EXISTENCE AND UNIQUENESS OF SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS ON A CLOSED SUBSET OF A BANACH SPACE

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1. Introduction.

In an earlier work [5], sufficient conditions for the existence of solutions in a closed subset F of a Banach space E for the Cauchy problem

$$(1.1) \qquad x'(t) = f(t,x_t), \quad x_{t_0} = \phi_0 ,$$

where $f : R^+ \times C \to E$, $C = C[[-\tau,0],E]$, $\phi_0 \in C_F = \{\phi \in C : \phi(0) \in F\}$, are obtained by requiring f to satisfy (i) a compactness-type condition in terms of the Kuratowski measure of noncompactness and (ii) a boundary condition, namely,

$$(1.2) \qquad \liminf_{h \to 0^+} \frac{1}{h} d(\phi(0) + hf(t,\phi),F) = 0 ,$$

for every $\phi \in C_F$. In this paper, we wish to explore the other direction followed in the study of the Cauchy problem for differential equations in a Banach space [4,6,7], that is, finding dissipative type conditions which assure the existence as well as the uniqueness of solutions in F for the Cauchy problem (1.1). In [3], the existence and uniqueness of solutions of (1.1) has been proved in an open set when the function f satisfies a simple dissipative-type condition over a suitable subset of its domain.

In Section 4, we show (Theorem 4.3) that when the boundary condition (1.2) is satisfied and f satisfies a Lipschitz type condition in terms of a comparison function of the form $g(t,u,u_t)$, $u_t \in C^+ = C[[-\tau,0],R^+]$, the existence and uniqueness of solutions of (1.1) in F is established by making use of comparison theorems involving functional differential inequalities. Also, if the closed set F is assumed to be convex, the dissipative condition on f can be weakened by imposing, as in [3], that it has to be satisfied only over a suitable subset of the domain of f (see Theorem 4.1).

One may also be interested in determining the existence and uniqueness of solutions of (1.1) when the initial function ϕ_0 is such that $\phi_0(\theta) \in F$ for every $\theta \in [-\tau,0]$. In this case, in general, it is not enough to require that (1.2) be satisfied only for such functions, as the counterexample given in [5] shows. But, if we assume that (1.2) is satisfied for $\phi \in \hat{C}_F = \{\phi \in C : \phi \in C_F$ and $\phi(\theta) \in \overline{co}\, F\}$, where $\overline{co}\, F$ is the closed convex hull of F, then the aforesaid dissipative-type condition assures the existence and uniqueness of solutions of (1.1) in F, when $\phi_0 \in \hat{C}_F$. Thus, the result obtained by Seifert in [8, Theorem 2] for delay differential equations in R^n, can be generalized in two different ways when the delay is constant, namely, either by eliminating the convexity hypothesis on F or by relaxing the dissipative condition.

The problem of proving an existence and uniqueness result when the set F is not convex and f satisfies a general dissipative-type condition in terms of a Lyapunov-like function presents several difficulties, as in the case of differential equations without delay [3,5]. In Section 5, we show (Lemma 5.2) that if (a) f satisfies a dissipative-type condition in terms of a Lyapunov function belonging to a certain class, and (b) the boundary condition (1.2) holds, we can construct certain auxiliary sequences of continuous functions which are in some sense close to the polygonal approximate solutions for the Cauchy problem (1.1) obtained in [5, Lemma 3]. Then, by employing a new comparison result (Lemma 5.3) which involves a sequence of functions satisfying a certain functional differential inequality, we establish the existence and uniqueness of solutions of (1.1) in F.

2. Preliminaries.

Let E be a Banach space with norm $||\cdot||$ and $F \subset E$ be a closed set. Let $C = C[[-\tau,0],E]$, $\tau > 0$, be the space of continuous functions $\phi : [-\tau,0] \to E$, equipped with the norm $||\cdot||_0$ given by

$$||\phi||_0 = \sup_{\theta \in [-\tau,0]} ||\phi(\theta)|| .$$

Let us consider the subsets

$$C_F = \{\phi \in C : \phi(0) \in F\}$$

and

$$\hat{C}_F = C_F \cap \{\phi \in C : \phi(\theta) \in \overline{co}\, F \text{ , for every } \theta \in [-\tau,0]\}$$

where $\overline{co}\, F$ is the closed convex hull of F .

Let $a > 0$, $t_0 \in R^+$ and $\phi_0 \in C_F$. Following the notation in [5], for $b > 0$ and $t \in [t_0,t_0 + a]$, we define

$$C_F^t(b) = C_F \cap \{\phi \in C : ||\phi - y_t||_0 \leq b\}$$

where $y_t \in C$ is defined by $y_t(\theta) = y(t + \theta)$, $-\tau \leq \theta \leq 0$, for each t , such that

$$y(t) = \begin{cases} \phi_0(t-t_0) \text{ , if } t_0 - \tau \leq t \leq t_0 \text{ ,} \\ \phi_0(0) \text{ , if } t_0 \leq t \leq t_0 + a \text{ .} \end{cases}$$

If $f \in C[R^+ \times C_F,E]$, it is possible to show that there exists a $b > 0$ such that f is bounded [4] on the set $\tilde{C}_0(b) = \bigcup_{t \in [t_0,t_0 + a]} (\{t\} \times C_F^t(b))$.

For convenience of reference, we list below the following hypotheses:

(A_1) $t_0 \in R^+$, $\phi_0 \in C_F$, $f \in C[[t_0, t_0 + a] \times C_F, E]$, a, b and M are such that $\|f(t,\phi)\| \leq M - 1$ $(M \geq 1)$ on $\tilde{C}_0(b)$ and $\gamma = \min\left(a, \frac{b}{M}\right)$;

(A_2) $\liminf_{h \to 0^+} \frac{1}{h} d(\phi(0) + hf(t,\phi), F) = 0$, for every $(t,\phi) \in [t_0, t_0 + a] \times C_F$.

The following two results [5] guarantee the existence of a sequence of approximate solutions and the fact that the limit function, if it exists, is a solution of the Cauchy problem (1.1).

<u>Lemma 2.1</u>. Suppose that the hypotheses (A_1) and (A_2) are satisfied. If $\{\varepsilon_n\} \subset (0,1)$ is a nonincreasing sequence with $\lim_{n\to\infty} \varepsilon_n = 0$, then there exists a sequence $\{x_n(t)\}$ of ε_n-approximate solutions for the Cauchy problem (1.1), that is, for every n, there exists a function $x_n(t) : [t_0 - \tau, t_0 + \gamma] \to E$ with the following properties:

(i) there exists a sequence $\{t_i^n\}_{i=1}^{\infty}$ in $[t_0, t_0 + \gamma]$ such that $t_0^n = t_0$, $t_{i+1}^n = t_i^n + \delta_i^n$, $\delta_i^n > 0$ if $t_i^n < t_0 + \gamma$, $\lim_{i\to\infty} t_i^n = t_0 + \gamma$.

(ii) $x_n(t) = \phi_0(t - t_0)$ for $t \in [t_0 - \tau, t_0]$, $\|x_n(t) - x_n(s)\| \leq M|t-s|$ for $t, s \in [t_0, t_0 + \gamma]$;

(iii) for each $i \geq 0$, $\left(t_i^n, x_{n,t_i^n}\right) \in \tilde{C}_0(b)$ and $x_n(t)$ is linear on each of the intervals $[t_i^n, t_{i+1}^n]$;

(iv) if $t \in (t_i^n, t_{i+1}^n)$ and $t_i^n < t + \gamma$, then $\|x_n'(t) - f(t_i^n, x_{n,t_i^n})\| \leq \varepsilon_n$;

(v) δ_i^n can be chosen less than $\min\{\varepsilon_n, \delta_{\phi_0}(\eta_n/2), \eta_n/2M\}$ where $\delta_{\phi_0}(\eta_n)$ is the number associated with η_n by the uniform continuity of ϕ_0 on $[-\tau, 0]$ and η_n is such that $|t - t_i^n| < \eta_n$, $\|\phi - x_{n,t_i^n}\|_0 < \eta_n$ imply that

$||f(t,\phi) - f(t_i^n, x_n, t_i^n)|| \leq \varepsilon_n$;

Remark 2.1. Lemma 2.1 remains valid when in (A_1) and (A_2), the set C_F is replaced by $\hat{C}_F$.

Lemma 2.2. Let the assumptions of Lemma 2.1 hold. If the sequence $\{x_n(t)\}$ of ε_n-approximate solutions of the Cauchy problem (1.1) which exist by virtue of Lemma 2.1 is such that $\{x_n(t)\}$ converges to $x(t)$ uniformly on $[t_0-\tau, t_0+\gamma]$, then $x(t)$ is a solution of (1.1) such that $x(t) \in F$ for $t \in [t_0, t_0+\gamma]$.

3. Comparison results.

In this section, we give some known convergence and comparison lemmas [2]. Let us consider the following assumptions:

Suppose that

(S_1) $r(t) = r(t,t_0,u_0)$ is the maximal solution of the scalar differential equation

$$(3.1) \qquad u' = g(t,u) \ , \ u(t_0) = u_0 \geq 0 \ , \ t_0 \in R^+$$

on $[t_0,\infty)$, where $g \in C[R^+ \times R^+, R^+]$;

(S_2) $r(t) = r(t,t_0,\phi_0)$ is the maximal solution of the scalar functional differential equation

$$(3.2) \qquad u' = g(t,u,u_t) \ , \ u_{t_0} = \phi_0 \ , \ t \in R^+$$

on $[t_0,\infty)$, where $g \in C[R^+ \times R^+ \times C^+, R]$, $C^+ = C[[-\tau,0],R^+]$ and $g(t,u,\phi)$ is nondecreasing in ϕ for each (t,u) , that is, $\phi_1(\theta) \leq \phi_2(\theta)$ for every $\theta \in [-\tau,0]$ implies $g(t,u,\phi_1) \leq g(t,u,\phi_2)$ for each (t,u) .

Lemma 3.1. Let (S_1) $((S_2)$ respectively) hold. Then, for any interval $[t_0,t_1] \subset [t_0,\infty)$, there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t,\varepsilon) = r(t,t_0,u_0,\varepsilon)$ of

$$(3.3) \qquad u' = g(t,u) + \varepsilon \, , \; u(t_0) = u_0 + \varepsilon$$

(the maximal solution $r(t,\varepsilon) = r(t,t_0,\phi_0,\varepsilon)$ of

$$(3.4) \qquad u' = g(t,u,u_t) + \varepsilon \, , \; u_{t_0} = \phi_0 + \varepsilon \, ,$$

respectively) exists on $[t_0,t_1]$ and $\lim_{\varepsilon\to 0} r(t,\varepsilon) = r(t)$ uniformly on $[t_0,t_1]$.

Lemma 3.2. Assume that (S_1) holds. Let $m \in C[t_0-\tau,\infty),R^+]$ and for every $\hat{t} \in [t_0,\infty)\setminus S$ where S is a countable subset of $[t_0,\infty)$, the differential inequality

$$Dm(\hat{t}) \leq g(\hat{t},m(\hat{t}))$$

(with D being any one of the Dini derivatives D^-,D_-,D^+) be satisfied provided $\hat{t}$ is such that $m_{\hat{t}}(\theta) \leq m(\hat{t})$, $-\tau \leq \theta \leq 0$. Then $m(t) \leq r(t,t_0,u_0)$, $t \geq t_0$ whenever $m_{t_0}(\theta) \leq u_0$, $-\tau \leq \theta \leq 0$.

Proof. The proof of this lemma is quite well known [2] for $D = D_-$ (and therefore for $D = D^-$). We sketch the proof for the case $D = D^+$. In view of Lemma 3.1, it is enough to show that for every $\varepsilon > 0$, sufficiently small, $m(t) \leq u(t,\varepsilon)$, $t \in [t_0,\infty)\setminus S$, where $u(t,\varepsilon) \equiv u(t,t_0,u_0,\varepsilon)$ is any solution of (3.3). Suppose that $Z = \{t \in [t_0,\infty)\setminus S : m(t) > u(t,\varepsilon)\}$ is nonempty and let $t_1 = \inf Z$. Since $m(t_0) \leq u_0 < u_0 + \varepsilon = u(t_0,\varepsilon)$, we have $t_1 > t_0$, $m(t_1) = u(t_1,\varepsilon)$ and $m(t) \leq u(t,\varepsilon)$, $t \in [t_0,t_1]$.

As $u(t,\varepsilon)$ is an increasing function and $m_{t_0}(\theta) \leq u_0$, $-\tau \leq \theta \leq 0$,

$$m(t) \leq u(t_1,\varepsilon) = m(t_1)$$

for every $t \in [t_0-\tau,t_1]$. Hence by hypothesis on m ,

$$D^+m(t_1) - g(t_1,m(t_1)) \leq 0 . \tag{3.5}$$

Let $v(t) = m(t) - u(t,\varepsilon)$. Since Z is nonempty, there exists a sequence $\{h_n\}$, $h_n > 0$, $h_n \to 0$ as $n \to \infty$, such that

$$v(t_1 + h_n) = m(t_1 + h_n) - u(t_1 + h_n ,\varepsilon) > 0 .$$

Since $D^+v(t_1) \geq \limsup_{n \to \infty} \frac{v(t_1 + h_n) - v(t_1)}{h_n}$

$$= \limsup_{n \to \infty} \frac{v(t_1 + h_n)}{h_n} \geq 0 ,$$

We have $D^+m(t_1) - g(t_1,m(t_1)) - \varepsilon = D^+v(t_1) \geq 0$ which contradicts (3.5). The lemma is therefore proved.

The following known comparison result involving a functional differential inequality [2] is also used in the sequel.

Lemma 3.3. Assume that (S_2) holds. Let $m \in C[t_0,\infty),R^+]$ and for $t \in [t_0,\infty)\backslash S$, S being a countable subset of $[t_0,\infty)$,

$$Dm(t) \leq g(t,m(t),m_t)$$

(where D is any Dini derivative). Then, $m_{t_0}(\theta) \leq \phi_0(\theta)$, $-\tau \leq \theta \leq 0$ implies $m(t) \leq r(t,t_0,\phi_0)$, $t \geq t_0$.

4. Existence and uniqueness results.

In this section we will give some existence and uniqueness theorems for the solutions of (1.1) in the closed set F, which extend to delay differential equations some analogous results given in [4,6] for ordinary differential equations. These theorems also provide, in the case of a constant delay, two kinds of generalizations of a theorem given by Seifert [7] in which the closed set F is supposed to be convex. In fact, we prove that when the convexity hypothesis on F is assumed, the following dissipative-type condition on f, namely

(A_3) for $t \in [t_0, t_0 + a]$,

$$||f(t,\phi) = f(t,\psi)|| \leq g(t,||\phi(0) - \psi(0)||) \tag{4.1}$$

whenever $\phi,\psi \in C_F^t(b)$ satisfy the relation

$$||\phi(\theta) - \psi(\theta)|| \leq ||\phi(0) - \psi(0)|| ,$$

for every $\theta \in [-\tau,0]$ and $g \in C[R^+ \times R^+, R^+]$ is such that $g(t,0) \equiv 0$ and $u(t) \equiv 0$ is the unique solution of

$$u' = g(t,u) , \; u(t_0) = 0 , \tag{4.2}$$

will imply, together with (A_1) and (A_2), the existence and uniqueness of the solutions of (1.1) in F .

If no convexity of F is assumed, then condition (A_3) has to be strengthened in the following natural way:

(A_4) for $t \in [t_0, t_0 + a]$, $\phi,\psi \in C_F^t(b)$,

$$||f(t,\phi) - f(t,\psi)|| \leq g(t,||\phi(0) - \psi(0)|| , ||\phi(\cdot) - \psi(\cdot)||) \tag{4.3}$$

where $g \in C[R^+ \times R^+ \times C^+, R]$ is such that $g(t,0,0) \equiv 0$, $g(t,u,u_t)$ is nondecreasing in u and u_t for each (t,u_t) and (t,u) respectively and $u(t) \equiv 0$ is the unique solution of

(4.4) $$u'(t) = g(t,u,u_t) , \; u_{t_0} \equiv 0 .$$

The assumption (A_4) and consequently (A_3) is weaker than the condition (H_4) in Theorem 2 of [8]. Obviously when the delay is zero both the conditions (4.1) and (4.2) reduce to

$$||f(t,x) - f(t,y)|| \leq g(t,||x-y||)$$

which is assumed in Theorem 3.1 in [4].

Let us first consider

(a) the case when F is convex.

<u>Theorem 4.1</u>. *Assume that the hypotheses* (A_1), (A_2) *and* (A_3) *hold and that the set* F *is convex. Then the problem* (1.1) *has a unique solution* $x(t)$ *on* $[t_0-\tau, t_0 + \gamma]$ *such that* $x(t) \in F$ *for* $t \in [t_0, t_0 + \gamma]$.

<u>Proof</u>. Let n and m be positive integers and let $x_n(t)$, $x_m(t)$ be the ε_n , ε_m-approximate solutions of (1.1) assured by Lemma 2.1. we have that $x_n(t)$, $x_m(t)$ belong to F for every $t \in [t_0, t_0 + \gamma]$ as F is convex. Let us set

$$m(t) = ||x_n(t) - x_m(t)|| , \; t \in [t_0 - \tau, t_0 + \gamma] .$$

Let $t \in \left(t_i^n , t_{i+1}^n\right) \cap \left(t_j^m , t_{j+1}^m\right)$. We then have

$$\begin{aligned} D^+m(t) &\leq ||x_n'(t) - x_m'(t)|| \\ &\leq ||x_n'(t) - f(t_i^n , x_{n,t_i^n})|| + ||x_m'(t) - f(t_j^m , x_{m,t_j^m})|| \\ &+ ||f(t_i^n , x_{n,t_i^n}) - f(t,x_{n,t})|| + ||f(t_j^m , x_{m,t_j^m}) - \\ &\quad f(t,x_{m,t})|| + ||f(t,x_{n,t}) - f(t,x_{m,t})|| . \end{aligned}$$

Since $t - t_i^n < \delta_i^n$, $||x_{n,t_i^n} - x_{n,t}||_0 < \eta_n$, $t - t_j^m < \delta_j^m$, $||x_{m,t_j^m} - x_{m,t}||_0 < \eta_m$ we obtain, using (i) and (iv) and (v) of Lemma 2.1,

$$D^+m(t) \leq 2(\varepsilon_n + \varepsilon_m) + ||f(t,x_{n,t}) - f(t,x_{m,t})|| .$$

If t is such that $||x_{n,t}(\theta) - x_{m,t}(\theta)|| \leq ||x_n(t) - x_m(t)||$, then, by (A_3), the above inequality can be written as

$$D^+m(t) \leq 2(\varepsilon_n + \varepsilon_m) + g(t,m(t)) . \tag{4.5}$$

Thus, $m(t) : [t_0-\tau,t_0 + \gamma] \to R^+$ is a continuous function satisfying the differential inequality (4.5) whenever $t \in [t_0,t_0 + \gamma] \setminus S$ is such that $m_t(\theta) \leq m(t)$. Note also that $m_{t_0} \equiv 0$. Therefore, by Lemma 3.2, we have

$$m(t) \leq r_{n,m}(t,t_0,0) \ , \ t \in [t_0,t_0 + \gamma]$$

where $r_{n,m}(t,t_0,0)$ is the maximal solution of

$$u' = g(t,u) + 2(\varepsilon_n + \varepsilon_m) \ , \ u(t_0) = 0 .$$

By Lemma 3.1, $r_{n,m}(t,t_0,0)$ converges to $r(t,t_0,0)$ as $n,m \to \infty$, uniformly on $[t_0,t_0 + \gamma]$, where $r(t,t_0,0)$ is the maximal solution of (4.2) and by (A_3), $r(t,t_0,0) \equiv 0$. Hence, it follows that $||x_n(t) - x_m(t)|| \to 0$, as $n,m \to \infty$, uniformly for $t \in [t_0,t_0 + \gamma]$, that is, $\{x_n(t)\}$ converges uniformly on $[t_0-\tau,t_0 + \gamma]$. Then by Lemma 2.2, $x(t) = \lim_{n\to\infty} x_n(t)$ is a solution of (1.1) such that $x(t) \in F$ for every $t \in [t_0,t_0 + \gamma]$.

If $x(t)$, $\hat{x}(t) : [t_0-\tau,t_0 + \gamma] \to E$ are two solutions of (1.1) such that $x(t)$, $\hat{x}(t) \in F \cap B[\phi_0(0),b]$, $t \in [t_0,t_0 + \gamma]$, by setting $m(t) = ||x(t) - \hat{x}(t)||$, we obtain

$$D^+m(t) \leq ||f(t,x_t) - f(t,\hat{x}_t)|| \leq g(t,m(t))$$

whenever $t > t_0$ is such that $m_t(\theta) \leq m(t)$. Since $m_{t_0} \equiv 0$, Lemma 3.2 and (A_3) together imply that $m(t) \equiv 0$, and the uniqueness of solutions of (1.1) is established. The proof is complete.

Corollary 4.1. If, in Theorem 4.1, we suppose that (4.1) is satisfied for $\phi,\psi \in B[y_t,b]$ such that $||\phi(\theta) - \psi(\theta)|| \leq ||\phi(0) - \psi(0)||$ for every $\theta \in [-\tau,0]$, then there exists a unique solution of (1.1) and the set F is positively invariant with respect to the equation $x'(t) = f(t,x_t)$.

Remark 4.1. In view of Remark 2.1, we can replace the set C_F in the assumptions of Theorem 4.1 and Corollary 4.1 by the set $\hat{C}_F$ (which coincides with the set $\{\phi \in C : \phi(\theta) \in F$ for every $\theta \in [-\tau,0]\}$ since F is convex). Then, we obtain existence and uniqueness results for the Cauchy problem (1.1) whenever $\phi_0 \in \hat{C}_F$ and Corollary 4.1 is a generalization of Theorem 2 in [8] for $\alpha(t) = \tau =$ constant.

The following result gives sufficient conditions for the existence and uniqueness of solutions of (1.1) in terms of a general Lyapunov-like function when F is convex.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold with* (A_3) *replaced by*

(A_3') *there exists a continuous function* $V : [t_0-\tau,t_0 + a] \times B[0,\beta] \to R^+$, $\beta > 2b$ *and* $B[0,\beta] = \{x \in E : ||x|| \leq \beta\}$ *such that*

(i) $V(t,0) \equiv 0$, $V(t,x) > 0$ *if* $x \neq 0$ *and* $|V(t,x) - V(t,y)| \leq L||x-y||$, *for* $t \in [t_0,t_0 + a]$, $x,y \in B[0,\beta]$;

(ii) *for any sequence* $\{x_n\}$, $x_n \in B[0,\beta]$, $V(t,x_n) \to 0$ *as* $n \to \infty$ *implies* $\lim_{n\to\infty} ||x_n|| = 0$, *uniformly with respect to* t ;

(iii) *for* $t \in [t_0, t_0 + a]$, $\phi, \psi \in C_F^t(b)$,

$$(4.6)\begin{cases} D^+V(t,\phi,\psi) \equiv \limsup_{h \to 0^+} \frac{1}{h}\Big\{V(t + h , \phi(0) - \psi(0) + h(f(t,\phi) - f(t,\psi))) \\ \qquad - V(t,\phi(0) - \psi(0))\Big\} \leq g\Big[t,V(t,\phi(0) - \psi(0))\Big] \end{cases}$$

whenever $V(t + \theta,\phi(\theta) - \psi(\theta)) \leq V(t, \phi(0) - \psi(0))$, *for every* $\theta \in [-\tau,0]$ *where* $g \in C[R^+ \times R^+,R^+]$ *satisfies the same conditions as in* (A_3). *Then, the Cauchy problem (1.1) has a unique solution* $x(t)$ *on* $[t_0-\tau,t_0 + \gamma]$ *such that* $x(t) \in F$, $t \in [t_0,t_0 + \gamma]$.

<u>Proof</u>. By setting $m(t) = V(t,x_n(t) - x_m(t))$, $t \in [t_0-\tau,t_0 + \gamma]$ where $x_n(t)$, $x_m(t)$ are the ε_n , ε_m-approximate solutions of (1.1), we can proceed as in Theorem 4.1 to arrive at the differential inequality

$$D^+m(t) \leq g(t,m(t)) + 2L(\varepsilon_n + \varepsilon_m) ,$$

for every $t \in (t_i^n , t_{i+1}^n) \cap (t_j^m , t_{j+1}^m)$ such that $m_t(\theta) \leq m(t)$. The rest of the proof is parallel to that in Theorem 4.1 and hence is omitted.

We shall now study

(b) the case when F is not convex.

<u>Theorem 4.3</u>. *Suppose that* (A_1), (A_2) *and* (A_4) *hold. Then, the Cauchy problem (1.1) has a unique solution* $x(t)$ *on* $[t_0-\tau,t_0 + \gamma]$ *such that* $x(t) \in F$, $t \in [t_0,t_0 + \gamma]$.

<u>Proof</u>. Let n and m be positive integers and let

$$m(t) = ||x_n(t) - x_m(t)|| , \quad t \in [t_0-\tau,t_0 + \gamma] ,$$

where $x_n(t)$, $x_m(t)$ are ε_n , ε_m-approximate solutions of (1.1). If $t \in (t_i^n , t_{i+1}^n) \cap (t_j^m , t_{j+1}^m)$, we get

$$\begin{aligned}
D^+m(t) &\leq ||x_n'(t) - x_m'(t)|| \\
&\leq ||f(t,x_{n,t_i^n}) - f(t,x_{m,t_j^m})|| + ||x_n'(t) - f(t_i^n , x_{n,t_i^n})|| \\
&+ ||x_m'(t) - f(t_j^m , x_{m,t_j^m})|| + ||f(t,x_{n,t_i^n}) - f(t_i^n , x_{n,t_i^n})|| \\
&+ ||f(t,x_{m,t_j^m}) - f(t_j^m , x_{m,t_j^m})|| \\
&\leq g(t,||x_n(t_i^n) - x_m(t_j^m)|| , ||x_{n,t_i^n}(\cdot) - x_{m,t_j^m}(\cdot)||) \\
&+ 2(\varepsilon_n + \varepsilon_m) ,
\end{aligned}$$

because of (4.3), the continuity of f and the properties (i), (iv) and (v) of Lemma 2.1. Now, using the property (ii) of Lemma 2.1, we have

$$\begin{aligned}
||x_n(t_i^n) - x_m(t_j^m)|| &\leq ||x_n(t) - x_m(t)|| + ||x_n(t) - x_n(t_i^n)|| \\
&+ ||x_m(t) - x_m(t_j^m)|| \leq m(t) + M(\varepsilon_n + \varepsilon_m) .
\end{aligned}$$

Also, for each $\theta \in [-\tau,0]$, using (ii) of Lemma 2.1, we obtain

$$||x_{n,t_i^n}(\theta) - x_{m,t_j^m}(\theta)|| \leq m(t + \theta) + 2M(\varepsilon_n + \varepsilon_m) .$$

Hence, by the increasing character of $g(t,u,u_t)$ in u for each (t,u_t) we can now write

$$\text{(4.7)} \qquad D^+m(t) \leq g(t,m(t) + 2\beta_{m,n} , m_t(\cdot) + 2\beta_{m,n}) + \eta_{m,n}$$

where $\beta_{m,n} = M(\varepsilon_n + \varepsilon_m)$ and $\eta_{m,n} = 2(\varepsilon_n + \varepsilon_m)$.

Setting $v(t) = m(t) + 2\beta_{m,n}$, the differential inequality (4.7) reduces to

$$D^+v(t) \leq g(t,v(t),v_t) + \eta_{m,n} .$$

Also, $v_{t_0}(\theta) = m_{t_0}(\theta) + 2\beta_{m,n} = 2\beta_{m,n}$. Hence, by applying Lemma 3.3, we obtain

$$v(t) \leq r_{m,n}(t,t_0,2\beta_{m,n})$$

where $r_{m,n}(t,t_0,2\beta_{m,n})$ is the maximal solution of

$$u' = g(t,u,u_t) + \eta_{m,n} , u_{t_0} = 2\beta_{m,n} .$$

But in virtue of Lemma 3.1, $r_{m,n}(t,t_0,2\beta_{m,n}) \to r(t,t_0,0)$ as $m,n \to \infty$ uniformly on $[t_0,t_0 + \gamma]$, where $r(t,t_0,0)$ is the maximal solution of (4.4). Hence, $||x_n(t) - x_m(t)|| = m(t) \leq v(t) \to r(t,t_0,0)$ as $m,n \to \infty$, which in view of (A_4) yields that $\lim_{m,n\to\infty} ||x_n(t) - x_m(t)|| = 0$ uniformly for $t \in [t_0,t_0 + \gamma]$ and by Lemma 2.2, $\lim_{n\to\infty} x_n(t) = x(t)$ is a solution of (1.1), such that $x(t) \in F$ for every $t \in [t_0,t_0 + \gamma]$.

Let $x(t)$, $\hat{x}(t)$ be two solutions of (1.1) such that $x(t)$, $\hat{x}(t) \in F \cap B[\phi_0(0),b]$ and set

$$m(t) = ||x(t) - \hat{x}(t)|| , t \in [t_0-\tau,t_0 + \gamma] .$$

Then by (4.3),

$$D^+m(t) \leq ||x'(t) - \hat{x}'(t)|| = ||f(t,x_t) - f(t,\hat{x}_t)|| \leq g(t,m(t),m_t) ,$$

$t \in [t_0,t_0 + \gamma]$ and $m_{t_0} \equiv 0$. Therefore, by applying Lemma 3.3, we have $m(t) \leq r(t,t_0,0)$, which by (A_4), implies that $m(t) \equiv 0$ and this establishes the uniqueness of solutions of (1.1) in F .

<u>Corollary 4.2</u>. Suppose that the assumptions of Theorem 4.3 hold with the inequality (4.3) being satisfied for $\phi,\psi \in B[y_t,b]$. Then there exists a unique solution of (1.1) and the set F is positively invariant with respect to the equation $x'(t) = f(t,x_t)$.

<u>Remark 4.2</u>. Theorem 4.3 and Corollary 4.2 with the set C_F replaced by the set $\hat{C}_F$ in conditions (A_1), (A_2) and (A_4) give existence and uniqueness results for (1.1) whenever $\phi_0 \in \hat{C}_F$ and Corollary 4.2 (with this modification) is also a generalization of Theorem 2 of [8] for $\alpha(t\ ) = t$ = constant. In fact, in Theorem 2 of [8], F is assumed to be convex and the Lipschitz condition satisfied by f is a special case of the dissipative condition (4.3) with $g(t,u,u_t) = L||u_t||_0$.

The weakening of the dissipative condition (4.3) in Theorem 4.3 presents several difficulties. However, a special variant of (4.3) in terms of a general Lyapunov-like function is given in the following result.

<u>Theorem 4.4</u>. *Assume that* (A_1) *and* (A_2) *hold. Suppose that* (A_3') *is verified with* (*iii*) *replaced by*

$$D^+V(t,x,y,\phi_1,\phi_2) \equiv \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h),x+hf(s_1,\phi_1) - y-hf(s_2,\phi_2)) - V(t,x-y)]$$

$$\leq g(t,V(t,x-y),V(t+\cdot,\phi_1(\cdot)-\phi_2(\cdot))) + p(|s_1-s_2| +$$

$$||x-\phi_1(0)|| + ||y-\phi_2(0)||)$$

for $s_1,s_2 \in [t_0,t_0 + a]$, $x,y \in E$ *such that* $x-y \in B(0,2b)$, *and* $\phi_1 \in C_F^{s_1}(b)$, $\phi_2 \in C_F^{s_2}(b)$, *where* $p : R^+ \to R^+$ *is a continuous nondecreasing function with* $\lim_{u \to 0} p(u) = 0$ *and* $g \in C[R^+ \times R^+ \times C^+,R]$ *satisfies the same conditions as in*

(A_4). *Then the Cauchy problem (1.1) has a unique solution* $x(t)$ *on* $[t_0-\tau, t_0+\gamma]$ *such that* $x(t) \in F$, $t \in [t_0, t_0+\gamma]$.

Proof. Let n and m be positive integers and as before, define

$$m(t) = V(t, x_n(t) - x_m(t)) \ , \ t \in [t_0-\tau, t_0+\gamma] \ ,$$

where $x_n(t)$, $x_m(t)$ are $\varepsilon_n, \varepsilon_m$-approximate solutions of (1.1). Then, for $t \in (t_i^n , t_{i+1}^n) \cap (t_j^m , t_{j+1}^m)$, we have

$$D^+m(t) \leq \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x_n(t+h) - x_m(t+h)) - V(t, x_n(t) - x_m(t))]$$

$$\leq g(t, m(t), V(t, x_{n,t_i^n}(\cdot) - x_{m,t_j^m}(\cdot))) + L(\varepsilon_n + \varepsilon_m)$$

$$+ p(|t_i^n - t_j^m| + ||x_n(t) - x_n(t_i^n)|| + ||x_m(t) - x_m(t_j^m)||).$$

Since $V(t+\theta, x_n(t_i^n+\theta) - x_m(t_j^m+\theta)) \leq V(t+\theta, x_n(t+\theta) - x_m(t+\theta))$

$$+ 2LM(\varepsilon_n + \varepsilon_m) \ ,$$

and $|t_i^n - t_j^m| \leq \varepsilon_n + \varepsilon_m$, using the property (ii) of Lemma 2.1, and the nondecreasing character of $g(t,u,u_t)$ in u_t , the above inequality can be written as

$$D^+m(t) \leq g(t, m(t) \ , \ m_t + 2LM(\varepsilon_n + \varepsilon_m)) + \eta_{m,n}$$

where $\eta_{m,n} = p((1+M)(\varepsilon_n + \varepsilon_m)) + L(\varepsilon_n + \varepsilon_m)$.

Setting $v(t) = m(t) + 2LM(\varepsilon_n + \varepsilon_m)$ and using the nondecreasing property of $g(t,u,u_t)$ in u for each (t,u_t) , we get

$$D^+v(t) \leq g(t, v(t), v_t) + \eta_{m,n} \ .$$

Since $v_{t_0} = 2LM(\varepsilon_n + \varepsilon_m) \equiv \beta_{m,n}$, by applying Lemma 3.3, we obtain

$$m(t) \leq v(t) \leq r_{m,n}(t,t_0,\beta_{m,n}) \ , \ t \geq t_0$$

where $r_{m,n}(t,t_0,\beta_{m,n})$ is the maximal solution of

$$u' = g(t,u,u_t) + \eta_{m,n} \ , \ u_{t_0} = \beta_{m,n} \ .$$

However, by virtue of Lemma 3.1, $r_{m,n}(t,t_0,\beta_{m,n}) \to r(t,t_0,0)$ as $m,n \to \infty$, uniformly on $[t_0,t_0 + \gamma]$. By the hypothesis on g and property (ii) of V in (A_3'), we therefore conclude that $\{x_n(t)\}$ is uniformly Cauchy on $[t_0,t_0 + \gamma]$. We omit the rest of the proof.

5. A general existence and uniqueness result.

We shall now consider the problem of generalizing Theorem 4.3 under a dissipative-type assumption weaker than (4.3). It can be accomplished by means of a Lyapunov-like function $V(x)$ for which there exists an upper semicontinuous function $M[x,y]$ such that for any differentiable function $x(t)$ we have

$$D^+V(x(t)) \leq M[x(t),x'(t)] \ .$$

Clearly, this condition is satisfied when $V(x)$ is taken as either $||x||$ or $||x||^2$ and the corresponding $M[x,y]$ is

$$[x,y]_+ = \limsup_{h \to 0^+} \frac{||x + hy|| - ||x||}{h}$$

or

$$(x,y)_+ = ||x|| \ [x,y]_+$$

respectively. (For an equivalent definition of $(x,y)_+$, see [4,6]).

In fact, for any differentiable function $x(t)$,

$$D^+(||x(t)||) = \limsup_{h \to 0^+} \frac{||x(t+h)|| - ||x(t)||}{h}$$

$$\leq \limsup_{h \to 0^+} \frac{||x(t) + hx'(t) + 0(h)|| - ||x(t)||}{h}$$

$$\leq \limsup_{h \to 0^+} \frac{||x(t) + hx'(t)|| - ||x(t)||}{h}$$

$$= [x(t), x'(t)]_+$$

and in a similar way, we obtain

$$D^+(||x(t)||^2) \leq 2(x(t), x'(t))_+ .$$

Let $[x,y]$ be a mapping from $B[0,2b] \times E \to R$ with the properties

(i) $M[x,y]$ is upper semicontinuous, that is, if $\lim_{n\to\infty} (x_n, y_n) = (x,y)$, then $\limsup_{n \to \infty} M[x_n, y_n] \leq M[x,y]$;

(ii) $M[x,\lambda y] \leq \lambda M[x,y]$, $\lambda \geq 0$, $x \in B[0,2b]$, $y \in E$;

(iii) $M[x,y_1 + y_2] \leq M[x,y_1] + N||x|| \, ||y_2||$, $N > 0$, $x \in B[0,2b]$, $y_1, y_2 \in E$.

Let us list for convenience the following assumptions:

(A_5) $V \in C[B[0,2b], R^+]$, $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, if $\lim_{n\to\infty} V(x_n) = 0$ then $\lim_{n\to\infty} ||x_n|| = 0$ for any sequence $\{x_n\}$ with $x_n \in B[0,2b]$, and for $x, x+y \in B[0,2b]$

$$V(x+y) - V(x) \leq M[x,y] + 0(||y||),$$

where $M[x,y]$ is the function described above;

(A_6) for $t \in [t_0,t_0 + a]$, $\phi,\psi \in C_F^t(b)$,

$$M[\phi(0) - \psi(0), f(t,\phi) - f(t,\psi)] \leq g(t,V(\phi(0) - \phi(0)),V(\phi(\cdot) - \psi(\cdot))),$$

where $g \in C[R^+ \times R^+ \times C^+,R]$ is the function having the same properties as in (A_4).

In order to prove an existence and uniqueness result for the Cauchy problem (1.1) under the general dissipativity assumption (A_6), we need the following lemmas.

<u>Lemma 5.1</u>. Let (A_5) hold. Then for any differentiable function $x : [t_0 - \tau, t_0 + a] \to B[0,2b]$,

$$D^+V(x(t)) \leq M[x(t),x'(t)], \quad t \in [t_0,t_0 + a].$$

For a proof of this Lemma, see [1].

<u>Lemma 5.2</u>. Let (A_1), (A_2), (A_5) and (A_6) hold. Let n,m be positive integers and let $\varepsilon_n,\varepsilon_m \subset (0,1)$ be sequences such that $\varepsilon_n,\varepsilon_m \to 0$ as $n,m \to \infty$ respectively. Let $x_n(t),x_m(t)$ be the sequences of ε_n, ε_m-approximate solutions of (1.1) respectively and let $\{t_k\}$ be the minimal refinement of the two sequences $\{t_i^n\}$ and $\{t_j^m\}$ constructed in Lemma 2.1. Then, for every $k = 1,2,\ldots,$ there exist two continuous functions $y_p^{(k)} : [t_0 - \tau,t_k] \to E$, $p = n,m$, such that

(i) $y_{p,t_0}^{(k)} = \phi_0$, $||y_p^{(k)}(t) - y_p^{(k)}(s)|| \leq M|t - s|$, $t,s \in [t_0,t_k]$;

(ii) $\left(t_k,y_{p,t_k}^{(k)}\right) \in \tilde{C}_0(b)$;

(iii) for $t \in (t_{k-1} , t_k)\setminus S$, where S is a countable subset of

$[t_0, t_0 + a]$, $(y_p^{(k)}(t))'$ exists and

$$M[y_n^{(k)}(t) - y_m^{(k)}(t), (y_n^{(k)}(t))' - y_m^{(k)}(t))'] \leq g(t, V(y_n^{(k)}(t) - y_m^{(k)}(t)),$$

$$V(y_{n,t}^{(k)}(\cdot) - y_{m,t}^{(k)}(\cdot))) + (1 + N||y_n^{(k)}(t) - y_m^{(k)}(t)||)(\varepsilon_n + \varepsilon_m);$$

(iv) if $t_i^n, t_j^m \leq t_{k-1} \leq t_k \leq t_{i+1}^n, t_{j+1}^m$,

$$||y_n^{(k)}(t) - x_n(t)|| \leq 3(t - t_i^n)\varepsilon_n, \quad t \in [t_{k-1}, t_k],$$

and

$$||y_m^{(k)}(t) - x_m(t)|| \leq 3(t - t_j^m)\varepsilon_m, \quad t \in [t_{k-1}, t_k],$$

Though the proof of the lemma is quite similar to the proof of Lemma 1 [6] or of Lemma 2.4 [4], we give below the complete proof in order to exhibit, all the needed technical modifications in adapting those proofs for the delay case.

Proof. For $k = 0$, define $y_p^{(0)}(t) = \phi_0(t - t_0)$. Obviously, the properties (i)-(iv) are satisfied. Let us suppose that for $k \geq 0$, there exist two functions $y_p^{(k)}(t)$, $p = m,n$, defined on $[t_0 - \tau, t_k]$ satisfying (i)-(iv) and prove that there exist $y_p^{(k+1)}(t)$, $p = m,n$, defined on $[t_0 - \tau, t_{k+1}]$ such that (i)-(iv) hold.

Let $t_i^n, t_j^m \leq t_k \leq t_{i+1}^n, t_{j+1}^m$ and define for $t \in [t_0 - \tau, t_k]$,

$$y_n^{(k+1)}(t) = \begin{cases} x_n(t), & \text{if } t_k = t_i^n, \\ y_n^{(k)}(t), & \text{if } t_k > t_i^n, \end{cases}$$

$$y_m^{(k+1)}(t) = \begin{cases} x_m(t), & \text{if } t_k = t_j^m, \\ y_m^{(k)}(t), & \text{if } t_k > t_j^m. \end{cases}$$

To define $y_p^{(k+1)}(t)$, $p = m,n$, on $(t_k,t_{k+1}]$, let us proceed by induction. Suppose that $y_p^{(k+1)}(t)$ is defined on $[t_0-\tau,s_q]$ such that (i)-(iv) are satisfied, $s_q \in [t_k,t_{k+1}]$, and $y_{p,s_q}^{(k+1)}$ $C_F^{s_q}(b)$, and show that it is possible to define $\gamma_q \geq 0$ and the functions $y_p^{(k+1)}(t)$ on $[t_0-\tau,s_{q+1}]$, $p = m,n$, $s_{q+1} = s_q + \gamma_q$ such that (i)-(iv) are satisfied and $y_{p,s_{q+1}}^{(k+1)} \in C_F^{s_{q+1}}(b)$. After this is done, the proof will be complete if we show that the sequence $\{s_q\}$ converges to t_{k+1}.

Let us define $\gamma_q = 0$ if $s_q = t_{k+1}$ and if $s_q < t_{k+1}$, choose $\gamma_q > 0$ as the largest number such that

(1) $s_q + \gamma_q \leq t_{k+1}$,

(2) $d(y_p^{(k+1)}(s_q) + \gamma_q f(s_q,y_{p,s_q}^{(k+1)}),F) \leq \gamma_q \frac{\varepsilon_p}{2}$,

(3) $\gamma_q \leq \min\left\{\delta_{\phi_0}\left(\frac{\delta_1}{2}\right), \frac{\delta_1}{2M}, \delta_2\right\}$, where $\delta_1,\delta_2 > 0$

are the largest numbers such that

$$(5.2)\quad \begin{cases} M[y_n^{(k+1)}(s_q) + x(0) - (y_m^{(k+1)}(s_q) + y(0)),f(s_q,y_{n,s_q}^{(k+1)}) - f(s_q,y_{m,s_q}^{(k+1)})] \\ \leq g(s_q + \sigma,V(y_n^{(k+1)}(s_q) + x(0) - (y_m^{(k+1)}(s_q) + y(0))) , \\ V(y_{n,s_q}^{(k+1)}(\cdot) + x(\cdot) - (y_{m,s_q}^{(k+1)}(\cdot) + y(\cdot)))) + \varepsilon_n + \varepsilon_m , \end{cases}$$

whenever $||x(\theta)||,||y(\theta)|| \leq \delta_1$ and $0 < \sigma \leq \delta_2$.

The boundary condition (A_2), the upper semicontinuity of $M[x,y]$, the continuity of $g(t,u,u_t)$ and the dissipativity assumption (A_6) imply that

$\gamma_q > 0$. Using (2), we can define $y_p^{(k+1)}(s_{q+1}) \in F$ such that

$$||y_p^{(k+1)}(s_q) + \gamma_q f(s_q, y_{p,s_q}^{(k+1)}) - y_p^{(k+1)}(s_{q+1})|| \leq \gamma_q \varepsilon_p$$

and for $t \in [s_q, s_{q+1}]$,

$$y_p^{(k+1)}(t) = \frac{y_p^{(k+1)}(s_{q+1}) - y_p^{(k+1)}(s_q)}{s_{q+1} - s_q}(t - s_q) + y_p^{(k+1)}(s_q) .$$

Thus we have

$$y_{p,t_0}^{(k+1)} = \phi_0 ,$$

$$||y_p^{(k+1)}(t) - y_p^{(k+1)}(s)|| = ||y_p^{(k+1)}(s_{q+1}) - y_p^{(k+1)}(s_q)||\ |t-s|/\gamma_q$$

$$\leq (||f(s_q, y_{p,s_q}^{(k+1)})|| + \varepsilon_p)\ |t - s|$$

$$\leq M|t - s|, \quad \text{for } t,s \in [s_q, s_{q+1}]$$

and by induction hypothesis,

$$||y_p^{(k+1)}(t) - y_p^{(k+1)}(s)|| \leq M|t - s|, \quad \text{for } t,s \in [t_0, s_{q+1}].$$

Also, we have

$$||y_{p,s_{q+1}}^{(k+1)} - y_{s_{q+1}}||_0 = \sup_{\theta \in [-\tau,0]} ||y_p^{(k+1)}(s_{q+1} + \theta) - y(s_{q+1} + \theta)|| \leq b$$

and therefore, $(s_{q+1}, y_{p,s_{q+1}}^{(k+1)}) \in \tilde{C}_0(b)$. Moreover, for $t \in (s_q, s_{q+1})$

$$||y_p^{(k+1)}(t))' - f(s_q, y_{p,s_q}^{(k+1)})|| \leq \varepsilon_p .$$

Since, $t-s_q \leq \gamma_q \leq \delta_2$, $\sup_{\theta\in[-\tau,0]} ||y_{p,t}^{(k+1)}(\theta) - y_{p,s_q}^{(k+1)}(\theta)|| \leq \delta_1$, for $t \in (s_q, s_{q+1})$, we have by (5.2),

$$M[y_n^{(k+1)}(t) - y_m^{(k+1)}(t), (y_n^{(k+1)}(t))' - (y_m^{(k+1)}(t))']$$

$$= M[y_n^{(k+1)}(t) - y_m^{(k+1)}(t), f(s_q, y_{n,s_q}^{(k+1)}) - f(s_q, y_{m,s_q}^{(k+1)}) + (y_n^{(k+1)}(t))'$$

$$- f(s_q, y_{n,s_q}^{(k+1)}) - (y_m^{(k+1)}(t))' + f(s_q, y_{m,s_q}^{(k+1)})]$$

$$\leq M[y_n^{(k+1)}(t) - y_m^{(k+1)}(t), f(s_q, y_{n,s_q}^{(k+1)}) - f(s_q, y_{m,s_q}^{(k+1)})]$$

$$+ N||y_n^{(k+1)}(t) - y_m^{(k+1)}(t)||(\varepsilon_n + \varepsilon_m)$$

$$\leq g(t, V(y_n^{(k+1)}(t) - y_m^{(k+1)}(t)), V(y_{n,t}^{(k+1)}(\cdot) - y_{m,t}^{(k+1)}(\cdot)))$$

$$+ (1 + N||y_n^{(k+1)}(t) - y_m^{(k+1)}(t)||)(\varepsilon_n + \varepsilon_m) .$$

Let us now verify that $y_n^{(k+1)}(t)$ and $y_m^{(k+1)}(t)$ satisfy (iv) for $t \in [s_q, s_{q+1}]$. If $t_i^n, t_j^m \leq t_k \leq t_{i+1}^n, t_{j+1}^m$, we have for $t \in [s_q, s_{q+1}]$,

$$||y_n^{(k+1)}(t) - x_n(t)|| \leq ||y_n^{(k+1)}(s_q) - x_n(s_q)|| + \int_{s_q}^t ||(y_n^{(k+1)}(s))' - x_n'(s)||ds$$

$$\leq 3(s_q - t_i^n)\varepsilon_n + \int_{s_q}^t [||f(s_q, y_{n,s_q}^{(k+1)}) - f(t_i^n, x_{n,t_i^n})||$$

$$+ ||(y_n^{(k+1)}(s))' - f(s_q, y_{n,s_q}^{(k+1)})|| + ||x_n'(s) - f(t_i^n, x_{n,t_i^n})||]ds$$

$$\leq 3(s_q - t_i^n)\varepsilon_n + 3(t - s_q)\varepsilon_n = 3(t - t_i^n)\varepsilon_n ,$$

in view of the continuity of f, property (iv) of the approximate solution $x_n(t)$ of Lemma 2.1 and the property (iv) of this lemma. In fact, by taking into account the definition of δ_i^n (See Lemma 2.1), we have

$$\left|\left|f(s_q, y_{n,s_q}^{(k+1)}) - f(t_i^n, x_{n,t_i^n})\right|\right| \leq \varepsilon_n .$$

Finally, let us show that $\lim_{q\to\infty} s_q = t_{k+1}$. Assume, for contradiction, that $\lim_{q\to\infty} s_q = \beta < t_{k+1}$. Let $\xi_p^{(k+1)} = \lim_{q\to\infty} y_{p,s_q}^{(k+1)}$, which exists because we have for $r \geq q$,

$$\left|\left|y_{p,s_q}^{(k+1)} - y_{p,s_r}^{(k+1)}\right|\right| = \sup_\theta \left|\left|y_p^{(k+1)}(s_q + \theta) - y_p^{(k+1)}(s_r + \theta)\right|\right| ,$$

and

$$\left|\left|y_p^{(k+1)}(s_q + \theta) - y_p^{(k+1)}(s_r + \theta)\right|\right| \leq \begin{cases} M|s_r - s_q|, & \text{if } s_q + \theta \geq t_0 , \\ ||\phi_0(s_q + \theta - t_0) - \phi_0(s_r + \theta - t_0)|| , & \\ \qquad \text{if } s_r + \theta \leq t_0 , & \\ M|s_r - s_q| + ||\phi_0(s_q + \theta - t_0) - \phi_0(0)|| , & \\ \qquad \text{if } s_q + \theta \leq t_0 \leq s_r + \theta . & \end{cases}$$

By using the uniform continuity of ϕ_0, we have

$$\left|\left|y_{p,s_q}^{(k+1)} - y_{p,s_r}^{(k+1)}\right|\right|_0 \to 0 \quad \text{as} \quad r,q \to \infty .$$

Since $\tilde{C}_0(b)$ is closed, we also have $(\beta, \xi_p^{(k+1)}) \in \tilde{C}_0(b)$.

Let $\bar{\delta}_1, \bar{\delta}_2 > 0$ be the largest numbers such that

$$M[\xi_n^{(k+1)}(0) + x(0) - \xi_m^{(k+1)}(0) - y(0), f(\beta,\xi_n^{(k+1)}) + z_n - f(\beta,\xi_m^{(k+1)}) - z_m]$$

$$\leq g(\beta + \sigma, V(\xi_n^{(k+1)}(0) + x(0) - \xi_m^{(k+1)}(0) - y(0)), V(\xi_n^{(k+1)}(\cdot) + x(\cdot)$$

$$- \xi_m^{(k+1)}(\cdot) - y(\cdot))) + \tfrac{1}{2}(\varepsilon_n + \varepsilon_m),$$

whenever $||x(\theta)||, ||y(\theta)|| \leq 2\bar{\delta}_1$, $||z_n||, ||z_m|| \leq 2\bar{\delta}_1$ and $0 \leq \sigma \leq \bar{\delta}_2$. It is easy to see that there exists a positive integer N such that, for $q \geq N$, (5.2) is satisfied whenever $||x(\theta)||, ||y(\theta)|| \leq \bar{\delta}_1$, $0 \leq \sigma \leq \bar{\delta}_2$. Let $\eta > 0$ be such that

1') $\eta < t_{k+1} - \beta$

2') $d(\xi_p^{(k+1)}(0) + \eta f(\beta,\xi_p^{(k+1)}), F) \leq \eta\varepsilon_p/8$

3') $\eta \leq \min\left\{\delta_{\phi_0}\left(\frac{\bar{\delta}_1}{2}\right), \frac{\bar{\delta}_1}{2}, \bar{\delta}_2\right\}$.

We have, for q large enough,

1) $s_q + \eta \leq t_{k+1}$

2) $d(y_p^{(k+1)}(s_q) + \eta f(s_q, y_{p,s_q}^{(k+1)}), F)$

$$\leq ||y_p^{(k+1)}(s_q) - \xi_p^{(k+1)}(0)|| + \eta||f(s_q, y_{p,s_q}^{(k+1)}) - f(\beta,\xi_p^{(k+1)})||$$

$$+ d(\xi_p^{(k+1)}(0) + \eta f(\beta,\xi_p^{(k+1)}), F) \leq \eta \frac{\varepsilon_p}{2}$$

and, because γ_q is the largest number satisfying (1),(2),(3), $\gamma_q \geq \eta$. But this contradicts the fact that $\gamma_q \to 0$ as $q \to \infty$, and the proof is complete.

We need the following comparison lemma in order to prove the general existence result.

<u>Lemma 5.3</u>. Let $\{t_k\}$ be a sequence of points of the interval $[t_0, t_0 + a)$ such that for every $k \geq 1$, $t_{k+1} \geq t_k \geq t_0$ and $t_k \to t_0 + a$ as $k \to \infty$. Let $v^{(k)}(t), v^{(k)} : [t_0-\tau, t_k] \to R^+$ be a sequence of continuous functions such that for every $k \geq 1$ the following conditions hold:

(i) $v^{(k)}(t_0 + \theta) \leq \phi_0(\theta)$ for every $\theta \in [-\tau, 0]$ and

$$|v^{(k+1)}(t) - v^{(k)}(t)| \leq \lambda_k , \quad t \in [t_0-\tau, t_k],$$

where $\sum_{k=1}^{\infty} \lambda_k$ is convergent;

(ii) $D^+v^{(k)}(t) \leq g(t, v^{(k)}(t), v_t^{(k)})$, $t \in (t_{k-1}, t_k)$ where $g \in C[R^+ \times R^+ \times C^+, R]$ is such that $g(t,u,u_t)$ is nondecreasing in u, u_t for each t.

Then, for every $t \in [t_{k-1} - \tau, t_k]$,

$$v^{(k)}(t) \leq r(t, t_0, \phi_0 + \sum_{i=1}^{k-1} \lambda_i) \leq r(t, t_0, \phi_0 + \sum_{k=1}^{\infty} \lambda_k), \tag{5.3}$$

where $r(t, t_0, \phi_0)$ is the maximal solution of

$$u' = g(t, u, u_t), \quad u_{t_0} = \phi_0 ,$$

existing for $t \geq t_0$.

<u>Proof</u>. By using Lemma 3.3, we have

$$v^{(1)}(t) \leq r(t, t_0, \phi_0), \quad t \in [t_0-\tau, t_1].$$

We shall assume that (5.3) is true for a given $k \geq 1$ and show that it is also valid for $k + 1$, thus proving the lemma by induction on k. Therefore. by (i) and (ii) of hypotheses, we have, for $k \geq 1$,

$$v_{t_k}^{(k+1)}(\theta) \leq v_{t_k}^{(k)}(\theta) + \lambda_k \leq r(t_k + \theta, t_0, \phi_0 + \sum_{i=1}^{k-1} \lambda_i) + \lambda_k$$

for every $\theta \in [-\tau, 0]$ and

$$D^+ v^{(k+1)}(t) \leq g(t, v^{(k+1)}(t), v_t^{(k+1)}) ,$$

for every $t \in (t_k, t_{k+1})$.

Again, applying Lemma 3.3, we get

$$v^{(k+1)}(t) \leq r(t, t_k, r_{t_k}^{(k)} + \lambda_k), \quad t \in [t_k - \tau, t_{k+1}]$$

where

$$r^{(k)}(t) = r(t, t_0, \phi_0 + \sum_{i=1}^{k-1} \lambda_i), \quad t \in [t_0 - \tau, t_k].$$

Let us set

$$\rho(t) = \begin{cases} r^{(k)}(t) + \lambda_k , & t \in [t_0 - \tau, t_k], \\ r(t, t_k, r_{t_k}^{(k)} + \lambda_k) , & t \in [t_k, t_{k+1}] . \end{cases}$$

With this definition of $\rho(t)$, we have

$$\begin{aligned} \rho'(t) &= g(t, r^{(k)}(t), r_t^{(k)}) \\ &\leq g(t, r^{(k)}(t) + \lambda_k, r_t^{(k)} + \lambda_k) \\ &= g(t, \rho(t), \rho_t) , \quad t \in [t_0, t_k), \end{aligned}$$

and

$$\rho'(t) = g(t, \rho(t), \rho_t) , \quad t \in [t_k, t_{k+1}) .$$

Also,

$$\rho_{t_0}(\theta) = r_{t_0}^{(k)}(\theta) + \lambda_k = \phi_0(\theta) + \sum_{i=1}^{k-1}\lambda_i + \lambda_k = \phi_0(\theta) + \sum_{i=1}^{k}\lambda_i , \quad \theta \in [-\tau,0].$$

Hence, by Lemma 3.3, we obtain

$$\rho(t) \leq r(t,t_0,\phi_0 + \sum_{i=1}^{k}\lambda_i), \quad t \in [t_0-\tau,t_{k+1})$$

and

$$v^{(k+1)}(t) \leq \rho(t) \leq r(t,t_0,\phi_0 + \sum_{i=1}^{k}\lambda_i), \quad t \in [t_k-\tau,t_{k+1}],$$

which shows that (5.3) is true for $k + 1$. The lemma is proved.

Theorem 5.1. *Let the assumptions* $(A_1),(A_2),(A_5)$ *and* (A_6) *hold and let the function* V *in* (A_5) *also satisfy the Lipshchitz condition, i.e., for* $x_1,x_2 \in B[0,2b]$,

$$|V(x_1) - V(x_2)| \leq L||x_1 - x_2|| .$$

Then there exists a unique solution $x(t)$ *of (1.1) on* $[t_0-\tau,t_0 + \gamma]$ *such that* $x(t) \in F$, $t \in [t_0,t_0 + \gamma]$.

Proof. Let $\{\varepsilon_n\} \subset]0,1[$ be a nondecreasing sequence such that $\varepsilon_n \to 0$ as $n \to +\infty$ and let $\{x_n(t)\}$ be the sequence of ε_n-approximate solutions of (1.1) existing by Lemma 2.1. In order to prove the existence of a solution of (1.1) which lies in F, it is enough, by Lemma 2.2, to prove that the sequence $\{x_n(t)\}$ is Cauchy uniformly on $[t_0,t_0 + \gamma]$. Let m,n be positive integers and let $\{y_p^{(k)}(t)\}$, $p = m,n$, be the sequences of functions constructed in Lemma 5.2. Set

$$v^{(k)}(t) \equiv v_{m,n}^{(k)}(t) = V(y_n^{(k)}(t) - y_m^{(k)}(t)), \quad t \in [t_0-\tau,t_k].$$

Using Lemma 5.1 and the property (iii) of $y_p^{(k)}(t)$ in Lemma 5.2, we obtain

$$D^+v^k(t) \equiv D^+v^{(k)}_{m,n}(t) \leq M[y^{(k)}_n(t) - y^{(k)}_m(t),(y^{(k)}_n(t))' - (y^{(k)}_m(t))']$$

$$\leq g(t,v^{(k)}(t),v^{(k)}_t) + (1 + N||y^{(k)}_n(t) - y^{(k)}_m(t)||)(\varepsilon_n + \varepsilon_m)$$

for $t \in [t_k,t_{k+1})$. Note that for $t \in [t_0,t_k]$ and any $k = 1,2,\ldots,$

$$||y^{(k)}_n(t) - y^{(k)}_m(t)|| \leq ||y^{(k)}_n(t) - y^k_n(t_0)|| + ||y^{(k)}_m(t) - y^{(k)}_m(t_0)|| \leq 2b.$$

Also, for each $k \geq 1$, the Lipschitzian property of V yields

$$|v^{(k+1)}(t) - v^{(k)}(t)| = |V(y^{(k+1)}_n(t) - y^{(k+1)}_m(t)) - V(y^{(k)}_n(t) - y^{(k)}_m(t))|$$

$$\leq L[||y^{(k+1)}_n(t) - y^{(k+1)}_m(t) - y^{(k)}_n(t) + y^{(k)}_m(t)||]$$

$$\leq L[||y^{(k+1)}_n(t) - y^{(k)}_n(t)|| + ||y^{(k+1)}_m(t) - y^{(k)}_m(t)||].$$

But for $t^n_i,t^m_j \leq t_k \leq t_{k+1} \leq t^n_{i+1},t^m_{j+1}$, the property (iv) of Lemma 5.2 gives

$$||y^{(k+1)}_n(t) - y^{(k)}_n(t)|| \leq \begin{cases} 0, & \text{if } t_k > t^n_i , \\ 3(t^n_{i+1} - t^n_i)\varepsilon_n, & \text{if } t_k = t^n_i \end{cases}$$

on $[t_0-\tau,t_k]$ and similarly

$$||y^{(k+1)}_m(t) - y^{(k)}_m(t)|| \leq \begin{cases} 3(t^m_{j+1} - t^m_j)\varepsilon_m , & \text{if } t_k = t^m_j \\ 0 , & \text{if } t_k > t^m_j , \end{cases}$$

on $[t_0-\tau,t_k]$. Hence

$$\sum_{k=1}^{\infty}|v^{(k+1)}(t) - v^{(k)}(t)| \leq 3L[\sum_{i=1}^{\infty}(t^n_{i+1} - t^n_i)\varepsilon_n + \sum_{j=1}^{\infty}(t^m_{j+1} - t^m_j)\varepsilon_m]$$

$$= 3L\, a\,(\varepsilon_n + \varepsilon_m) \equiv \eta_{n,m} .$$

Moreover, we have $v^{(k)}(t_0 + \theta) = V(y_n^{(k)}(t_0 + \theta) - y_m^{(k)}(t_0 + \theta)) = 0$. Therefore, from Lemma 5.3, we obtain

$$v^{(k)}(t) \leq r_{n,m}(t,t_0,\eta_{n,m}), \quad t \in [t_k,t_{k+1}],$$

where $r_{n,m}(t,t_0,\eta_{n,m})$ is the maximal solution of

$$u' = g(t,u,u_t) + (1 + 2Nb)(\varepsilon_n + \varepsilon_m), \quad u_{t_0} \equiv \eta_{n,m} .$$

However, by Lemma 3.1, we have $r_{n,m}(t,t_0,\eta_{n,m}) \to r(t,t_0,0)$ as $n,m \to \infty$ uniformly on $[t_0-\tau,t_0 + a]$ and by hypothesis (A_6), $r(t,t_0,0) \equiv 0$.

Now, let $t \in [t_0,t_0 + \gamma]$. Let $\varepsilon > 0$ and let N be sufficiently large such that $r_{n,m}(t,t_0,\eta_{n,m}) < \frac{\varepsilon}{3}$ for $n,m \geq N$ and every $t \in [t_0,t_0 + \gamma]$. For some $k = \hat{k}$, $t \in [t_{\hat{k}},t_{\hat{k}+1}]$ where $\{t_k\}$ is the minimal refinement of the partition sequences $\{t_i^n\},\{t_j^m\}$. Hence

$$v^{(\hat{k})}(t) \equiv v_{m,n}^{(\hat{k})}(t) < \frac{\varepsilon}{3} , \quad n,m \geq N.$$

In order to prove that the sequence $\{x_n(t)\}$ is uniformly Cauchy, let us consider $V(x_n(t) - x_m(t))$. The above inequality and the Lipschitz condition on V yield

$$\begin{aligned} V(x_n(t) - x_m(t)) &= V(x_n(t) - x_m(t)) - V(y_n^{(\hat{k})}(t) - y_m^{(\hat{k})}(t)) + v^{(\hat{k})}(t) \\ &< L[||x_n(t) - x_m(t) - y_n^{(\hat{k})}(t) + y_m^{(\hat{k})}(t)|| + \frac{\varepsilon}{3} \\ &\leq L[||x_n(t) - y_n^{(\hat{k})}(t)|| + ||x_m(t) - y_m^{(\hat{k})}(t)||] + \frac{\varepsilon}{3} , \end{aligned}$$

which in view of the property (iv) of Lemma 5.2 leads to

$$V(x_n(t) - x_m(t)) \leq L[3(t - t_i^n)\varepsilon_n + 3(t - t_j^m)\varepsilon_m] + \frac{\varepsilon}{3} \leq 3L[\varepsilon_n^2 + \varepsilon_m^2] + \frac{\varepsilon}{3}$$

If the positive integer N is also such that for $n,m \geq N$, $3\varepsilon_n^2 < \frac{\varepsilon}{3L}$ and $3\varepsilon_m^2 < \frac{\varepsilon}{3L}$, then we have $V(x_n(t) - x_m(t)) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we have $V(x_n(t) - x_m(t)) \to 0$ as $n,m \to \infty$ uniformly for $t \in [t_0, t_0 + \gamma]$ which by (A_5) implies $\lim_{n,m\to\infty} ||x_n(t) - x_m(t)|| = 0$ uniformly for $t \in [t_0, t_0 + \gamma]$. The uniqueness of $x(t)$ follows from standard arguments. The proof of the theorem is complete.

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