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EXISTENCE AND SYMMETRY RESULTS FOR A SCHRÖDINGER TYPE PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN

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This paper deals with the following class of nonlocal Schrödinger equations

$$(-\Delta)^s u + u = |u|^{p-1}u \ \text{ in } \mathbb{R}^N, \quad \text{for } s \in (0,1).$$

We prove existence and symmetry results for the solutions u in the fractional Sobolev space $H^s(\mathbb{R}^N)$. Our results are in clear accordance with those for the classical local counterpart, that is when $s = 1$.

1. Introduction

We consider the following problem

$$\begin{cases} -\Delta u + \eta u = \lambda |u|^{p-1}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ u \not\equiv 0, \end{cases} \tag{1}$$

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where λ and η are fixed positive constants and $p > 1$.

The equation in (1) has been widely studied in the last decades, since it is the basic version of some fundamental models arising in various applications (e. g., stationary states in nonlinear equations of Schrödinger type). One of the first contributions to the analysis of problem (1) was given by Pohozaev in [19], where he proved that there exists a solution u of (1) if and only if $1 < p < 2^*-1$, being $2^* = 2N/(N-2)$ the so-called Sobolev critical exponent. In [19] also a by-now classical "identity" appears, in order to prove that there are no solutions to (1) when p is greater or equal than 2^*-1.

Another important contribution to the analysis of problem (1) has been given in [4] (see also [5]), in which the authors consider an extension of the equation in (1) by replacing the nonlinearity $-\eta u + \lambda |u|^{p-1}u$ by a wider class of odd continuous functions $g = g(u)$ satisfying $g(0) = 0$ and some superlinear and growth assumptions. Among other results, in [4] it has been shown the existence of a solution u to (1), with some properties of symmetry and a precise decay at infinity. It is worth pointing out that the method to prove the existence of solutions to (1) relies on a variational approach $\big($the *constrained minimization method*, see [4, Section 3]$\big)$, by working directly with the energy functional related to (1).

A natural question could be whether or not this method can be adapted to deal with a nonlocal version of the problem above. In this respect, the aim of the present paper is to extend the existence and symmetry results in [4] for the nonlocal analogue of problem (1) by replacing the standard Laplacian operator by the fractional Laplacian operator $(-\Delta)^s$, where, as usual, for any $s \in (0,1)$, $(-\Delta)^s$ denotes the s-power of the Laplacian operator and, omitting a multiplicative constant $C = C(N,s)$, we have

$$(-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \lim_{\varepsilon\to 0}\int_{\mathscr{C}B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy. \qquad (2)$$

Here $B_\varepsilon(x)$ denotes the N-dimensional ball of radius ε, centered at $x \in \mathbb{R}^N$, $\mathscr{C}$ denotes the complementary set, and "*P.V.*" is a commonly used abbreviation for "in the principal value sense".

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as, among the others, obstacle problems, financial market, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science, water waves, etc... The literature is really too wide to attempt any reasonable comprehensive treatment in a single paper[1]. We would just cite some very recent papers which analyze fractional elliptic equations involving the critical Sobolev exponent, [2, 6, 9, 16, 23, 24, 26].

Let us come back to the present paper. We will deal with the following problem

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-1}u \ \text{ in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \ u \not\equiv 0, \end{cases} \tag{3}$$

where $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space; we immediately refer to Section 2.2 for the definitions of the space $H^s(\mathbb{R}^N)$ and of variational solutions to (3).

Precisely, we are interested in existence and symmetry properties of the variational solutions u to (3), as stated in the following

Theorem 1.1. *Let $s \in (0,1)$ and $p \in (1, (N+2s)/(N-2s))$, with $N > 2s$. There exists a solution $u \in H^s(\mathbb{R}^N)$ to problem* (3) *which is positive and spherically symmetric.*

Note that the upperbound on the exponent p is exactly $2^*_s + 1$, where $2^*_s = 2N/(N-2s)$ is the critical Sobolev exponent of the embedding $H^s \hookrightarrow L^p$. This fractional Sobolev exponent also plays a role for the nonlinear analysis methods for equations in bounded domains; see [23]. As in the classical case, the threshold given by this exponent is essentially optimal, since non-existence results may be obtained from a fractional Pohozaev identity (see, e.g., Lemma 5.1 in [9], and Theorem 1.1 in [22]).

The proof of Theorem 1.1 extends part of that of Theorem 2 in [4]; in particular, we will apply the variational approach by the constrained method mentioned above, for the energy functional related to (3), that is

$$\begin{aligned} \mathscr{E}(u) &:= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{1}{2}|u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right)dx. \end{aligned} \tag{4}$$

It is worth mentioning that the results in Theorem 1.1 for $N = 1$ have been obtained in [28], where modulation stability of ground states solitary wave

[1] For an elementary introduction to this topic and a wide, but still not fully comprehensive, list of related references, we refer to [7].

solutions of nonlinear Schrödinger equations has been studied, via an *unconstrained* variational approach within the “concentration-compactness” framework of P.L. Lions ([11, 12]). Also, in the more recent papers [13] and [14], an alternative approach has been presented, which permits to handle a very general context, also including the equations we are dealing with (see, in addition, [15], where the decay of solutions is analyzed in the case $s = 1/2$).

Here, we will present a very simple proof, whose general strategy will follow the original argument in [4]. The method used here (and in [4]) relies on the selection of a specific minimizing sequence composed of radial functions: though this idea is now classical, we thought it was interesting to point out that this argument also works in the case of the fractional Laplacian. Clearly, we need to operate various technical modifications due to the non-locality of the fractional Laplacian operator $\big($and of the correspondent norm $H^s(\mathbb{R}^N)\big)$. Moreover, we will need some energy estimates and preliminary results, also including the analogue of the classical Polya-Szegö inequality, as given in the forthcoming Section 2.3. Given the elementary nature of this note, we put an effort in making all the arguments as transparent as possible.

As for the precise decay of the solution found, a precise bound may be obtained via the construction of exact barriers (see Lemma 3.1 in [21] and, also, Lemma 8 in [17]). Also, it could be taken into account to extend all the results above in order to investigate a problem of type (3) by substituting the nonlinearity with an odd continuous function satisfying standard growth assumptions, in the same spirit of [4].[2]

The paper is organized as follows. In Section 2 below, we fix notation and we state and prove some preliminary results. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

In this section, we state and prove a few preliminary results that we will need in the rest of the paper. First, we will recall some definitions involving the fractional Laplacian operator and we give the definition of the solutions to the problem we are dealing with.

2.1. Notation

In the present paper we follow the usual convention of denoting by C a general positive constant, possibly varying from line to line. Relevant dependencies on

[2] After completing this project, we have heard of an interesting work, where related results have been presented by using different techniques (see [8]).

parameters will be emphasized by using parentheses; special constants will be denoted by C_1, C_2, ...

We consider the Schwartz space $\mathscr{S}$ of rapidly decaying C^∞ functions in $\mathbb{R}^N$, with the corresponding topology generated by the seminorms

$$\Pi_M(\varphi) = \sup_{x\in\mathbb{R}^N} (1+|x|)^M \sum_{|\alpha|\leq M} |D^\alpha \varphi(x)|, \quad M = 0,1,2,\dots,$$

where $\varphi \in \mathscr{S}(\mathbb{R}^N)$. Let $\mathscr{S}'(\mathbb{R}^N)$ be the set of all tempered distributions, that is the topological dual of $\mathscr{S}(\mathbb{R}^N)$. As usual, for any $\varphi \in \mathscr{S}(\mathbb{R}^N)$, we denote by

$$\mathscr{F}\varphi(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} \varphi(x)\, dx$$

the Fourier transform of φ and we recall that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^N)$ to $\mathscr{S}'(\mathbb{R}^N)$.

For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \;:\; \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}, \tag{5}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2\, dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx\, dy \right)^{\frac{1}{2}},$$

where the term

$$[u]_{H^s(\mathbb{R}^N)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx\, dy \right)^{\frac{1}{2}} \tag{6}$$

is the so-called *Gagliardo semi-norm* of u.

2.2. A few basic results on the fractional Laplacian and setting of the problem

In the following, we make use of equivalent definitions of the fractional Laplacian and the Gagliardo semi-norm via the Fourier transform. Indeed, the fractional Laplacian $(-\Delta)^s$ can be seen as a pseudo-differential operator of symbol $|\xi|^s$, as stated in the following

Proposition 2.1. (see, e.g., *[7, Proposition 3.3]* or *[27, Section 3]*). *Let* $s \in (0,1)$ *and let* $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^N)$ *be the fractional operator defined by* (2). *Then, for any* $u \in \mathscr{S}$,

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)) \quad \forall \xi \in \mathbb{R}^N,$$

up to a multiplicative constant.

Analogously, one can see that the fractional Sobolev space $H^s(\mathbb{R}^N)$, given by (5), can be defined via the Fourier transform as follows

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} (1+|\xi|^{2s})|\mathscr{F}u(\xi)|^2 \, d\xi < +\infty \right\}. \tag{7}$$

This is a natural consequence of the equivalence stated in the following proposition, whose proof relies on the Plancherel formula.

Proposition 2.2. (see, e.g., *[7, Proposition 3.4]*). *Let* $s \in (0,1)$. *For any* $u \in H^s(\mathbb{R}^N)$

$$[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u(\xi)|^2 \, d\xi, \tag{8}$$

up to a multiplicative constant.

Finally, we recall the definition of variational solutions $u \in H^s(\mathbb{R}^N)$ to

$$(-\Delta)^s u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \quad u \not\equiv 0, \tag{9}$$

where $p > 1$.

For any $s \in (0,1)$, a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is a variational solution to (9) if

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}} \, dx\, dy + \int_{\mathbb{R}^N} u(x)\varphi(x)\, dx = \int_{\mathbb{R}^N} |u(x)|^{p-1}u(x)\varphi(x)\, dx, \tag{10}$$

for any function $\varphi \in C^1_0(\mathbb{R}^N)$.

As stated in the Introduction, a natural method to solve (9) is to look for critical points of the related energy functional $\mathscr{E}$ on the space $H^s(\mathbb{R}^N)$ defined in (4), that is

$$\mathscr{E}(u) := \frac{1}{2}[u]^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} G(u)\, dx, \tag{11}$$

where $[u]_{H^s}$ is defined by (6) and we denoted by G the function

$$G(u) := \frac{1}{p+1}|u|^{p+1} - \frac{1}{2}|u|^2. \tag{12}$$

Therefore, from now on we will focus on the following variational problem

$$\min\left\{[u]^2_{H^s(\mathbb{R}^N)} : u \in H^s(\mathbb{R}^N),\ \int_{\mathbb{R}^N} G(u)\,dx = 1\right\}. \tag{13}$$

2.3. Tools

For any measurable function u consider the corresponding symmetric radial decreasing rearrangement u^*, whose classical definition and basic properties can be found, for instance, in [10, Chapter 2]. As in the classic case (i.e., the Polya-Szegö inequality [20]), also in the fractional framework the energy of u^* decreases with respect to that of u. Again, by using the Fourier characterization of $[u]_{H^s(\mathbb{R}^N)}$ given by Proposition 2.2, one can plainly apply the symmetrization lemma by Beckner ([3]; see also [1]) to obtain the following

Lemma 2.3. (see, e.g., *[18, Theorem 1.1]*). *Let $s \in (0,1)$. For any $u \in H^s(\mathbb{R}^N)$, the following inequality holds*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{N+2s}}dx\,dy \le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dx\,dy, \tag{14}$$

where u^* denotes the symmetric radial decreasing rearrangement of u.

Next we recall two results which we will use in the proof of Theorem 1.1 (see, in particular, Step 2 there). The first one is the following *radial lemma*.

Lemma 2.4. *Let $u \in L^2(\mathbb{R}^N)$ be a nonnegative radial decreasing function. Then*

$$|u(x)| \le \left(\frac{N}{\omega_{N-1}}\right)^{1/2} |x|^{-N/2}\|u\|_{L^2(\mathbb{R}^N)}, \quad \forall x \neq 0,$$

where ω_{N-1} is the Lebesgue measure of the unit sphere in $\mathbb{R}^N$.

Proof. Setting $r = |x|$, we have that, for every $r > 0$,

$$\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u(x)|^2\,dx \ge \omega_{N-1}\int_0^R |u(r)|^2 r^{N-1}\,dr \ge \omega_{N-1}|u(R)|^2\frac{R^N}{N},$$

where in the last inequality we used the fact that u is decreasing. □

The second result is a *compactness lemma* due to Strauss [25] (see also [4, Theorem A.I] for a simple proof).

Lemma 2.5. *Let $P,Q:\mathbb{R}\to\mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(t)}{Q(t)}\to 0,\quad \text{as } |t|\to+\infty. \tag{15}$$

Let $u_n:\mathbb{R}^N\to\mathbb{R}$ be a sequence of measurable functions such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))|\,dx<+\infty, \tag{16}$$

and

$$P(u_n(x))\to v(x)\quad \text{a. e. in } \mathbb{R}^N \quad \text{as } n\to+\infty. \tag{17}$$

Then, for every bounded Borel set B, we have

$$\int_B |P(u_n(x))-v(x)|\,dx\to 0\quad \text{as } n\to+\infty. \tag{18}$$

If we further assume that

$$\frac{P(t)}{Q(t)}\to 0\quad \text{as } t\to 0, \tag{19}$$

and

$$u_n(x)\to 0\quad \text{as } |x|\to+\infty,\quad \text{uniformly with respect to } n, \tag{20}$$

then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n\to+\infty$.

We conclude this section with the following Lemma 2.6, in which we state and prove some H^s estimates, which, in turn, imply that there exists a nontrivial competitor for the variational problem (13), as described in the subsequent Remark 2.8.

Lemma 2.6. *Let ζ, $R>0$. For any $t\geq 0$ let*

$$v_R(t):=\begin{cases}\zeta & \text{if } t\in[0,R],\\ \zeta\,(R+1-t) & \text{if } t\in(R,R+1),\\ 0 & \text{if } t\in[R+1,+\infty).\end{cases}$$

For any $x\in\mathbb{R}^N$, let $w_R(x):=v_R(|x|)$.

Then, $w_R\in H^s(\mathbb{R}^N)$ for any $s\in(0,1)$ and there exists $C(N,s,R)>0$ such that $\|w_R\|_{H^s(\mathbb{R}^N)}\leq C(N,s,R)\,\zeta$.

Proof. We take $\zeta := 1$ (the general case follows by multiplication by ζ). Notice that w_R is uniformly Lipschitz and vanishes outside B_{R+1}. In particular $w_R \in H^1(B_{R+1})$. Also, if $x \in B_{R+1} \setminus B_R$ and $y \in B_{R+2} \setminus B_{R+1}$, we have

$$|w_R(x) - w_R(y)| = R + 1 - |x| \leq |y| - |x| \leq |x - y|,$$

therefore

$$\begin{aligned}
&\iint_{B_{R+1} \times (\mathbb{R}^n \setminus B_{R+1})} \frac{|w_R(x) - w_R(y)|^2}{|x-y|^{n+2s}} \, dx\, dy \\
&\qquad \leq \iint_{(B_{R+1} \setminus B_R) \times (B_{R+2} \setminus B_{R+1})} \frac{|w_R(x) - w_R(y)|^2}{|x-y|^{n+2s}} \, dx\, dy + C_1(N, s, R) \\
&\qquad \leq C_2(N, s, R).
\end{aligned}$$

Hence, by Proposition 2.2 in [7],

$$\begin{aligned}
\|w_R\|_{H^s(\mathbb{R}^n)} &\leq C \left(\iint_{B_{R+1} \times (\mathbb{R}^n \setminus B_{R+1})} \frac{|w_R(x) - w_R(y)|^2}{|x-y|^{n+2s}} \, dx\, dy + \|w_R\|_{H^s(B_{R+1})} \right) \\
&\leq C_3(N, s, R) \left(1 + \|w_R\|_{H^1(B_{R+1})} \right) \leq C_4(N, s, R),
\end{aligned}$$

which proves the desired result. □

Remark 2.7. Here is another proof of Lemma 2.6 based on an interpolation inequality: given $u \in H^1(\mathbb{R}^N)$, by Proposition 2.2, using the Hölder inequality with exponents $1/s$ and $1/(1-s)$, we have

$$\begin{aligned}
[u]_{H^s(\mathbb{R}^N)} &= \sqrt{\int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F} u(\xi)|^{2s} |\mathscr{F} u(\xi)|^{2(1-s)} \, d\xi} \\
&\leq \left(\int_{\mathbb{R}^N} |\xi|^2 |\mathscr{F} u(\xi)|^2 \, d\xi \right)^{s/2} \left(\int_{\mathbb{R}^N} |\mathscr{F} u(\xi)|^2 \, d\xi \right)^{(1-s)/2} \\
&= [u]^s_{H^1(\mathbb{R}^N)} \|u\|^{1-s}_{L^2(\mathbb{R}^N)},
\end{aligned}$$

which clearly implies Lemma 2.6 by choosing $u := w_R$.

Remark 2.8. By Lemma 2.6, the set in the minimum problem (13) is not empty. Indeed, if $w_R \in H^s(\mathbb{R}^N)$ is defined as in Lemma 2.6, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} G(w_R(x)) \, dx &= \int_{B_{R+1}} G(w_R(x)) \, dx \\
&= \int_{B_R} G(w_R(x)) \, dx + \int_{B_{R+1} \setminus B_R} G(w_R(x)) \, dx \\
&\geq G(\zeta) |B_R| - |B_{R+1} \setminus B_R| \left(\max_{t \in [0, \zeta]} |G(t)| \right),
\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure. This implies that there exist two positive constants C_1 and C_2 (possibly depending on the fixed ζ) such that

$$\int_{\mathbb{R}^N} G(w_R(x))\,dx \geq C_1 R^N - C_2 R^{N-1},$$

and so we can choose $R > 0$ large enough such that $\int_{\mathbb{R}^N} G(w_R(x))\,dx > 0$.

Now we make the scale change $w_{R,\sigma}(x) = w_R(x/\sigma)$, and a suitable choice of $\sigma > 0$, so that

$$\int_{\mathbb{R}^N} G(w_{R,\sigma}(x))\,dx = \sigma^N \int_{\mathbb{R}^N} G(w_R(x))\,dx = 1.$$

3. Proof of Theorem 1.1

In the same spirit of the proof of Theorem 2 in [4], we divide that of Theorem 1.1 in a few steps. For the reader's convenience, we will give full details of the proof, by taking into account the preliminary results in Section 2.3 together with the modifications due to the presence of the fractional Sobolev spaces.

Proof.
Step 1 - A minimizing sequence u_n. Consider a sequence $\{u_n\} \subseteq H^s(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(u_n)\,dx = 1$ and

$$\lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} = \inf\left\{ [u]^2_{H^s(\mathbb{R}^N)} : u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u)\,dx = 1 \right\} \geq 0. \quad (21)$$

By the triangle inequality,

$$\big| |u_n(x)| - |u_n(y)| \big| \leq |u_n(x) - u_n(y)|,$$

thus the Gagliardo semi-norm of $|u_n|$ is not bigger than the one of u_n. Therefore, without loss of generality, we may suppose that u_n is nonnegative.

Let u_n^* denote the symmetric radial decreasing rearrangement of u_n. Then

$$\int_{\mathbb{R}^N} G(u_n^*)\,dx = \int_{\mathbb{R}^N} G(u_n)\,dx = 1,$$

and so, in view of Lemma 2.3, we have that $\{u_n^*\}$ is also a minimizing sequence.

These observations imply that we can select a sequence $\{u_n\}$ in such a way that, for every $n \in \mathbb{N}$, u_n is nonnegative, spherically symmetric and decreasing in $r = |x|$.

Step 2 - A priori estimates for u_n. We want to obtain bounds uniform in n on $\|u_n\|_{L^q(\mathbb{R}^N)}$, for every $2 \le q \le 2N/(N-2s)$, and on $\|u_n\|_{H^s(\mathbb{R}^N)}$.

We begin with $\|u_n\|_{H^s(\mathbb{R}^N)}$. Clearly, by (21), $[u_n]^2_{H^s(\mathbb{R}^N)} \le C$ for some positive constant C (recall also Remark 2.8). Therefore, it remains to prove that $\|u_n\|_{L^2(\mathbb{R}^N)}$ is bounded. To do this, we set

$$g_1(t) := |t|^{p-1}t, \quad g_2(t) := t, \quad G_1(t) := \frac{1}{p+1}|t|^{p+1} \quad \text{and} \quad G_2(t) := \frac{1}{2}|t|^2.$$

Then

$$g(t) = g_1(t) - g_2(t),$$

and so

$$G(z) = \int_0^z g(t)dt = \int_0^z g_1(t)dt - \int_0^z g_2(t)dt = G_1(z) - G_2(z), \quad \forall z \ge 0. \tag{22}$$

Since $1 < p < (N+2s)/(N-2s)$, we have that for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$g_1(t) \le C_\varepsilon |t|^{\frac{N+2s}{N-2s}} + \varepsilon g_2(t) \qquad \text{for any } t \ge 0. \tag{23}$$

To check this, we distinguish two cases.
If $0 \le t \le \varepsilon^{1/(p-1)}$, we have that $|t|^{p-1} = t^{p-1} \le \varepsilon$ and so $g_1(t) \le \varepsilon t = \varepsilon g_2(t)$, and we are done. Conversely, if $t \ge \varepsilon^{1/(p-1)}$, we use the Young inequality with $A := (N+2s)/(p(N-2s))$ and B its conjugated exponent: that is, if $\eta := (A\varepsilon)^{1/A}$, we see that

$$g_1(t) \le \frac{(\eta|t|^p)^A}{A} + \frac{(1/\eta)^B}{B} \le \varepsilon|t|^{(N+2s)/(N-2s)} + \frac{(1/\eta)^B}{B\varepsilon^{1/(p-1)}}|t|,$$

which proves (23). This implies that $G_1(z) \le C_\varepsilon|z|^{\frac{2N}{N-2s}} + \varepsilon G_2(z)$ for any $z \ge 0$. Choosing $\varepsilon = 1/2$, we get

$$G_1(z) \le C|z|^{\frac{2N}{N-2s}} + \frac{1}{2}G_2(z). \tag{24}$$

Now, the condition $\int_{\mathbb{R}^N} G(u_n)\,dx = 1$ can be written in the following form

$$\int_{\mathbb{R}^N} G_1(u_n)\,dx = \int_{\mathbb{R}^N} G_2(u_n)\,dx + 1. \tag{25}$$

Putting together (24) and (25), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N} G_2(u_n)\,dx + 1 \le C\int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2s}}\,dx. \tag{26}$$

Now we use the fractional Sobolev embedding theorem (see, e.g., [7, Theorem 6.5]) to say that

$$\|u_n\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \leq C[u_n]_{H^s(\mathbb{R}^N)},$$

where the constant C does not depend on n. Thus, since u_n is a minimizing sequence, the boundedness of $[u_n]^2_{H^s(\mathbb{R}^N)}$ yields that of $\|u_n\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}$. By the definition of G_2, the inequality in (26) implies that

$$\frac{1}{2}\int_{\mathbb{R}^N} u_n^2\,dx = \int_{\mathbb{R}^N} G_2(u_n)\,dx \leq C,$$

and thus we bound $\|u_n\|^2_{L^2(\mathbb{R}^N)}$ (and so $\|u_n\|^2_{H^s(\mathbb{R}^N)}$) uniformly in n.

Finally, by the bounds on $\|u_n\|_{L^2(\mathbb{R}^N)}$ and $\|u_n\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}$, using the Hölder inequality, we obtain that $\|u_n\|_{L^q(\mathbb{R}^N)} \leq C$ for every $2 \leq q \leq 2N/(N-2s)$. More explicitly, fixed $q \in \big(2, 2N/(N-2s)\big)$, we define

$$\tau := \frac{\frac{2N}{N-2s} - q}{\frac{2N}{N-2s} - 2} \in (0,1).$$

In this way, the Hölder inequality with exponents $1/\tau$ and $1/(1-\tau)$ gives

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^q\,dx &= \int_{\mathbb{R}^n} |u|^{2\tau+((2N)/(N-2s))(1-\tau)}\,dx \\ &\leq \left(\int_{\mathbb{R}^n} |u|^2\,dx\right)^{\tau} \cdot \left(\int_{\mathbb{R}^n} |u|^{2N/(N-2s)}\,dx\right)^{1-\tau}, \end{aligned}$$

which is finite.

Step 3 - Passage to the limit and conclusion of the proof. Since $u_n \in L^2(\mathbb{R}^N)$ is a sequence of nonnegative radial decreasing functions, we can apply Lemma 2.4 to get

$$|u_n(x)| \leq \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{2}} |x|^{-N/2}\|u_n\|_{L^2(\mathbb{R}^N)}. \tag{27}$$

From the previous step we have that u_n is uniformly bounded in $L^2(\mathbb{R}^N)$; then $|u_n(x)| \leq C|x|^{-N/2}$, with C independent of n. This implies that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Now, since u_n is bounded in $H^s(\mathbb{R}^N)$, we can extract a subsequence of u_n, again denoted by u_n, such that u_n converges weakly in $H^s(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$ to a function $\overline{u}$. Moreover, by construction, $\overline{u} \in H^s(\mathbb{R}^N)$ is spherically symmetric and decreasing in r.

Now, in order to apply Lemma 2.5 (with $P := G_1$), consider the polynomial function Q defined by

$$Q(t) := t^2 + |t|^{\frac{2N}{N-2s}}.$$

Since the sequence u_n is uniformly bounded in $L^2(\mathbb{R}^N)$ and in $L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, we have that Q satisfies

$$\int_{\mathbb{R}^N} |Q(u_n(x))|\, dx = \int_{\mathbb{R}^N} \left(u_n^2(x) + |u_n(x)|^{\frac{2N}{N-2s}}\right) dx \le C, \quad \text{for every } n \in \mathbb{N}.$$

Moreover, if G_1 is defined as in the previous step, by the fact that $p \in \left(1, \frac{N+2s}{N-2s}\right)$ we derive

$$\frac{G_1(t)}{Q(t)} \to 0, \quad \text{as } t \to +\infty \text{ and } t \to 0.$$

Since u_n converges almost everywhere in $\mathbb{R}^N$ to $\overline{u}$, we have that also $G_1(u_n)$ converges $G_1(\overline{u})$. Finally, $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n. Therefore Lemma 2.5 holds, getting

$$\int_{\mathbb{R}^N} G_1(u_n(x))\, dx \to \int_{\mathbb{R}^N} G_1(\overline{u}(x))\, dx \quad \text{as } n \to +\infty.$$

Thus, using Fatou's Lemma in (25), we obtain that

$$\int_{\mathbb{R}^N} G_1(\overline{u}(x))\, dx \ge \int_{\mathbb{R}^N} G_2(\overline{u}(x))\, dx + 1, \tag{28}$$

that is

$$\int_{\mathbb{R}^N} G(\overline{u}(x))\, dx \ge 1.$$

On the other hand, using (6) and Fatou's Lemma once more, we have that

$$\begin{aligned} [\overline{u}]^2_{H^s(\mathbb{R}^N)} &\le \lim_{n\to+\infty} [u_n]^2_{H^s(\mathbb{R}^N)} \\ &= \inf\left\{ [u]^2_{H^s(\mathbb{R}^N)} : u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u)\, dx = 1 \right\}. \end{aligned} \tag{29}$$

Now, suppose by contradiction that $\int_{\mathbb{R}^N} G(\overline{u}(x))\, dx > 1$. Then, by the scale change $\overline{u}_\sigma(x) = \overline{u}(x/\sigma)$, we have

$$\int_{\mathbb{R}^N} G(\overline{u}_\sigma(x))\, dx = \sigma^N \int_{\mathbb{R}^N} G(\overline{u}(x))\, dx = 1 \tag{30}$$

for some

$$\sigma \in (0,1). \tag{31}$$

Moreover, from (6) we have

$$\begin{aligned} [\overline{u}_\sigma]^2_{H^s(\mathbb{R}^N)} &= \sigma^{N-2s} [\overline{u}]^2_{H^s(\mathbb{R}^N)} \\ &\le \sigma^{N-2s} \inf\left\{ [u]^2_{H^s(\mathbb{R}^N)} : u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u)\, dx = 1 \right\}, \end{aligned} \tag{32}$$

due to (29), and

$$\inf\left\{[u]^2_{H^s(\mathbb{R}^N)}\,:\,u\in H^s(\mathbb{R}^N),\,\int_{\mathbb{R}^N}G(u)\,dx=1\right\}\leq[\overline{u}_\sigma]^2_{H^s(\mathbb{R}^N)},$$

thanks to (30). Combining the last two inequalities and recalling (31), we get

$$\inf\left\{[u]^2_{H^s(\mathbb{R}^N)}\,:\,u\in H^s(\mathbb{R}^N),\,\int_{\mathbb{R}^N}G(u)\,dx=1\right\}=0,$$

hence also $[\overline{u}]^2_{H^s(\mathbb{R}^N)}=0$, thanks to (32). Then $\overline{u}\equiv 0$, which is in contradiction with (28). Therefore, $\int_{\mathbb{R}^N}G(\overline{u}(x))\,dx=1$ and so

$$[\overline{u}]_{H^s(\mathbb{R}^N)}=\inf\left\{[u]_{H^s(\mathbb{R}^N)}\,:\,u\in H^s(\mathbb{R}^N),\,\int_{\mathbb{R}^N}G(u)\,dx=1\right\};$$

that is, $\overline{u}$ solves the minimization problem (13). □

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