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Existence criteria for special locally conformally Kähler metrics

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Abstract
We investigate the relation between holomorphic torus actions on complex manifolds of locally conformally Kähler (LCK) type and the existence of special LCK metrics. We show that if the group of biholomorphisms of such a manifold (M, J) contains a compact torus which is not totally real, then there exists a Vaisman metric on the manifold, generalising a result of Kamishima–Ornea. Also, we obtain a new obstruction to the existence of LCK structures on a given complex manifold in terms of its automorphism group. As an application, we obtain a classification of manifolds of LCK type among all the manifolds having the structure of a holomorphic principal torus bundle. Moreover, we show that if the group of biholomorphisms contains a compact torus whose dimension is half the real dimension of M, then (M, J) admits an LCK metric with positive potential. Finally, we obtain new non-existence results for LCK metrics on certain products of complex manifolds.

Keywords Locally conformally Kähler metric · Vaisman metric · Torus action · Lee field

Mathematics Subject Classification 53A30 · 53C25 · 53B35

1 Introduction

Locally conformally Kähler (LCK) metrics are natural conformal analogues of Kähler metrics. Namely, a Hermitian metric on a complex manifold (M, J) with fundamental form Ω is LCK if $d\Omega = \theta \wedge \Omega$ for some closed form θ, called the Lee form. On the minimal cover of M on which the pullback of θ becomes exact, given by $p : \hat{M} \to M$ with $p^*\theta = d\varphi$, $\varphi \in \mathcal{C}^\infty(\hat{M}, \mathbb{R})$, there exists a global Kähler metric $\Omega_K = e^{-\varphi} p^*\Omega$, and $(\hat{M}, J, \Omega_K)$ is called the minimal Kähler cover of the LCK structure.

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Any LCK metric on a compact manifold of Kähler type is globally conformal to a Kähler metric ([27]). For this reason, we will always assume tacitly that our manifolds are not of Kähler type, in order to study only strict LCK metrics. In this setting, a first obstruction appears for manifolds of LCK type, namely: $0 < b_1 < 2h^{0,1}$, where $h^{0,1} = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M)$ and $b_1 = \dim_{\mathbb{R}} H^1(M, \mathbb{R})$ (see Sect. 2.2). As a matter of fact, this is the only cohomological obstruction known for a general LCK manifold. Vaisman had conjectured that such a manifold should always have b_{2k+1} odd for some $k \in \mathbb{N}$; however, this was disproved by the Oeljeklaus-Toma manifolds [16].

There are a few special LCK metrics which are better understood. The most important one is a *Vaisman metric*, defined by the condition $\nabla^g \theta = 0$, where ∇^g is the Levi–Civita connection determined by g. It can be seen that a Vaisman metric (Ω, θ) on (M, J) has the form

$$\Omega = -adJ\theta + \theta \wedge aJ\theta, \quad a \in \mathbb{R}_{>0}, \tag{1}$$

and the corresponding Kähler metric on $\hat{M}$ satisfies $\Omega_K = dd^c(a\mathrm{e}^{-\varphi})$. Thus Ω_K has a positive potential. This was first noted by Verbitsky [30], and as a consequence Ornea–Verbitsky [18] introduced and started the study of the more general notion of a *LCK metric with (positive) potential*. These are LCK metrics whose Kähler form on $\hat{M}$ satisfies

$$\Omega_K = dd^c(p^* f \mathrm{e}^{-\varphi}), \quad f \in \mathcal{C}^\infty(M, \mathbb{R}). \tag{2}$$

This class of metrics has the advantage of being closed under small deformations of the complex structure ([8,18]), while the Vaisman manifolds are not (see [1]). Even more general than this is the notion of an *exact LCK metric*, which is an LCK metric whose Kähler metric has the form:

$$\Omega_K = d(\mathrm{e}^{-\varphi}\eta), \quad \eta \in \mathcal{E}^1(M, \mathbb{R}). \tag{3}$$

The main objective of the present paper is to study the relation between the existence of special LCK metrics on a compact complex manifold and the group of biholomorphisms of the manifold. It turns out that this problem translates into certain properties of a compact torus acting homolorphically on the manifold. In order to give the precise statements, let us first specify these properties.

Let M be smooth compact manifold and let $\mathbb{T} \subset \mathrm{Aut}(M, J)$ be a compact torus with Lie algebra $\mathfrak{t} \subset \mathcal{C}^\infty(TM)$. Moreover, let $[\theta] \in H^1(M, \mathbb{R})$ and let $\hat{M}_{[\theta]}$ be the minimal cover of M on which θ becomes exact.

Definition 1 (a) We say that $\mathbb{T}$ is *horizontal* with respect to $[\theta]$ if the action of $\mathbb{T}$ on M lifts to an action of $\mathbb{T}$ on $\hat{M}_{[\theta]}$. Otherwise, we say that it is *vertical* with respect to $[\theta]$.
(b) If there exists a complex structure J on M so that $\mathbb{T} \subset \mathrm{Aut}(M, J)$, we say that $\mathbb{T}$ is *totally real* if $\mathfrak{t} \cap J\mathfrak{t} = 0$.

We then have the following results:

Theorem 1 *Let (M, J) be a compact complex manifold of LCK type, and let $\mathbb{T} \subset \mathrm{Aut}(M, J)$ be a torus which is not totally real. Then (M, J) admits a Vaisman metric.*

This result generalises a criterion of Kamishima and Ornea in [11] for deciding whether a given LCK conformal class is Vaisman or not, in terms of the presence of a complex Lie group with certain properties. As a corollary of the proof of our result, we obtain an obstruction to the existence of a general LCK metric:

Corollary 1 *Let (M, J) be a compact complex manifold. If there exists a torus $\mathbb{T} \subset \mathrm{Aut}(M, J)$ so that $\dim_{\mathbb{R}}(\mathfrak{t} \cap J\mathfrak{t}) > 2$ then (M, J) admits no LCK metrics.*

Moreover, we give an alternative proof (Sect. 4) of the following result of Ornea–Verbitsky, in which we construct explicitly a positive potential by means of an ODE:

Theorem 2 ([19] and [20]) *Let (M, J) be a compact complex manifold, and let $\tau \in H^1(M, \mathbb{R})$ be the de Rham class of a Lee form of an LCK structure on (M, J). If there exists $\mathbb{S}^1 \subset \mathrm{Aut}(M, J)$ which is vertical with respect to τ, then there exists $\theta \in \tau$ and Ω so that (Ω, θ) is an LCK structure with positive potential.*

As a corollary of these results and a previous one concerning toric LCK manifolds [10] we also obtain:

Theorem 3 *Let (M, J) be a compact complex manifold of complex dimension n. Suppose that the group of biholomorphisms of (M, J) contains an n-dimensional compact torus $\mathbb{T}^n$. Then, for any $\tau \in H^1(M, \mathbb{R})$ which is the class of a Lee form of an LCK metric, there exists an LCK metric with positive potential (Ω, θ) so that $\theta \in \tau$.*

We should note here that the dimension hypothesis on the torus is necessary, as the Inoue–Bombieri surface S^+ ([9]) admits an effective holomorphic action of $\mathbb{S}^1$, but no exact LCK metric ([22, Proposition 4.14]).

We also tackle the following problem, related to the above results. Any LCK metric (Ω, θ) defines two natural vector fields B and $A = JB$, the Lee and anti-Lee vector fields, via:

$$\iota_A \Omega = -\theta, \quad \iota_B \Omega = J\theta. \tag{4}$$

It is well known that, for a Vaisman metric, these vector fields are real holomorphic. It is natural to ask whether the converse holds, or under which conditions. In the recent paper [15], Ornea–Moroianu–Moroianu find additional properties ensuring that an LCK metric with real holomorphic Lee vector field is Vaisman, namely: if the metric has harmonic Lee form (i.e. is Gauduchon), or if B has constant norm. Moreover, they construct an example of a non-Vaisman LCK metric with real holomorphic Lee vector field. In the present paper, we show:

Proposition 1 *Let (M, J) be a compact complex manifold endowed with an LCK structure (Ω, θ) of the form (1), whose corresponding Lee vector field is real holomorphic. Then Ω is Vaisman.*

This criterion should be particularly useful when constructing examples, as it is easy to check. Moreover, we note that the metric constructed in [15] can be chosen with positive potential, so the above result is the sharpest statement one can get. Finally, let us mention that the example of non-Vaisman metric with holomorphic Lee field is constructed on a manifold of Vaisman type. Thus the question remains open whether a manifold admitting an LCK metric with holomorphic Lee field is of Vaisman type.

As a direct application of Theorem 1, we obtain in Sect. 7 a classification of manifolds of LCK type among all the manifolds having the structure of a holomorphic torus principal bundle. This is analogous to the result of Blanchard [2] in the Kähler context.

In the last part, we discuss the issue of irreducibility in LCK geometry. From early time [27], it was known that if one takes two compact Hermitian manifolds (M_i, g_i), $i = 1, 2$, the product metric is not LCK on $M_1 \times M_2$. However, whether there might exist some other LCK metric on $M_1 \times M_2$ has remained an open question, and in Sect. 8, we extend the known cases ([26, Corollary 3.3], [17, Corollary 2]) in which this fails. More precisely, we show that $M_1 \times M_2$ admits no LCK structure if M_1 is of Vaisman type (Theorem 7) or if M_1 is a Riemann surface and M_2 admits no LCK metrics with positive potential (Proposition 7).

2 LCK metrics

In this section, we fix a complex manifold (M, J). A metric g on (M, J) is *Hermitian* if $g(J\cdot, J\cdot) = g$. In this case, g induces a fundamental form $\Omega = g(J\cdot, \cdot)$ of bidegree $(1, 1)$ with respect to J. Conversely, a $(1, 1)$-form $\Omega \in \mathcal{E}^{1,1}(M, \mathbb{R})$ is called positive and we write $\Omega > 0$ if the symmetric tensor $\Omega(\cdot, J\cdot) =: g$ is positive definite, in which case g is a Hermitian metric. This one-to-one correspondence will be used implicitly throughout the present text.

We begin with the equivalent definitions of a locally conformally Kähler (LCK) metric. Let g be a Hermitian metric with fundamental form Ω on (M, J).

Definition 2 The metric g is called LCK if one of the following equivalent facts holds:

1. There exists a real closed one-form θ on M, called *the Lee form*, for which we have:
$$d\Omega = \theta \wedge \Omega. \tag{5}$$
2. M is covered by open sets $\{U_\alpha\}_{\alpha \in I}$ so that for each $\alpha \in I$ there exists a Kähler metric g_α on (U_α, J) and a real function $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R})$ so that:
$$g_{|U_\alpha} = \mathrm{e}^{\varphi_\alpha} g_\alpha. \tag{6}$$
3. There exists a Kähler metric Ω_K on the universal cover with the induced complex structure $\pi : (\tilde{M}, J) \to (M, J)$ on which $\pi_1(M)$, seen as the deck group of π, acts by homotheties:
$$\gamma^* \Omega_K = \rho(\gamma)^{-1} \Omega_K, \quad \gamma \in \pi_1(M), \quad \rho(\gamma) \in \mathbb{R}_{>0}. \tag{7}$$

It is not difficult to see that indeed all the above conditions are equivalent, and for the details, one can consult the monograph [4]. The Lee form is given on the open sets U_α by $\theta|_{U_\alpha} = d\varphi_\alpha$. The pullback of the Kähler metrics $\{g_\alpha\}_{\alpha \in I}$ to $\tilde{M}$ glue up to a global Kähler metric, which corresponds precisely to Ω_K. Moreover, it is easy to check that the constants given in (7) form a group morphism $\rho : \pi_1(M) \mapsto (\mathbb{R}_{>0}, \cdot), \gamma \mapsto \rho(\gamma)$. The kernel of ρ is a normal subgroup of $\pi_1(M)$, so one can consider $\Gamma := \pi_1(M)/\ker \rho$ and the corresponding Galois cover $p : \hat{M} \to M$ of deck group Γ. By definition, Ω_K is $\ker \rho$-invariant, so descends to a Kähler metric on $\hat{M}$.

In fact, $\hat{M}$ is the minimal cover of M on which the pullback of Ω is globally conformal to a Kähler metric. For this reason, we will call $(\hat{M}, \Omega_K)$ *the minimal Kähler cover* corresponding to Ω. Moreover, the pullback of θ becomes exact on $\hat{M}$ and one has:
$$p^*\Omega = \mathrm{e}^\varphi \Omega_K, \quad p^*\theta = d\varphi, \quad \varphi \in \mathcal{C}^\infty(\hat{M}, \mathbb{R}).$$

Note that the notion of an LCK metric is conformal in nature. Thus, any relevant definition concerning a general LCK structure should be conformally invariant. We will denote by $[\Omega] = \{\mathrm{e}^f \Omega | f \in \mathcal{C}^\infty(M, \mathbb{R})\}$ the conformal class of an LCK metric. Among the objects defined above, the de Rham class $[\theta]$, the morphism ρ and the half-line of Kähler metrics $\mathbb{R}_{>0}\Omega_K$ are indeed univoquely defined by the conformal class $[\Omega]$.

For later use, let us introduce the set of de Rham classes of Lee forms of LCK structures:
$$\mathcal{L}(M, J) := \left\{[\theta] \in H^1(M, \mathbb{R}) | \exists \Omega \in \mathcal{E}^{1,1}(M, \mathbb{R}),\ \Omega > 0,\ d\Omega = \theta \wedge \Omega\right\}.$$
We will say that (M, J) is of LCK type if $\mathcal{L}(M, J)$ is not empty.

In LCK geometry, an important role plays the differential operator:
$$d_\theta : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$$
$$d_\theta \eta = d\eta - \theta \wedge \eta.$$

This operator naturally appears when one is led to consider ρ^{-1}-equivariant forms on $\hat{M}$, such as the Kähler form. Indeed, equivariant forms are exactly pullbacks of forms from M multiplied by $\mathrm{e}^{-\varphi}$, and under this operation d on $\hat{M}$ corresponds to d_θ on M, as for any smooth form α on M, one has the relation:

$$d(\mathrm{e}^{-\varphi} p^*\alpha) = \mathrm{e}^{-\varphi} p^*(d_\theta \alpha).$$

In the same manner appears also the operator $d_\theta^c := d^c - J\theta \wedge \cdot$. A simple, but very useful fact is the following lemma, cf. [29, Proposition 2.1]:

Lemma 1 *Let M be a connected differentiable manifold and θ a real-valued closed 1-form on M. Then $d_\theta : \mathcal{C}^\infty(M) \to \mathcal{E}^1(M)$ is injective if and only if θ is not exact.*

Most of the special LCK metrics defined in the introduction can also be given equivalent definitions in terms of the operator d_θ.

Definition 3 Let (Ω, θ) be an LCK structure on (M, J).

(a) It is called *an exact LCK structure* if there exists a one-form η on M so that $\Omega = d_\theta \eta$. This is equivalent to (3).
(b) It is called *an LCK structure with potential* if there exists $f \in \mathcal{C}^\infty(M, \mathbb{R})$ so that $\Omega = d_\theta d_\theta^c f$. This is equivalent to (2). Moreover, it is called *with positive potential* if f can be chosen positive. The function f will be called *a θ-potential* of Ω.

Remark that these definitions are invariant by conformal transformations, so can also be used for conformal classes of metrics.

2.1 Vaisman metrics

On the other hand, Vaisman metrics cannot be defined by the operator d_θ alone, but it is true that they admit constant θ-potential. First of all, the Lee and anti-Lee vector fields A and B of a Vaisman structure (g, Ω, θ) on (M, J), defined by (4), have remarkable properties. The defining condition $\nabla^g \theta = 0$ is also equivalent to $\nabla^g B = 0$, as B is the metric dual of θ. This immediately implies that B (and so also A) is of constant norm. Moreover, it is not difficult to see that A and B are real holomorphic and Killing. Finally, this also implies that B is symplectic, which is equivalent to Ω admitting $f = \frac{1}{\|B\|^2} \in \mathbb{R}$ as a θ-potential:

$$\Omega = \frac{1}{\|B\|^2}(-dJ\theta + \theta \wedge J\theta) = d_\theta d_\theta^c \left(\frac{1}{\|B\|^2}\right).$$

Moreover, a Vaisman metric is a *Gauduchon metric*, meaning that its Lee form is d^*-closed (and thus harmonic), where d^* is the co-differential with respect to g. This is easy to see whether one writes $d^* = -\sum_{j=1}^{2n} \iota_{e_j} \nabla_{e_j}$, with $\{e_1, \dots e_{2n}\}$ a local orthonormal real basis of TM. In particular, a Vaisman metric inherits the property of Gauduchon metrics of being unique in their conformal class up to multiplication by a positive constant, cf. [6]. For this reason, we will usually normalise a Vaisman metric to verify $\|B\| = 1$, which then implies that $\Omega = d_\theta d_\theta^c 1$.

2.2 Cohomological obstructions to the existence of LCK metrics

Unlike the Kähler case, we lack cohomological or even topological obstructions to the existence of LCK structures. Note that an obvious one is the fact that $b_1 \neq 0$ for a strict LCK manifold.

The only other cohomological obstruction that we are aware of is a direct consequence of [5, Proposition 3] and [27, Theorem 2.1]. Indeed, Vaisman's proof implies that if (Ω, θ) is some strict LCK structure on a compact complex manifold (M, J), then the Bott–Chern class $[dJ\theta]_{BC} \in H^{1,1}_{BC}(M, \mathbb{R})$ cannot vanish. We recall here the definition of the real (1, 1)-Bott–Chern cohomology group:

$$H^{1,1}_{BC}(M, \mathbb{R}) := \frac{\left\{\alpha \in \mathcal{E}^{1,1}_M(M, \mathbb{R}) | d\alpha = 0\right\}}{i\partial\bar{\partial}(\mathcal{C}^\infty(M, \mathbb{R}))}.$$

Note on the other hand that there exists a natural map $F : H^{1,1}_{BC}(M, \mathbb{R}) \to H^{1,1}(M, \mathbb{R})$ and $[dJ\theta]_{BC} \in \ker F$, where:

$$H^{1,1}(M, \mathbb{R}) := \frac{\left\{\alpha \in \mathcal{E}^{1,1}_M(M, \mathbb{R}) | d\alpha = 0\right\}}{d(\mathcal{C}^\infty(M, \mathbb{R}))}.$$

But by [5, Proposition 3], we have:

$$\ker F \cong \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{R})}.$$

Thus, it follows that for a non-Kähler manifold of LCK type, we have the strict inequality:

$$0 < b_1 < 2h^{0,1}$$

also equivalent to the non-compactness of the Picard variety

$$\mathrm{Pic}_0(M) := \ker(c_1 : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z})) \cong \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})}.$$

3 The Lee vector field

It is easy to see that, if the Lee vector field of an LCK metric is Killing, then the metric is Vaisman. Moreover, in the compact setting, the same conclusion holds if the Lee vector field preserves the fundamental form, by a result of [14]. However, it is not true that the holomorphicity of the Lee vector field implies the Vaisman condition. It was recently shown:

Theorem 4 ([15]) *Let $(M, J, g, \Omega, \theta)$ be a compact LCK manifold with holomorphic Lee vector field. If B has constant norm, or if g is Gauduchon, then g is a Vaisman metric.*

Moreover, we have the following simple result, which holds also in the non-compact setting:

Proposition 2 *Let (M, J) be a complex manifold endowed with an LCK metric (Ω, θ) with constant θ-potential $\Omega = -adJ\theta + a\theta \wedge J\theta$, $a \in \mathbb{R}^*$. If Ω has real holomorphic Lee vector field B, then Ω is Vaisman.*

Proof Without loss of generality, we can suppose that $a = 1$. If B is real holomorphic, then also $A = JB$ is. The Cartan formula and $\mathcal{L}_A\theta = 0$ imply:

$$\begin{aligned} 0 = \mathcal{L}_A J\theta &= d\iota_A J\theta + \iota_A dJ\theta \\ &= -d(\theta(JA)) + \iota_A(\theta \wedge J\theta - \Omega) \\ &= d(\|B\|^2) - \theta\|B\|^2 + \theta \end{aligned}$$

$$= d_\theta(\|B\|^2 - 1).$$

Thus Lemma 1 implies that $\|B\|^2 = 1$. Using again Cartan's formula and the form of Ω, we obtain:

$$\begin{aligned}\mathcal{L}_B\Omega &= d\iota_B\Omega + \iota_B(\theta \wedge \Omega)\\ &= dJ\theta + \|B\|^2\Omega - \theta \wedge J\theta\\ &= -\Omega + \Omega = 0.\end{aligned}$$

Finally, since B preserves both the complex structure and the symplectic form, it also preserves the metric. This implies that $\nabla^g\theta$ is antisymmetric. We therefore obtain: $0 = d\theta = 2\nabla^g\theta$, i.e. g is Vaisman. □

In the paper [15], the authors also construct an example of an LCK metric which is not Vaisman, but which has real holomorphic Lee vector field, thus showing that one needs some additional hypotheses on Ω to ensure that it is Vaisman. We now present this example, with the remark that in the original construction, the metric can in fact be chosen with positive potential. This shows that the hypotheses in Proposition 2 cannot be relaxed.

Example 1 ([15]) Let (M, J, Ω, θ) be a compact Vaisman manifold with $\|\theta\|^2 = 1$, and let B be its Lee vector field. Suppose there exists a non-constant smooth function $f \in \mathcal{C}^\infty(M, \mathbb{R})$ verifying $f > -1$ everywhere on M and such that df is colinear with θ. After taking the interior product with B, this last condition is more precisely $df = B(f)\theta$. Such functions exist whenever B generates an $\mathbb{S}^1$-action on M, for instance on the standard Hopf manifold.

Consider next the form:

$$\Omega' := \Omega + f\theta \wedge J\theta = d_{(1+f)\theta}(-dJ\theta).$$

As $f > -1$, Ω' is a strictly positive real $(1, 1)$-form on M and verifies $d\Omega' = (1+f)\theta\wedge\Omega'$. Thus $(\Omega', \theta' = (1 + f)\theta)$ is an LCK structure with real holomorphic Lee field equal to B:

$$\iota_B\Omega' = (1 + f)J\theta = J\theta'.$$

As noted in [15], Ω' is not Vaisman as the norm of B is non-constant: $\Omega'(B, JB) = \theta'(B) = 1 + f$. In fact we have more:

Lemma 2 *The metric Ω' is not conformal to any Vaisman metric.*

Proof Suppose that there exists a Vaisman metric Ω'' on M so that $\Omega'' = e^h\Omega'$. By a result of K. Tsukada [25, Corollary 2.7], the Lee vector field of a Vaisman metric is unique on the manifold M up to multiplication by a constant. Thus, we can suppose right from the beginning that the Lee vector field of Ω'' is also B. Now this reads:

$$e^h J\theta' = e^h\iota_B\Omega' = \iota_B\Omega'' = J\theta' + d^c h$$

that is: $dh + \theta'(1 - e^h) = 0$, or also, after multiplying by $-e^{-h}$: $d_{\theta'}(e^{-h} - 1) = 0$. As θ' has no zero, it is non-exact, so Lemma 1 implies that $e^{-h} = 1$, i.e. $h = 0$ and Ω' is Vaisman. But this last fact is impossible as already noted. □

Lemma 3 *Suppose moreover that we have a diffeomorphism $M \cong N \times \mathbb{S}^1$ with N a smooth manifold, so that B is given by $B = \frac{d}{dt}$, with t a local coordinate on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and θ is the pullback of a one-form from $\mathbb{S}^1$. Then for any function $f \in \mathcal{C}^\infty(M)$ with $f > -1$ and df colinear with θ, $\Omega' = d_{(1+f)\theta}(-J\theta)$ is an LCK form with positive potential on M.*

Proof As df is colinear with θ, f is induced by a 2π-periodic function on $\mathbb{R}$. We are looking for a positive function $h : \mathbb{R} \to \mathbb{R}$, also 2π-periodic, verifying, when seen as a function on M:

$$\Omega' = d_{\theta'} d^c_{\theta'} h. \tag{8}$$

The function h verifies that both dh and $d\mathcal{L}_B h$ are colinear with θ, which implies the following relations:

$$dh = \mathcal{L}_B h \cdot \theta, \quad d^c h = \mathcal{L}_B h \cdot J\theta, \quad dd^c h = \mathcal{L}_B^2 h \cdot \theta \wedge J\theta + \mathcal{L}_B h \cdot dJ\theta.$$

With this in mind, (8) writes:

$$\begin{aligned} -dJ\theta + (1+f)\cdot\theta\wedge J\theta = (\mathcal{L}_B h - h(1+f))dJ\theta \\ + \left(\mathcal{L}_B^2 h - \mathcal{L}_B f \cdot h - 2(1+f)\mathcal{L}_B h + h(1+f)^2\right)\theta\wedge J\theta. \end{aligned}$$

Now, the two forms $-dJ\theta$ and $\theta \wedge J\theta$ are linearly independent, which implies that in the above equation, the corresponding coefficients preceding them must be equal. Seeing f and h as functions on $\mathbb{R}$, (8) now becomes equivalent to:

$$\frac{\mathrm{d}}{\mathrm{d}t}h - h(1+f) + 1 = 0 \tag{9}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}h - 2(1+f)\frac{\mathrm{d}}{\mathrm{d}t}h - h\frac{\mathrm{d}}{\mathrm{d}t}f + h(1+f)^2 - (1+f) = 0. \tag{10}$$

By differentiating the first equation, one obtains the second one, while the first ODE has a solution of the form:

$$h(t) = \left(c - \int_0^t \mathrm{e}^{-F(s)}\mathrm{d}s\right)\mathrm{e}^{F(t)}, \quad \text{with } F(t) = a + \int_0^t (f(s)+1)\mathrm{d}s, \quad a, c \in \mathbb{R}.$$

Thus a solution h of the above system exists, and now it is left for us to show that we can choose the constants a and c such that h is moreover strictly positive and 2π-periodic.

Let us note that, because f is 2π-periodic, we have, for any $t \in \mathbb{R}$:

$$F(t+2\pi) = F(t) + b, \quad \text{where } b = \int_0^{2\pi} (f(s)+1)\mathrm{d}s > 0.$$

Thus we obtain:

$$\begin{aligned} h(t+2\pi) &= \left(c - \int_0^{2\pi} \mathrm{e}^{-F(s)}\mathrm{d}s - \int_{2\pi}^{2\pi+t} \mathrm{e}^{-F(s)}\mathrm{d}s\right)\mathrm{e}^{F(t)}\mathrm{e}^b \\ &= \left(c - K - \int_0^t \mathrm{e}^{-F(u)}\mathrm{e}^{-b}\mathrm{d}u\right)\mathrm{e}^{F(t)}\mathrm{e}^b \\ &= h(t) + \mathrm{e}^{F(t)}((c-K)\mathrm{e}^b - c) \end{aligned}$$

where $K = \int_0^{2\pi} \mathrm{e}^{-F(s)} > 0$ and, for the second equality, we made the change of variable $s = u + 2\pi$. Thus, in order for h to be 2π-periodic, we take $c := \frac{K\mathrm{e}^b}{\mathrm{e}^b - 1} > 0$. Finally, we need to see that h is in fact positive, which is also equivalent to saying that $v(t) := c - \int_0^t \mathrm{e}^{-F(s)}ds$ is positive. Note that $\frac{d}{dt}v(t) = -\mathrm{e}^{-F(t)} < 0$, so v can change sign at most once, and the same is then true for the function h. On the other hand, h is periodic and $h(0) = c\mathrm{e}^a > 0$, thus h is indeed everywhere positive. □

Note that, although the above example shows that there can exist non-Vaisman metrics with holomorphic Lee vector field, it is however constructed out of a Vaisman metric. So one can still ask the following question:

Question 1 Let (M, J, Ω) be a compact LCK manifold with holomorphic Lee vector field. Does there exist an LCK metric on M, not necessarily conformal to Ω, which is Vaisman?

Also, recall that the Lee vector field of any Vaisman metric on a compact manifold is uniquely determined up to multiplication by a positive constant, by [25]. A related question is then:

Question 2 Suppose that the Lee vector field of an LCK metric on a compact manifold of Vaisman type is holomorphic. Is it then the Lee vector field of a Vaisman metric?

4 Existence of LCK metrics with positive potential

Let us start by reviewing the notion of a *vertical* action of a torus. For our discussion, it is enough to consider $\mathbb{S}^1$-actions. In what follows, we fix M a compact smooth manifold and $\tau \in H^1(M, \mathbb{R})$ a de Rham class. By the universal coefficient theorem, we can also view $\tau \in \mathrm{Hom}(\pi_1(M), \mathbb{R})$. Then $\ker \tau$ is a normal subgroup of $\pi_1(M)$, so we can take $\hat{M}_\tau := \tilde{M}/\ker \tau$, which is a normal cover of M. If $\theta \in \mathcal{C}^\infty(T^*M)$ is a smooth representative of τ, then $\hat{M}_\tau$ is the minimal cover of M on which θ becomes exact.

Suppose $\mathbb{S}^1$ acts on M with fundamental vector field C, and let Φ_t denote the corresponding 1-periodic flow. By averaging θ to an $\mathbb{S}^1$-invariant form: $\theta' := \int_0^1 \Phi_t^* \theta dt$, the de Rham class does not change, i.e. $[\theta'] = [\theta] = \tau$, so we can just suppose that θ is $\mathbb{S}^1$-invariant. Now we have $0 = \mathcal{L}_C \theta = d(\theta(C))$, so $\theta(C) = a \in \mathbb{R}$. Moreover, the value a only depends on the de Rham class τ: it is in fact τ evaluated on the homotopy class of an orbit of $\mathbb{S}^1$.

We have the following simple characterisation of vertical actions:

Lemma 4 *The action $\mathbb{S}^1$ is vertical for $\tau \in H^1(M, \mathbb{R})$ if and only if $\theta(C) \neq 0$ for some (and so any) $\mathbb{S}^1$-invariant representative $\theta \in \tau$.*

Proof In any case, C lifts to a vector field, also denoted by C, to $\hat{M}_\tau$, generating an $\mathbb{R}$-action on $\hat{M}$. Let us denote by $\hat{\Phi}_t$ the corresponding flow. We want to show the equivalence: $\hat{\Phi}_1 = \mathrm{id}_{\hat{M}_\tau} \Leftrightarrow \int_\gamma \tau \neq 0$, where $\gamma = [\mathbb{S}^1.x] \in \pi_1(M, x)$ is the homotopy class of an $\mathbb{S}^1$-orbit through an arbitrary point $x \in M$.

Denote by $\hat{\pi} : \hat{M}_\tau \to M$ the covering of deck group $\Gamma := \pi_1(M, x)/\ker \tau \subset \mathrm{Aut}(\hat{M}_\tau)$, and let $p : \pi_1(M, x) \to \Gamma$ be the natural projection. Then $\int_\gamma \tau \neq 0$ if and only if $p\gamma \neq id_{\hat{M}_\tau}$. But $p\gamma = \hat{\Phi}_1$: indeed, for any $\hat{x} \in \hat{\pi}^{-1}(x)$, the curve $[0, 1] \ni t \mapsto \hat{\Phi}_t(\hat{x})$ is the unique lift from $\hat{x}$ of the loop $[0, 1] \ni t \mapsto \Phi_t(x)$ representing γ. Thus the conclusion follows. □

Hence, given a de Rham class $\tau \in H^1(M, \mathbb{R})$, a torus action $\mathbb{T}^n$ on M lifts to an action of $\mathbb{T}^n$ on $\hat{M}_\tau$ if and only if $\mathfrak{Lie}(\mathbb{T}^n) \subset \ker \theta$ for any smooth $\mathbb{T}^n$-invariant closed one-form $\theta \in \tau$.

Proof of Theorem 2 The beginning of the proof is exactly as in [19]; however, we give it here for the sake of completeness. We recall that, by hypotheses, we have a compact LCK manifold (M, J, Ω, θ) and a vertical $\mathbb{S}^1$-action on M with respect to $\tau = [\theta]$. Let $(\hat{M}, J, \omega)$ be the minimal Kähler cover of $(M, J, [\Omega])$. We denote by D the real holomorphic vector field on

M generating the $\mathbb{S}^1$-action, as well as its lift to $\hat{M}$. By a standard average argument which does not change the de Rham class of θ, we can suppose that both Ω and θ are preserved by D. In particular, $\mathcal{L}_D\theta = 0$ implies, by Lemma 4, that $\theta(D) = \lambda \in \mathbb{R}^*$, as the action is vertical. Let $C := \frac{1}{\lambda}D$, so that $\theta(C) = 1$.

Let $\theta = d\varphi$ on $\hat{M}$, so that the Kähler form writes $\omega = \exp(-\varphi)\Omega$. Then we have:

$$\mathcal{L}_C\omega = -\theta(C)\omega = -\omega. \tag{11}$$

Let us denote by η the real one-form on $\hat{M}$ defined by $\iota_C\omega = \eta$. Then (11) together with Cartan's formula implies:

$$\omega = -(d\iota_C + \iota_C d)\omega = -d\eta. \tag{12}$$

At the same time, using the fact that $\eta(JC) = \omega(C, JC) = \|C\|^2_\omega := f$, we have:

$$\mathcal{L}_{JC}\eta = d\iota_{JC}\eta + \iota_{JC}d\eta = df - J\eta,$$

from which it follows:

$$\begin{aligned}\mathcal{L}_{JC}\omega &= -d(df - J\eta) = dJ\eta\\ \mathcal{L}^2_{JC}\omega &= dJ\mathcal{L}_{JC}\eta = dd^c f + d\eta = dd^c f - \omega.\end{aligned}$$

If we let Φ_t denote the one-parameter group generated by JC and denote by $\omega_t := \Phi_t^*\omega$ and by $f_t = \Phi_t^* f$, the last equation reads:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\omega_t = -\omega_t + dd^c f_t. \tag{13}$$

Let now h_t be the real-valued functions on $\hat{M}$ defined by the second-order linear differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}h_t + h_t = f_t, \quad h_0 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}h_t = 0. \tag{14}$$

We want to show that $\omega_t = \cos t\omega + \sin t dJ\eta + dd^c h_t$. For this, consider the forms $\beta_t := \omega_t - (\cos t\omega + \sin t dJ\eta + dd^c h_t)$, $t \in \mathbb{R}$. Using (13) and the definition (14) of the functions h_t, we have:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\beta_t &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\omega_t + \cos t\omega + \sin t dJ\eta - dd^c\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}h_t\right)\\ &= -\omega_t + dd^c f_t + \cos t\omega + \sin t dJ\eta - dd^c f_t + dd^c h_t\\ &= -\beta_t.\end{aligned}$$

Thus, the forms β_t verify the following homogeneous second-order linear differential equation with the initial conditions:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\beta_t + \beta_t = 0, \quad \beta_0 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\beta_t = 0.$$

By the uniqueness of the solution, we have then that for all $t \in \mathbb{R}$, β_t vanishes identically, and so:

$$\omega_t = \cos t\omega + \sin t dJ\eta + dd^c h_t, \quad t \in \mathbb{R}. \tag{15}$$

Define now, using (15), a new form $\hat{\omega}$ by:

$$\hat{\omega} := \frac{1}{2\pi}\int_0^{2\pi}\Phi_t^*\omega\mathrm{d}t = dd^c\frac{1}{2\pi}\int_0^{2\pi}h_t\mathrm{d}t$$

and let us denote by h the function $1/2\pi \int_0^{2\pi} h_t dt$. As $\{\Phi_t\}_{t\in\mathbb{R}}$ is a subgroup of biholomorphism of $\hat{M}$, $\hat{\omega}$ is a Kähler form on $\hat{M}$. We wish to show that h is a strictly positive function on $\hat{M}$.

Note first that, as $\theta(C) = 1$, C has no zeroes so the function f is everywhere positive. Moreover, as JC is real holomorphic, we have $[C, JC] = 0$, so Φ_t preserves both C and JC. This gives, for any $x \in \hat{M}$:

$$f_t(x) = \omega_{\Phi_t(x)}(C, JC) = \omega_{\Phi_t(x)}((d_x\Phi_t)C, (d_x\Phi_t)JC) = (\Phi_t^*\omega)_x(C, JC)$$

thus also the function f_t is strictly positive for any $t \in \mathbb{R}$.

Fix $x \in \hat{M}$ and define the functions $f_x, h_x : \mathbb{R} \to \mathbb{R}$ by $f_x(t) = f_t(x)$ and $h_x(t) = h_t(x)$. By (14), they satisfy:

$$h_x'' + h_x = f_x, \quad h_x(0) = 0, \quad h_x'(0) = 0. \tag{16}$$

Then we have:

$$\int_0^{2\pi} h_x(t)\mathrm{d}t = \int_0^{2\pi} f_x(t)\mathrm{d}t - \int_0^{2\pi} h_x''(t)\mathrm{d}t = \int_0^{2\pi} f_x(t)\mathrm{d}t - h_x'(2\pi). \tag{17}$$

On the other hand, integrating by parts and using (16) we compute:

$$\begin{aligned} h_x'(2\pi) &= h_x'(t)\cos t\Big|_0^{2\pi} \\ &= \int_0^{2\pi} h_x''(t)\cos t\mathrm{d}t + \int_0^{2\pi} h_x'(t)(-\sin t)\mathrm{d}t \\ &= \int_0^{2\pi} h_x''(t)\cos t\mathrm{d}t - \left(h_x(t)\sin t\Big|_0^{2\pi} - \int_0^{2\pi} h_x(t)\cos t\mathrm{d}t\right) \\ &= \int_0^{2\pi} f_x(t)\cos t\mathrm{d}t. \end{aligned}$$

Thus, it follows from (17):

$$\int_0^{2\pi} h_x(t)\mathrm{d}t = \int_0^{2\pi} f_x(t)(1-\cos t)\mathrm{d}t > 0$$

implying that the function h is indeed everywhere positive.

Hence we can define $\hat{\theta} := d\ln h$. Note that by the uniqueness of the solution of (14), the functions h_t have the same Γ-equivariance as the functions f_t, or also as the function f. Here, Γ denotes the deck group of the cover $\hat{M} \to M$. Also we should note that, as C and JC are Γ-invariant, being lifts of vector fields from M, then the Γ-equivariance of $f = \omega(C, JC)$ is exactly the equivariance of ω. Thus it follows that $\hat{\theta}$ has the same Γ-equivariance as θ, and so the two one-forms are cohomologous. Hence the form

$$\hat{\Omega} := h^{-1}dd^c h \tag{18}$$

descends to M to an LCK metric with positive potential with Lee form $\hat{\theta}$, and the proof is finished. □

Remark 1 Let us note that the above construction of an LCK metric with potential is natural and only depends on Ω and on C. In particular, if Ω is already JC-invariant, which will imply that the metric is Vaisman, then we have $f_t = f$ and the solution of (14) is then $h_t = (1-\cos t)f$, so in particular the potential $h = f$ remains unchanged.

Remark 2 On the other hand, for a metric Ω which is not JC-invariant, the above construction gives us a countable set of metrics with potential associated to the de Rham class of θ. Indeed, we considered the potential $h_{[1]} := h$, but for any $n \in \mathbb{N}^*$, the potential $h_{[n]} := 1/2n\pi \int_0^{2n\pi} h_t dt$ works as well.

5 Existence of Vaisman metrics

In this section, we are interested in giving a proof of Theorem 1 and of Corollary 1. We start by giving the main proposition, which will directly imply the general criterion.

Let (M, J, Ω, θ) be a compact Vaisman manifold with corresponding fundamental vector fields B and $A = JB$. Then $A, B \in \mathfrak{aut}(M, J, \Omega)$ generate a holomorphic $\mathbb{R}^2$ action on M, and we will denote by G the image of $\mathbb{R}^2$ in $\mathrm{Aut}(M, J, \Omega)$. Since the Lie group $\mathrm{Aut}(M, J, \Omega)$ is compact, we can take the closure of G in it, obtaining thus a compact torus $\mathbb{T} \subset \mathrm{Aut}(M, J, \Omega)$. The torus $\mathbb{T}$ is not totally real, since both A and B are in $\mathfrak{t} \cap J\mathfrak{t}$. In fact, we have:

Proposition 3 *Let $(M, J, [\Omega], [\theta]_{dR})$ be a compact complex manifold endowed with a strict LCK structure and let $\mathbb{T} \subset \mathrm{Aut}(M, J, [\Omega])$ be a compact torus. If $\mathbb{T}$ is not totally real, then $[\Omega]$ is Vaisman and $\mathfrak{t} \cap J\mathfrak{t} = \mathbb{R}\{A, B\}$, where $B = -JA$ is the Lee vector field of some Vaisman metric in $[\Omega]$.*

Proof Choose a $\mathbb{T}$-invariant LCK structure (Ω, θ) in the conformal class $[\Omega]$, so that for any $X \in \mathfrak{t}$, $d(\theta(X)) = \mathcal{L}_X\theta = 0$. Let $0 \neq C \in \mathfrak{t}$ with $D := JC \in \mathfrak{t}$. Then both $\theta(C)$ and $\theta(D)$ are constant. However, we cannot have $\theta(C) = \theta(D) = 0$. Indeed, if it was the case, then:

$$\begin{aligned} 0 &= \iota_{[C,D]}\Omega = \mathcal{L}_C\iota_D\Omega - \iota_D\mathcal{L}_C\Omega \\ &= d\iota_C\iota_D\Omega + \iota_C d\iota_D\Omega = \\ &= d(-\|C\|^2) + \theta\iota_C\iota_D\Omega = d_\theta(-\|C\|^2) \end{aligned}$$

implying, by Lemma 1, that $\|C\|^2 = 0$, contradiction. Hence, if $\theta(C) = a$ and $\theta(D) = b$, then $X := aD - bC \neq 0$ still verifies $X \in \mathfrak{t}$ and $JX \in \mathfrak{t}$ and, moreover, $\theta(X) = 0$, so $\theta(JX) \neq 0$. Therefore, we can suppose from the beginning that $\theta(C) = 1$ and $\theta(D) = 0$.

Let $f := \|C\|^2_\Omega$, which is an everywhere positive function since C cannot have any zeros. Take $\Omega' := \frac{1}{f}\Omega$, with corresponding Lee form $\theta' = \theta - d\ln f$. Then, since f is preserved by both C and D, we still have $\theta'(C) = 1$ and $\theta'(D) = 0$, and $C, D \in \mathfrak{aut}(M, J, \Omega')$.

Let $\eta := \iota_C\Omega'$. Then we have:

$$d\eta = \mathcal{L}_C\Omega' - \iota_C d\Omega' = -\theta'(C)\Omega' + \theta' \wedge \eta$$

or also $\Omega' = d_{\theta'}(-\eta)$. Since D preserves both C and Ω', it also preserves η. Moreover, we have $1 = \|C\|^2_{\Omega'} = \eta(D)$. Hence we get:

$$\begin{aligned} 0 &= \mathcal{L}_D\eta = d\iota_D\eta + \iota_D d\eta = \iota_D(-\Omega' + \theta' \wedge \eta) \\ &= -J\eta + \theta'(D)\eta - \theta'\eta(D) = -J\eta - \theta'. \end{aligned}$$

Finally, this implies that $\eta = J\theta'$, so that C is actually the Lee vector field B of Ω'. Since C is real holomorphic and preserves Ω', it is also Killing, so $2\nabla\theta' = d\theta' = 0$, that is, Ω' is Vaisman.

Finally, since a Vaisman metric is unique in its conformal class up to multiplication by constants, it follows that $\mathfrak{t} \cap J\mathfrak{t} = \mathbb{R}\{C, D\} = \mathbb{R}\{A, B\}$. □

Note that this immediately implies Corollary 1.

Proof of Theorem 1 As we already noted at the beginning of the section, if M admits a Vaisman metric then the corresponding real holomorphic vector fields B and $A = JB$ sit in the Lie algebra of a torus in $\mathrm{Aut}(M, J)$.

Conversely, suppose $\mathbb{T} \subset \mathrm{Aut}(M, J)$ is not totally real. Take any LCK metric (Ω, θ) and average it over $\mathbb{T}$, in order to get a $\mathbb{T}$-invariant LCK metric. Hence we have $\mathbb{T} \subset \mathrm{Aut}(M, J, \Omega)$, and we can apply Proposition 3 in order to get the conclusion. □

6 Maximal torus actions

The main goal of this section is to give a proof of Theorem 3, as a consequence of the previous results, together with our result concerning toric LCK manifolds of [10].

Let $(M, J, [\Omega], [\theta])$ be a compact LCK manifold. There are two natural Lie algebras of vector fields one can consider in this context, which we present next.

Definition 4 A vector field $X \in \Gamma(TM)$ is called *horizontal* for $([\Omega], [\theta])$ if $\mathcal{L}_X \Omega = \theta(X)\Omega$ for some (and hence any) form $\Omega \in [\Omega]$. We denote by $\mathfrak{aut}'(M, [\Omega])$ the set of horizontal vector fields, and note that:

$$\mathfrak{aut}'(M, [\Omega]) \subset \mathfrak{aut}(M, [\Omega]) := \left\{X \in \Gamma(TM) | \mathcal{L}_X \Omega = f_X \Omega, \quad f_X \in \mathcal{C}^\infty(M)\right\}.$$

Moreover, $\mathfrak{aut}'(M, [\Omega])$ inherits the structure of a Lie subalgebra of $\Gamma(TM)$.

Remark 3 If a vector field generates an $\mathbb{S}^1$-action and is a horizontal vector field for $([\Omega], [\theta])$, then the $\mathbb{S}^1$-action is horizontal with respect to $[\theta]$ in the sense of Definition 1.

Inside $\mathfrak{aut}'(M, [\Omega])$ there is another natural Lie algebra, namely the one given by twisted Hamiltonian vector fields.

Definition 5 A vector field $X \in \Gamma(TM)$ is called *twisted Hamiltonian* for $([\Omega], [\theta])$ if for some (and hence any) representative $\Omega \in [\Omega]$, there exists $f_X \in \mathcal{C}^\infty(M)$ so that $\iota_X \Omega = d_\theta f_X$. Twisted Hamiltonian vector fields form a Lie subalgebra of horizontal vector fields, denoted by $\mathfrak{ham}(M, [\Omega])$.

Note that these definitions are conformally invariant. The claims that are made above are easy to check, but for the complete proofs and for a motivation of these definitions, one can consult the paper of Vaisman [29], where they were first considered.

In particular, we call an action of a connected Lie group G on $(M, J, [\Omega])$ twisted Hamiltonian if $\mathfrak{Lie}(G) \subset \mathfrak{ham}(M, [\Omega])$. Moreover, if n is the complex dimension of M, the manifold $(M, J, [\Omega])$ together with a torus $\mathbb{T}^n$ that acts on the manifold effectively by biholomorphisms and in a twisted Hamiltonian way is called a *toric LCK manifold*. These kind of manifolds were studied recently in [13,23] and [10].

Remark 4 It was shown in [13, Proposition 3.9], although not explicitly stated, that the orbits of a horizontal torus action of $\mathbb{T}^n$ on $(M, [\Omega])$ are isotropic for Ω. The main point is that, if (Ω, θ) is $\mathbb{T}^n$-invariant, then for any $X, Y \in \mathfrak{Lie}(\mathbb{T}^n) \subset \mathfrak{aut}(M)$, as $X, Y \subset \ker\theta$, we have:

$$0 = \iota_{[X,Y]}\Omega = \mathcal{L}_X \iota_Y \Omega - \iota_Y \mathcal{L}_X \Omega = d\iota_X \iota_Y \Omega + \iota_X d\iota_Y \Omega = d_\theta(\Omega(Y, X)).$$

This implies, by Lemma 1, that $\Omega(Y, X) = 0$.

It is not difficult to see that if $[\Omega]$ is exact in the sense of Definition 3, then horizontal actions of compact tori coincide with twisted Hamiltonian ones, see the above references for details. Also, as shown in [13, Lemma 3.7], this is also the case if $\hat{M}$ is simply connected. In fact, when the dimension of the torus is maximal, we do not need any hypothesis for this equivalence to hold. The proof of this follows the lines of the one from [10], for this matter we will skip some of the details:

Theorem 5 *Let $(M, J, [\Omega])$ be a compact LCK manifold of complex dimension n and let $\mathbb{T}^n$ be a torus that acts effectively by biholomorphisms on the manifold. Then the action is twisted Hamiltonian if and only if it is horizontal.*

Proof Clearly, we only need to show the if direction, so let us suppose that the action is horizontal. Let us fix $\Omega \in [\Omega]$ which is $\mathbb{T}^n$ invariant, with Lee form θ, so that $\mathfrak{t} := \mathfrak{Lie}(\mathbb{T}^n) \subset \ker\theta$. Here and in all that follows, we identify $\mathfrak{t}$ with a Lie subalgebra of $\mathfrak{aut}(M, J)$. Also let $(\hat{M}, J, \Omega_K)$ be the corresponding minimal Kähler cover. Then, by Lemma 4, we have a lifted action by biholomorphisms of $\mathbb{T}^n$ on $(\hat{M}, J)$, and as it is not difficult to see, this action is also symplectic with respect to Ω_K.

By the principal orbit theorem, see for instance [3], there exists a dense connected open subset $M_0 \subset M$ on which $\mathbb{T}^n$ acts locally freely. Moreover, as $\mathbb{T}^n$ is abelian and acts effectively on M, it acts in fact freely on M_0. The preimage $\hat{M}_0$ of M_0 in $\hat{M}$ is exactly the dense connected open subset of $\hat{M}$ on which $\mathbb{T}^n$ acts freely.

Now the proof of Proposition 3 shows that any horizontal torus is totally real, i.e. $\mathfrak{t} \subset \ker\theta$ implies that $\mathfrak{t} \cap J\mathfrak{t} = \{0\}$. Thus we have a complex linear injection $\mathfrak{t} \oplus J\mathfrak{t} \to \mathfrak{aut}(M, J)$ generating a holomorphic action of $\mathbb{T}^c := (\mathbb{C}^*)^n$ on M and on $\hat{M}$. As for any $\xi \in \mathfrak{t}$, ξ has no zeroes on M_0, and as at any point of M_0, $\mathfrak{t}$ is orthogonal to $J\mathfrak{t}$ with respect to the metric $g := \Omega(\cdot, J\cdot)$ by Remark 4, it follows that the action of $\mathbb{T}^c$ is locally free on M_0, and so also on $\hat{M}_0$.

Let us fix $x \in \hat{M}_0$ and let $H = \{a \in \mathbb{T}^c | a.x = x\}$, which by the above discussion is a closed discrete subgroup of $\mathbb{T}^c$. We have a holomorphic embedding $F : \mathbb{T}^c/H \to \hat{M}_0$, $a \mapsto a.x$. As $\dim_{\mathbb{C}} \mathbb{T}^c/H = \dim_{\mathbb{C}} \hat{M}_0$, F must be an open embedding, and as $\hat{M}_0$ is connected, F is thus a biholomorphism. In particular, as $\hat{M}_0$ is dense in $\hat{M}$, H acts trivially on the whole of $\hat{M}$, so we have a well-defined effective action of $\mathbb{T}^c/H$ on $\hat{M}$.

Now, as Γ commutes with $\mathbb{T}^c$ and preserves $\hat{M}_0$, it follows easily that $\Gamma \subset \mathbb{T}^c/H$. Thus, given $\mathrm{id}_{\hat{M}} \neq \gamma \in \Gamma$, there exists a subgroup $\mathbb{R} \cong G \subset \mathbb{T}^c/H$ containing $< \gamma >$ as a subgroup. G acts by biholomorphisms on $\hat{M}$ and this action clearly commutes with Γ, so descends to an effective $\mathbb{S}^1$ action on M. By definition, this action is vertical with respect to $[\theta]$. One can average (Ω, θ) over this $\mathbb{S}^1$-action to obtain an exact $\mathbb{T}^n$-invariant LCK metric, which is in particular toric for the given action of $\mathbb{T}^n$. This is exactly the construction of [10, Lemma 5.1], where the details can be found.

Finally, by applying the result of [10], it follows that there exists a Vaisman structure on (M, J) with Lee class $[\theta]$. But by [12, Theorem 4.5] any d_θ-closed form on M is d_θ-exact. Therefore, any LCK form on (M, J) is exact, and so the torus action for the initial LCK form $[\Omega]$ was twisted Hamiltonian. □

Now, summing up, we get:

Proof of Theorem 3 Let $\mathbb{T}^n \subset \mathrm{Aut}(M, J)$, let $\tau \in \mathcal{L}(M, J)$ and let (Ω, θ) be a $\mathbb{T}^n$-invariant LCK structure with $\theta \in \tau$. If $\mathfrak{t} \subset \ker\theta$, then the above result implies that (M, J) admits a Vaisman structure (Ω', θ') with $\theta' \in \tau$.

If not, then identify $\mathbb{T}^n$ with $(\mathbb{S}^1)^n$ and let $\xi_1, \ldots, \xi_n$ be the fundamental vector fields generating each of the $\mathbb{S}^1$-actions on M. As θ does not vanish on the whole of $\mathfrak{t}$, there exists at least one $\xi = \xi_k$, $k \in \{1, \dots, n\}$ generating a vertical $\mathbb{S}^1$-action with respect to $[\theta]$. Thus, by applying Theorem 2, it follows that θ is the Lee form of an LCK metric with positive potential. □

Remark 5 All hypotheses in Theorem 5 are necessary. Indeed, on the one hand, the non-diagonal Hopf surfaces provide examples of LCK manifolds admitting an effective holomorphic action of a torus of dimension 2, but which are not toric, by [13, Theorem 7.2]. This shows that we need to impose the action of the torus to be horizontal in our result.

On the other hand, consider the Inoue surfaces of type S_t^+ ([9]) with $t \in \mathbb{R}$. These admit an LCK structure (Ω, θ), by [24]. As noted in [21, Example 5.7], these surfaces have a $[\theta]$-horizontal holomorphic action of $\mathbb{S}^1$ which is not twisted Hamiltonian. This example then shows that we need also to impose the dimension hypothesis on the torus in Theorem 5.

7 Holomorphic torus principal bundles

Let $\mathbf{T} = \mathfrak{t}/\Lambda$ be a compact complex torus of dimension n, let N be a compact complex manifold and let $\pi : M \to N$ be a holomorphic $\mathbf{T}$-principal bundle over N. Its Chern class is an element:

$$c^{\mathbb{Z}}(\pi) \in H^2(N, \Lambda) \cong H^2(N, \mathbb{Z}) \otimes \Lambda.$$

The inclusion $\Lambda \subset \mathfrak{t}$ induces a natural map $H^2(N, \Lambda) \to H^2(N, \mathfrak{t}) \cong H^2(N, \mathbb{C}) \otimes \mathfrak{t}$, and we will denote by $c(\pi)$ the image of $c^{\mathbb{Z}}(\pi)$ under this map. The class $c(\pi)$ has a well-defined rank. If we choose $\mathbb{C}$-basis for both $\mathfrak{t}$ and $H^2(N, \mathbb{C})$, then $c(\pi)$ can be represented by a $2n \times b_2(N)$ matrix over $\mathbb{C}$, and then the rank of $c(\pi)$ is the rank of this matrix.

Note that if the rank of $c(\pi)$ is 1, then there exists a minimal element $a \in \Lambda$, unique modulo sign, such that the non-torsion part of $c^{\mathbb{Z}}(\pi)$ writes $c^{\mathbb{Z}}(\pi)_0 = c_1^{\mathbb{Z}}(\pi) \otimes a$ with $c_1^{\mathbb{Z}}(\pi) \in H^2(N, \mathbb{Z})$. If $c_1(\pi)$ is the image of $c_1^{\mathbb{Z}}(\pi)$ under $H^2(N, \mathbb{Z}) \to H^2(N, \mathbb{C})$, then we will have $c(\pi) = c_1(\pi) \otimes a$, and again $c_1(\pi)$ is uniquely defined modulo sign. So it makes sense to ask weather $c_1(\pi)$ is a positive or negative class, i.e. weather $c_1(\pi)$ or $-c_1(\pi)$ can be represented by a Kähler form on N. In the affirmative case, we will call the class $c(\pi)$ *definite*.

By a theorem of Blanchard [2], when N is of Kähler type, M carries a Kähler metric if and only if the rank of $c(\pi)$ is 0. On the other hand, a theorem of Vuletescu [31] states that if $n = 1$ and the rank of $c(\pi)$ is 2, then M cannot admit LCK metrics.

As a direct application of our existence criterion for Vaisman metrics and of Corollary 1, we obtain a characterisation of manifolds of LCK type among all the compact torus principal bundles over compact complex manifolds.

Proposition 4 *Let* $\mathbf{T}$ *be a complex compact* n*-dimensional torus and* $\pi : M \to N$ *be a* $\mathbf{T}$*-principal bundle over a compact complex manifold* N*. Then* M *admits a strict LCK metric if and only if* $n = 1$ *and the Chern class of* π *is of rank* 1 *and definite. In this case,* M *is of Vaisman type.*

Proof Suppose that M admits a strict LCK metric. The complex torus $\mathbf{T}$ acts holomorphically and effectively on M, so, by Theorem 1, M admits a Vaisman metric (Ω, θ). Let B be the Lee vector field with $\theta(B) = 1$ and $A := JB$. By Proposition 3, $n = 1$ and $\mathfrak{t} = \mathfrak{Lie}(\mathbf{T})$ is spanned by A and B. Here, we identify $\mathfrak{t}$ with its isomorphic image as a subalgebra of $\Gamma(TM)$.

Since the $\mathbf{T}$-invariant 1-forms $\theta_1 = J\theta$ and $\theta_2 = \theta$ verify $\theta_i(X_j) = \delta_{ij}$, for $i, j = 1, 2$, where $X_1 = A$ and $X_2 = B$, there will exist some linear combination of them giving a connection form $\alpha \in \mathcal{C}^\infty(T^*M \otimes \mathfrak{t})$ in π. More precisely, if we denote by ξ_1, ξ_2 the fundamental vector fields of the action, and let $G = (g_{ij})$ be the matrix of $\{X_1, X_2\}$ in the basis $\{\xi_1, \xi_2\}$ of $\mathfrak{t}$, then the connection form will be given by:

$$\alpha := (g_{11}\theta_1 + g_{21}\theta_2) \otimes \xi_1 + (g_{12}\theta_1 + g_{22}\theta_2) \otimes \xi_2.$$

Indeed, it is $\mathbf{T}$-invariant and we have $\alpha(\xi_i) = \xi_i$ for $i = 1, 2$. Moreover, since $d\theta = 0$, its curvature is:

$$\Theta := d\alpha = d\theta_1 \otimes g_{11}\xi_1 + d\theta_1 \otimes g_{12}\xi_2 = dJ\theta \otimes A.$$

It is a basic form, so given by $\Theta = \pi^*\eta \otimes A$, with $\eta \in \Omega^2(N)$, and $\eta \otimes A$ represents the Chern class $c(\pi) \in H^2(N, \mathfrak{t})$. Then clearly $c(\pi)$ is of rank 1, and moreover, it is definite since the form $-\eta$ is a Kähler form on N. The last assertion comes from the fact that, as Ω is Vaisman, we have $-dJ\theta = \Omega - \theta \wedge J\theta$, so the $(1, 1)$-form $-dJ\theta$ is strictly positive on $Q := \ker\theta \cap \ker J\theta \subset TM$. But Q is exactly the horizontal distribution given by the connection α, and so identifies with TN via π_*.

The converse statement is given in [28, Theorem 3.5], see also [26]. □

8 Analytic irreducibility of complex manifolds of LCK type

It is not very difficult to see that a product metric cannot be LCK ([27]), but whether an LCK manifold must be analytically irreducible is still an open question. Under additional hypotheses, the answer is known to be positive ([26, Corollary 3.3], [17, Corollary 2]). In this section, we wish to enlarge the list of hypotheses implying the analytic irreducibility of the manifold.

One of the results in this direction is due to Tsukada, which we can also obtain as a direct consequence of Theorem 1:

Proposition 5 ([26, Corollary 3.3]) *Let M_1 and M_2 be two compact complex manifolds of Vaisman type. Then $M := M_1 \times M_2$ admits no LCK metric.*

Proof By Theorem 1, the groups of biholomorphisms $\mathrm{Aut}(M_i)$ contain tori $\mathbb{T}_i$ which are not totally real, for $i = 1, 2$. Then the Lie algebra $\mathfrak{t}$ of the torus $\mathbb{T} := \mathbb{T}_1 \times \mathbb{T}_2 \subset \mathrm{Aut}(M)$ verifies $\dim_{\mathbb{C}} \mathfrak{t} \cap J\mathfrak{t} = 2$. Hence, by Corollary 1, M cannot admit an LCK metric. □

Tsukada obtained Proposition 5 as a corollary to the following result:

Theorem 6 ([26, Theorem 3.2]) *Let (M, Ω) be a compact Vaisman manifold and let $\mathcal{F}$ be the canonical foliation on M generated by the Lee and the anti-Lee vector fields. Then $\mathcal{F}$ has a compact leaf.*

We can further exploit this and obtain the following, more general, result:

Theorem 7 *Let M_1, M_2 be two compact complex manifolds and suppose that M_1 is of Vaisman type. Then $M := M_1 \times M_2$ admits no LCK metric.*

Proof Suppose M admits some LCK metric. Then, for any $x \in M_1$, this metric restricted to $\{x\} \times M_2 \cong M_2$ gives an LCK metric on M_2.

Since M_1 is of Vaisman type, there exists $\mathbb{T}_1 \subset \mathrm{Aut}(M_1)$ whose Lie algebra $\mathfrak{t}_1$ verifies $\dim_{\mathbb{C}} \mathfrak{t}_1 \cap J\mathfrak{t}_1 = 1$. The induced torus $\mathbb{T} = \mathbb{T}_1 \times \{\mathrm{id}_{M_2}\} \subset \mathrm{Aut}(M)$ is still not totally real, so by Theorem 1, M is of Vaisman type and $\mathfrak{t} := \mathfrak{Lie}(\mathbb{T})$ contains the corresponding Lee vector field B.

Let Ω be a Vaisman metric on M which, possibly after averaging, is $\mathbb{T}$-invariant. Then for any $y \in M_2$, Ω restricted to $M_1 \times \{y\} \cong M_1$ must be Vaisman. Indeed, by construction, the Lee vector field B is tangent to M_1, and [28, Theorem 5.1] states that any complex submanifold of a Vaisman manifold that is tangent to the Lee vector field is again Vaisman with the induced metric. Let now $E \subset M_1$ be a closed leaf of the canonical foliation on the Vaisman manifold M_1, as in the above theorem. Clearly, after choosing $O \in E$, E has the structure of an elliptic curve whose tangent bundle is generated by B and JB restricted to E. Hence, the submanifold $i : Y = E \times M_2 \to M$ together with $i^*\Omega$ is Vaisman. At the same time, $Y \to M_2$ is a trivial E-principal bundle, so we arrive at a contradiction via Proposition 4. □

Also, using the result which states that a compact complex submanifold of a Vaisman manifold must contain the leaves of the canonical foliation, one has:

Proposition 6 *A compact complex manifold of Vaisman type is holomorphically irreducible.*

Proof Let $M = M_1 \times M_2$ be the compact complex manifold with the product complex structure, and suppose it admits a Vaisman metric Ω with corresponding canonical foliation $\mathcal{F}$ generated by B, JB. Then, by [25, Thm 3.2], for any $(x_1, x_2) \in M$, both the submanifolds $M_1 \times \{x_2\}$ and $\{x_1\} \times M_2$ of M contain the leaves of $\mathcal{F}$, which is impossible. □

On the other extreme, we have the following result of Ornea, Parton and Vuletescu:

Theorem 8 ([17, Corollary 2]) *Let M_1, M_2 be two compact connected complex manifolds, and suppose that M_1 verifies the $\partial\bar{\partial}$-lemma. Moreover, if M_1 is a Riemann surface, then suppose that its genus is* 0 *or* 1. *Then $M := M_1 \times M_2$ admits no (strict) LCK metric.*

Remark 6 In [17], the authors claim a proof of Theorem 8 also for the case when M_1 is a Riemann surface of genus ≥ 2, but we believe that their argument does not hold. However, we are only able to find restrictions on the manifold M_2 under the hypothesis that $M_1 \times M_2$ admits an LCK metric:

Proposition 7 *Let M_1 be a compact complex curve, let M_2 be a compact complex manifold and suppose that $M := M_1 \times M_2$ admits an LCK metric. Then M_2 admits an LCK metric with positive potential.*

Proof Let (Ω, θ) be an LCK structure on M. Denote by $p_i : M \to M_i$, $i = 1, 2$ the canonical projections. We have, by the Künneth formula, an isomorphism $p_1^* \oplus p_2^* : H^1(M_1, \mathbb{R}) \oplus H^1(M_2, \mathbb{R}) \to H^1(M, \mathbb{R})$, meaning that there exist two closed forms $\theta_i \in \mathcal{C}^\infty(T^*M_i)$, $i = 1, 2$, such that θ is cohomologuous to $p_1^*\theta_1 + p_2^*\theta_2$. After a conformal change of Ω, we can suppose that $\theta = p_1^*\theta_1 + p_2^*\theta_2$. Moreover, as M_1 is Kählerian, up to a conformal change of Ω we can choose θ_1 to be the real part of a holomorphic one-form, so that $dJ\theta_1 = 0$, where J is the product complex structure on M.

The algebra of differential forms on M, $\mathcal{C}^\infty(\bigwedge T^*M)$, has two compatible gradings: one given by the degree of the forms, and the second one induced by the splitting $T^*M = p_1^*T^*M_1 \oplus p_2^*T^*M_2$. With respect to this second splitting, write the differential $d = d_1 + d_2$, and write $\Omega = \Omega_1 + \Omega_{12} + \Omega_2 \in \mathcal{C}^\infty(\bigwedge^2 T^*M)$, where:

$$\bigwedge^2 T^*M = \bigwedge^2 p_1^*T^*M_1 \oplus p_1^*T^*M_1 \otimes p_2^*T^*M_2 \oplus \bigwedge^2 p_2^*T^*M_2.$$

Then the equation $d\Omega = \theta \wedge \Omega$ gives, in the homogeneous part $\bigwedge^2 p_1^* T^* M_1 \otimes \bigwedge^1 p_2^* T^* M_2$:

$$d_1\Omega_{12} + d_2\Omega_1 = \theta_2 \wedge \Omega_1 + \theta_1 \wedge \Omega_{12}. \tag{19}$$

Extend J as a derivation acting on forms, and let $d^c = i(\bar\partial - \partial)$. Then, on M we have the commutation relation:

$$[J, d] = d^c. \tag{20}$$

The formula $Jd\Omega = J(\theta \wedge \Omega)$, together with $J\Omega = 0$ and (20) gives, on the $\bigwedge^1 T^* M_1 \otimes \bigwedge^2 T^* M_2$-part:

$$d_2^c \Omega_{12} + d_1^c \Omega_2 = J\theta_1 \wedge \Omega_2 + J\theta_2 \wedge \Omega_{12}. \tag{21}$$

On the other hand, the compactness of M_1 implies that p_2 is a proper submersion, so it induces a push forward map on forms given by fiberwise integration:

$$(p_2)_* : \mathcal{C}^\infty \left(\bigwedge^{2n} p_1^* T^* M_1 \otimes \bigwedge^k p_2^* T^* M_2 \right) \to \mathcal{C}^\infty (\bigwedge^k T^* M_2)$$

$$((p_2)_* \alpha)_y := \int_{M_1 \times \{y\}} \alpha, \quad y \in M_2.$$

We apply the map $(p_2)_*$ to Eq. (19) and Stoke's theorem in order to obtain:

$$(p_2)_* d_2 \Omega_1 = (p_2)_* (\theta_2 \wedge \Omega_1 + \theta_1 \wedge \Omega_{12}).$$

If we denote by h the strictly positive function on M_2 given by $(p_2)_* \Omega_1$, this also reads:

$$d_{\theta_2} h = (p_2)_* (\theta_1 \wedge \Omega_{12}). \tag{22}$$

We apply d^c to this identity and use Eq. (21) together with (22) to get:

$$\begin{aligned} d^c d_{\theta_2} h &= -(p_2)_* (\theta_1 \wedge d_2^c \Omega_{12}) \\ &= -(p_2)_* (\theta_1 \wedge J\theta_1 \wedge \Omega_2) + J\theta_2 \wedge d_{\theta_2} h + (p_2)_* (\theta_1 \wedge d_1^c \Omega_2). \end{aligned}$$

Since we chose θ_1 so that $dJ\theta_1 = 0$, equation (20) implies that $d_1^c \theta_1 = 0$, hence the above simply gives:

$$d^c d_{\theta_2} h - J\theta_2 \wedge d_{\theta_2} h = -(p_2)_* (\Omega_2 \wedge \theta_1 \wedge J\theta_1).$$

Note that $\alpha := \Omega_2 \wedge \theta_1 \wedge J\theta_1$ is a semipositive $(2, 2)$-form on M which is strictly positive on a non-empty open subset of M of the form $U \times M_2$, where $U \subset M_1$ is the open set where θ_1 does not vanish. Then $\eta := (p_2)_* \alpha$ is a strictly positive $(1, 1)$-form on M_2 verifying:

$$\eta = d_{\theta_2} d_{\theta_2}^c h. \tag{23}$$

Thus (η, θ_2) is an LCK metric with positive potential on M_2. □

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