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Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations

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Abstract

In this manuscript, we examine the existence and the Ulam stability of solutions for a class of boundary value problems for nonlinear implicit fractional differential equations with instantaneous impulses in Banach spaces. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. We provide some examples to indicate the applicability of our results.

Keywords: Boundary value problem; Generalized Hilfer fractional derivative; Fractional integral; Impulses; Fixed point; Measure of noncompactness; Implicit fractional differential equations; Ulam–Hyers–Rassias stability

1 Introduction

As it is known very well, the roots of fixed point theory go to the method of successive approximations (or Picard's iterative method) that is used to solve certain differential equations. Roughly speaking, Banach derived the fixed point theorem from the method of successive approximations. In the last decades fixed point theory has been enormously and independently from the differential equations. But, recently, fixed point results turn to be the tools for the solutions of the differential equation. In this paper, we shall involve two interesting fixed point theorems (Darbo's fixed point theorem and Mönch's fixed point theorem) in the setting of "measure of noncompactness" to solve the boundary value problem for nonlinear implicit fractional differential equations with instantaneous impulses.

Differential equations of fractional order have been recently proved to be a powerful tool to study many phenomena in various fields of science and engineering such as electrochemistry, electromagnetics, viscoelasticity, finance, and so on. In the literature, it is very common to propose a solution for fractional differential equations by involving different kinds of fractional derivatives, see e.g. [1–10, 12, 13, 21, 22, 36]. On the other hand, there a few results that deal with the boundary value problems for fractional differential equations. The aim of the present paper is to underline the importance of the theory of impulsive differential equations. Further, by the help of these observations, we aim to understand several phenomena that are not clarified by the non-impulsive equations (see e.g. [15, 16, 18, 32]).

In 1940, Ulam [34, 35] raised the following problem of the stability of the functional equation (of group homomorphisms): "*Under what conditions does it exist an additive mapping near an approximately additive mapping?*"

Let G_1 be a group, and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\epsilon > 0$, does there exist $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

A partial answer was given by Hyers [20] in 1941, and between 1982 and 1998 Rassias [28, 29] established the Hyers–Ulam stability of linear and nonlinear mappings. Subsequently, many works have been published in order to generalize Hyers results in various directions, see for example [24, 25, 30, 31, 33].

Inspired by the papers mentioned above, we examine the existence results to the boundary value problem with nonlinear implicit generalized Hilfer-type fractional differential equation with instantaneous impulses:

$$\left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t) = f\left(t, u(t), \left({}^{\rho}D^{\alpha,\beta}_{t_k^+}u\right)(t)\right); \quad t \in J_k, k = 0, \dots, m, \tag{1}$$

$$\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}u\right)\left(t_k^+\right) = \left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_{k-1}^+}u\right)\left(t_k^-\right) + \varpi_k\left(u\left(t_k^-\right)\right); \quad k = 1, \dots, m, \tag{2}$$

$$c_1\left({}^{\rho}\mathcal{J}^{1-\gamma}_{a^+}u\right)\left(a^+\right) + c_2\left({}^{\rho}\mathcal{J}^{1-\gamma}_{t_m^+}u\right)(b) = c_3, \tag{3}$$

where

- ${}^{\rho}D^{\alpha,\beta}_{t_k^+}$ is the generalized Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and $\rho > 0$;
- ${}^{\rho}\mathcal{J}^{1-\gamma}_{t_k^+}$ is the generalized Hilfer fractional integral of order $1-\gamma$, ($\gamma = \alpha + \beta - \alpha\beta$);
- c_1, c_2 are reals with $c_1 + c_2 \neq 0$, $J_k := (t_k, t_{k+1}]$; $k = 0, \dots, m$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b < \infty$;
- $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right-hand and left-hand limits of $u(t)$ at $t = t_k$, $c_3 \in E$;
- $f : (a, b] \times E \times E \to E$ is a given function over a Banach space $(E, \|\cdot\|)$;
- $\varpi_k : E \to E$; $k = 1, \dots, m$, are given continuous functions.

2 Preliminaries

In this section, we recall and recollect the basic notion, notations together with some fundamental results that will be necessary in the main results. Throughout the paper, $(E, \|\cdot\|)$ represents a Banach space. Set $J = [a, b]$ where $0 < a < b$. The letter C is reserved to represent the Banach space which consists of all continuous functions $u : J \to E$ where the norm is

$$\|u\|_\infty = \sup\left\{\left\|u(\tau)\right\| : \tau \in J\right\}.$$

In what follows, we pay attention to the weighted spaces of continuous functions

$$C_{\gamma,\rho}(J) = \left\{u : (a, b] \to E : \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} u(t) \in C(J, \mathbb{R})\right\},$$

where $0 \leq \gamma < 1$,

$$C^n_{\gamma,\rho}(J) = \{u \in C^{n-1} : u^{(n)} \in C_{\gamma,\rho}(J)\}, \quad n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J),$$

with the norms

$$\|u\|_{C_{\gamma,\rho}} = \sup_{t\in J}\left\|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} u(t)\right\|$$

and

$$\|u\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1}\|u^{(k)}\|_\infty + \|u^{(n)}\|_{C_{\gamma,\rho}}.$$

Consider the Banach space

$$\begin{aligned} PC(J) = \{&u : (a,b] \to E : u(t) \in C(J_k); k = 0,\dots,m, \text{ and there exist } u(t_k^-) \\ &\text{and } ({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+}u)(t_k^+); k = 0,\dots,m, \text{ with } u(t_k^-) = u(t_k)\}, \quad 0 \leq \gamma < 1. \end{aligned}$$

Also, we consider the weighted space

$$PC_{\gamma,\rho}(J) = \left\{u(t) : \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} u(t) \in PC(J)\right\}, \quad 0 \leq \gamma < 1,$$

and

$$PC^n_{\gamma,\rho}(J) = \{u \in PC^{n-1} : u^{(n)} \in PC_{\gamma,\rho}(J)\}, \quad n \in \mathbb{N},$$
$$PC^0_{\gamma,\rho}(J) = PC_{\gamma,\rho}(J),$$

equipped with the norm

$$\|u\|_{PC_{\gamma,\rho}} = \sup_{t\in J}\left\|\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} u(t)\right\|.$$

The letter $L^1(J)$ indicates the space of Bochner-integrable functions $f : J \longrightarrow E$ with the norm

$$\|f\|_1 = \int_a^b \|f(t)\|\,dt.$$

Definition 2.1 ([23]) Let $\alpha \in \mathbb{R}_+$ and $g \in L^1(J)$. The generalized Hilfer fractional integral of order α is

$$({}^\rho\mathcal{J}^\alpha_{a^+}g)(t) = \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}\,ds, \quad \rho > 0, t > a,$$

with $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$, $\alpha > 0$ (the Euler gamma function)

Definition 2.2 ([23] Generalized Hilfer fractional derivative) Set $\rho > 0$ and $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$. The generalized Hilfer fractional derivative ${}^\rho D^\alpha_{a^+}$ of order α is defined by

$$\begin{aligned}\left({}^\rho D^\alpha_{a^+} g\right)(t) &= \delta^n_\rho\left({}^\rho \mathcal{J}^{n-\alpha}_{a^+} g\right)(t)\\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1}\frac{g(s)}{\Gamma(n-\alpha)}\,ds, \quad t > a, \rho > 0,\end{aligned}$$

where $n = [\alpha] + 1$ and $\delta^n_\rho = (t^{1-\rho}\frac{d}{dt})^n$.

Theorem 2.3 ([23]) *Let $\alpha > 0$, $\beta > 0$, $1 \le \rho \le \infty$, $0 < a < b < \infty$. Then, for $g \in L^1(J)$, we have*

$$\left({}^\rho \mathcal{J}^\alpha_{a^+}\, {}^\rho \mathcal{J}^\beta_{a^+} g\right)(t) = \left({}^\rho \mathcal{J}^{\alpha+\beta}_{a^+} g\right)(t).$$

Lemma 2.4 ([23, 27]) *Let $\alpha > 0$ and $0 \le \gamma < 1$. Then ${}^\rho \mathcal{J}^\alpha_{a^+}$ lies between $PC_{\gamma,\rho}(J)$ and $PC_{\gamma,\rho}(J)$.*

Lemma 2.5 ([27]) *Suppose that $0 \le \gamma < 1$, $0 < a < b < \infty$, $\alpha > 0$, and $u \in PC_{\gamma,\rho}(J)$. If $\alpha > 1 - \gamma$, then ${}^\rho \mathcal{J}^\alpha_{a^+} u$ is continuous on J and*

$$\left({}^\rho \mathcal{J}^\alpha_{a^+} u\right)(a) = \lim_{t\to a^+}\left({}^\rho \mathcal{J}^\alpha_{a^+} u\right)(t) = 0.$$

Lemma 2.6 ([11]) *Let $t > a$. Then, for $\alpha \ge 0$ and $\beta > 0$, we have*

$$\begin{aligned}&\left[{}^\rho \mathcal{J}^\alpha_{a^+}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1},\\ &\left[{}^\rho D^\alpha_{a^+}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right](t) = 0 \quad (for)\ 0 < \alpha < 1.\end{aligned}$$

Lemma 2.7 ([27]) *Suppose that $g \in PC_\gamma[a,b]$ with $0 \le \gamma < 1$ and $\alpha > 0$. Then we have*

$$\left({}^\rho D^\alpha_{a^+}\, {}^\rho \mathcal{J}^\alpha_{a^+} g\right)(t) = g(t) \quad \textit{for all } t \in (a,b].$$

Lemma 2.8 ([27]) *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $g \in PC_{\gamma,\rho}[a,b]$ and ${}^\rho \mathcal{J}^{1-\alpha}_{a^+} g \in PC^1_{\gamma,\rho}[a,b]$, then*

$$\left({}^\rho \mathcal{J}^\alpha_{a^+}\, {}^\rho D^\alpha_{a^+} g\right)(t) = g(t) - \frac{({}^\rho \mathcal{J}^{1-\alpha}_{a^+} g)(a)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} \quad \textit{for all } t \in (a,b].$$

Definition 2.9 ([27]) For a function $g \in PC_{\gamma,\rho}[a,b]$ with $\rho > 0$, the generalized Hilfer-type fractional derivative is defined by

$$\begin{aligned}\left({}^\rho D^{\alpha,\beta}_{a^+} g\right)(t) &= \left({}^\rho \mathcal{J}^{\beta(n-\alpha)}_{a^+}\left(t^{\rho-1}\frac{d}{dt}\right)^n {}^\rho \mathcal{J}^{(1-\beta)(n-\alpha)}_{a^+} g\right)(t)\\ &= \left({}^\rho \mathcal{J}^{\beta(n-\alpha)}_{a^+}\delta^n_\rho\, {}^\rho \mathcal{J}^{(1-\beta)(n-\alpha)}_{a^+} g\right)(t),\end{aligned}$$

where type β with $0 \le \beta \le 1$, and order α with $n - 1 < \alpha < n$ for $n \in \mathbb{N}$.

Since $0 < \alpha < 1$, we shall focus only on the case $n = 1$.

Property 2.10 ([27]) The operator ${}^{\rho}D_{a^+}^{\alpha,\beta}$ can be written as

$$ {}^{\rho}D_{a^+}^{\alpha,\beta} = {}^{\rho}\mathcal{J}_{a^+}^{\beta(1-\alpha)}\delta_{\rho}\,{}^{\rho}\mathcal{J}_{a^+}^{1-\gamma} = {}^{\rho}\mathcal{J}_{a^+}^{\beta(1-\alpha)}\,{}^{\rho}D_{a^+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta. $$

Property 2.11 ([27]) The fractional derivative ${}^{\rho}D_{\alpha^+}^{\alpha,\beta}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$), Hilfer–Hadamard ($\rho \to 0^+$), generalized ($\beta = 0$), Caputo-type ($\beta = 1$), Riemann–Liouville ($\beta = 0, \rho \to 1$), Hadamard ($\beta = 0, \rho \to 0^+$), Caputo ($\beta = 1, \rho \to 1$), Caputo–Hadamard ($\beta = 1, \rho \to 0^+$), Liouville ($\beta = 0, \rho \to 1, a = 0$), and Weyl ($\beta = 0, \rho \to 1, a = -\infty$).

Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha, \beta, \gamma < 1$. Construct the space

$$ PC_{\gamma,\rho}^{\alpha,\beta}(J) = \left\{u \in PC_{\gamma,\rho}(J), {}^{\rho}D_{t_k^+}^{\alpha,\beta}u \in PC_{\gamma,\rho}(J)\right\} $$

and

$$ PC_{\gamma,\rho}^{\gamma}(J) = \left\{u \in PC_{\gamma,\rho}(J), {}^{\rho}D_{t_k^+}^{\gamma}u \in PC_{\gamma,\rho}(J)\right\}, $$

where $k = 0, \dots, m$.

Since ${}^{\rho}D_{t_k^+}^{\alpha,\beta}u = {}^{\rho}\mathcal{J}_{t_k^+}^{\gamma(1-\alpha)}\,{}^{\rho}D_{t_k^+}^{\gamma}u$, it follows from Lemma 2.4 that

$$ PC_{\gamma,\rho}^{\gamma}(J) \subset PC_{\gamma,\rho}^{\alpha,\beta}(J) \subset PC_{\gamma,\rho}(J). $$

Lemma 2.12 ([27]) *Let $\gamma = \alpha + \beta - \alpha\beta$ with $0 < \alpha < 1$, $0 \leq \beta \leq 1$. If $u \in PC_{\gamma,\rho}^{\gamma}(J)$, then*

$$ {}^{\rho}\mathcal{J}_{a^+}^{\gamma}\,{}^{\rho}D_{a^+}^{\gamma}u = {}^{\rho}\mathcal{J}_{a^+}^{\alpha}\,{}^{\rho}D_{a^+}^{\alpha,\beta}u $$

and

$$ {}^{\rho}D_{a^+}^{\gamma}\,{}^{\rho}\mathcal{J}_{a^+}^{\alpha}u = {}^{\rho}D_{a^+}^{\beta(1-\alpha)}u. $$

Definition 2.13 ([14]) Let X be a Banach space, and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \longrightarrow [0,\infty)$ defined by

$$ \mu(M) = inf\left\{\epsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, diam(M_j) \leq \epsilon\right\}, $$

where $M \in \Omega_X$.

For all $M, M_1, M_2 \in \Omega_X$, the map μ satisfies the following properties:

- M is relatively compact ($\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact).
- $\mu(M) = \mu(\overline{M})$.
- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
- $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

- $\mu(cM) = |c|\mu(M)$, $c \in \mathbb{R}$.
- $\mu(convM) = \mu(M)$.

Lemma 2.14 ([19]) *Let $D \subset PC_{\gamma,\rho}(J)$ be a bounded and equicontinuous set, then*
(*i*) *the function $t \to \mu(D(t))$ is continuous on $(a,b]$, and*

$$\mu_{PC_{\gamma,\rho}}(D) = \sup_{t\in J} \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} D(t)\right);$$

(*ii*) $\mu(\int_a^b u(s)\,ds : u \in D) \leq \int_a^b \mu(D(s))\,ds$, *where*

$$D(t) = \{u(t) : u \in D\}, \quad t \in (a,b].$$

Lemma 2.15 (Theorem 4.1, [27]) *Let $f : (a,b] \times E \to E$ be a function such that $f(\cdot, u(\cdot), {}^\rho D_{t_k^+}^{\alpha,\beta} u(\cdot)) \in C_{\gamma,\rho}(J)$ for any $u \in PC_{\gamma,\rho}(J)$. Then $u \in PC_{\gamma,\rho}^{\gamma}(J)$ is a solution of the differential equation*

$$\left({}^\rho D_{t_k^+}^{\alpha,\beta} u\right)(t) = f\left(t, u(t), {}^\rho D_{t_k^+}^{\alpha,\beta} u(t)\right), \quad \text{for each } t \in J_k, k = 0, \dots, m, 0 < \alpha, \beta < 1,$$

if and only if u satisfies the following Volterra integral equation:

$$u(t) = \frac{({}^\rho \mathcal{J}_{t_k^+}^{1-\gamma} u)(t_k^+)}{\Gamma(\gamma)} \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^\rho D_{t_k^+}^{\alpha,\beta} u(s)\right) ds,$$

where $\gamma = \alpha + \beta - \alpha\beta$.

Theorem 2.16 (Mönch's theorem [26]) *Suppose that D is a closed, bounded, and convex subset of a Banach space X such that $0 \in D$. Let T be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} T(V), \quad \text{or} \quad V = T(V) \cup \{0\} \quad \Rightarrow \quad \mu(V) = 0 \tag{4}$$

holds for every subset V of D, then T has a fixed point.

Theorem 2.17 (Darbo's theorem [17]) *Suppose that D is a nonempty, closed, bounded, and convex subset of a Banach space X. Let T be a continuous mapping of D into itself such that, for any nonempty subset C of D,*

$$\mu\left(T(C)\right) \leq k\mu(C), \tag{5}$$

where $0 \leq k < 1$, and μ is the Kuratowski measure of noncompactness. Then T has a fixed point in D.

3 The existence of solutions

We start this section by stating the following linear fractional differential equation:

$$\left({}^\rho D_{t_k^+}^{\alpha,\beta} u\right)(t) = \psi(t), \quad t \in J_k, k = 0, \dots, m, \tag{6}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\rho > 0$, with the conditions

$$\big({}^{\rho}\mathcal{J}_{t_k^+}^{1-\gamma}u\big)\big(t_k^+\big) = \big({}^{\rho}\mathcal{J}_{t_{k-1}^+}^{1-\gamma}u\big)\big(t_k^-\big) + \varpi_k\big(u\big(t_k^-\big)\big); \quad k = 1,\dots,m, \tag{7}$$

and

$$c_1\big({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + c_2\big({}^{\rho}\mathcal{J}_{t_m^+}^{1-\gamma}u\big)(b) = c_3, \tag{8}$$

where $\gamma = \alpha + \beta - \alpha\beta$, $c_3 \in E$, $c_1, c_2 \in \mathbb{R}$ with $c_1 + c_2 \neq 0$ and $\xi_1 = \frac{c_2}{c_1+c_2}$, $\xi_2 = \frac{c_3}{c_1+c_2}$. The following theorem shows that problem (6)–(8) has a unique solution given by

$$u(t) = \begin{cases} \frac{1}{\Gamma(\gamma)}(\frac{t^{\rho}-a^{\rho}}{\rho})^{\gamma-1}[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}({}^{\rho}\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i) \\ \quad - \xi_1({}^{\rho}\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi)(b)] + \frac{1}{\Gamma(\alpha)}\int_a^t(\frac{t^{\rho}-s^{\rho}}{\rho})^{\alpha-1}s^{\rho-1}\psi(s)\,ds \quad \text{if } t \in J_0, \\ \frac{1}{\Gamma(\gamma)}(\frac{t^{\rho}-t_k^{\rho}}{\rho})^{\gamma-1}[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}({}^{\rho}\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i) \\ \quad - \xi_1({}^{\rho}\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi)(b) + \sum_{i=1}^{k}\varpi_i(u(t_i^-)) + \sum_{i=1}^{k}({}^{\rho}\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i)] \\ \quad + ({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}\psi)(t) \quad \text{if } t \in J_k, k = 1,\dots,m. \end{cases} \tag{9}$$

Theorem 3.1 *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $\psi : (a,b] \to E$ is a function such that $\psi(\cdot) \in C_{\gamma,\rho}(J)$, then $u \in PC_{\gamma,\rho}^{\gamma}(J)$ satisfies problem (6)–(8) if and only if it satisfies (9).*

Proof Assume that u satisfies (6)–(8). If $t \in J_0$, then

$$\big({}^{\rho}D_{a^+}^{\alpha,\beta}u\big)(t) = \psi(t).$$

Lemma 2.15 implies we have the solution that can be written as

$$u(t) = \frac{({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u)(a^+)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)\,ds. \tag{10}$$

If $t \in J_1$, then Lemma 2.15 implies

$$\begin{aligned} u(t) &= \frac{({}^{\rho}\mathcal{J}_{t_1^+}^{1-\gamma}u)(t_1^+)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-t_1^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)\,ds \\ &= \frac{({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u)(t_1^-) + \varpi_1(u(t_1^-))}{\Gamma(\gamma)}\left(\frac{t^{\rho}-t_1^{\rho}}{\rho}\right)^{\gamma-1} + \big({}^{\rho}\mathcal{J}_{t_1^+}^{\alpha}\psi\big)(t) \\ &= \frac{(t^{\rho}-t_1^{\rho})^{\gamma-1}}{\Gamma(\gamma)\rho^{\gamma-1}}\big[\big({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + \varpi_1\big(u\big(t_1^-\big)\big) + \big({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\big)(t_1)\big] + \big({}^{\rho}\mathcal{J}_{t_1^+}^{\alpha}\psi\big)(t). \end{aligned}$$

If $t \in J_2$, then Lemma 2.15 implies

$$\begin{aligned} u(t) &= \frac{({}^{\rho}\mathcal{J}_{t_2^+}^{1-\gamma}u)(t_2^+)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-t_2^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)\,ds \\ &= \frac{({}^{\rho}\mathcal{J}_{t_1^+}^{1-\gamma}u)(t_2^-) + \varpi_2(u(t_2^-))}{\Gamma(\gamma)}\left(\frac{t^{\rho}-t_2^{\rho}}{\rho}\right)^{\gamma-1} + \big({}^{\rho}\mathcal{J}_{t_2^+}^{\alpha}\psi\big)(t) \end{aligned}$$

$$= \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_2^\rho}{\rho}\right)^{\gamma-1}\Big[\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + \varpi_1\big(u\big(t_1^-\big)\big) + \varpi_2\big(u\big(t_2^-\big)\big)$$
$$+\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi\big)(t_1) + \big({}^\rho\mathcal{J}_{t_1^+}^{1-\gamma+\alpha}\psi\big)(t_2)\Big] + \big({}^\rho\mathcal{J}_{t_2^+}^{\alpha}\psi\big)(t).$$

Repeating the process in this way, the solution $u(t)$ for $t \in J_k$, $k = 1,\dots,m$, can be written as

$$u(t) = \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\left[\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + \sum_{i=1}^{k}\varpi_i\big(u\big(t_i^-\big)\big) + \sum_{i=1}^{k}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\big)(t_i)\right] \tag{11}$$
$$+ \big({}^\rho\mathcal{J}_{t_k^+}^{\alpha}\psi\big)(t).$$

Applying ${}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}$ on both sides of (11), using Lemma 2.6, and taking $t = b$, we obtain

$$\begin{aligned}\big({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}u\big)(b) &= \big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + \sum_{i=1}^{m}\varpi_i\big(u\big(t_i^-\big)\big) + \sum_{i=1}^{m}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\big)(t_i)\\ &\quad+ \big({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\big)(b).\end{aligned} \tag{12}$$

Multiplying both sides of (12) by c_2 and using condition (8), we obtain

$$\begin{aligned}c_3 - c_1\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) &= c_2\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) + c_2\sum_{i=1}^{m}\varpi_i\big(u\big(t_i^-\big)\big)\\ &\quad+ c_2\sum_{i=1}^{m}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\big)(t_i) + c_2\big({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\big)(b),\end{aligned}$$

which implies that

$$\begin{aligned}\big({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\big)\big(a^+\big) &= \xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i\big(u\big(t_i^-\big)\big) - \xi_1\sum_{i=1}^{m}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\big)(t_i)\\ &\quad- \xi_1\big({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi\big)(b).\end{aligned} \tag{13}$$

Substituting (13) into (11) and (10), we obtain (9).

Reciprocally, applying ${}^\rho\mathcal{J}_{t_k^+}^{1-\gamma}$ on both sides of (9) and using Lemma 2.6 and Theorem 2.3, we get

$$\big({}^\rho\mathcal{J}_{t_k^+}^{1-\gamma}u\big)(t) = \begin{cases}\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i)\\ \quad - \xi_1({}^\rho\mathcal{J}_{(t_m)^+}^{1-\gamma+\alpha}\psi)(b) + ({}^\rho\mathcal{J}_{a^+}^{1-\gamma+\alpha}\psi)(t) \quad \text{if } t \in J_0,\\ \xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i(u(t_i^-)) - \xi_1\sum_{i=1}^{m}({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i)\\ \quad - \xi_1({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi)(b) + \sum_{i=1}^{k}\varpi_i(u(t_i^-)) + \sum_{i=1}^{k}({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi)(t_i)\\ \quad + ({}^\rho\mathcal{J}_{t_k^+}^{1-\gamma+\alpha}\psi)(t) \quad \text{if } t \in J_k, k = 1,\dots,m.\end{cases} \tag{14}$$

Next, taking the limit $t \to a^+$ of (14) and using Lemma 2.5, with $1-\gamma < 1-\gamma+\alpha$, we obtain

$$\begin{aligned}\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)\left(a^+\right) = \xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i\left(u\left(t_i^-\right)\right) - \xi_1\sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i) \\ - \xi_1\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi\right)(b).\end{aligned} \tag{15}$$

Now, taking $t = b$ in (14), we get

$$\begin{aligned}\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}u\right)(b) = \xi_2 + (1-\xi_1)\left(\sum_{i=1}^{m}\varpi_i\left(u\left(t_i^-\right)\right) + \sum_{i=1}^{m}\left({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\psi\right)(t_i)\right. \\ \left. + \left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\psi\right)(b)\right).\end{aligned} \tag{16}$$

From (15) and (16), we find that

$$c_1\left({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u\right)\left(a^+\right) + c_2\left({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}u\right)(b) = c_3,$$

which shows that the boundary condition $c_1({}^\rho\mathcal{J}_{a^+}^{1-\gamma}u)(a^+) + c_2({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma}u)(b) = c_3$ is satisfied. Next, we apply operator ${}^\rho D_{t_k^+}^{\gamma}$ on both sides of (9), where $k = 0,\dots,m$. Then, from Lemma 2.6 and Lemma 2.12, we obtain

$$\left({}^\rho D_{t_k^+}^{\gamma}u\right)(t) = \left({}^\rho D_{t_k^+}^{\beta(1-\alpha)}\psi\right)(t). \tag{17}$$

Since $u \in PC_{\gamma,\rho}^{\gamma}(J)$ and by definition of $PC_{\gamma,\rho}^{\gamma}(J)$, we have ${}^\rho D_{t_k^+}^{\gamma}u \in PC_{\gamma,\rho}(J)$, then (17) implies that

$$\left({}^\rho D_{t_k^+}^{\gamma}u\right)(t) = \left(\delta_\rho{}^\rho\mathcal{J}_{t_k^+}^{1-\beta(1-\alpha)}\psi\right)(t) = \left({}^\rho D_{t_k^+}^{\beta(1-\alpha)}\psi\right)(t) \in PC_{\gamma,\rho}(J). \tag{18}$$

As $\psi(\cdot) \in C_{\gamma,\rho}(J)$ and from Lemma 2.4, it follows

$$\left({}^\rho\mathcal{J}_{t_k^+}^{1-\beta(1-\alpha)}\psi\right) \in PC_{\gamma,\rho}(J). \tag{19}$$

Regarding the definition of the space $PC_{\gamma,\rho}^{n}(J)$, together with (18), (19), we derive that

$$\left({}^\rho\mathcal{J}_{t_k^+}^{1-\beta(1-\alpha)}\psi\right) \in PC_{\gamma,\rho}^{1}(J).$$

Applying operator ${}^\rho\mathcal{J}_{t_k^+}^{\beta(1-\alpha)}$ on both sides of (17) and using Lemma 2.8, Lemma 2.5, and Property 2.10, we have

$$\begin{aligned}\left({}^\rho D_{t_k^+}^{\alpha,\beta}u\right)(t) = {}^\rho\mathcal{J}_{t_k^+}^{\beta(1-\alpha)}\left({}^\rho D_{t_k^+}^{\gamma}u\right)(t) &= \psi(t) - \frac{\left({}^\rho\mathcal{J}_{t_k^+}^{1-\beta(1-\alpha)}\psi\right)(t_k)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \psi(t),\end{aligned}$$

that is, (6) holds.

Also, we can easily show that

$$\left({}^{\rho}\mathcal{J}_{t_k^+}^{1-\gamma}u\right)\left(t_k^+\right) = \left({}^{\rho}\mathcal{J}_{t_{k-1}^+}^{1-\gamma}u\right)\left(t_k^-\right) + \varpi_k\left(u\left(t_k^-\right)\right); \quad k = 1,\dots,m.$$

This completes the proof. □

As a consequence of Theorem 3.1, we have the following result.

Lemma 3.2 *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 \le \beta \le 1$ and $0 < \alpha < 1$. Suppose that $f : (a,b] \times E \times E \to E$ is a function so that $f(\cdot,u(\cdot),w(\cdot)) \in C_{\gamma,\rho}(J)$ for each $w,u \in PC_{\gamma,\rho}(J)$.*

If $u \in PC_{\gamma,\rho}^{\gamma}(J)$, then u satisfies problem (1)–(3) iff u is the fixed point of $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined by

$$\begin{aligned}\Psi u(t) = {}& \frac{1}{\Gamma(\gamma)}\left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{\gamma-1}\Bigg[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i\left(u\left(t_i^-\right)\right) - \xi_1\sum_{i=1}^{m}\left({}^{\rho}\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}h\right)(t_i) \\ & - \xi_1\left({}^{\rho}\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}h\right)(b) + \sum_{a<t_k<t}\varpi_k\left(u\left(t_k^-\right)\right) + \sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}h\right)(t_k)\Bigg] \\ & + \left({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}h\right)(t), \quad t \in J_k, k = 0,\dots,m,\end{aligned} \tag{20}$$

where $h : (a,b] \to \mathbb{R}$ is defined as

$$h(t) = f\left(t,u(t),h(t)\right).$$

Evidently, $h \in C_{\gamma,\rho}(J)$. Also, by Lemma 2.4, $\Psi u \in PC_{\gamma,\rho}(J)$.

In the sequel, we shall use the following hypotheses efficiently:

($Ax1$) The mapping $t \mapsto f(t,u,w)$ is measurable on $(a,b]$ for each $u,w \in E$, and the functions $u \mapsto f(t,u,w)$ and $w \mapsto f(t,u,w)$ are continuous on E for a.e. $t \in (a,b]$, and

$$f\left(\cdot,u(\cdot),w(\cdot)\right) \in PC_{\gamma,\rho}^{\beta(1-\alpha)} \quad \text{for any } u,w \in PC_{\gamma,\rho}(J).$$

($Ax2$) There exists a continuous function $p : J \longrightarrow [0,\infty)$ such that

$$\left\|f(t,u,w)\right\| \le p(t) \quad \text{for a.e. } t \in (a,b] \text{ and for each } u,w \in E.$$

($Ax3$) For each bounded set $B \subset E$ and for each $t \in (a,b]$, we have

$$\mu\left(f\left(t,B,\left({}^{\rho}D_{a^+}^{\alpha,\beta}B\right)\right)\right) \le \left(\frac{t^{\rho}-t_k^{\rho}}{\rho}\right)^{1-\gamma} p(t)\mu(B),$$

with ${}^{\rho}D_{a^+}^{\alpha,\beta}B = \{{}^{\rho}D_{a^+}^{\alpha,\beta}w : w \in B\}$ and $k = 1,\dots,m$.

($Ax4$) The functions $\varpi_k : E \longrightarrow E$ are continuous and there exists $\eta^* > 0$ such that

$$\left\|\varpi_k(u)\right\| \le \eta^*\|u\| \quad \text{for each } u \in E, k = 1,\dots,m.$$

(*Ax5*) For any bounded set $B \subset E$ and for any $t \in (a,b]$, we find

$$\mu\big(\varpi_k(B)\big) \leq \eta^* \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \mu(B), \quad k = 1,\dots,m.$$

Set $p^* = \sup_{t\in J} p(t)$.

We are now in a position to investigate the existence result for problem (1)–(3) based on the fixed point theorem of Mönch.

Theorem 3.3 *Assume that* (Ax1)–(Ax5) *hold. If*

$$\mathfrak{L} := \frac{m\eta^*}{\Gamma(\gamma)} + p^* \left(\frac{1}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} < 1, \tag{21}$$

then problem (1)–(3) *has at least one solution in* $PC^{\gamma}_{\gamma,\rho}(J) \subset PC^{\alpha,\beta}_{\gamma,\rho}(J)$.

Proof Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in (20) and the ball $B_R := B(0,R) = \{w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R\}$.

For any $u \in B_R$ and any $t \in (a,b]$, we find

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t) \right\| \\
&\quad\leq \frac{1}{\Gamma(\gamma)} \Bigg[\|\xi_2\| + |\xi_1| \sum_{i=1}^m \big\|\varpi_i\big(u(t_i^-)\big)\big\| + |\xi_1| \sum_{i=1}^m \big({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+} \big\|h(s)\big\|\big)(t_i) \\
&\qquad + |\xi_1| \big({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+} \|h\|\big)(b) + \sum_{a<t_k<t} \big\|\varpi_k\big(u(t_k^-)\big)\big\| + \sum_{a<t_k<t} \big({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} \big\|h(s)\big\|\big)(t_k) \Bigg] \\
&\qquad + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \big({}^\rho\mathcal{J}^{\alpha}_{t_k^+} \big\|h(s)\big\|\big)(t) \\
&\quad\leq \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)} \big(m\eta^* R + mp^* \big({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{i-1})^+}(1)\big)(t_i)\big) + \frac{|\xi_1| p^*}{\Gamma(\gamma)} \big({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{t_m^+}(1)\big)(b) \\
&\qquad + p^* \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} \big({}^\rho\mathcal{J}^{\alpha}_{t_k^+}(1)\big)(t).
\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma} (\Psi u)(t) \right\| \\
&\quad\leq \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)} \left(ml^* R + \frac{mp^*}{\Gamma(2-\gamma+\alpha)} \left(\frac{t_i^\rho - t_{i-1}^\rho}{\rho}\right)^{1-\gamma+\alpha} \right) \\
&\qquad + \frac{|\xi_1| p^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)} \left(\frac{b^\rho - t_m^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{p^*}{\Gamma(\alpha+1)} \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma+\alpha}.
\end{aligned}$$

Hence, for all $u \in PC_{\gamma,\rho}(J)$ and each $t \in (a,b]$, we get

$$\begin{aligned}\|(\Psi u)\|_{PC_{\gamma,\rho}} &\leq \frac{\|\xi_2\|}{\Gamma(\gamma)} + \frac{|\xi_1|+1}{\Gamma(\gamma)}\left(m\eta^*R + \frac{mp^*}{\Gamma(2-\gamma+\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right) \\ &\quad + \left(\frac{|\xi_1|p^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)} + \frac{p^*}{\Gamma(\alpha+1)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ &\leq R.\end{aligned}$$

Hence, we conclude that Ψ transforms the ball B_R into itself. Now, we prove that the operator $\Psi : B_R \to B_R$ fulfills all conditions of Theorem 2.16. For the sake of transparency, we shall divide the proof in four steps.

Step 1: We shall prove that the operator $\Psi : B_R \to B_R$ is continuous.

Suppose that the sequence $\{u_n\}$ converges to u in $PC_{\gamma,\rho}(J)$.

Then we get

$$\begin{aligned}&\left\|\big((\Psi u_n)(t) - (\Psi u)(t)\big)\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\right\| \\ &\quad\leq \frac{1}{\Gamma(\gamma)}\Bigg[|\xi_1|\sum_{i=1}^{m}\big\|\varpi_i\big(u_n(t_i^-)\big) - \varpi_i\big(u(t_i^-)\big)\big\| + |\xi_1|\big({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}\|h_n(s) - h(s)\|\big)(b) \\ &\qquad + |\xi_1|\sum_{i=1}^{m}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}\|h_n(s) - h(s)\|\big)(t_i) + \sum_{a<t_k<t}\big\|\varpi_k\big(u_n(t_k^-)\big) - \varpi_k\big(u(t_k^-)\big)\big\| \\ &\qquad + \sum_{a<t_k<t}\big({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\|h_n(s) - h(s)\|\big)(t_k)\Bigg] + \left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}\big({}^\rho\mathcal{J}_{t_k^+}^{\alpha}\|h_n(s) - h(s)\|\big)(t)\end{aligned}$$

for each $t \in (a,b]$, where $h_n, h \in C_{\gamma,\rho}(J)$ in a way that

$$\begin{aligned}h(t) &= f\big(t, u(t), h(t)\big), \\ h_n(t) &= f\big(t, u_n(t), h_n(t)\big).\end{aligned}$$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in (a,b]$. So we find that

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \quad \text{as } n \to \infty$$

by the Lebesgue dominated convergence theorem.

Step 2: We shall indicate that $\Psi(B_R)$ is equicontinuous and bounded.

Indeed, $\Psi(B_R)$ is bounded since $\Psi(B_R) \subset B_R$ and B_R is bounded. Next, let $\epsilon_1, \epsilon_2 \in J$, $\epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then

$$\begin{aligned}&\left\|\left(\frac{\epsilon_1^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_1) - \left(\frac{\epsilon_2^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(\epsilon_2)\right\| \\ &\quad\leq \frac{1}{\Gamma(\gamma)}\left[\sum_{\epsilon_1<t_k<\epsilon_2}\big\|\varpi_k\big(u(t_k^-)\big)\big\| + \sum_{\epsilon_1<t_k<\epsilon_2}\big({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\|h(s)\|\big)(t_k)\right]\end{aligned}$$

$$+\frac{p^*}{\Gamma(\alpha+1)}\left|\left(\frac{\epsilon_1^\rho-t_k^\rho}{\rho}\right)^{1-\gamma+\alpha}-\left(\frac{\epsilon_2^\rho-t_k^\rho}{\rho}\right)^{1-\gamma+\alpha}\right|$$
$$\to 0 \quad \text{as } \epsilon_1\to\epsilon_2.$$

Consequently, we conclude that $\Psi(B_R)$ is bounded and equicontinuous.

Step 3: The implication (4) of Theorem 2.16 holds.

Now let D be an equicontinuous subset of B_R such that $D\subset\overline{\Psi(D)}\cup\{0\}$, therefore the function $t\longrightarrow d(t)=\mu(D(t))$ is continuous on J. Regarding the properties of the measure μ, together with $(Ax3)$ and $(Ax5)$, for each $t\in(a,b]$, we find

$$\begin{aligned}
&\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}d(t)\\
&\le\mu\left(\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\cup\{0\}\right)\\
&\le\mu\left(\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right)\\
&\le\frac{1}{\Gamma(\gamma)}\left[\sum_{a<t_k<t}\eta^*\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\mu\big(D(t)\big)\right.\\
&\quad\left.+\sum_{a<t_k<t}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+}\left(\frac{s^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}p(s)\mu\big(D(s)\big)\right)(t_k)\right]\\
&\quad+\left(\frac{t^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^{\alpha}_{t_k^+}\left(\frac{s^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}p(s)\mu\big(D(s)\big)\right)(t)\\
&\le\frac{m\eta^*}{\Gamma(\gamma)}\|d\|_{PC_{\gamma,\rho}}+p^*\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho\mathcal{J}^{\alpha}_{a^+}\left(\frac{s^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}d(s)\right)(t)\\
&\quad+\frac{mp^*}{\Gamma(\gamma)}\left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{a^+}\left(\frac{s^\rho-t_k^\rho}{\rho}\right)^{1-\gamma}d(s)\right)(t)\\
&\le\left[\frac{m\eta^*}{\Gamma(\gamma)}+\frac{p^*}{\Gamma(\alpha+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right.\\
&\quad\left.+\frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right]\|d\|_{PC_{\gamma,\rho}}.
\end{aligned}$$

Thus

$$\|d\|_{PC_{\gamma,\rho}}\le\mathfrak{L}\|d\|_{PC_{\gamma,\rho}}.$$

From (21), we get $\|d\|_{PC_{\gamma,\rho}}=0$, that is, $d(t)=\mu(D(t))=0$, for each $t\in J_k$, $k=0,\dots,m$, and then $D(t)$ is relatively compact in E. In view of the Ascoli–Arzela theorem, D is relatively compact in B_R. Applying now Theorem 2.16, we conclude that Ψ has a fixed point $u^*\in PC_{\gamma,\rho}(J)$, which is a solution of problem (1)–(3).

Step 4: We show that such a fixed point $u^*\in PC_{\gamma,\rho}(J)$ lies in $PC^{\gamma}_{\gamma,\rho}(J)$.

Regarding the fact that u^* is the only fixed point of the mapping Ψ at $PC_{\gamma,\rho}(J)$, for any $t \in J_k$, with $k = 0, \ldots, m$, we have

$$u^*(t) = \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\Bigg[\xi_2 - \xi_1\sum_{i=1}^{m}\varpi_i\big(u(t_i^-)\big) - \xi_1\sum_{i=1}^{m}\big({}^\rho\mathcal{J}_{(t_{i-1})^+}^{1-\gamma+\alpha}h\big)(t_i)$$
$$- \xi_1\big({}^\rho\mathcal{J}_{t_m^+}^{1-\gamma+\alpha}h\big)(b) + \sum_{a<t_k<t}\varpi_k\big(u(t_k^-)\big) + \sum_{a<t_k<t}\big({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}h\big)(t_k)\Bigg]$$
$$+ \big({}^\rho\mathcal{J}_{t_k^+}^{\alpha}h\big)(t),$$

where $h \in C_{\gamma,\rho}(J)$ in a way that

$$h(t) = f\big(t, u^*(t), h(t)\big).$$

Applying ${}^\rho D_{t_k^+}^{\gamma}$ to both sides and by Lemma 2.6 and Lemma 2.12, we have

$$\begin{aligned}{}^\rho D_{t_k^+}^{\gamma}u^*(t) &= \big({}^\rho D_{t_k^+}^{\gamma}\,{}^\rho\mathcal{J}_{t_k^+}^{\alpha}f\big(s, u^*(s), h(s)\big)\big)(t)\\ &= \big({}^\rho D_{t_k^+}^{\beta(1-\alpha)}f\big(s, u^*(s), h(s)\big)\big)(t).\end{aligned}$$

On account of $\gamma \geq \alpha$, and (*Ax1*), it lies in $PC_{\gamma,\rho}(J)$. Consequently, ${}^\rho D_{t_k^+}^{\gamma}u^* \in PC_{\gamma,\rho}(J)$, which implies that $u^* \in PC_{\gamma,\rho}^{\gamma}(J)$. Regarding Theorem 3.3, as an outcome of Step 1 to Step 4, we deduce that problem (1)–(3) has at least one solution in $PC_{\gamma,\rho}^{\gamma}(J)$. □

Our second existence result for problem (1)–(3) is based on Darbo's fixed point theorem.

Theorem 3.4 *Assume that* (Ax1)–(Ax5) *and* (21) *hold. Then problem* (1)–(3) *has at least one solution in* $PC_{\gamma,\rho}^{\gamma}(J) \subset PC_{\gamma,\rho}^{\alpha,\beta}(J)$.

Proof Consider the operator Ψ defined in (20). We know that $\Psi : B_R \longrightarrow B_R$ is bounded and continuous and that $\Psi(B_R)$ is equicontinuous, we need to prove that the operator Ψ is a $\mathfrak{L}$-contraction.

Let $D \subset B_R$ and $t \in J$. Then we have

$$\begin{aligned}&\mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi D)(t)\right)\\ &\quad= \mu\left(\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}(\Psi u)(t) : u \in D\right)\\ &\quad\leq \frac{1}{\Gamma(\gamma)}\Bigg[\sum_{a<t_k<t}\eta^*\mu\left(\left\{\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}u(t), u \in D\right\}\right)\\ &\qquad+ \sum_{a<t_k<t}\left\{\left({}^\rho\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}p^*\mu\left(\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}u(s)\right)\right)(t_k), u \in D\right\}\Bigg]\\ &\qquad+ \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left\{\left({}^\rho\mathcal{J}_{t_k^+}^{\alpha}p^*\mu\left(\left(\frac{s^\rho - t_k^\rho}{\rho}\right)^{1-\gamma}u(s)\right)\right)(t), u \in D\right\}.\end{aligned}$$

By Lemma, we have

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq \left[\frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\right]\mu_{PC_{\gamma,\rho}}(D).$$

Therefore

$$\mu_{PC_{\gamma,\rho}}(\Psi D) \leq \mathfrak{L}\mu_{PC_{\gamma,\rho}}(D).$$

So, by (21), the operator Ψ is a $\mathfrak{L}$-contraction. □

As a consequence of Theorem 2.17 and using Step 4 of the last result, we deduce that Ψ possesses a fixed point. In particular, this fixed point forms a solution for problem (1)–(3).

Example 3.5 Consider the Banach space

$$E = l^1 = \left\{u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty\right\}$$

with the norm

$$\|u\| = \sum_{n=1}^{\infty} |u_n|.$$

We examine the following impulsive boundary value problem of the generalized Hilfer FDE:

$$\left({}^1D_{t_k^+}^{\frac{1}{2},0} u_n\right)(t) = \frac{3t^2 - 20}{213e^{-t+3}(1 + |u_n(t)| + |{}^1D_{t_k^+}^{\frac{1}{2},0} u_n(t)|)}, \quad t \in J_k, k = 0, \dots, 9, \tag{22}$$

$$\left({}^1\mathcal{J}_{t_k^+}^{\frac{1}{2}} u_n\right)\left(t_k^+\right) - \left({}^1\mathcal{J}_{t_{(k-1)}^+}^{\frac{1}{2}} u_n\right)\left(t_k^-\right) = \frac{|u_n(t_k^-)|}{10(k+3) + |u_n(t_k^-)|}, \quad k = 1, \dots, 9, \tag{23}$$

$$\left({}^1\mathcal{J}_{1^+}^{\frac{1}{2}} u_n\right)\left(1^+\right) + 2\left({}^1\mathcal{J}_{\frac{9}{5}^+}^{\frac{1}{2}} u_n\right)(3) = 0, \tag{24}$$

where $J_k = (t_k, t_{k+1}]$, $t_k = 1 + \frac{k}{5}$ for $k = 0, \dots, 9$, $m = 9$, $a = t_0 = 1$, and $b = t_{10} = 3$.

Set

$$f(t, u, w) = \frac{3t^2 - 20}{213e^{-t+3}(1 + \|u\| + \|w\|)}, \quad t \in (1, 3], u, w \in E.$$

We have

$$PC_{\gamma,\rho}^{\beta(1-\alpha)}([1,3]) = PC_{\frac{1}{2},1}^{0}([1,3]) = \{g : (1,3] \to \mathbb{R} : (\sqrt{t - t_k})g \in PC([1,3])\},$$

where $\alpha = \gamma = \frac{1}{2}$, $\beta = 0$, $\rho = 1$, and $k = 0, \dots, 9$. Clearly, the continuous function $f \in PC_{\frac{1}{2},1}^{0}([1,2])$.

As a result, condition $(Ax1)$ is fulfilled.

For each $u, w \in E$ and $t \in (1,3]$,

$$\|f(t,u,w)\| \le \frac{3t^2-20}{213e^{-t+3}}.$$

Hence condition $(Ax2)$ is satisfied with $p^* = \frac{7}{213}$.

And let

$$\varpi_k(u) = \frac{\|u\|}{10(k+3)+\|u\|}, \quad k=1,\dots,9, u \in E.$$

Let $u \in E$. Then we have

$$\|\varpi_k(u)\| \le \frac{1}{40}\|u\|, \quad k=1,\dots,9,$$

and so condition $(Ax4)$ is satisfied with $\eta^* = \frac{1}{40}$.

The condition (21) of Theorem 3.3 is satisfied for

$$\begin{aligned}\mathfrak{L} &:= \frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ &= \frac{9}{40\sqrt{\pi}} + 2\left(\frac{17}{213\sqrt{\pi}} + \frac{63}{213\Gamma(2)\sqrt{\pi}}\right) \\ &\approx 0.55074703829 < 1.\end{aligned}$$

Then problem (22)–(24) has at least one solution in $PC^{\frac{1}{2}}_{\frac{1}{2},1}([1,3]) \subset PC^{\frac{1}{2},0}_{\frac{1}{2},1}([1,3])$.

4 Ulam-type stability

Now, we consider the Ulam stability for problem (1)–(3). Let $u \in PC_{\gamma,\rho}(J)$, $\epsilon > 0$, $\tau > 0$, and $\vartheta : (a,b] \longrightarrow [0,\infty)$ be a continuous function. We consider the following inequality:

$$\begin{cases} \|({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t) - f(t,u(t),({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t))\| \le \epsilon\vartheta(t), & t \in J_k, k=0,\dots,m, \\ \|({}^\rho \mathcal{J}^{1-\gamma}_{t_k^+} u)(t_k^+) - ({}^\rho \mathcal{J}^{1-\gamma}_{t_{k-1}^+} u)(t_k^-) - \varpi_k(u(t_k^-))\| \le \epsilon\tau, & k=1,\dots,m. \end{cases} \tag{25}$$

Definition 4.1 Problem (1)–(3) is Ulam–Hyers–Rassias (U-H-R) stable with respect to (ϑ,τ) if there exists a real number $a_{f,m,\vartheta} > 0$ such that, for each $\epsilon > 0$ and for each solution $u \in PC_{\gamma,\rho}(J)$ of inequality (25), there exists a solution $w \in PC_{\gamma,\rho}(J)$ of (1)–(3) with

$$\|u(t) - w(t)\| \le \epsilon a_{f,m,\vartheta}(\vartheta(t)+\tau), \quad t \in (a,b].$$

Remark 4.2 A function $u \in PC_{\gamma,\rho}(J)$ is a solution of inequality (25) if and only if there exist $\sigma \in PC_{\gamma,\rho}(J)$ and a sequence σ_k, $k=0,\dots,m$, such that

1 $\|\sigma(t)\| \le \epsilon\vartheta(t)$ and $\|\sigma_k\| \le \epsilon\tau$, $t \in J_k$, $k=1,\dots,m$;
2 $({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t) = f(t,u(t),({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t)) + \sigma(t)$, $t \in J_k$, $k=0,\dots,m$;
3 $({}^\rho \mathcal{J}^{1-\gamma}_{t_k^+} u)(t_k^+) = ({}^\rho \mathcal{J}^{1-\gamma}_{t_{k-1}^+} u)(t_k^-) + \varpi_k(u(t_k^-)) + \sigma_k$, $k=1,\dots,m$.

Now we are concerned with the Ulam–Hyers–Rassias stability of our problem (1)–(3).

Theorem 4.3 *Assume that in addition to* (Ax1)–(Ax5) *and* (21), *the following hypotheses hold*:

(*Ax6*) *There exist a nondecreasing function* $\vartheta \in PC_{\gamma,\rho}(J)$ *and* $\lambda_\vartheta > 0$ *such that, for each* $t \in (a,b]$, *we have*

$$\left({}^\rho\mathcal{J}^\alpha_{a^+}\vartheta\right)(t) \le \lambda_\vartheta \vartheta(t).$$

(*Ax7*) *There exists a continuous function* $\chi : [a,b] \longrightarrow [0,\infty)$ *such that, for each* $t \in J_k$; $k = 0,\dots,m$, *we have*

$$p(t) \le \chi(t)\vartheta(t).$$

Then equation (1) *is U-H-R stable with respect to* (ϑ,τ).

Set $\chi^* = \sup_{t\in[a,b]} \chi(t)$.

Proof Consider the operator Ψ defined in (20). Let $u \in PC_{\gamma,\rho}(J)$ be a solution of inequality (25), and let us assume that w is the unique solution of the problem

$$\begin{cases} ({}^\rho D^{\alpha,\beta}_{t_k^+} w)(t) = f(t, w(t), ({}^\rho D^{\alpha,\beta}_{t_k^+} w)(t)); & t \in J_k, k = 0,\dots,m, \\ ({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} w)(t_k^+) = ({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+} w)(t_k^-) + \varpi_k(w(t_k^-)); & k = 1,\dots,m, \\ c_1({}^\rho\mathcal{J}^{1-\gamma}_{a^+} w)(a^+) + c_2({}^\rho\mathcal{J}^{1-\gamma}_{t_m^+} w)(b) = c_3, \\ ({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} w)(t_k^+) = ({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} u)(t_k^+); & k = 0,\dots,m. \end{cases}$$

By Theorem 3.1 and Lemma 3.2, we obtain for each $t \in (a,b]$

$$w(t) = \frac{({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} w)(t_k^+)}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho\mathcal{J}^\alpha_{t_k^+} h\right)(t), \quad t \in J_k, k = 0,\dots,m,$$

where $h : (a,b] \to E$ is a function satisfying the functional equation

$$h(t) = f\bigl(t, w(t), h(t)\bigr).$$

Since u is a solution of inequality (25), by Remark 4.2, we have

$$\begin{cases} ({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t) = f(t, u(t), ({}^\rho D^{\alpha,\beta}_{t_k^+} u)(t)) + \sigma(t), & t \in J_k, k = 0,\dots,m; \\ ({}^\rho\mathcal{J}^{1-\gamma}_{t_k^+} u)(t_k^+) = ({}^\rho\mathcal{J}^{1-\gamma}_{t_{k-1}^+} u)(t_k^-) + \varpi_k(u(t_k^-)) + \sigma_k, & k = 1,\dots,m. \end{cases} \tag{26}$$

Clearly, the solution of (26) is given by

$$\begin{aligned} u(t) = {}& \frac{1}{\Gamma(\gamma)}\left(\frac{t^\rho - t_k^\rho}{\rho}\right)^{\gamma-1}\Biggl[\left({}^\rho\mathcal{J}^{1-\gamma}_{a^+} u\right)\left(a^+\right) + \sum_{a<t_k<t} \varpi_k\bigl(u\bigl(t_k^-\bigr)\bigr) + \sum_{a<t_k<t} \sigma_k \\ & + \sum_{a<t_k<t} \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} g\right)(t_k) + \sum_{a<t_k<t} \left({}^\rho\mathcal{J}^{1-\gamma+\alpha}_{(t_{k-1})^+} \sigma\right)(t_k)\Biggr] \end{aligned}$$

$$+\left({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}g\right)(t)+\left({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}\sigma\right)(t),\quad t\in J_k, k=0,\dots,m,$$

where $g:(a,b]\to E$ is a function satisfying the functional equation

$$g(t)=f\big(t,u(t),g(t)\big).$$

We have, for each $t\in J_k$, $k=0,\dots,m$,

$$\begin{aligned}\left({}^{\rho}\mathcal{J}_{t_k^+}^{1-\gamma}u\right)\left(t_k^+\right)&=\left({}^{\rho}\mathcal{J}_{a^+}^{1-\gamma}u\right)\left(a^+\right)+\sum_{a<t_k<t}\varpi_k\left(u\left(t_k^-\right)\right)+\sum_{a<t_k<t}\sigma_k\\&\quad+\sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}g\right)(t_k)+\sum_{a<t_k<t}\left({}^{\rho}\mathcal{J}_{(t_{k-1})^+}^{1-\gamma+\alpha}\sigma\right)(t_k).\end{aligned}$$

Hence, for each $t\in(a,b]$, we have

$$\left\|u(t)-w(t)\right\|\le\left({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}\left|g(s)-h(s)\right|\right)(t)+\left({}^{\rho}\mathcal{J}_{t_k^+}^{\alpha}\left|\sigma(s)\right|\right)(t).$$

Thus,

$$\begin{aligned}\left\|u(t)-w(t)\right\|&\le\left({}^{\rho}\mathcal{J}_{a^+}^{\alpha}\left\|g(s)-h(s)\right\|\right)(t)+\left({}^{\rho}\mathcal{J}_{a^+}^{\alpha}\left\|\sigma(s)\right\|\right)\\&\le\epsilon\lambda_\vartheta\vartheta(t)+\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{2\chi(t)\vartheta(t)}{\Gamma(\gamma)}\,ds\\&\le\epsilon\lambda_\vartheta\vartheta(t)+2\chi^*\left({}^{\rho}\mathcal{J}_{a^+}^{\alpha}\vartheta\right)(t)\\&\le\left(\epsilon+2\chi^*\right)\lambda_\vartheta\vartheta(t)\\&\le\left(1+\frac{2\chi^*}{\epsilon}\right)\lambda_\vartheta\epsilon\left(\tau+\vartheta(t)\right)\\&\le a_\vartheta\epsilon\left(\tau+\vartheta(t)\right),\end{aligned}$$

where $a_\vartheta=(1+\frac{2\chi^*}{\epsilon})\lambda_\vartheta$. Hence, equation (1) is U-H-R stable with respect to (ϑ,τ). □

Example 4.4 Consider the Banach space

$$E=l^1=\left\{u=(u_1,u_2,\dots,u_n,\dots),\sum_{n=1}^{\infty}|u_n|<\infty\right\}$$

with the norm

$$\|u\|=\sum_{n=1}^{\infty}|u_n|,$$

and let the following impulsive anti-periodic boundary value problem hold:

$$\left({}^{\frac{1}{2}}D_{t_k^+}^{\frac{1}{2},0}u_n\right)(t)=\frac{(3t^3+5e^{-3})|u_n(t)|}{144e^{-t+e}(1+\|u(t)\|+\|{}^{\frac{1}{2}}D_{t_k^+}^{\frac{1}{2},0}u(t)\|)}\quad\text{for each } t\in J_0\cup J_1,\tag{27}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}_{2^+}^{\frac{1}{2}}u_n\right)\left(2^+\right)-\left({}^{\frac{1}{2}}\mathcal{J}_{1^+}^{\frac{1}{2}}u_n\right)\left(2^-\right)=\frac{|u_n(2^-)|}{77e^{-t+4}+2},\tag{28}$$

$$\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{1^+}u\right)(1^+) = -\left({}^{\frac{1}{2}}\mathcal{J}^{\frac{1}{2}}_{2^+}u\right)(e), \tag{29}$$

where $J_0 = (1,2]$, $J_1 = (2,e]$, $t_1 = 2$, $m = 1$, $a = t_0 = 1$, and $b = t_2 = e$.

Set

$$f(t,u,w) = \frac{(3t^3 + 5e^{-3})\|u\|}{144e^{-t+e}(1 + \|u\| + \|w\|)}, \quad t \in (1,e], u,w \in E.$$

We have

$$PC^{\beta(1-\alpha)}_{\gamma,\rho}([1,2]) = PC^0_{\frac{1}{2},\frac{1}{2}}([1,e]) = \left\{g : (1,e] \to E : \sqrt{2}(\sqrt{t} - \sqrt{t_k})^{\frac{1}{2}} g \in C([1,e])\right\},$$

with $\gamma = \alpha = \frac{1}{2}$, $\rho = \frac{1}{2}$, $\beta = 0$, and $k \in \{0,1\}$. Clearly, the continuous function $f \in PC^0_{\frac{1}{2},\frac{1}{2}}([1,e])$.

Hence condition (*Ax1*) is satisfied.

For each $u,w \in E$ and $t \in (1,e]$,

$$\|f(t,u,w)\| \le \frac{(3t^3 + 5e^{-3})}{144e^{-t+e}}.$$

Hence condition (*Ax2*) is satisfied with

$$p(t) = \frac{(3t^3 + 5e^{-3})}{144e^{-t+e}}$$

and

$$p^* = \frac{(3e^3 + 5e^{-3})}{144}.$$

And let

$$\varpi_1(u) = \frac{\|u\|}{77e^{-t+4} + 2}, \quad u \in E.$$

Let $u \in E$. Then we have

$$\|\varpi_k(u)\| \le \frac{1}{77e^{-t+4} + 2}\|u\|,$$

and so condition (*Ax4*) is satisfied with $\eta^* = \frac{1}{77e^{4-e}+2}$.

The condition (21) of Theorem 3.3 is satisfied for

$$\begin{aligned}\mathfrak{L} &:= \frac{m\eta^*}{\Gamma(\gamma)} + \left(\frac{p^*}{\Gamma(\alpha+1)} + \frac{mp^*}{\Gamma(\gamma)\Gamma(2-\gamma+\alpha)}\right)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha} \\ &= \frac{1}{(77e^{4-e}+2)\sqrt{\pi}} + (2\sqrt{e} - 2)\left(\frac{6e^3 + 10e^{-3}}{144\sqrt{\pi}} + \frac{3e^3 + 5e^{-3}}{144\sqrt{\pi}\Gamma(2)}\right) \\ &\approx 0.92473323802 < 1.\end{aligned}$$

Then problem (27)–(29) has at least one solution in $PC^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}([1,e]) \subset PC^{\frac{1}{2},0}_{\frac{1}{2},\frac{1}{2}}([1,e])$. Also, hypothesis (*Ax6*) is satisfied with $\tau = 1$, $\vartheta(t) = e^3$, and $\lambda_\vartheta = 3$. Indeed, for each $t \in (1,e]$, we get

$$\left({}^{\frac{1}{2}}\mathcal{J}_{1^+}^{\frac{1}{2}}\vartheta\right)(t) \le \frac{2e^3}{\Gamma(\frac{3}{2})} \le \lambda_\vartheta \vartheta(t).$$

Let the function $\chi : [1,e] \longrightarrow [0,\infty)$ be defined by

$$\chi(t) = \frac{(3e^{-3}t^3 + 5e^{-6})}{144e^{-t+e}},$$

then, for each $t \in (1,e]$, we have

$$p(t) = \chi(t)\vartheta(t)$$

with $\chi^* = p^* e^{-3}$. Hence, condition $(Ax7)$ is satisfied. Consequently, Theorem 4.3 implies that equation (27) is U-H-R stable.

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