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Existence and uniqueness of Stoneley waves

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SUMMARY

The basic existence-uniqueness theory for Stoneley waves propagating along a plane interface between different isotropic elastic media is re-examined, using a matrix formulation of the secular equation. The resulting development is appreciably simpler than previous treatments of the theory. The domain of existence of Stoneley waves and the limiting curves forming its outer boundary are characterized in terms of coordinates β_1^2/β_2^2 and μ_2/μ_1 where μ_1, μ_2 are the shear moduli and β_1, β_2 the speeds of transverse plane waves in the constituent media. The equations of the limiting curves are given explicitly and exemplified numerically.

Key words: interfacial waves, isotropic elastic media, limiting curves, secular equation.

1 INTRODUCTION

The possibility of transmitting a harmonic wave along the plane interface between dissimilar isotropic elastic solids was first considered by Robert Stoneley in 1924. He arrived at the secular equation governing the speed of propagation and showed, by examining special cases, that there are combinations of materials that admit an interfacial wave and others that do not. The common use of the term *Stoneley wave* recognizes the pioneering nature of his work.

The domain of existence of Stoneley waves was subsequently investigated by a number of authors. Sezawa & Kanai (1939) correctly characterized the limiting curves that bound this domain and carried out numerical calculations for two particular cases. However, the equations of the limiting curves are not recorded in their paper, no formal proof of existence is given and the question of the uniqueness of the interfacial wave is not addressed. Scholte (1942) gave an elaborate derivation of the equations of the limiting curves and proved that they bound the domain of existence. Again, no explicit proof of existence and no discussion of uniqueness were provided. In a later contribution, Scholte (1947) restated the equations of the limiting curves and gave some further information about them, including the transformation which maps one curve into the other. He also corrected and extended numerical results presented in his earlier work. Cagniard (1962, pp. 42–49) approached the basic theory of Stoneley waves from a different standpoint. Applying the principle of the argument to a complex function that reproduces the secular equation when evaluated on the real axis and equated to zero, he verified the existence of a unique interfacial wave within the domain bounded by the limiting curves and non-existence elsewhere.

The studies of Sezawa & Kanai, Scholte and Cagniard supply the ingredients for a complete account of the existence and uniqueness of Stoneley waves. The methods are recondite, however, and the details complicated. Not surprisingly, the topic is skimped in most texts on elastic-wave theory.

During the past 20 years considerable progress has been made in clarifying the behaviour of interfacial waves along the join of two *anisotropic* elastic bodies differing in composition or orientation. Notably, Barnett *et al.* (1985) have deduced an existence-uniqueness theorem for subsonic interfacial waves from properties of a Hermitian interface impedance matrix. These developments might be expected to yield, by specialization, a simplified treatment of the existence and uniqueness of Stoneley waves, and our present purpose is to confirm that this is indeed the case.

The line of argument runs as follows. After summarizing in Section 2 the solution of the displacement equations of motion describing a Stoneley wave, we introduce in Section 3 two real symmetric 2×2 matrices, M_1 and M_2, related to the surface impedance matrices of the abutting semi-infinite isotropic elastic bodies. The vanishing of the determinant of $M_1 + M_2$ reconstructs the Stoneley-wave secular equation. Definiteness properties of M_1 and M_2 are established in Section 4 and used in Section 5 to answer the fundamental questions of existence and uniqueness. Limiting curves represented by explicit equations are discussed in Section 6 and illustrative numerical results outlined.

2 THE STONELEY-WAVE SOLUTION

We are concerned with a composite medium consisting of two homogeneous semi-infinite isotropic elastic bodies, B_1 and B_2, which, on the plane interface, are in welded contact. The density and the shear modulus of the material composing B_1 are denoted by ρ_1 and μ_1 respectively and the speeds of propagation of longitudinal and transverse plane waves in this material by α_1 and β_1. The corresponding quantities for B_2 are ρ_2, μ_2 and α_2, β_2. The dimensionless material constants

$$\Lambda_i = \beta_i^2/\alpha_i^2$$

occur naturally in the ensuing theory, the values 1 and 2 of the subscript i referring throughout to B_1 and B_2. The bulk and shear moduli of both materials are taken to be positive, so that

$$\mu_i > 0, \qquad 0 < \Lambda_i < \tfrac{3}{4}. \tag{1}$$

Let $\mathbf{n}$ be a unit vector directed along the interface and $\mathbf{m}$ the unit normal to the interface pointing towards B_1 (see Fig. 1). Then a harmonic Stoneley wave in the composite medium gives rise to displacement fields

$$\left.\begin{aligned} \mathbf{u} &= A \exp\{ik(\mathbf{n}\cdot\mathbf{x} - v_S t)\}\{\exp(-kp_1\mathbf{m}\cdot\mathbf{x})(\mathbf{n} + ip_1\mathbf{m}) \\ &\quad - n_1 \exp(-kq_1\mathbf{m}\cdot\mathbf{x})(q_1\mathbf{n} + i\mathbf{m})\}, \\ \mathbf{u} &= mA \exp\{ik(\mathbf{n}\cdot\mathbf{x} - v_S t)\}\{\exp(kp_2\mathbf{m}\cdot\mathbf{x})(\mathbf{n} - ip_2\mathbf{m}) \\ &\quad + n_2 \exp(kq_2\mathbf{m}\cdot\mathbf{x})(q_2\mathbf{n} - i\mathbf{m})\} \end{aligned}\right\} \tag{2}$$

in B_1 and B_2 respectively. Here $\mathbf{x}$ is the position vector relative to an origin in the interface and t the time: k is the wave number and A a length that is arbitrary within the overriding restriction to infinitesimal deformations. The speed of propagation v_S is a positive real root of the secular equation

$$\begin{aligned} &\kappa^2\{1 - p_1(\gamma_1)q(\gamma_1)\}\{1 - p_2(\gamma_2)q(\gamma_2)\} \\ &\quad - 2\kappa[\rho_1\{1 - p_2(\gamma_2)q(\gamma_2)\} - \rho_2\{1 - p_1(\gamma_1)q(\gamma_1)\}]v^2 \\ &\quad + [(\rho_1 - \rho_2)^2 - \{\rho_1 p_2(\gamma_2) + \rho_2 p_1(\gamma_1)\}\{\rho_1 q(\gamma_2) \\ &\quad + \rho_2 q(\gamma_1)\}]v^4 = 0, \end{aligned} \tag{3}$$

in which $\kappa = 2(\mu_1 - \mu_2)$ and

$$p_i(\gamma_i) = (1 - \Lambda_i\gamma_i)^{1/2}, \qquad q(\gamma_i) = (1 - \gamma_i)^{1/2}, \tag{4}$$

with

$$\gamma_i = v^2/\beta_i^2 = \rho_i v^2/\mu_i. \tag{5}$$

The numbers p_i and q_i controlling the decay of the displacement with increasing distance from the interface are the values of $p_i(\gamma_i)$ and $q(\gamma_i)$ at $v = v_S$ and the remaining constants in eqs (2) are

$$\left.\begin{aligned} n_1 &= \frac{\kappa(1 - p_1q_2) - (\rho_1 - \rho_2)v_S^2}{\kappa(q_1 - q_2) + (\rho_1 q_2 + \rho_2 q_1)v_S^2}, \\ n_2 &= \frac{\kappa(1 - p_2q_1) - (\rho_1 - \rho_2)v_S^2}{\kappa(q_1 - q_2) + (\rho_1 q_2 + \rho_2 q_1)v_S^2}, \\ m &= \frac{-\kappa q_2(1 - p_1q_1) + \rho_1(q_1 + q_2)v_S^2}{\kappa q_1(1 - p_2q_2) + \rho_2(q_1 + q_2)v_S^2}, \end{aligned}\right\} \tag{6}$$

(*cf.* Chadwick 1976, Section 3). Eqs (3) and (6) stem from the interface conditions, requiring continuity of displacement and traction on the plane $\mathbf{m}\cdot\mathbf{x} = 0$. The six scalar conditions reduce to four because of the displacement being everywhere coplanar with $\mathbf{m}$ and $\mathbf{n}$.

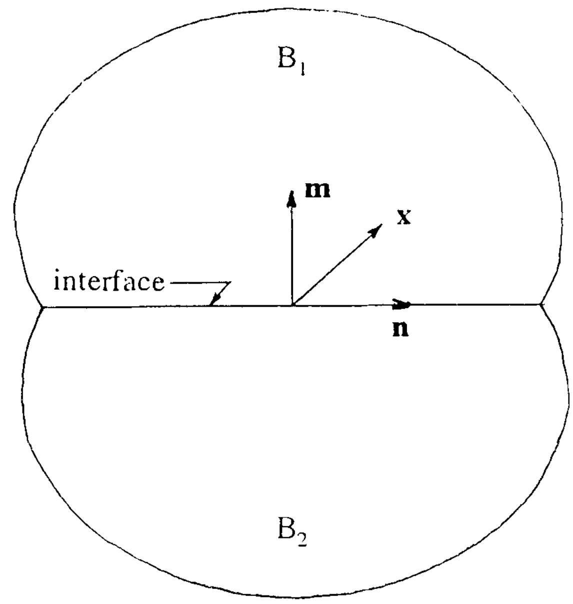

Figure 1. The composite isotropic elastic medium.

3 A REFORMULATION OF THE SECULAR EQUATION

We introduce the real symmetric matrices

$$M_i(\gamma_i) = \mu_i \begin{bmatrix} A_i(\gamma_i) & (-1)^{i+1}C_i(\gamma_i) \\ (-1)^{i+1}C_i(\gamma_i) & B_i(\gamma_i) \end{bmatrix} \tag{7}$$

and

$$M(v) = M_1(\gamma_1) + M_2(\gamma_2), \tag{8}$$

where

$$\left.\begin{aligned} A_i(\gamma_i) &= p_i(\gamma_i)P_i(\gamma_i), \qquad B_i(\gamma_i) = q(\gamma_i)P_i(\gamma_i), \\ C_i(\gamma_i) &= P_i(\gamma_i) - 2, \end{aligned}\right\} \tag{9}$$

$$P_i(\gamma_i) = \{1 - p_i(\gamma_i)q(\gamma_i)\}^{-1}\gamma_i, \tag{10}$$

(*cf.* Barnett *et al.* 1985, Section 4). Eqs (7) to (10) yield

$$\det M(v) = -\{1 - p_1(\gamma_1)q(\gamma_1)\}^{-1}\{1 - p_2(\gamma_2)q(\gamma_2)\}^{-1}S(v), \tag{11}$$

with

$$\begin{aligned} S(v) &= \mu_1^2\{1 - p_2(\gamma_2)q(\gamma_2)\}\{(2 - \gamma_1)^2 - 4p_1(\gamma_1)q(\gamma_1)\} \\ &\quad - \mu_1\mu_2[2\{2 - \gamma_1 - 2p_1(\gamma_1)q(\gamma_1)\} \\ &\quad \times \{2 - \gamma_2 - 2p_2(\gamma_2)q(\gamma_2)\} \\ &\quad + \gamma_1\gamma_2\{p_1(\gamma_1)q(\gamma_2) + p_2(\gamma_2)q(\gamma_1)\}] \\ &\quad + \mu_2^2\{1 - p_1(\gamma_1)q(\gamma_1)\}\{(2 - \gamma_2)^2 - 4p_2(\gamma_2)q(\gamma_2)\}. \end{aligned} \tag{12}$$

The result of setting $\rho_i v^2 = \mu_i\gamma_i$ in eq. (3), in accordance with the definitions (5), and invoking (12) is

$$S(v) = 0, \tag{13}$$

which is consequently the secular equation for Stoneley waves. This form was first given by Sezawa & Kanai (1939, eq. 1).

It is evident from the solution (2) that a Stoneley wave exists only if p_1, q_1, p_2 and q_2 are all real and positive. From (4), (5) and $(1)_2$,

$$0 < p_i(\gamma_i) < 1, \qquad 0 < q(\gamma_i) < 1 \quad \forall \quad 0 < v < \beta, \tag{14}$$

where

$$\beta = \min(\beta_1, \beta_2), \tag{15}$$

and at least one of $q(\gamma_i)$ is zero or imaginary for all $v \geq \beta$. The speed of propagation v_S therefore satisfies the inequalities

$$0 < v_S < \beta. \tag{16}$$

4 PROPERTIES OF $M_i(\gamma_i)$ AND $M(v)$

It is a simple matter to deduce from eqs (4) the identities

$$\{p_i(\gamma_i) + q(\gamma_i)\}\{1 - p_i(\gamma_i)q(\gamma_i)\} = \gamma_i\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\} \tag{17}$$

and

$$\{p_i(\gamma_i) - q(\gamma_i)\}\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}\{p_i^2(\gamma_i) - \Lambda_i q^2(\gamma_i)\} = (1 - \Lambda_i)^2\{1 - p_i(\gamma_i)q(\gamma_i)\}. \tag{18}$$

With the help of (17) the definition (10) can be recast as

$$P_i(\gamma_i) = \{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}^{-1}\{p_i(\gamma_i) + q(\gamma_i)\}. \tag{19}$$

It follows from eqs (7), (9), (19) and (4) that

$$M_i(0) = 2\mu_i(1 + \Lambda_i)^{-1}\begin{bmatrix} 1 & (-1)^i\Lambda_i \\ (-1)^i\Lambda_i & 1 \end{bmatrix}.$$

Owing to the inequalities (1), $M_i(0)$ are positive definite.

In order to differentiate the matrices $M_i(\gamma_i)$ we use the formulae

$$\left.\begin{aligned} p_i'(\gamma_i) &= -\Lambda_i\{2p_i(\gamma_i)\}^{-1}, \qquad q'(\gamma_i) = -\{2q(\gamma_i)\}^{-1}, \\ P_i'(\gamma_i) &= -(1 - \Lambda_i)^2\{2p_i(\gamma_i)q(\gamma_i)\}^{-1}\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}^{-2}, \end{aligned}\right\} \tag{20}$$

derived from eqs (4) and (19). Throughout this section a prime denotes differentiation with respect to the argument, γ_i or v. From eqs (7), (9), (19) and (20),

$$\begin{aligned} \operatorname{tr} M_i'(\gamma_i) &= \mu_i[\{p_i(\gamma_i) + q(\gamma_i)\}P_i'(\gamma_i) + \{p_i'(\gamma_i) + q'(\gamma_i)\}P_i(\gamma_i)] \\ &= -\mu_i\{2p_i(\gamma_i)q(\gamma_i)\}^{-1}\{p_i(\gamma_i) + q(\gamma_i)\} \\ &\quad \times [1 + (1 - \Lambda_i)^2\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}^{-2}], \end{aligned}$$

$$\begin{aligned} \det M_i'(\gamma_i) &= \mu_i^2[p_i'(\gamma_i)q'(\gamma_i)P_i^2(\gamma_i) \\ &\quad + \{p_i(\gamma_i)q'(\gamma_i) + p_i'(\gamma_i)q(\gamma_i)\}P_i(\gamma_i)P_i'(\gamma_i) \\ &\quad - \{1 - p_i(\gamma_i)q(\gamma_i)\}\{P_i'(\gamma_i)\}^2] \\ &= \mu_i^2\{4p_i(\gamma_i)q(\gamma_i)\}^{-1}\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}^{-2} \\ &\quad \times [2(1 - \Lambda_i)^2 + \Lambda_i\{p_i(\gamma_i) + q(\gamma_i)\}^2], \end{aligned} \tag{21}$$

use being made of the identity (18) in reaching (21). On account of the inequalities (1) and (14),

$$\operatorname{tr} M_i'(\gamma_i) < 0, \qquad \det M_i'(\gamma_i) > 0 \quad \forall \quad 0 < v < \beta,$$

and $M_i'(\gamma_i)$ are therefore negative definite in this interval.

The properties of $M_i(0)$ and $M_i'(\gamma_i)$ secured above imply, through eqs (8) and (5), that $M(0)$ is positive definite and $M'(v)$ negative definite for all $0 < v < \beta$. This means that the eigenvalues of $M(v)$, necessarily real, are positive at $v = 0$ and decrease monotonically in $(0, \beta)$ (*cf.* Chadwick & Smith 1977, Section VIII, A, 1). Eqs (7) to (9) and (19) give

$$\operatorname{tr} M(v) = \sum_{i=1}^{2} \mu_i\{p_i(\gamma_i) + \Lambda_i q(\gamma_i)\}^{-1}\{p_i(\gamma_i) + q(\gamma_i)\}^2,$$

and, by virtue of (1) and (14), the expression on the right is positive for all $0 < v < \beta$. At most one eigenvalue of $M(v)$ can vanish at most once, therefore, in the interval $(0, \beta)$.

5 EXISTENCE-UNIQUENESS CONSIDERATIONS

We infer from (11), (13) and (14) that a Stoneley wave exists if and only if an eigenvalue of $M(v)$ has a zero in $(0, \beta)$. It has been proved in Section 4 that the two eigenvalues of $M(v)$ decrease monotonically from positive values as v increases from 0 towards β and that at most one eigenvalue can pass through zero. A necessary and sufficient condition for the existence of a Stoneley wave is thus

$$\det M(\beta) < 0, \tag{22}$$

and such a wave is unique whenever it exists. We see from eqs (11) and (14) that the condition

$$S(\beta) > 0 \tag{23}$$

is equivalent to (22). Moreover, $S(v)$ has at most one zero, v_S, in $(0, \beta)$, so that

$$S(v)\begin{cases} <0 & \forall \quad 0 \leq v < v_S, \\ >0 & \forall \quad v_S < v \leq \beta, \end{cases} \tag{24}$$

when a Stoneley wave exists.

We introduce at this point the additional dimensionless quantities

$$\xi = \beta_1^2/\beta_2^2, \qquad \eta = \mu_2/\mu_1, \tag{25}$$

to be treated as coordinates in Section 6 below. Eq. (12), in conjunction with (4) and (5), gives

$$\mu_1^{-2}S(\beta_1) = a\eta^2 - b\eta + c, \qquad \mu_1^{-2}S(\beta_2) = c^*\eta^2 - b^*\eta + a^*, \tag{26}$$

where

$$a = (2 - \xi)^2 - 4(1 - \xi)^{1/2}(1 - \Lambda_2\xi)^{1/2}, \tag{27}$$

$$b = 2\{1 - (1 - \xi)^{1/2}\}^2 + (1 - \xi)^{1/2}[(1 - \Lambda_1)^{1/2}\xi + 4\{1 - (1 - \Lambda_2\xi)^{1/2}\}], \tag{28}$$

$$c = 1 - (1 - \xi)^{1/2}(1 - \Lambda_2\xi)^{1/2}, \tag{29}$$

and an asterisk signifies that the replacements

$$\xi \to \xi^{-1}, \qquad \Lambda_1 \to \Lambda_2, \qquad \Lambda_2 \to \Lambda_1 \tag{30}$$

have been made. Since

$$S(\beta) = \begin{cases} S(\beta_1) \text{ when } \beta_1 \leq \beta_2 \text{ (i.e. } 0 < \xi \leq 1), \\ S(\beta_2) \text{ when } \beta_1 \geq \beta_2 \text{ (i.e. } \xi \geq 1), \end{cases} \tag{31}$$

the condition (23) and eqs (26) lead immediately to the following.

Existence-uniqueness theorem

Given the material constants ρ_1, ρ_2 and μ_1, μ_2, the numbers ξ and η are defined by eqs (25), *with $\beta_i = (\mu_i/\rho_i)^{1/2}$, and a, b, c, a^*, b^*, c^* by* (27) *to* (30). *A Stoneley wave exists, and is unique, if and only if either*

$$0 < \xi \le 1, \ \ a\eta^2 - b\eta + c > 0 \ \ or \ \ \xi \ge 1, \ \ c^*\eta^2 - b^*\eta + a^* > 0.$$

If the bodies B_1 and B_2 are separated and their plane boundaries left traction free, the inequalities (1) are sufficient conditions for the existence of Rayleigh waves in B_1 and B_2. The speeds of propagation of these waves are $\beta_i\gamma_{Ri}^{1/2}$ where γ_{Ri} is the unique real root in (0, 1) of the secular equation

$$R_i(\gamma_i) := \gamma_i^{-1}\{(2-\gamma_i)^2 - 4p_i(\gamma_i)q(\gamma_i)\} = 0 \tag{32}$$

(e.g. Chadwick 1976, Section 2e). The properties

$$R_i(0) = -2(1-\Lambda_i), \qquad R_i(1) = 1,$$

$$R_i'(\gamma_i) = 1 + 2\{\gamma_i^2 p_i(\gamma_i)q(\gamma_i)\}^{-1}\{p_i(\gamma_i) - q(\gamma_i)\}^2,$$

drawn from eqs (32) and (4), show that when Λ_i is held fixed, $R_i(\gamma_i)$ increases monotonically as γ_i increases, from a negative value at $\gamma_i = 0$ to a positive value at $\gamma_i = 1$ (see Fig. 2).

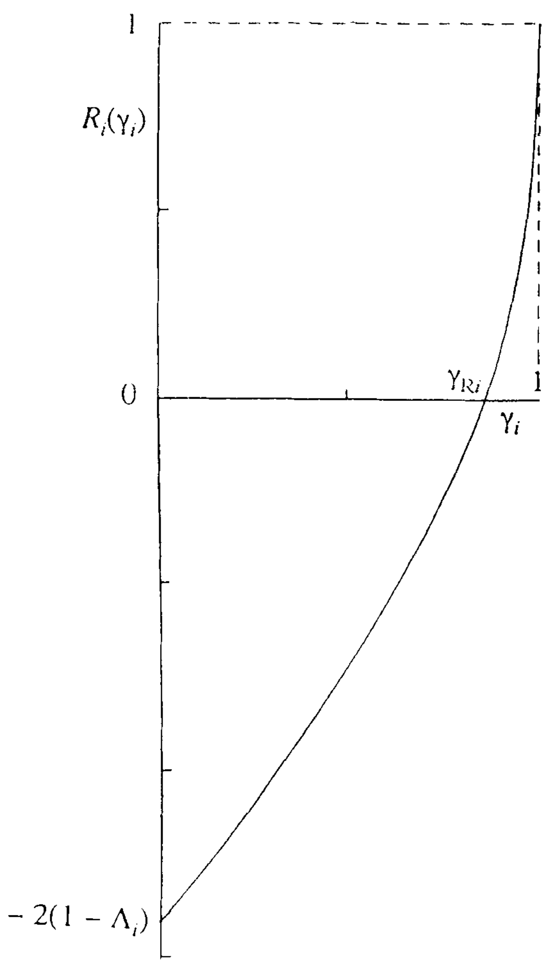

Figure 2. Variation with γ_i of the function $R_i(\gamma_i)$ defined by eqs (32) and (4). Here, $\Lambda_i = 0.3$.

With the aid of the definitions (32) and (4), eq. (12) can be written as

$$\begin{aligned} S(v) = {} & \mu_1^2\gamma_1\{1 - p_2(\gamma_2)q(\gamma_2)\}R_1(\gamma_1) \\ & - \mu_1\mu_2[2[\{1 - p_1(\gamma_1)q(\gamma_1)\}^2 + \Lambda_1\gamma_1 q^2(\gamma_1)] \\ & \times [\{1 - p_2(\gamma_2)q(\gamma_2)\}^2 + \Lambda_2\gamma_2 q^2(\gamma_2)] \\ & + \gamma_1\gamma_2\{p_1(\gamma_1)q(\gamma_2) + p_2(\gamma_2)q(\gamma_1)\}] \\ & + \mu_2^2\gamma_2\{1 - p_1(\gamma_1)q(\gamma_1)\}R_2(\gamma_2). \end{aligned} \tag{33}$$

Let $\{j, k\}$ be the permutation of $\{1, 2\}$ such that $\beta_j\gamma_{Rj}^{1/2} \le \beta_k\gamma_{Rk}^{1/2}$ and let

$$v_R = \beta_j\gamma_{Rj}^{1/2}.$$

This speed lies between 0 and β: otherwise we would have $\beta = \beta_k$ (as $v_R < \beta_j$) and

$$\beta_k \le v_R \le \beta_k\gamma_{Rk}^{1/2} < \beta_k,$$

which is impossible. When $v = v_R$, the coefficient of μ_j^2 in eq. (33) is zero, by (32). Since $\beta_k^{-2}v_R^2 \le \gamma_{Rk}$, it is seen from Fig. 2 that $R_k(\beta_k^{-2}v_R^2)$, and hence the coefficient of μ_k^2 in (33), is non-positive. The inequalities (1) and (14) ensure that the coefficient of $\mu_i\mu_j$ is negative, so that $S(v_R) < 0$. It follows from (24) that when a Stoneley wave exists in the composite body its speed of propagation exceeds v_R, the smaller of the Rayleigh-wave speeds in the constituent bodies B_1 and B_2 (*cf.* Barnett *et al.* 1985, Theorem 3). The bounds (16) on the Stoneley-wave speed can therefore be tightened to

$$v_R < v_S < \beta. \tag{34}$$

There is some confusion in the literature about the bounds (34). Koppe (1948, Section 2) asserted that v_S lies between the speeds of the Rayleigh wave and the transverse wave in the medium of greater acoustic density. This statement is repeated by Ewing, Jardetzky & Press (1957, p. 112) and misquoted by Owen (1964, Section 5), while Ginzbarg & Strick (1958, p. 53) replace the acoustic density by the density. Eqs (32), (4) and (5) show that the Rayleigh-wave speed in B_i depends on β_i and Λ_i, so no association between the lower bound and a single material constant can be generally valid.

6 LIMITING CURVES

It is clear from (26) to (30) that the equations $S(\beta_i) = 0$ represent curves in the (ξ, η) plane, and, in view of (31), the condition (23) implies that, for specified values of Λ_1 and Λ_2, the domain bounded by these curves in the first quadrant constitutes the set of all pairs (ξ, η) for which a Stoneley wave exists. We refer to

$$\left.\begin{aligned} L^-&: \ S(\beta_1) = 0, \qquad 0 \le \xi \le 1, \qquad \eta \ge 0, \\ L^+&: \ S(\beta_2) = 0, \qquad \xi \ge 1, \qquad \eta \ge 0, \end{aligned}\right\} \tag{35}$$

as the *limiting curves* (*cf.* Sezawa & Kanai 1939, Section 3).

Owing to eqs (26) being quadratic in η, each limiting curve consists of two arcs represented by the factors equated to zero. We write $L^\pm = L_-^\pm \cup L_+^\pm$ and the equations of the arcs, derived from (26) and (35), are

$$\left.\begin{aligned} L_\pm^-&: \ \eta = \eta_\pm^- = (2a)^{-1}\{b \pm (b^2 - 4ac)^{1/2}\}, \\ L_\pm^+&: \ \eta = \eta_\pm^+ = 2a^*\{b^* \mp (b^{*2} - 4a^*c^*)^{1/2}\}^{-1}, \end{aligned}\right\} \tag{36}$$

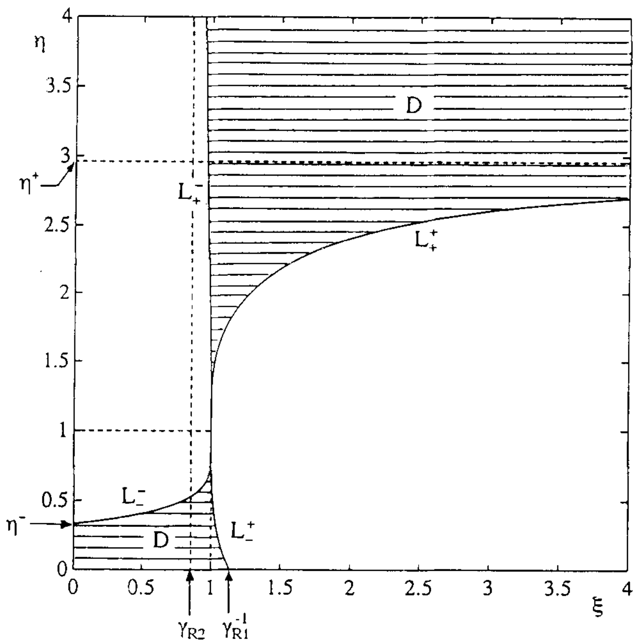

Figure 3. The limiting curves $L_\pm^-$ and $L_\pm^+$ and the domain of existence D in the (ξ, η) plane for a composite medium with dimensionless material constants $\Lambda_1 = 0.2$, $\Lambda_2 = 0.3$.

(*cf.* Scholte 1947, Section 2). It may readily be checked from eqs (36) and (26) to (29) that the point $\xi = 1$, $\eta = 1$ lies on each of $L_\pm^-$ and $L_\pm^+$, and it is confirmed below that the limiting curves have the forms shown in Fig. 3.

First, however, we verify that the interior of the region D, bounded by the limiting curves and hatched in Fig. 3, is the domain of existence of Stoneley waves. With reference to the condition (23) and eqs (31), it suffices to show that $S(\beta_1)$ and $S(\beta_2)$ are positive somewhere in the parts of D in which they are defined. From (26) to (30),

$$S(\beta_1)|_{\xi=1} = S(\beta_2)|_{\xi=1} = \mu_1^2(\eta - 1)^2,$$

so the positivity requirement is satisfied everywhere on the intersection of D with the line $\xi = 1$.

Returning now to the limiting curves, we consider first the arcs $L_\pm^-$. We find from eqs (27) to (29) that

$$\begin{aligned} b^2 - 4ac = {} & \xi(1-\xi)^{1/2}[4\xi(1-\Lambda_2\xi)^{1/2} \\ & \times \{1 - (1-\Lambda_1)^{1/2}(1-\xi)^{1/2}\} \\ & + (1-\Lambda_1)^{1/2}[(1-\Lambda_1)^{1/2}\xi(1-\xi)^{1/2} \\ & + 4(2-\xi)\{1-(1-\xi)^{1/2}(1-\Lambda_2\xi)^{1/2}\}]], \end{aligned} \quad (37)$$

$$\begin{aligned} b - 2a = {} & (1-\xi)^{1/2}[(1-\Lambda_1)^{1/2}\xi + 2\xi(1-\xi)^{1/2} \\ & + 4\{(1-\Lambda_2\xi)^{1/2} - (1-\xi)^{1/2}\}], \end{aligned} \quad (38)$$

$$\begin{aligned} b - a - c = {} & (1-\xi)^{1/2}[(1-\Lambda_1)^{1/2}\xi + \xi(1-\xi)^{1/2} \\ & + \{(1-\Lambda_2\xi)^{1/2} - (1-\xi)^{1/2}\}], \end{aligned}$$

and the inequalities $(1)_2$ guarantee that each of these expressions, and the right-hand sides of eqs (28) and (29), are positive for all $0 < \xi < 1$. From $(36)_1$ and (37), $\eta_\pm^-$ are therefore real in this interval, and since

$$\eta_-^- = 2c\{b + (b^2 - 4ac)^{1/2}\}^{-1},$$

$$1 - \eta_-^- = 2(b - a - c)\{b - 2a + (b^2 - 4ac)^{1/2}\}^{-1},$$

$0 < \eta_-^- < 1$. Eq. $(36)_1$ and (27) to (29), expanded to the first power in ξ, give the value of η_-^- at $\xi = 0$ as

$$\begin{aligned} \eta^- = {} & \{4(1-\Lambda_2)\}^{-1}[\{5 - \Lambda_1 + 4(1-\Lambda_1)^{1/2}\Lambda_2\}^{1/2} \\ & - (1-\Lambda_1)^{1/2} - 2\Lambda_2]. \end{aligned} \quad (39)$$

The arc L_-^- is thus confined to the square $0 \le \xi \le 1$, $0 \le \eta \le 1$ and has the form shown in Fig. 3. From eqs $(36)_1$, (28), (37) and (38),

$$\eta_+^- = (2a)^{-1}\{b + (b^2 - 4ac)^{1/2}\}$$

and

$$\eta_+^- - 1 = (2a)^{-1}\{b - 2a + (b^2 - 4ac)^{1/2}\}$$

have the same sign as a. Eqs (27) and (32) lead to $a = \xi R_2(\xi)$ whence, by inspection of Fig. 2, a is negative when $0 < \xi < \gamma_{R2}$ and positive when $\gamma_{R2} < \xi \le 1$. In the first quadrant of the (ξ, η) plane, L_+^- is therefore restricted to the strip $\gamma_{R2} < \xi \le 1$, $\eta \ge 1$ and approaches asymptotically the line $\xi = \gamma_{R2}$ (where $a = 0$) as $\eta \to \infty$, as shown in Fig. 3.

Eqs (36) indicate that the transformation (30) maps, by interchange of Λ_1 and Λ_2, the connection between η and ξ on $L_\mp^-$ into the connection between η^{-1} and ξ^{-1} on $L_\pm^+$. The forms of the latter arcs are thereby deducible from those of the former, already described. The arc L_-^+ lies in the rectangle $1 \le \xi \le \gamma_{R1}^{-1}$, $0 \le \eta \le 1$ and, corresponding to the asymptote $\xi = \gamma_{R2}$ of L_+^-, L_-^+ meets the ξ axis at $\xi = \gamma_{R1}^{-1}$. The arc L_+^+ lies in the quarter-plane $\xi \ge 1$, $\eta \ge 1$,

Table 1. Numbers defining the intercepts and asymptotes of the limiting curves for various values of Λ_1 and Λ_2.

Λ_1	Λ_2	γ_{R1}^{-1}	η^-	γ_{R2}	η^+
0	0	1.095744	0.309017	0.912622	3.236068
0	0.1	1.095744	0.312164	0.899137	3.162278
0	0.2	1.095744	0.315100	0.881076	3.085318
0	0.3	1.095744	0.317850	0.855931	3.004608
0	0.4	1.095744	0.320436	0.819359	2.919358
0	0.5	1.095744	0.322876	0.763932	2.828427
0.1	0.1	1.112177	0.319174	0.899137	3.133085
0.1	0.2	1.112177	0.321928	0.881076	3.059545
0.1	0.3	1.112177	0.324513	0.855931	2.982241
0.1	0.4	1.112177	0.326947	0.819359	2.900375
0.1	0.5	1.112177	0.329246	0.763932	2.812798
0.2	0.2	1.134975	0.329404	0.881076	3.035789
0.2	0.3	1.134975	0.331809	0.855931	2.961534
0.2	0.4	1.134975	0.334078	0.819359	2.882715
0.2	0.5	1.134975	0.336225	0.763932	2.798178
0.3	0.3	1.168319	0.339873	0.855931	2.942279
0.3	0.4	1.168319	0.341961	0.819359	2.866222
0.3	0.5	1.168319	0.343942	0.763932	2.784458
0.4	0.4	1.220467	0.350783	0.819359	2.850766
0.4	0.5	1.220467	0.352580	0.763932	2.771541
0.5	0.5	1.309017	0.362404	0.763932	2.759348

and, as counterpart to L_-^- intersecting the η axis at $\eta = \eta^-$, given by eq. (39), L_+^+ approaches asymptotically, as $\xi \to \infty$, the line $\eta = \eta^+$, where

$$\eta^+ = (1 + \Lambda_1)^{-1}[\{5 + 4\Lambda_1(1 - \Lambda_2)^{1/2} - \Lambda_2{}^{1/2} + 2\Lambda_1 + (1 - \Lambda_2)^{1/2}].$$

To illustrate the preceding analysis we have computed the limiting curves for the 21 pairs of values of Λ_1 and Λ_2 set out in the first two columns of Table 1. The results for the pair $\Lambda_1 = 0.2$, $\Lambda_2 = 0.3$ are displayed in Fig. 3 and the values of γ_{R1}^{-1}, η^- and γ_{R2}, η^+, specifying the intercepts of $L_-^\pm$ on the axes and the asymptotes to $L_+^\pm$, are listed for all 21 combinations in the last four columns of Table 1. It is clear from these entries that none of the 21 plots differs very much from Fig. 3, the proportional differences between the greatest and least values being only 19 per cent for γ_{R1}^{-1} and γ_{R2} and 17 per cent for η^- and η^+. In connection with this choice of data, it should be noted, first, that the pairs (Λ_1, Λ_2) and (Λ_2, Λ_1) are equivalent in the sense that relabelling B_1 as B_2 and B_2 as B_1 converts the latter into the former, and, second, that $\Lambda_i = \frac{1}{2}(1 - \nu_i)^{-1}(1 - 2\nu_i)$ where ν_i is Poisson's ratio. The extreme values $\Lambda_i = 0$ ($\nu_i = 0.5$) and $\Lambda_i = 0.5$ ($\nu_i = 0$) correspond to materials which are incompressible and transversely rigid respectively.

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