Existence and uniqueness of solutions of stochastic functional differential equations

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Abstract

Using a variant of the Euler-Murayama scheme for stochastic functional differential equations with bounded memory driven by Brownian motion we show that only weak one-sided local Lipschitz (or 'monotonicity') conditions are sufficient for local existence and uniqueness of strong solutions. In case of explosion the method yields the maximal solution up to the explosion time. We also provide a weak growth condition which prevents explosions to occur. In an appendix we formulate and prove four lemmas which may be of independent interest: three of them can be viewed as rather general stochastic versions of Gronwall's Lemma, the final one provides tail bounds for Hölder norms of stochastic integrals.

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1 Introduction

There is by now a rather comprehensive mathematical literature on the mathematical theory and on applications of stochastic functional (or delay) differential equations driven by Brownian motion. Existence and uniqueness of global solutions have been established under global Lipschitz conditions on the coefficients (e.g. [10]) or under local Lipschitz and linear growth conditions (e.g. [9, 12]). On the other hand it is common knowledge for non-delay (stochastic) differential equations that only one sided Lipschitz conditions are sufficient for local existence of solutions. This distinction becomes particularly relevant in infinite dimensions where the drift in (stochastic) evolution equations is unbounded and discontinuous in almost all interesting cases but nevertheless satisfies a one-sided Lipschitz i.e. 'monotonicity'/'dissipativity' condition, cf. e.g. [11]. In this paper we show that monotonicity of the coefficients guarantees local existence of solutions to delay equations with bounded memory, thereby closing a systematic gap in the existing literature.

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We choose the classical framework of the space of continuous functions as a natural state space of the equation. Note that, due to the absence of an inner product on this space, the right formulation of monotonicity is not obvious in this case. The proposed condition (M) below fits well to our needs, since it recovers the classical monotonicity condition for the non-delay case as a limit and yet is weak enough to cover a rather big set of equations.

In our proof we define a specific Euler-Murayama scheme, which is generally a very powerful tool in the Markovian case [1, 6, 7]. Other variants have been treated for the numerical simulation of stochastic delay equations under Lipschitz conditions in e.g. [4, 8, 5] and most recently [3]. We point out that our method yields an approximation in the strong sense even in the case of an explosion. In particular our proof below shows how the explosion time can be recovered numerically, which seems to be a question typically neglected in the literature.

As for the proofs, note that the left hand side of condition (M) is quite weak w.r.t. the C^0-norm. As a consequence the standard two-step Burkholder-Davis-Gundy and Gronwall argument cannot be applied to obtain the crucial contraction estimates. We overcome this difficulty by what we call stochastic Gronwall lemmas and which are presented in the appendix. We think that they may be of independent interest. These lemmas are also crucial for the global existence assertion which holds under a rather familiar growth (or 'coercivity', [11]) condition (C), which is again weak in the C^0-topology.

2 Set Up and Main Results

For $r > 0$, let $\mathcal{C}$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[-r, 0]$ endowed with the sup-norm $\|.\|$. For a function or a process X defined on $[t-r, t]$ we write $X_t(s) := X(t+s)$, $s \in [-r, 0]$. Consider the stochastic functional differential equation

$$\begin{cases} \mathrm{d}X(t) = f(X_t)\ \mathrm{d}t + g(X_t)\,\mathrm{d}W(t), \\ \quad X_0 = \varphi, \end{cases} \tag{1}$$

where W is an $\mathbb{R}^m$-valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the augmented Brownian filtration $\mathcal{F}_t^W = \sigma\left(W(u), 0 \le u \le t\right) \vee \mathcal{N} \subset \mathcal{F}$, where $\mathcal{N}$ denotes the null-sets in $\mathcal{F}$, φ is an $(\mathcal{F}_t^W)$-independent $\mathcal{C}$-valued random variable and $f : \mathcal{C} \to \mathbb{R}^d$, $g : \mathcal{C} \to \mathbb{R}^{d\times m}$ are continuous maps.

We will suppose throughout this work the following monotonicity assumption on f and g.

For each compact subset $C \subset \mathcal{C}$, there exists a number K_C and some $r_C \in]0, r]$ such that for all $x, y \in C$ with $x(s) = y(s)\,\forall\, s \in [-r, -r_C]$

$$2\left\langle f(x) - f(y), x(0) - y(0)\right\rangle + |||g(x) - g(y)|||^2 \le K_C\ \|x - y\|^2, \tag{M}$$

where $\langle ., . \rangle$ denotes the standard inner product on $\mathbb{R}^d$ and $|||M|||^2 = \mathrm{tr}(MM^*)$ for $M \in \mathbb{R}^{d\times m}$.

As an example in $d = 1$ take $f(x) = \varphi(\sum_{i=1}^N w_i\, x(t_i))$, where $t_i \in [-r, 0], w_i \ge 0$, $i = 1, \ldots, N$ and $\varphi \in C(\mathbb{R})$ is a non-increasing continuous (not necessarily Lipschitz) function, e.g. $\varphi(s) = -\mathrm{sign}(s)\sqrt{|s|}$ and g locally Lipschitz on $\mathcal{C}$. Another example is $f = f_1 + f_2 + f_3$ with f_1 locally Lipschitz on $\mathcal{C}$, $f_2(x) = \int_{-r}^{-r_0} \psi(x(s))k(s)ds$ for some $0 < r_0 < r$, $k, \psi \in C(\mathbb{R})$ and $f_3(x) = \varphi(x(0))$ with $\varphi \in C(\mathbb{R})$ non-increasing as above.

Our first result is a local existence and uniqueness statement for solutions to (1) for which we recall some basic notions. Given any filtration $(\mathcal{F}_t)$ on Ω, an $(\mathcal{F}_t)$-stopping time $\sigma : \Omega \to \overline{\mathbb{R}}_{\geq 0}$ is called *predictable* if there exists a sequence of ('announcing') stopping times σ_n such that $\sigma_n < \sigma$ and $\sigma_n \nearrow \sigma$ $\mathbb{P}$-almost surely. A tuple $X = (X, \sigma)$ of a predictable stopping time σ and a map $X : \Omega \times ([-r, 0] \cup [0, \sigma[) \to \mathbb{R}^d$ is called a *local $(\mathcal{F}_t)$-semimartingale up to time σ starting from $\varphi \in \mathcal{C}$*, if $X_0 = \varphi$ holds $\mathbb{P}$-almost surely and for any (announcing) stopping time $\sigma_n < \sigma$, the process $(X^{\sigma_n}(t))_{t\geq 0}$ with $X^{\sigma_n}(t) = X(t \wedge \sigma_n)$ is an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-adapted semimartingale.

Definition 2.1 (Local Solution). *Let $\mathcal{F}_t = \mathcal{F}_t^W \vee \sigma(\varphi)$. A local $(\mathcal{F}_t)$-semimartingale (X, σ) up to a predictable stopping time σ is called a* local strong solution *to equation (1) if $X_0 = \varphi$ and for any stopping time $\sigma_n < \sigma$ and any $t \geq 0$*

$$X(t \wedge \sigma_n) = X(0) + \int_0^{t\wedge\sigma_n} f(X_u)\, \mathrm{d}u + \int_0^{t\wedge\sigma_n} g(X_u)\, \mathrm{d}W(u) \quad \mathbb{P}\text{-}a.s.$$

The pair (X, σ) is called maximal strong solution *if in addition (X_t) eventually leaves any compact set $K \subset \mathcal{C}$ for $t \to \sigma$, $\mathbb{P}$-almost surely on $\{\sigma < \infty\}$; i.e.*

$$\mathbb{P}\left(\{\exists \text{ a compact set } K \subset \mathcal{C} \text{ and } t_i \nearrow \sigma \text{ s.t. } X_{t_i} \in K\} \cap \{\sigma < \infty\}\right) = 0.$$

Theorem 2.2. *Equation* (1) *admits a unique maximal strong solution (X, σ) provided* (M) *holds.*

Theorem 2.3. *In addition to the assumptions of Theorem 2.2 let f and g be bounded on bounded subsets of $\mathcal{C}$ and let the pair (f, g) be weakly coercive in the sense that there exists a non-decreasing function $\rho : [0, \infty[\to]0, \infty[$ such that $\int_0^\infty 1/\rho(u)\, \mathrm{d}u = \infty$ and for all $x \in \mathcal{C}$*

$$2 \langle f(x), x(0)\rangle + |||g(x)|||^2 \leq \rho(\|x\|^2). \tag{C}$$

Then X is globally defined, i.e. $\sigma = \infty$ $\mathbb{P}$-almost surely.

3 Proof of Theorem 2.2

The proof of Theorem 2.2 is based on an iteration of Lemma 3.1 below, which requires some auxiliary notation. For $\Phi \subset \mathcal{C}$ and $R > 0$ let

$$C_{\Phi,R} = \{\eta \in \mathcal{C} |\, \exists\, \varphi \in \Phi, r_0 \in [0, r] : \eta(u) = \varphi(u + r_0), u \in] - r, -r_0], \|\eta - \varphi(0)\|_{1/4;[-r_0,0]} \leq R\} \subset \mathcal{C},$$

where

$$\|\eta\|_{\alpha;[a,b]} = \sup_{a\leq u<v\leq b} \left(|\eta(v) - \eta(u)|/(v - u)^\alpha\right) + \sup_{a\leq u\leq b} |\eta(u)|$$

denotes the Hölder-α-norm on $C([a, b], \mathbb{R}^d)$, $\alpha \in (0, 1)$. Note that $C_{\Phi,R}$ is compact in $\mathcal{C}$ provided Φ is.

Below we drop the subscript Φ whenever this causes no confusion.

Lemma 3.1. *In addition to the conditions of Theorem 2.2 assume there is a compact subset $\Phi \subset \mathcal{C}$ such that $\varphi \in \Phi$ $\mathbb{P}$-almost surely. For $R > 0$, let $r_R = r_C$ be the constant appearing in* (M) *for choosing $C = C_{\Phi,R}$. Then there exists a stopping time $0 < \sigma_R \leq r_R$ and a unique (up to indistinguishability) $(\mathcal{F}_t)$-adapted process $X(t)$, $t \in [0, \sigma_R]$ such that $X_t \in C_R$ for all $t \in [0, \sigma_R]$ which solves* (1) *up to time σ_R. Moreover,*

$$\|X(.) - \varphi(0)\|_{1/4;[0,\sigma_R]} \geq \frac{R}{2} \quad \mathbb{P}\text{-}a.s. \text{ on } \{\sigma_R < r_R\}. \tag{2}$$

Proof. The proof is inspired by the arguments for finite dimensional monotone SDEs in [7], cf. e.g. [11]. For $n \in \mathbb{N}$, we define an Euler-like approximation to (1) with step size $\frac{1}{n}$ by

$$\begin{cases} \mathrm{d}X^n(t) = f(\overline{X}_t^n)\,\mathrm{d}t + g(\overline{X}_t^n)\,\mathrm{d}W(t), \\ \quad X_0^n = \varphi, \end{cases} \tag{3}$$

where we define $\overline{X}_s^n(.) \in \mathcal{C}$, $s \geq 0$ by

$$\overline{X}_s^n(u) = X^n\big((s+u) \wedge \tfrac{\lfloor ns \rfloor}{n}\big), \quad u \in [-r, 0].$$

Equation (3) admits a global in time solution via the recursion $X_0^n = \varphi$ and

$$X^n(t) = X^n\big(\tfrac{\lfloor nt \rfloor}{n}\big) + \int_{\lfloor nt \rfloor/n}^t f\big(\overline{X}_s^n\big)\,\mathrm{d}s + \int_{\lfloor nt \rfloor/n}^t g\big(\overline{X}_s^n\big)\,\mathrm{d}W(s).$$

The process $t \mapsto X^n(t)$ is adapted and continuous, hence

$$t \mapsto p_t^n(.) := \overline{X}_t^n(.) - X_t^n(.),\ t \geq 0$$

defines an adapted $\mathcal{C}$-valued process (which is càdlàg). With this, (3) is equivalent to $X_0^n = \varphi$ and

$$X^n(t) = \varphi(0) + \int_0^t f(X_s^n + p_s^n)\,\mathrm{d}s + \int_0^t g(X_s^n + p_s^n)\,\mathrm{d}W(s).$$

Without loss of generality, we may assume that the set Φ has the property that $0 \in \Phi$ and $\eta \in \Phi$, $s \in [-r, 0)$ implies that the function $u \mapsto \eta(u \wedge s)$, $u \in [-r, 0]$ also belongs to Φ. Then, $X_t^n \in C_R$ implies $\overline{X}_t^n \in C_R$, hence $p_t^n \in \widetilde{C}_R = \{\eta_1 - \eta_2 \,|\, \eta_i \in C_R\}$ provided

$$t \leq \tau_R^n := \inf\{t > 0 | X_t^n \notin C_R\}.$$

Since $\widetilde{C}_R \subset \mathcal{C}$ is again compact,

$$\widetilde{\rho}(R) = \sup_{x \in \widetilde{C}_R} \|x\| < \infty \tag{4}$$

and the continuity of f and g ensures that

$$C_1(R) := \sup_{x \in \widetilde{C}_R} \{|f(x)| + |||g(x)|||\} < \infty. \tag{5}$$

Fix $n, m \in \mathbb{N}$ and let $0 \leq \tau$ be a finite stopping time. Then, by Itô's formula,

$$\begin{aligned} |X^n(\tau) - X^m(\tau)|^2 &= 2\int_0^\tau \langle X^n(u) - X^m(u), (g(X_u^n + p_u^n) - g(X_u^m + p_u^m))\,\mathrm{d}W(u)\rangle \\ &+ \int_0^\tau \Big(2\langle f(X_u^n + p_u^n) - f(X_u^m + p_u^m), X^n(u) - X^m(u)\rangle + \Big|\Big|\Big|g(X_u^n + p_u^n) - g(X_u^m + p_u^m)\Big|\Big|\Big|^2\Big)\,\mathrm{d}u. \end{aligned}$$

In order to use condition (M), note that by construction for $s > 0$ and $s + u \leq 0$

$$\overline{X}_s^m(u) = \overline{X}_s^n(u) = \varphi(s+u).$$

Hence, together with (4) and (5), the second term on the r.h.s. can be estimated from above by

$$\int_0^\tau \Big(2\langle f(X_u^n+p_u^n)-f(X_u^m+p_u^m), p_u^m(0)-p_u^n(0)\rangle + K_R\big\|X_u^n+p_u^n-(X_u^m+p_u^m)\big\|^2\Big)\,\mathrm{d}u$$
$$\le \int_0^\tau \Big(4C_1(R)\big(|p_u^n(0)|+|p_u^m(0)|\big)+4K_R\big(\|p_u^n\|^2+\|p_u^m\|^2\big)+2K_R\|X_u^n-X_u^m\|^2\Big)\,\mathrm{d}u$$
$$\le \int_0^\tau \big[4C_1(R)+4K_R\widetilde{\rho}(R)\big]\big(\|p_u^n\|+\|p_u^m\|\big)+2K_R\sup_{v\in[0,u]}|X^n(v)-X^m(v)|^2\,\mathrm{d}u$$

provided $\tau\le\tau_R^m\wedge\tau_R^n\wedge r_R=:\kappa$. Hence we may apply Lemma 5.4 to $Z(s):=|X^n(s\wedge\kappa)-X^m(s\wedge\kappa)|^2$ with $M(s):=2\int_0^{s\wedge\kappa}\langle X^n(u)-X^m(u),(g(X_u^n+p_u^n)-g(X_u^m+p_u^m))\,\mathrm{d}W(u)\rangle$, $H(s)=\int_0^{s\wedge\kappa}\big[4C_1(R)+4K(R)\widetilde{\rho}(R)\big]\big(\|p_u^n\|+\|p_u^m\|\big)\,\mathrm{d}u$ and $T=r_R$. Once we have shown that some moment of $H^*(T):=\sup_{0\le s\le T}H(s)$ converges to 0 as $n,m\to\infty$, Lemma 5.4 implies that for all $\varepsilon>0$,

$$\lim_{m,n\to\infty}\mathbb{P}\Big\{\sup_{s\in[0,\tau_R^m\wedge\tau_R^n\wedge r_R]}|X^m(s)-X^n(s)|\ge\varepsilon\Big\}=0. \tag{6}$$

Since $H^*(T)$ is bounded uniformly in ω,n,m, it suffices to show that $H^*(T)$ converges to zero in probability as $m,n\to\infty$ which can be verified as follows:

$$p_s^n(u)=\begin{cases}0 & \text{for } u\ge -r, u+s\le\frac{\lfloor ns\rfloor}{n}\\ -\int_{\lfloor ns\rfloor/n}^{s+u}f(\overline{X}_t^n)\,\mathrm{d}t-\int_{\lfloor ns\rfloor/n}^{s+u}g(\overline{X}_t^n)\,\mathrm{d}W(t) & \text{for } u+s\ge\frac{\lfloor ns\rfloor}{n}, u\le 0\end{cases}$$

implies

$$\|p_s^n\|\le\sup_{\lfloor ns\rfloor/n\le t\le s}\Big|\int_{\lfloor ns\rfloor/n}^t f(\overline{X}_u^n)\,\mathrm{d}u\Big|+\sup_{\lfloor ns\rfloor/n\le t\le s}\Big|\int_{\lfloor ns\rfloor/n}^t g(\overline{X}_u^n)\,\mathrm{d}W(u)\Big|,$$

and hence – since f and g are bounded on C_R –

$$\mathbb{E}1\!\!1_{\{\tau_R^n\ge s\}}\|p_s^n\|\to 0 \text{ as } n\to\infty, \text{ uniformly in } [0,r_R].$$

Therefore, $\mathbb{E}H^*(T)$ converges to 0 and (6) follows. By definition of $\overline{X}^m$ this also yields

$$\lim_{m,n\to\infty}\mathbb{P}\Big\{\sup_{s\in[0,\tau_R^m\wedge\tau_R^n\wedge r_R]}\|\overline{X}_s^m-\overline{X}_s^n\|\ge\varepsilon\Big\}=0. \tag{7}$$

Since f,g are uniformly continuous on the compact set C_R

$$\lim_{m,n\to\infty}\mathbb{P}\Big\{\sup_{s\in[0,\tau_R^m\wedge\tau_R^n\wedge r_R]}\big\{|f(\overline{X}_s^m)-f(\overline{X}_s^n)|\vee|\!|\!|g(\overline{X}_s^m)-g(\overline{X}_s^n)|\!|\!|\big\}\ge\varepsilon\Big\}=0. \tag{8}$$

To further improve this statement, we apply Lemma 5.5 to

$$X^n(s\wedge\tau_R^m\wedge\tau_R^n\wedge r_R)-X^m(s\wedge\tau_R^m\wedge\tau_R^n\wedge r_R)=\int_0^{s\wedge\tau_R^m\wedge\tau_R^n\wedge r_R}(F^n-F^m)(u)\,\mathrm{d}Z(u),$$

where for simplicity we write $Z(u)=(u,W(u))\in\mathbb{R}^{m+1}$ and $F^n(u)=\big(f(\overline{X}_u^n),g(\overline{X}_u^n)\big)$. Together with (8) this allows to conclude that for all $\varepsilon>0$

$$\lim_{m,n\to\infty}\mathbb{P}\big\{\|X^m(.)-X^n(.)\|_{1/4;[0,\tau_R^m\wedge\tau_R^n\wedge r_R]}\ge\varepsilon\big\}=0. \tag{9}$$

Let us select a subsequence, which will again be denoted by X^n such that

$$\mathbb{P}\{\|X^k(.) - X^l(.)\|_{1/4;[0,\tau_R^k\wedge\tau_R^l\wedge r_R]} \geq 2^{-(l\wedge k)}\} \leq 2^{-(l\wedge k)}, \tag{10}$$

and define

$$\tau_R = \liminf_{n\to\infty} \tau_R^n.$$

Due to (10), there is an $(\mathcal{F}_t)$-adapted process X defined in $[0,\tau_R[\cap[0,r_R]$ to which X^n converges $\mathbb{P}$-almost surely locally in $C^{1/4}([0,\tau_R[\cap[0,r_R];\mathbb{R}^d)$. From (3), (6) and (8) and the continuity of f and g we infer that X must be a solution to equation (1) on $[0,\tau_R[\cap[0,r_R[$.

We remark that $\tau_R > 0$ almost surely, which can be seen as follows. For any $\varepsilon > 0$, using (10) we choose n_0 such that the set

$$A = \{\omega|\sup_{k\geq n_0}\|X^{n_0}(.) - X^k(.)\|_{1/4;[0,\tau_R^k\wedge\tau_R^{n_0}\wedge r_R]} < \frac{R}{2}\}$$

satisfies $\mathbb{P}(A) \geq 1-\varepsilon$. From $\overline{X}_s^{n_0}(.) = \varphi((s+\cdot)\wedge 0) \in \Phi$ for $s \in [0,\frac{1}{n_0}]$, using Lemma 5.5 for the SDE (3) solved by X^{n_0}, it follows that $\eta_{R/2}^{n_0} := \inf\{t \geq 0\,|\,\|X^{n_0}(.) - \varphi(0)\|_{1/4;[0,t]} \geq \frac{R}{2}\} \wedge r_R$ is strictly positive. By construction of A it holds on A that $\tau_R^n \wedge r_R \geq \eta_{R/2}^{n_0} \wedge r_R$ for all $n \geq n_0$, hence in particular $\tau_R > 0$.

Next, we show that almost surely one of the two following events occur:

$$\{\tau_R \geq r_R\} \qquad \text{or} \qquad \{\tau_R < r_R\} \cap \{\sup_{t<\tau_R}\|X(.) - \varphi(0)\|_{1/4;[0,t]} \geq \frac{3R}{4}\}. \tag{11}$$

In case $\{\tau_R \geq r_R\}$, using (1) for $X(.)$ on $[0,r_R[$ and the uniform boundedness of the coefficients on C_R we may extend $X(.)$ on the closed interval $[0,r_R]$ by setting

$$X(r_R) := X(0) + \int_0^{r_R} f(X_s)ds + \int_0^{r_R} g(X_s)dW(s).$$

Together with (11) for

$$\sigma_R := \inf\{t \in [0,\tau_R[\cap[0,r_R]\,|\,\|X(.) - \varphi(0)\|_{1/4;[0,t]} \geq \frac{R}{2}\} \wedge r_R$$

this gives a well defined process $t \mapsto X(t)$ for $t \in [0,\sigma_R]$ which solves (1) in up to time σ_R in the sense of Definition 2.1. Moreover, (2) holds by construction.

To prove (11) we show that the set

$$B := \{\tau_R < r_R\} \cap \{\sup_{t<\tau_R}\|X(.) - \varphi(0)\|_{1/4;[0,t]} < \frac{3R}{4}\}.$$

has vanishing $\mathbb{P}$-measure. Assume the contrary, i.e. $\mathbb{P}(B) = p > 0$. Then by (10) and the definition of τ_R we find some $n_0 \in \mathbb{N}$ such that $\mathbb{P}(A) > \frac{p}{2}$, where

$$A := \{\omega|\sup_{k\geq n_0}\|X^{n_0}(.) - X^k(.)\|_{1/4;[0,\tau_R^k\wedge\tau_R^{n_0}\wedge r_R]} < \frac{R}{16};\ \inf_{n\geq n_0}\tau_R^n < r_R;\ \sup_{t<\tau_R}\|X(.) - \varphi(0)\|_{1/4;[0,t]} < \frac{3R}{4}\}.$$

We show that in fact $\mathbb{P}(A) = 0$. To this aim note that w.l.o.g. we may assume that X^n converges to X locally in $C^{1/4}([0, \tau_R[)$ and $[0, r_R \wedge \tau_R^m] \ni t \mapsto \|X^m(.)\|_{1/4;[0,t]}$ is continuous for all $m \in \mathbb{N}$, for *all* $\omega \in A$, where the latter is again a consequence of Lemma 5.5. Now for $\omega \in A$ choose $m = m(\omega) \geq n_0$ such that $\tau_R^m < r_R$. Let $\eta_R^m := \inf\{t \geq 0 \,|\, \|X^m(.) - \varphi(0)\|_{1/4;[0,t]} \geq R\} \leq \tau_R^m$, then by continuity $\eta_{7R/8}^m < \eta_{15R/16}^m \leq \tau_R^n$ for all $n \geq n_0$, hence $\eta_{7R/8}^m < \tau_R$. Again by continuity, $\sup_{t<\eta_{7R/8}^m} \|X^n(.) - \varphi(0)\|_{1/4;[0,t]} \geq \frac{3R}{4}$ for all $n \geq n_0$ satisfying $\tau_R^n > \eta_{7R/8}^m$. In view of the convergence of X^n to X in $C^{1/4}[0, \eta_{7R/8}^m]$ for $n \to \infty$ this yields a contradiction to $\sup_{t<\tau_R} \|X(.) - \varphi(0)\|_{1/4;[0,t]} < \frac{3R}{4}$. Hence $A = \emptyset$ almost surely which proves (11).

To show uniqueness of a local solution, assume X and $\tilde{X}$ are two solutions defined up to a stopping time $\tilde{\sigma} \leq \sigma_R$. Applying Itô's formula to the square of the norm of the difference of the solutions and using condition (M), Lemma 5.2 (with $C = 0$) shows that the solutions agree on $[0, \tilde{\sigma}]$ almost surely. This completes the proof of Lemma 3.1. □

Proof of Theorem 2.2. First we remark that it is sufficient to prove both the existence and uniqueness assertion of the theorem under the stronger assumption that $\mathbb{P}(\varphi \in \Phi) = 1$ for any fixed compact subset $\Phi \subset \mathcal{C}$. In fact, since any probability measure on the Polish space $\mathcal{C}$ is tight, in both cases the general statement follows by approximation in $\mathbb{P}$-measure by initial conditions $\varphi_n = 1_{\Phi_n}(\varphi) \cdot \varphi$, where e.g. the compact subsets $\Phi_n \subset \mathcal{C}$ are chosen such that $\mathbb{P}(\varphi \notin \Phi_n) \leq \frac{1}{n}$.

The proof of the existence statement is based on iterative use of Lemma 3.1. Recall for $R > 0$, r_R denotes the constant r_C in condition (M) when $C = C_{\Phi,R}$. We may assume w.l.o.g. that the function $R \mapsto r_R$ is non-increasing and we may select a sequence $R^{(k)} \nearrow \infty$, $k \in \mathbb{N}$, such that $\sum_k r_{R^{(k)}} = \infty$.

Lemma 3.1 with $\Phi =: \Phi^{(1)}$ and $R := R^{(1)}$ for initial condition $\varphi =: \varphi^{(1)} \in \Phi^{(1)}$ guarantees the existence of a process $t \mapsto X(t) =: X^{(1)}(t)$, $t \in [0, \sigma^{(1)}]$ with an $\mathcal{F}_.$-stopping time $\sigma^{(1)} := \sigma_{R^{(1)}} \leq r_{R^{(1)}}$ which is a local solution to (1) on $[0, \sigma^{(1)}[$.

Next we may apply Lemma 3.1 to the same equation (1), now in the situation when R and W are chosen to be $R^{(2)}$ and $W_t^{(2)} = W(\sigma^{(1)}+t) - W(\sigma^{(1)})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ respectively, with $\mathcal{F}_t^{(2)} = \mathcal{F}_t^{W^{(2)}} \vee \mathcal{N} \subset \mathcal{F}$, $(t \geq 0)$, and $\mathcal{F}_.^{(2)}$-independent initial condition $\varphi^{(2)} := X_{\sigma^{(1)}}^{(1)} \in C_{\Phi,R_1} =: \Phi^{(2)}$. This yields an $\mathcal{F}_.^{(2)}$-stopping time $\tilde{\sigma}^{(2)} \leq r_{R^{(2)}}$ and a process $t \mapsto \tilde{X}^{(2)}$, $[0, \tilde{\sigma}^{(2)}]$ solving (1) on $t \in [0, \tilde{\sigma}^{(2)}[$. (Note that here we have used the simple fact that $C_{C_{\Phi,R_1},R_2} = C_{\Phi,R_2}$ for $R_2 \geq R_1$.) Hence, by continuation

$$t \mapsto X^{(2)}(t) = \begin{cases} X^{(1)}(t) & \text{if } t \in [-r, \sigma^{(1)}] \\ \tilde{X}^{(2)}(t - \sigma^{(1)}) & \text{if } t \in]\sigma^{(1)}, \sigma^{(1)} + \tilde{\sigma}^{(2)}] \end{cases}$$

we obtain an $\mathcal{F}_.$-adapted process which is a local solution to equation (1) up to the $\mathcal{F}_.$-stopping time $\sigma^{(2)} = \sigma^{(1)} + \tilde{\sigma}^{(2)}$ in the sense of Definition 2.1.

For general n this construction is repeated inductively, furnishing a local solution (X, σ) to equation (1) in the sense of Definition 2.1 where

$$\sigma = \lim_{n\to\infty} \sigma^{(n)}.$$

To prove that (X, σ) is maximal using the continuity of f and g it suffices to prove that the set

$$\Sigma = \Big\{ \sup_{t\in[0,\sigma[} (|f(X_t)| \vee |||g(X_t)|||) < \infty \Big\} \cap \{\sigma < \infty\} \tag{12}$$

has zero $\mathbb{P}$-measure. Now from the second statement in Lemma 3.1, from the construction of X and from the property $\sum_k r_{R^{(k)}} = \infty$ it follows that

$$\sup_{s\in[0,\sigma[} \left\| X(.) - X(\sigma^{(k-1)}) \right\|_{1/4;[0,s]} \geq \frac{R^{(k)}}{2} \quad \mathbb{P}\text{-a.s.}$$

for infinitely many $k \in \mathbb{N}$ on $\{\sigma < \infty\}$, i.e.

$$\mathbb{P}(\Sigma) = \mathbb{P}\Big(\sigma < \infty; \sup_{s\in[0,\sigma[} (|f(X_s)| \vee |||g(X_s)|||) < \infty; \sup_{s\in[0,\sigma[} \|X(.)\|_{1/4;[0,s]} = \infty\Big).$$

Since X solves (1), due to e.g. Lemma 5.5, the r.h.s. is zero.

As for the uniqueness statement let (Y, τ) be another maximal solution with an associated sequence of announcing stopping times $\tau^{(n)}$. The construction of X above yields a sequence of announcing stopping times $\sigma^{(n)}$ for σ and compact sets $C_n \subset \mathcal{C}$ such that $X_{t\wedge\sigma^{(n)}} \in C_n$. Hence, by the same argument as in the proof of Lemma 3.1 one obtains that $X_{\sigma^{(n)}\wedge\tau^{(n)}\wedge.}$ and $Y_{\sigma^{(n)}\wedge\tau^{(n)}\wedge.}$ are indistinguishable. Moreover, the maximality of the pair (Y, τ) implies that $\sigma^{(n)} < \tau$ for all $n \in \mathbb{N}$, i.e. $\sigma \leq \tau$ almost surely. Conversely, the maximality of σ implies $\sigma > \tau^n$, i.e $\sigma \geq \tau$, which completes the proof. □

4 Proof of Theorem 2.3

Proof of Theorem 2.3. Let (X, σ) be the maximal strong solution of equation (1). We want to show that $\sigma = \infty$ almost surely. Since f and g are bounded on bounded subsets of $\mathcal{C}$, it follows from (12) that $\limsup_{t\nearrow\sigma} |X(t)| = \infty$ almost surely on the set $\{\sigma < \infty\}$. For a stopping time $0 \leq \tau < \sigma$, Itô's formula implies that

$$\begin{aligned} X^2(\tau) - X^2(0) &= \int_0^\tau 2\langle f(X_u), X(u)\rangle + |||g(X_u)|||^2 \,\mathrm{d}u + 2\int_0^\tau \langle X(u), g(X_u)\,\mathrm{d}W(u)\rangle \\ &\leq \int_0^\tau \rho(\|X_u\|^2)\,\mathrm{d}u + M(\tau), \end{aligned}$$

where M is a continuous local martingale. Applying Lemma 5.1 to $Z(t) := X^2(t)$ finishes the proof. □

5 Appendix

We start by proving three lemmas which could be called *stochastic Gronwall lemmas*. We use them in the proof of Theorems 2.2 and 2.3. Then we prove a result about the tails of Hölder norms of stochastic integrals which we owe to Steffen Dereich (TU Berlin). We believe that all these results are of independent interest. In all lemmas, we assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is given and that it satisfies the usual conditions. Throughout, we will use the notation $Z^*(T) = \sup_{0\leq t\leq T} Z(t)$ for a real-valued process Z.

Lemma 5.1. *Let $\sigma > 0$ be a stopping time and let Z be an adapted non-negative stochastic process with continuous paths defined on $[0, \sigma[$ which satisfies the inequality*

$$Z(t) \leq \int_0^t \rho(Z^*(u))\,\mathrm{d}u + M(t) + C,$$

and $\lim_{t\uparrow\sigma} Z^*(t) = \infty$ *on* $\{\sigma < \infty\}$ *almost surely. Here,* $C \geq 0$ *and* M *is a continuous local martingale defined on* $[0, \sigma[$, $M(0) = 0$ *and* $\rho : [0, \infty[\to]0, \infty[$ *is non-decreasing, and* $\int_0^\infty 1/\rho(u)\,\mathrm{d}u = \infty$*. Then* $\sigma = \infty$ *almost surely.*

Proof. Let Y be the unique (maximal) solution of the equation

$$Y(t) = \int_0^t \rho(Y^*(u))\,\mathrm{d}u + M(t) + C.$$

Clearly, $Y(t) \geq Z(t)$ for all t for which Y is defined and therefore it suffices to prove the claim for Y instead of Z. For $a > C$, define $\tau_a := \inf\{t \geq 0 | Y(t) \geq a\}$. For $C < a < b$ and $\delta > 0$ we get

$$\mathbb{P}\{\tau_b - \tau_a \leq \delta | \mathcal{F}_{\tau_a}\} \leq \mathbb{P}\{b - a \leq \delta\rho(b) + \sup_{t\in[\tau_a, \tau_b\wedge(\tau_a+\delta)]} M(t) - M(\tau_a) | \mathcal{F}_{\tau_a}\}$$

on the set $\{\tau_a < \infty\}$. Note that on $\{\tau_a < \infty\}$ we have

$$M(t) - M(\tau_a) \geq Y(t) - Y(\tau_a) - (t - \tau_a)\rho(b) \geq -a - \delta\rho(b) \tag{13}$$

for $\tau_a \leq t \leq \tau_b \wedge (\tau_a + \delta)$ since Y is non-negative. For

$$\tau := \inf\{t \geq \tau_a | M(t) - M(\tau_a) \geq b - a - \delta\rho(b)\} \wedge \tau_b \wedge (\tau_a + \delta)$$

we therefore get

$$0 = \mathbb{E}(M(\tau) - M(\tau_a) | \mathcal{F}_{\tau_a}) \geq (b - a - \delta\rho(b))p - (a + \delta\rho(b))(1 - p),$$

where $p := \mathbb{P}\{M(\tau) - M(\tau_a) \geq b - a - \delta\rho(b) | \mathcal{F}_{\tau_a}\}$. Hence

$$\mathbb{P}\{\tau_b - \tau_a \leq \delta | \mathcal{F}_{\tau_a}\} \leq p \leq \frac{a + \delta\rho(b)}{b} \text{ on } \{\tau_a < \infty\}. \tag{14}$$

Fix $a > C$. Then

$$\sigma = \tau_a + \sum_{k=1}^{\infty} \big(\tau_{2^k a} - \tau_{2^{k-1}a}\big).$$

We show that the sum diverges almost surely. To ease notation, we write τ_k instead of $\tau_{2^k a}$. For $\delta_k > 0$, $k \in \mathbb{N}$, (14) implies that

$$\mathbb{P}\{\tau_k - \tau_{k-1} \geq \delta_k | \mathcal{F}_{\tau_{k-1}}\} \geq \frac{1}{2} - \delta_k \frac{\rho(2^k a)}{2^k a}$$

on the set $\{\tau_{k-1} < \infty\}$. Now

$$\sigma \geq \sum_{k=1}^{\infty} \tau_k - \tau_{k-1} \geq \sum_{k=1}^{\infty} \delta_k 1_{\{\tau_k - \tau_{k-1} \geq \delta_k\}}. \tag{15}$$

We choose

$$\delta_k := \frac{1}{4} \frac{2^k a}{\rho(2^k a)} \; k \in \mathbb{N}.$$

Since ρ is non-decreasing we have

$$\sum_{k=1}^{\infty} \delta_k \geq \frac{1}{4} \int_{2a}^{\infty} \frac{1}{\rho(u)}\,\mathrm{d}u = \infty$$

and

$$\mathbb{P}\{\tau_k - \tau_{k-1} \geq \delta_k | \mathcal{F}_{\tau_{k-1}}\} \geq \frac{1}{4} \text{ on } \{\tau_{k-1} < \infty\}.$$

It follows (e.g. from Kolmogorov's three series theorem) that the right hand side of (15) diverges on the set $\{\tau_k < \infty \text{ for all } k \in \mathbb{N}\}$. On the complement of this set, σ is also infinite, i.e. the proof of the lemma is complete. □

While the previous lemma was concerned with non-blow up of Z, the following lemma shows that Z remains small it case the initial condition is small. In principle we could formulate the following lemma also using a function ρ as in the previous one but we prefer not to in order to obtain a reasonably explicit formula for moments of $Z^*(T)$.

Lemma 5.2. *Let Z be an adapted non-negative stochastic process with continuous paths defined on $[0,\infty)$ which satisfies the inequality*

$$Z(t) \leq K \int_0^t Z^*(u)\,\mathrm{d}u + M(t) + C,$$

where $C \geq 0$, $K > 0$ and M is a continuous local martingale with $M(0) = 0$. Then for each $0 < p < 1$, there exist universal finite constants $c_1(p), c_2(p)$ (not depending on K, C, T and M) such that

$$\mathbb{E}(Z^*(T))^p \leq C^p c_2(p) \exp\{c_1(p)KT\} \text{ for every } T \geq 0.$$

Proof. Let Y be the unique solution of the equation

$$Y(t) = K \int_0^t Y^*(u)\,\mathrm{d}u + M(t) + C.$$

Clearly, $Y(t) \geq Z(t)$ for all $t \geq 0$ and therefore it suffices to prove the claim for Y instead of Z. Let $\tau_a := \inf\{t \geq 0 : Y(t) \geq a\}$. Like in the proof of Lemma 5.1, we obtain for $\beta \in (0,1)$ and $b > a \geq C$

$$\mathbb{P}\{\tau_b - \tau_a \leq \frac{\beta}{K} | \mathcal{F}_{\tau_a}\} \leq \frac{a + \beta b}{b} \text{ on } \{\tau_a < \infty\}. \tag{16}$$

For $T > 0$, $m \in \mathbb{N}$, $\gamma > (1-\beta)^{-1}$ we get

$$\mathbb{P}\{Y^*(T) \geq \gamma^m C\} = \mathbb{P}\{\tau_{\gamma^m C} \leq T\} = \mathbb{P}\{\sum_{i=1}^m \tau_{\gamma^i C} - \tau_{\gamma^{i-1} C} \leq T\}.$$

By (16), the last sum is stochastically larger than β/K times a binomial variable V with parameters m and $\alpha := 1 - \frac{1}{\gamma} - \beta$. Therefore, for $\lambda > 0$ and $N := \lceil \frac{KT}{\beta} \rceil$ we get

$$\mathbb{P}\{Y^*(T) \geq \gamma^m C\} \leq \mathbb{P}\{V \leq N\} = \mathbb{P}\{\mathrm{e}^{-\lambda V} \geq \mathrm{e}^{-\lambda N}\}.$$

Applying Markov's inequality, representing V as a sum of m independent Bernoulli(α) variables and optimizing over $\lambda > 0$ as usual, we obtain for $m \geq \lceil \frac{N}{\alpha} \rceil =: m_0$

$$\mathbb{P}\{Y^*(T) \geq \gamma^m C\} \leq \exp\{(m-N)\log\frac{m}{m-N} + (m-N)\log(1-\alpha) + N\log\alpha + N\log\frac{m}{N}\}.$$

Assume that $p\log\gamma+\log(1-\alpha)<0$ (which requires $p<1$ since $1-\alpha=\frac{1}{\gamma}+\beta>\frac{1}{\gamma}$) and fix $q>0$ such that $p\log\gamma+\log(1-\alpha)+q^{-1}<0$. Then

$$\begin{aligned}\mathbb{E}Y^*(T)^p &= \int_0^\infty \mathbb{P}\{Y^*(T)\geq s^{1/p}\}\,\mathrm{d}s\\ &\leq \gamma^{m_0p}C^p+\sum_{m=m_0}^\infty C^p\gamma^{pm}(\gamma-1)\exp\Big\{(m-N)\log\frac{m}{m-N}+(m-N)\log(1-\alpha)\\ &\qquad +N\log\alpha+N\log\frac{m}{N}\Big\}\\ &\leq \gamma^{m_0p}C^p+C^p(\gamma-1)\exp\{N\log\frac{\alpha q}{1-\alpha}\}\sum_{m=m_0}^\infty \exp\{m(p\log\gamma+\log(1-\alpha)+q^{-1})\}\\ &= C^p\Big(\gamma^{m_0p}+(\gamma-1)\exp\{N\log\frac{\alpha q}{1-\alpha}\}\frac{\exp\{m_0(p\log\gamma+\log(1-\alpha)+q^{-1})\}}{1-\exp\{p\log\gamma+\log(1-\alpha)+q^{-1}\}}\Big),\end{aligned}$$

where we used the inequalities $\log(1+x)\leq x$ (for $x=\frac{N}{m-N}$) and $\log x\leq\log q+q^{-1}(x-q)$ (for $x=\frac{m}{N}$) in the last "$\leq$". Observing that $m_0\leq(\frac{kT}{\beta}+1)\frac{1}{\alpha}+1$ and $N\leq\frac{KT}{\beta}+1$, the claim follows. □

Remark 5.3. It is clear that the previous lemma does not hold for $p>1$: just consider a scalar geometric Brownian motion starting with C. Its p^{th} moment for $p>1$ at time 1 (say) is unbounded with respect to the volatility σ. We don't know whether the lemma holds true for $p=1$ but we conjecture that it doesn't.

Lemma 5.4. *Let Z be an adapted non-negative stochastic process with continuous paths defined on $[0,\infty[$ which satisfies the inequality*

$$Z(t)\leq K\int_0^t Z^*(u)\,\mathrm{d}u+M(t)+H(t),$$

where $K>0$, M is a continuous local martingale with $M(0)=0$, and H is an adapted process with continuous paths satisfying $H(0)=0$. Then, for each $0<p<1$ and $\alpha>\frac{1+p}{1-p}$, there exist constants c_3,c_4 depending on p,α only such that

$$\mathbb{E}(Z^*(T))^p\leq c_3\exp\{c_4KT\}(\mathbb{E}H^*(T)^\alpha)^{p/\alpha}\ \textit{for every } T\geq 0.$$

Proof. Fix $T>0$ and for $i\in\mathbb{N}$ let X_i be the unique solution of

$$X_i(t)=K\int_0^t X_i^*(u)\,\mathrm{d}u+M(t)+i.$$

Hence, $Z\leq X_i$ on $[0,T]\times\Omega_i$ where

$$\Omega_i:=\{\omega:\sup_{0\leq t\leq T}H(t)\leq i\}.$$

Let $s\in]\frac{1}{1-p},\frac{\alpha}{1+p}[$ and let $r>1$ be defined by $r^{-1}+s^{-1}=1$. Then $pr<1$ and Lemma 5.2 and Hölder's inequality imply

$$\begin{aligned}\mathbb{E}(Z^*(T))^p &\leq \sum_{i=1}^\infty \mathbb{E}((X_i^*(T))^p\mathbb{1}_{\Omega_i\backslash\Omega_{i-1}})\leq\sum_{i=1}^\infty(\mathbb{E}(X_i^*(T))^{pr})^{1/r}\mathbb{P}\{\Omega_i\backslash\Omega_{i-1}\}^{1/s}\\ &\leq \sum_{i=1}^\infty i^pc_2(pr)^{1/r}\exp\{KTc_1(pr)/r\}\mathbb{P}\{H^*(T)\geq i-1\}^{1/s}\\ &\leq \exp\{KTc_1(pr)/r\}c_2(pr)^{1/r}\Big((\mathbb{E}H^*(T)^\alpha)^{1/s}\sum_{i=2}^\infty i^p(i-1)^{-\alpha/s}+1\Big),\end{aligned}$$

where we used Markov's inequality in the last step.

For each $\xi > 0$, the inequality in the assumption of the lemma remains true if H, M, and Z are multiplied by ξ. Therefore, the inequality

$$\mathbb{E}(Z^*(T))^p \le \exp\{KTc_1(pr)/r\}c_2(pr)^{1/r}\Big(\xi^{\frac{\alpha}{s}-p}(\mathbb{E}H^*(T)^\alpha)^{1/s}\sum_{i=2}^{\infty} i^p(i-1)^{-\alpha/s} + \xi^{-p}\Big)$$

follows. Optimizing the right hand side over $\xi > 0$ yields the assertion of the lemma. □

Lemma 5.5 (**S. Dereich**). *For $m, d \in \mathbb{N}$, $\alpha \in]0, \frac{1}{2}[$ and $t_0 > 0$ there exist some universal strictly positive constants $c_i = c_i(d, m, \alpha, t_0), i = 1, 2, 3$ such that for $Z(t) = (t, W(t)) \in \mathbb{R}^{m+1}$ with an $\mathbb{R}^m$-valued Brownian motion W*

$$\mathbb{P}\Big(\big\|\int_\sigma^{(.)} F\,\mathrm{d}Z\big\|_{\alpha;[\sigma,\tau]} \ge u\Big) \le c_1 e^{-c_2u^2/v^2T} \qquad \textit{for } \frac{u}{v(T+T^{1-\alpha})} \ge c_3,\ T \ge t_0$$

for any pair $\sigma \le \tau$ of finite $(\mathcal{F}_t)$-stopping times with $\tau - \sigma \le T$ and any $(\mathcal{F}_t)$-predictable $\mathbb{R} \times \mathbb{R}^{d\times m}$-valued process $(F(t))$ satisfying $\sup_{s\in[\sigma,\tau]} |||F(s)||| \le v$ $\mathbb{P}$-almost surely.

Proof. It suffices to treat the case when $\sigma = 0$ and $m = d = 1$, where we have to deal with real-valued semimartingales of the form

$$t \mapsto \int_0^t F(s)\,\mathrm{d}s =: A(t) \qquad \text{or} \qquad t \mapsto \int_0^t F(s)\,\mathrm{d}W(s) =: M(t)$$

with integrands satisfying $\sup_{s\in[0,T]} |F(s)| \le v$ almost surely. The first case is easy: the map $t \mapsto A(t)$ is Lipschitz with constant (at most) v and therefore $\|A(.)\|_{\alpha;[0,\tau]} \le v\,(T + T^{1-\alpha})$ almost surely, so the claim follows in this case. Let us consider M. The Gaussian isoperimetric inequality, cf. e.g. [2, Section 4.3], implies the existence of some universal positive constants $k_i = k_i(\alpha), i = 1, 2$ such that

$$\mathbb{P}\Big(\big\|W(.)\big\|_{\alpha;[0,1]} \ge u\Big) \le k_1 e^{-k_2u^2} \qquad \text{for } u \ge 0.$$

We choose an independent Brownian motion W' and let $F'(s) = \sqrt{v^2 - F^2(s)}$. Then both processes

$$t \mapsto B^{(j)}(t) = \int_0^t F(s)\,\mathrm{d}W(s) - (-1)^j \int_0^t F'(s)\,\mathrm{d}W'(s), \quad j = 1, 2,$$

have the same distribution as $t \mapsto vW(t)$. From $B^{(1)}(t) + B^{(2)}(t) = 2\int_0^t F(s)\,\mathrm{d}W(s)$ and the triangle inequality in C^α one gets

$$\begin{aligned}\mathbb{P}\Big(\big\|\int_0^{(.)} F(s)\,\mathrm{d}W(s)\big\|_{\alpha;[0,\tau]} \ge u\Big) &\le 2\mathbb{P}\Big(\big\|vW(.)\big\|_{\alpha;[0,T]} \ge u\Big) \le 2\mathbb{P}\Big(\big\|W(.)\big\|_{\alpha;[0,1]} \ge \frac{u}{v\sqrt{T}(T^{-\alpha}\vee 1)}\Big)\\ &\le 2k_1 \exp\Big\{-k_2\frac{u^2}{v^2T(T^{-\alpha}\vee 1)^2}\Big\},\end{aligned}$$

which yields the claim of the lemma. □

Remark: Alternatively, the previous lemma can be proved using the fact that each continuous local martingale starting at 0 can be represented as a time-changed Brownian motion.

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