution on $J$ and by Theorem 4.2, the problem (21)-(22) is Ulam-Hyers stable.

**Remark 5.1.** *The main results of Example 2 stay available when*

$$g(t)=\frac{1}{4}\left(\frac{|y(t)|}{1+|y(t)|}\right)$$

*and*

$$L+\frac{KT^{\alpha}}{\left(1-\overline{K}\right)\Gamma\left(\alpha+1\right)}=\frac{1}{4}+\frac{1}{9\Gamma\left(\frac{3}{2}\right)}=\frac{9\sqrt{\pi}+8}{36\sqrt{\pi}}<1.$$

**Acknowledgement.** The authors are grateful to the referees for the careful reading of the paper.



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