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# Existence for Nonlinear Fourth-Order Two-Point Boundary Value Problems

**Ravi Agarwal [1,*], Gabriela Mihaylova [2] and Petio Kelevedjiev [3]**

[1] Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363-8202, USA
[2] Department of Electrical Engineering, Electronics and Automation, Faculty of Engineering and Pedagogy of Sliven, Technical University of Sofia, 8800 Sliven, Bulgaria
[3] Department of Qualification and Professional Development of Teachers of Sliven, Technical University of Sofia, 8800 Sliven, Bulgaria
* Correspondence: agarwal@tamuk.edu

**Abstract:** The present paper is devoted to the solvability of various two-point boundary value problems for the equation $y^{(4)} = f(t, y, y', y'', y''')$, where the nonlinearity $f$ may be defined on a bounded set and is needed to be continuous on a suitable subset of its domain. The established existence results guarantee not just a solution to the considered boundary value problems but also guarantee the existence of monotone solutions with suitable signs and curvature. The obtained results rely on a basic existence theorem, which is a variant of a theorem due to A. Granas, R. Guenther and J. Lee. The a priori bounds necessary for the application of the basic theorem are provided by the barrier strip technique. The existence results are illustrated with examples.

**Keywords:** nonlinear differential equation; fourth-order; two-point boundary conditions; solvability; barrier strips

**MSC:** 34B15; 34B16

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## 1. Introduction

This paper studies the solvability in $C^4[0,1]$ of boundary value problems (BVPs) for the equation

$$y^{(4)} = f(t, y, y', y'', y'''),\ t \in (0,1), \tag{1}$$

where $f(t, y, u, v, w)$ is a scalar function defined on $[0,1] \times D_y \times D_u \times D_v \times D_w$, and $D_y, D_u, D_v, D_w \subseteq \mathbf{R}$.

We show sufficient conditions for the existence of solutions of (1) satisfying one of the following boundary conditions (BCs)

$$y'''(0) = A, y''(1) = B, y'(0) = C, y(0) = D, \tag{2}$$

$$y'''(0) = A, y''(0) = B, y(0) = C, y(1) = D, \tag{3}$$

$$y'''(0) = A, y'(0) = B, y'(1) = C, y(1) = D, \tag{4}$$

or

$$y'''(0) = A, y'(0) = B, y(0) = C, y(1) = D, \tag{5}$$

where $A, B, C, D \in \mathbf{R}$. It is established that the considered problems have positive or non-negative, monotone, convex or concave solutions.

It is well known that boundary value problems for fourth-order differential equations arise as models studying the deformations of an elastic beam, which is one of the basic structures in architecture, used often in the design of bridges and various structures.

The solvability of fourth-order BVPs with various two-point BCs has been studied by many authors.

Various BVPs for equations of the type

$$y^{(4)} = f(t, y),\ t \in (0, 1),$$

have been studied by A. Cabada et al. [1], J. Caballero et al. [2], J. Cid et al. [3], G. Han and Z. Xu [4], J. Harjani et al. [5], G. Infante and P. Pietramala [6], J. Li [7] (here, the nonlinearity $f(t, y)$ may be singular at the ends of the interval and at $y = 0$), B. Yang [8] and C. Zhai and C. Jiang [9].

J. Liu and W. Xu [10] and D. O'Regan [11] (in this work, the function $f(t, y, u)$ admit singularities at the ends $t = 0, 1$, at $y = 0$ and/or at $u = 0$) and Q. Yao [12] has studied boundary value problems for equations of the form

$$y^{(4)} = f(t, y, y').$$

In [11], the homogeneous conditions (3) are among the considered boundary conditions.

Many authors have considered BVPs for equations of the type

$$y^{(4)} = f(t, y, y''),\ t \in (0, 1),$$

see Z. Bai et al. [13], D. Brumley et al. [14], M. Del Pino and R. Manasevich [15], A. El-Haffaf [16] (with homogeneous boundary conditions (2)), P. Habets and M. Ramalho [17], R. Ma [18] and D. O'Regan [19]. In the last work, the function $f(t, y, v)$ may be singular at the ends of the interval, at $y = 0$ and/or at $v = 0$.

The solvability of boundary value problems for the more general equations

$$y^{(4)} = f(t, y, y', y''),\ t \in (0, 1),$$

has been studied in [16] (with homogeneous boundary conditions (2)), [20–22], where the main nonlinearity $f(t, y, u, v)$ may be singular at $t = 0, 1,\ y = 0, u = 0$ and $v = 0$.

BVPs for equations of the form (1) with various two-point boundary conditions have been considered by R. Agarwal [23], Z. Bai [24], C. De Coster et al. [25], J. Ehme et al. [26], D. Franco et al. [27], A. Granas et al. [28], Y. Li and Q. Liang [29], Y. Liu and W. Ge [30], R. Ma [31], F. Minhós et al. [32], B. Rynne [33], F. Sadyrbaev [34] and Q. Yao [35]. Moreover, the BCs in [23,34] are

$$y(0) = A, y'(0) = B, y(1) = C, y''(1) = D,$$

in the work [24], they are of the form

$$y(0) = y'(1) = y''(0) = y''(1) = 0,$$

the authors of [24,31] consider the conditions

$$y(0) = y'(1) = y''(0) = y'''(1) = 0,$$

and those of [29,32,33] consider

$$y(0) = y(1) = y''(0) = y''(1) = 0. \tag{6}$$

The boundary conditions in [25] are periodic, and in [28,33], they are

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{7}$$

and of the form

$$y'(0) = y'(1) = y'''(0) = y'''(1) = 0$$

in [30]. BVPs with boundary conditions either (6), (7),

$$y(0) = y'(0) = y''(1) = y'''(1) = 0$$

or

$$y(0) = y'(0) = y'(1) = y'''(1) = 0$$

have been studied in [31]. In [34], the boundary conditions are generally nonlinear, and in [26,27], they are of the type

$$g_1(\bar{x}) = 0, g_2(\bar{x}) = 0, h_1(\tilde{x}) = 0, h_2(\tilde{x}) = 0,$$

where the functions $g_i, h_i, i = 1,2$, are continuous, $\bar{x} = (x(0), x(1), x'(0), x'(1), x''(0), x''(1))$ in both papers, $\tilde{x} = \bar{x}$ in [26], and $\tilde{x} = (x(0), x(1), x'(0), x'(1), x''(0), x''(1), x'''(0), x'''(1))$ in [27].

Results guaranteeing positive solutions can be found in [2,3,5–10,12,14,18,20,21]. In [9], the most recent of these articles, the following nonlinear fourth-order two-point boundary value problem

$$y^{(4)} = f(t, y'),\ t \in (0,1),$$
$$y(0) = y'(0) = y''(1) = y'''(1) + g(y(1)) = 0,$$

is considered under the assumption that there exist two suitable real numbers $b > a \geq 0$ and a non-negative function $l \in C(0,1) \cap L^1[0,1]$ such that the function $f : (0,1) \times [0,b] \to \mathbf{R}$ is continuous and $|f(t,y)| \leq l(t)$ for $(t,x) \in (0,1) \times [0,b]$, and the function $g : [0,b] \to (0,\infty)$ is continuous and increasing. The authors establish that this problem has two nontrivial solutions $x^*, y^* \in C[0,1]$ with $at^2 \leq x^* \leq y^* \leq bt^2, t \in [0,1]$, which are limits of sequences with first terms $x_0(t) = at^2$ and $y_0(t) = bt^2$, respectively.

A classic tool for studying the solvability of initial and boundary value problems is the lower and upper solutions technique. It was probably E. Picard [36], in 1893, who first used an initial version of this technique to study a first-order initial boundary value problem. This idea was further developed later by G. Scorca Dragoni [37]. The lower and upper solutions technique is often used together with so-called growth conditions imposed on the main nonlinearity of the differential equation. S. Bernstein [38], in 1912, first used such a condition to establish the solvability of a second-order boundary value problem with Dirichlet boundary conditions. Subsequently, his idea was further developed by a number of mathematicians, with M. Nagumo [39] being the first to do this in 1937. In a series of papers of recent decades, R. Agarwal and D. O'Regan, see for example [40], study the solvability of various nonsingular and singular initial and boundary value problems under the assumption that the main nonlinearity does not change its sign.

Except for [9], where the function $f(t,y)$ is defined and continuous on a bounded set of the form $(0,1) \times [a,b]$, in the mentioned works, the main nonlinearity is defined and continuous with respect to the dependent variables on unbounded sets, [1–5,7,8,10,11,13–30,32–34,41–43], or is a Carathéodory function on an unbounded set, see [6,31,35]. Various existence and uniqueness results are obtained under assumptions that the considered boundary value problem admits lower and upper solutions [1,7,13,16,17,24,26,27,32,34], under assumptions that the main nonlinearity is positive or non-negative [2,3,5–8,10,14,18–21], and under Nagumo-type growth conditions [24,26,27,32,34], non-resonance conditions [15,25], and monotone conditions [7,24]. Maximum principles and various applications of the Green function are used in [1,17] and [2,3,6,8,10,14,16,29,41,43,44], respectively.

We use other tools. In [45], under a barrier strips condition, we study the solvability of BVPs for (1) with BCs, including $y'''(1) = C$. In the present paper, we extend the list of BVPs considered in [45], imposing a different barrier strips condition, which is adapted to the new BC for the third derivative. Barrier strip conditions have also been used by W. Qin [42] for studying the solvability of a three-point BVP for Equation (1).

Our results rely on the following assumptions:

**Hypothesis 1 (H1).** *There exist constants $F_i, L_i, i = 1,2$, with the following properties:*

$$F_2 < F_1 \leq A \leq L_1 < L_2, [F_2, L_2] \subseteq D_w,$$

$$f(t,y,u,v,w) \leq 0 \text{ for } (t,y,u,v,w) \in [0,1] \times D_y \times D_u \times D_v \times [L_1, L_2], \tag{8}$$

$$f(t,y,u,v,w) \geq 0 \text{ for } (t,y,u,v,w) \in [0,1] \times D_y \times D_u \times D_v \times [F_2, F_1]. \tag{9}$$

**Hypothesis 2 (H2).** *There exist constants $m_k \leq M_k, k = \overline{0,3}$, with the properties: $[m_0 - \varepsilon, M_0 + \varepsilon] \subseteq D_y$, $[m_1 - \varepsilon, M_1 + \varepsilon] \subseteq D_u$, $[m_2 - \varepsilon, M_2 + \varepsilon] \subseteq D_v$, $[m_3 - \varepsilon, M_3 + \varepsilon] \subseteq D_w$, where $\varepsilon > 0$ is a sufficiently small and $f(t,y,u,v,w)$ is continuous on the set $[0,1] \times J$ with*

$$J = [m_0 - \varepsilon, M_0 + \varepsilon] \times [m_1 - \varepsilon, M_1 + \varepsilon] \times [m_2 - \varepsilon, M_2 + \varepsilon] \times [m_3 - \varepsilon, M_3 + \varepsilon].$$

In Lemma 1, we will see that the strips $[0,1] \times [L_1, L_2]$ and $[0,1] \times [F_2, F_1]$ from $(\mathbf{H}_1)$ control the behavior of $y'''(t)$ on $[0,1]$ and, in this way, guarantee a priori bounds for $y'''(t)$. These strips are called barrier ones—see P. Kelevedjiev [46]; more details on the nature of the barrier technique are presented in Section 5. Note also that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ allow the sets $D_y, D_u, D_v$ and $D_w$ to be bounded and the nonlinearity $f$ to be continuous only on a bounded subset of its domain.

The barrier idea can be used in various variants. One of the possibilities is to replace the constants $F_i, L_i, i = 1,2$, from $(\mathbf{H}_1)$ by continuous functions having suitable monotonicity. Such curvilinear strips have been used in P. Kelevedjiev [47] for second-order two-point boundary value problems. Of course, the strips $[0,1] \times [L_1, L_2]$ and $[0,1] \times [F_2, F_1]$ from $(\mathbf{H}_1)$ can be replaced by the segments $[0,1] \times \{L_1\}$ and $[0,1] \times \{F_1\}$; see again [47]. A disadvantage of barrier segments is that the right side of the equation must not become zero on them. Barrier segments have also been used by I. Rachůnková and S. Staněk [48] and R. Ma [49] for studying the solvability of various BVPs. Discontinuous barrier strips, curvilinear strips and barrier segments can also be useful; see [47].

Our basic existence result is stated in Section 2. There, we also give auxiliary results, which guarantee a priori bounds for each eventual solution $y(t) \in C^4[0,1]$ to the families of BVPs for

$$y^{(4)} = \lambda f(t,y,y',y'',y'''),\ \lambda \in [0,1],\ t \in (0,1), \tag*{$(1)_\lambda$}$$

with BCs either (2)–(4) or (5); these a priori bounds are necessary for the application of the basic existence theorem. Moreover, the barrier condition $(\mathbf{H}_1)$ first provides the a priori bound for $y'''(t)$, and those for $y(t), y'(t)$ and $y''(t)$ are a consequence of it. The existence results are stated in Section 3. They are based on the simultaneous use of $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and guarantee not just a solution to the considered boundary value problems but also solutions with important properties such as an invariant sign, increasing, decreasing, convexity, and concavity. In Section 4, we illustrate the application of the obtained existence theorems with examples.

## 2. Basic Existence Results, Auxiliary Results

Following [45], we first introduce the notation needed to formulate the basic existence theorem.

Consider the BVP

$$\begin{cases} y^{(4)} + s_3(t)y''' + s_2(t)y'' + s_1(t)y' + s_0(t)y = f(t,y,y',y'',y'''),\ t \in (0,1), \\ V_i(y) = r_i, i = \overline{1,4}, \end{cases} \tag{10}$$

where $s_i \in C[0,1], i = \overline{0,3}$, $f : [0,1] \times D_y \times D_u \times D_v \times D_w \to \mathbf{R}$,

$$V_i(y) \equiv \sum_{j=0}^{3}[a_{ij}y^{(j)}(0) + b_{ij}y^{(j)}(1)], \;\; r_i \in \mathbf{R}, \; i = \overline{1,4},$$

and the constants $a_{ij}$ and $b_{ij}$ are such that $\sum_{j=0}^{3}(a_{ij}^2 + b_{ij}^2) > 0$ for $i = \overline{1,4}$.

Consider also the family of BVPs

$$\begin{cases} y^{(4)} + s_3(t)y''' + s_2(t)y'' + s_1(t)y' + s_0(t)y = g(t, y, y', y'', y''', \lambda), t \in (0,1), \\ V_i(y) = r_i, i = \overline{1,4}, \end{cases} \tag{11}$$

where $\lambda \in [0,1]$, $g : [0,1] \times D_y \times D_u \times D_v \times D_w \times [0,1] \to \mathbf{R}$, and $s_i(t), i = \overline{0,3}, V_i, r_i$, $i = \overline{1,4}$, are as above.

Finally, introduce the sets

$$BC = \{y(t), t \in [0,1] : V_i(y) = r_i, i = \overline{1,4}\}, \;\; BC_0 = \{y(t), t \in [0,1] : V_i(y) = 0, i = \overline{1,4}\},$$

$$C^4_{BC}[0,1] = C^4[0,1] \cap BC, \;\; C^4_{BC_0}[0,1] = C^4[0,1] \cap BC_0.$$

We are ready to formulate our basic existence result. It is similar to Theorem 5.1 (Chapter I) and Theorem 1.2 (Chapter V) of [28].

**Theorem 1.** *Assume that:*

*(i)* *For $\lambda = 0$, the problem (10) has a unique solution $y_0 \in C^4[0,1]$.*

*(ii)* *Problems (10) and (11) are equivalent when $\lambda = 1$.*

*(iii)* *The map $\Lambda_h : C^4_{BC_0}[0,1] \to C[0,1]$ defined by the left side of (10), i.e.,*

$$\Lambda_h y = y^{(4)} + s_3(t)y''' + s_2(t)y'' + s_1(t)y' + s_0(t)y,$$

*is one-to-one.*

*(iv)* *Each solution $y \in C^4[0,1]$ to family (11) satisfies the bounds*

$$m_k \le y^{(k)}(t) \le M_k, t \in [0,1], k = \overline{0,4},$$

*where the constants $-\infty < m_k, M_k < \infty, k = \overline{0,4}$, are independent of $\lambda$ and $y$.*

*(v)* *There is a sufficiently small $\varepsilon > 0$ such that $[m_0 - \varepsilon, M_0 + \varepsilon] \subseteq D_y$, $[m_1 - \varepsilon, M_1 + \varepsilon] \subseteq D_u$, $[m_2 - \varepsilon, M_2 + \varepsilon] \subseteq D_v$, $[m_3 - \varepsilon, M_3 + \varepsilon] \subseteq D_w$, and the function $g(t, y, u, v, w, \lambda)$ is continuous for $(t, y, u, v, w, \lambda) \in [0,1] \times J \times [0,1]$, where the set $J$ is as in* **(H$_2$)***, and the constants $m_k, M_k$, $k = \overline{0,3}$, are as in (iv).*

*Then BVP (10) has at least one solution in $C^4[0,1]$.*

We skip the proof, it can be found in [45].

Our first auxiliary result guarantees a priori bounds for the third derivatives of all eventual $C^4[0,1]$-solutions to the families of BVPs $(1.1)_\lambda$, (2)–(5).

**Lemma 1.** *Let $y \in C^4[0,1]$ be a solution to some of the families of BVPs $(1)_\lambda$, (2)–(5) and* **(H$_1$)** *hold. Then*

$$F_1 \le y'''(t) \le L_1 \text{ for } t \in [0,1].$$

**Proof.** Suppose that

$$T_- = \{t \in [0,1] : L_1 < y'''(t) \le L_2\}$$

is a non-empty set. Then, bearing in mind that $y'''(0) \leq L_1$ and $y'''(t)$ is continuous on $[0,1]$, we conclude that there exists an $\alpha \in T_-$ such that

$$y^{(4)}(\alpha) > 0.$$

For $\alpha$, we have

$$y^{(4)}(\alpha) = \lambda f(\alpha, y(\alpha), y'(\alpha), y''(\alpha), y'''(\alpha)),$$

since $y(t)$ is a solution to $(1)_\lambda$. In addition,

$$(\alpha, y(\alpha), y'(\alpha), y''(\alpha), y'''(\alpha)) \in T_- \times D_y \times D_u \times D_v \times (L_1, L_2].$$

In view of (8), this means that

$$\lambda f(\alpha, y(\alpha), y'(\alpha), y''(\alpha), y'''(\alpha)) \leq 0 \text{ for } \lambda \in [0,1],$$

i.e.,

$$y^{(4)}(\alpha) \leq 0.$$

The obtained contradiction shows that the set $T_-$ is empty and so

$$y'''(t) \leq L_1 \text{ for } t \in [0,1].$$

Similarly, assuming, on the contrary, that

$$T_+ = \{t \in [0,1] : F_2 \leq y'''(t) < F_1\}$$

is a non-empty set and using (9), we arrive at a contradiction, which implies

$$F_1 \leq y'''(t), \ t \in [0,1].$$

□

**Lemma 2.** *Assume that* $(\mathbf{H}_1)$ *holds. Then the bounds*

$$|y(t)| \leq |B| + |C| + |D| + \max\{|F_1|, |L_1|\},$$

$$|y'(t)| \leq |B| + |C| + \max\{|F_1|, |L_1|\}, \tag{12}$$

$$|y''(t)| \leq |B| + \max\{|F_1|, |L_1|\} \tag{13}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* $(1)_\lambda$, (2).

**Proof.** By the Lagrange mean value theorem, for each $t \in [0,1)$, there exists a $\gamma \in (t,1)$ such that

$$y''(1) - y''(t) = y'''(\gamma)(1-t), \ t \in [0,1),$$
$$|y''(t)| \leq |y'''(\gamma)|(1-t) + |y''(1)|, \ t \in [0,1).$$

However, $|y'''(\gamma)| \leq \max\{|F_1|, |L_1|\}$, by Lemma 1, and $y''(1) = B$. Therefore, we obtain (13).

Using the mean value theorem again, we conclude that for each $t \in (0,1]$, there is a $\delta \in (0,t)$ such that

$$y'(t) - y'(0) = y''(\delta)t, \ t \in (0,1],$$
$$|y'(t)| \leq |y'(0)| + |y''(\delta)|t, \ t \in (0,1],$$

and using (13), we establish (12).

Finally, applying the mean value theorem on $y(t)$ at intervals $(0,t)$ for each $t \in (0,1]$, and using (12), we establish the bound for $|y(t)|$. $\square$

**Lemma 3.** *Assume that* $(\mathbf{H}_1)$ *holds. Then the bounds*

$$\begin{gathered}|y(t)| \leq |B| + |D-C| + |C| + \max\{|F_1|,|L_1|\},\\ |y'(t)| \leq |B| + |D-C| + \max\{|F_1|,|L_1|\},\\ |y''(t)| \leq |B| + \max\{|F_1|,|L_1|\}\end{gathered} \tag{14}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* $(1)_\lambda$, (3).

**Proof.** Following the proof of Lemma 2, we obtain the bound for $|y''(t)|$. Next, consider that there exists a $\nu \in (0,1)$ with the property $y'(\nu) = D - C$. Further, for each $t \in [0,\nu)$, there exists a $\gamma \in (t,\nu)$ such that

$$y'(\nu) - y'(t) = y''(\gamma)(\nu - t),\ t \in [0,\nu),$$

$$|y'(t)| \leq |y''(\gamma)|(\nu - t) + |y'(\nu)|,\ t \in [0,\nu),$$

from which, using the established estimate for $|y''(\gamma)|$, we obtain

$$|y'(t)| \leq |D-C| + |B| + \max\{|F_1|,|L_1|\},\ t \in [0,\nu].$$

We can proceed analogously to see that this bound is also valid in the interval $[\nu,1]$.
For each $t \in (0,1]$, again by the mean value theorem, there exists a $\delta \in (0,t)$ such that

$$y(t) - y(0) = y'(\delta)t,\ t \in (0,1],$$

$$|y(t)| \leq |y(0)| + |y'(\delta)|t,\ t \in (0,1],$$

and using (14), we establish the a priori bound for $|y(t)|$. $\square$

**Lemma 4.** *Assume that* $(\mathbf{H}_1)$ *holds. Then the bounds*

$$\begin{gathered}|y(t)| \leq |B| + |C-B| + |D| + \max\{|F_1|,|L_1|\},\\ |y'(t)| \leq |B| + |C-B| + \max\{|F_1|,|L_1|\},\\ |y''(t)| \leq |C-B| + \max\{|F_1|,|L_1|\}\end{gathered}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* $(1)_\lambda$, (4).

**Proof.** Clearly, there exists a $\nu \in (0,1)$ for which $y''(\nu) = C - B$. Further, for each $t \in [0,\nu)$ there exists a $\gamma \in (t,\nu)$ for which

$$y''(\nu) - y''(t) = y'''(\gamma)(\nu - t),\ t \in [0,\nu),$$

$$|y''(t)| \leq |y'''(\gamma)|(\nu - t) + |y''(\nu)|,\ t \in [0,\nu),$$

from where, using $|y'''(\gamma)| \leq \max\{|F_1|,|L_1|\}$, which Lemma 1 gives, we obtain

$$|y''(t)| \leq |C-B| + \max\{|F_1|,|L_1|\} \text{ for } t \in [0,\nu].$$

This estimate is also valid in the interval $[\nu,1]$, and it is established with similar reasoning.
Following the proof of Lemma 2, establish the assertion for $|y'(t)|$.
Finally, for each $t \in [0,1)$, there exists a $\delta \in (t,1)$ for which

$$y(1) - y(t) = y'(\delta)(1-t),\ t \in [0,1),$$

which gives the assertion for $|y(t)|$. $\square$

**Lemma 5.** *Assume that* $(\mathbf{H}_1)$ *holds. Then the bounds*

$$|y(t)| \leq |B| + |C| + |D - C - B| + \max\{|F_1|, |L_1|\},$$

$$|y'(t)| \leq |B| + |D - C - B| + \max\{|F_1|, |L_1|\},$$
$$|y''(t)| \leq |D - C - B| + \max\{|F_1|, |L_1|\}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* (1)$_\lambda$, (5).

**Proof.** By the Lagrange mean value theorem, there is a $\mu \in (0,1)$ for which $y'(\mu) = D - C$, and there is a $\nu \in (0,\mu)$ such that $y''(\nu) = D - C - B$. Further, again by the mean value theorem, for each $t \in [0,\nu)$, there is a $\gamma \in (t,\nu)$ such that

$$y''(\nu) - y''(t) = y'''(\gamma)(\nu - t),\ t \in [0,\nu),$$

$$|y''(t)| \leq |y'''(\gamma)|(\nu - t) + |y''(\nu)|,\ t \in [0,\nu).$$

However, from Lemma 1, we know that $|y'''(\gamma)| \leq \max\{|F_1|, |L_1|\}$. Consequently

$$|y''(t)| \leq |D - C - B| + \max\{|F_1|, |L_1|\},\ t \in [0,\nu].$$

By similar reasoning, one finds that this estimate also holds in the interval $t \in [\nu,1]$.

As in the proofs of Lemmas 2 and 3, respectively, establish the bounds for $|y'(t)|$ and $|y(t)|$. $\square$

**Lemma 6.** *Assume that* $A \leq 0, B, C, D \geq 0$ *and* $(\mathbf{H}_1)$ *holds for* $L_1 \leq 0$. *Then the bounds*

$$D \leq y(t) \leq B + C + D - F_1,$$

$$C \leq y'(t) \leq B + C - F_1, \tag{15}$$

$$B \leq y''(t) \leq B - F_1 \tag{16}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* (1)$_\lambda$, (2).

**Proof.** We have

$$F_1 \leq y'''(t) \leq L_1 \leq 0 \ \text{ for } t \in [0,1],$$

by Lemma 1. Then,

$$\int_t^1 F_1 ds \leq \int_t^1 y'''(s) ds \leq \int_t^1 L_1 ds,\ t \in [0,1),$$

from where we obtain consecutively

$$F_1(1-t) \leq y''(1) - y''(t) \leq L_1(1-t) \text{ for } t \in [0,1],$$

$$F_1 \leq B - y''(t) \leq 0, t \in [0,1],$$

which yields (16). Next, we integrate (16) from 0 to $t \in (0,1]$ and obtain (15). Finally, by an integration of (15) from 0 to $t \in (0,1]$, we obtain the assertion for $y(t)$. $\square$

**Lemma 7.** *Assume that* $A, B \leq 0, C, D \geq 0$ *and* $(\mathbf{H}_1)$ *holds for* $L_1 \leq 0$. *Then the bounds*

$$\min\{C, D\} \leq y(t) \leq C + |D - C| - B - F_1,$$

$$D + B - C + F_1 \le y'(t) \le D - B - C - F_1,$$

$$B + F_1 \le y''(t) \le B \tag{17}$$

*are valid in the interval* $[0,1]$ *for each solution* $y \in C^4[0,1]$ *to* $(1)_\lambda$, (3).

**Proof.** In view of Lemma 1, we have

$$F_1 \le y'''(t) \le L_1 \le 0 \text{ for } t \in [0,1].$$

Then

$$\int_0^t F_1 ds \le \int_0^t y'''(s)ds \le \int_0^t L_1 ds, \ t \in (0,1],$$

$$F_1 \le F_1 t \le y''(t) - B \le L_1 t \le 0, \ t \in (0,1],$$

which yields (17). Further, use the fact that there exists a $\nu \in (0,1)$ such that $y'(\nu) = D - C$ to establish consecutively

$$\int_t^\nu (B + F_1)ds \le \int_t^\nu y''(s)ds \le \int_t^\nu Bds, t \in [0,\nu),$$

$$(B + F_1)(\nu - t) \le y'(\nu) - y'(t) \le B(\nu - t), \ t \in [0,\nu],$$

$$B + F_1 \le y'(\nu) - y'(t) \le 0, \ t \in [0,\nu],$$

since $0 \le \nu - t \le 1$,

$$D - C \le y'(t) \le D - B - C - F_1, \ t \in [0,\nu].$$

Similarly from

$$\int_\nu^t (B + F_1)ds \le \int_\nu^t y''(s)ds \le \int_\nu^t Bds, t \in (\nu,1],$$

establish

$$B + D - C + F_1 \le y'(t) \le D - C, \ t \in [\nu,1].$$

As a result, keeping in mind that $B, F_1 \le 0$, we obtain

$$B + D - C + F_1 \le y'(t) \le D - B - C - F_1 \text{ for } t \in [0,1],$$

from where it follows

$$|y'(t)| \le \max\{|B + D - C + F_1|, |D - B - C - F_1|\} \le |B| + |D - C| + |F_1| = |D - C| - B - F_1$$

for $t \in [0,1]$.

Now, by the mean value theorem, for any $t \in (0,1]$ there exists a $\gamma \in (0,t)$ such that

$$y(t) - y(0) = y'(\gamma)t,$$

from where it follows

$$|y(t)| \le C + |D - C| - B - F_1, \ t \in [0,1].$$

Since $B \le 0$, $y(t)$ is concave on $[0,1]$ in view of (17). In addition, $y(0) = C \ge 0$ and $y(1) = D \ge 0$, which means

$$y(t) \ge \min\{C, D\} \ge 0 \text{ on } [0,1],$$

from where the assertion for $y(t)$ follows. □

**Lemma 8.** *Assume that* $A, D \geq 0, B, C \leq 0$ *and* $(\mathbf{H}_1)$ *holds for* $F_1 \geq 0$. *Then the bounds*

$$D \leq y(t) \leq D + |B| + |C - B| + L_1,$$

$$-(|B| + |C - B| + L_1) \leq y'(t) \leq \max\{B, C\},$$

$$C - B - L_1 \leq y''(t) \leq C - B + L_1$$

*are valid in the interval* $[0, 1]$ *for each solution* $y \in C^4[0, 1]$ *to* $(1)_\lambda$, (4).

**Proof.** For some $\nu \in (0, 1)$, we have $y''(\nu) = y'(1) - y'(0)$, i.e., $y''(\nu) = C - B$. Now using the bounds for $y'''(t)$ from Lemma 1, we obtain consecutively

$$\int_t^\nu F_1 ds \leq \int_t^\nu y'''(s) ds \leq \int_t^\nu L_1 ds, t \in [0, \nu), \tag{18}$$

$$F_1(\nu - t) \leq y''(\nu) - y''(t) \leq L_1(\nu - t),\ t \in [0, \nu],$$

$$0 \leq y''(\nu) - y''(t) \leq L_1,\ t \in [0, \nu],$$

$$C - B - L_1 \leq y''(t) \leq C - B,\ t \in [0, \nu].$$

Similarly, from

$$\int_\nu^t F_1 ds \leq \int_\nu^t y'''(s) ds \leq \int_\nu^t L_1 ds, t \in (\nu, 1], \tag{19}$$

establish

$$C - B \leq y''(t) \leq C - B + L_1,\ t \in [\nu, 1].$$

Thus,

$$C - B - L_1 \leq y''(t) \leq C - B + L_1 \text{ for } t \in [0, 1],$$

since $L_1 \geq 0$.

Now, for each $t \in (0, 1]$, by the mean value theorem, there is a $\gamma \in (0, t)$ such that

$$y'(t) - y'(0) = y''(\gamma)t,\ t \in (0, 1],$$

which means

$$|y'(t)| \leq |B| + |C - B| + L_1 \text{ for } t \in [0, 1].$$

However, from $y'''(t) \geq F_1 \geq 0$ on $[0, 1]$ it follows that $y'(t)$ is convex on $[0, 1]$, which means that $y'(t) \leq \max\{B, C\}$ on $[0, 1]$ and so

$$-(|B| + |C - B| + L_1) \leq y'(t) \leq \max\{B, C\} \leq 0 \text{ for } t \in [0, 1].$$

To establish the bound for $y(t)$, we integrate from $t \in [0, 1)$ to 1 the inequality

$$-(|B| + |C - B| + L_1) \leq y'(t) \leq 0.$$

□

**Lemma 9.** *Assume that* $A \leq 0, B, C \geq 0, D > C$ *and* $(\mathbf{H}_1)$ *holds for* $L_1 \leq 0$ *and* $D - C \geq B - F_1$. *Then the bounds*

$$C \leq y(t) \leq D - F_1,$$

$$B \leq y'(t) \leq D - C - F_1,$$

$$D - C - B + F_1 \leq y''(t) \leq D - C - B - F_1$$

*are valid in the interval* $[0, 1]$ *for each solution* $y \in C^4[0, 1]$ *to* $(1)_\lambda$, (5).

**Proof.** From the proof of Lemma 5, we know that there exists a $\nu \in (0,1)$ such that $y''(\nu) = D - C - B$. Now the integration (18) of the estimates for $y'''(t)$ that Lemma 1 guarantees gives us

$$F_1 \leq y''(\nu) - y''(t) \leq 0,\ t \in [0,\nu],$$

$$F_1 \leq D - C - B - y''(t) \leq 0,\ t \in [0,\nu],$$

and

$$D - C - B \leq y''(t) \leq D - C - B - F_1,\ t \in [0,\nu].$$

On the other hand, the integration (19) gives

$$D - C - B + F_1 \leq y''(t) \leq D - C - B,\ t \in [\nu,1].$$

Bearing in mind that $F_1 \leq 0$, on the whole interval $[0,1]$, we obtain

$$D - C - B + F_1 \leq y''(t) \leq D - C - B - F_1,$$

which means $y''(t) \geq 0$ for $t \in [0,1]$ because of the assumption $D - C \geq B - F_1$ and so $y'(t)$ is non-decreasing on the interval $[0,1]$. Then, in view of the boundary condition $y'(0) = B$, we have

$$y'(t) \geq B \text{ for } t \in [0,1].$$

Thus, the bound for $y'(t)$ from Lemma 5 takes the form

$$B \leq y'(t) \leq B + |D - C - B| + \max\{|F_1|, |L_1|\},\ t \in [0,1],$$

and

$$B \leq y'(t) \leq D - C - F_1,\ t \in [0,1],$$

because due to $F_1 \leq L_1 \leq 0$ the condition $D - C \geq B - F_1$ also implies $D - C - B \geq 0$.

Further, from $y'(t) \geq B \geq 0$, it follows that $y(t)$ is non-decreasing. This fact, together with the bound for $|y(t)|$ from Lemma 5, gives

$$C \leq y(t) \leq |B| + |C| + |D - C - B| + \max\{|F_1|, |L_1|\},\ t \in [0,1],$$

from where the assertion for $y(t)$ follows. $\square$

## 3. Existence Results

**Theorem 2.** *Assume that* $(\mathbf{H}_1)$ *holds, and* $(\mathbf{H}_2)$ *holds for*

$$M_0 = |B| + |C| + |D| + \max\{|F_1|, |L_1|\}, m_0 = -M_0,$$

$$M_1 = |B| + |C| + \max\{|F_1|, |L_1|\}, m_1 = -M_1,$$

$$M_2 = |B| + \max\{|F_1|, |L_1|\}, m_2 = -M_2, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (2) *has a solution in* $C^4[0,1]$.

**Proof.** We easily check that (i) of Theorem 1 holds for $(1)_0$, (2). Obviously, BVPs (1), (2) and $(1)_1$, (2) are the same. Thus, (ii) is also satisfied. To verify (iii) for the map $\Lambda_h = y'''$, we establish that for each $z \in C[0,1]$, the problem

$$y^{(4)} = z(t),\ t \in (0,1),$$

$$y'''(0) = 0, y''(1) = 0, y'(0) = 0, y(0) = 0,$$

has a unique solution $y(t)$ in $C^4[0,1]$. Next, according to Lemma 2 and Lemma 1, each solution $y \in C^4[0,1]$ to family $(1)_\lambda, (2)$ is such that

$$m_k \leq y^{(i)}(t) \leq M_k, t \in [0,1], k = 0,1,2,3. \tag{20}$$

Now, from the continuity of $f(t,y,u,v,w)$ on $[0,1] \times J$, it follows that there exist constants $m_4$ and $M_4$ for which

$$m_4 \leq \lambda f(t,y,u,v,w) \leq M_4 \text{ when } (t,y,u,v,w) \in [0,1] \times J \text{ and } \lambda \in [0,1].$$

In view of (20), for each solution $y \in C^4[0,1]$ to $(1)_\lambda, (2)$ we have $(y(t), y'(t), y''(t), y'''(t)) \in J$ for $t \in [0,1]$. Thus,

$$m_4 \leq \lambda f(y(t), y'(t), y''(t), y'''(t)) \leq M_4 \text{ when } \ t \in [0,1] \text{ and } \lambda \in [0,1]$$

and Equation $(1)_\lambda$ gives

$$m_4 \leq y^{(4)}(t) \leq M_4, t \in [0,1].$$

This and (20) imply that (iv) holds for $(1)_\lambda$, (2). Finally, *(v)* follows from the continuity of $f$ on the set $J$. Therefore, we can apply Theorem 1 to conclude that the assertion is true. □

**Theorem 3.** *Assume that* $A \leq 0, B, C, D \geq 0$ $(C, D > 0)$*. Assume also that* $(\mathbf{H}_1)$ *holds for* $L_1 \leq 0$ *and* $(\mathbf{H}_2)$ *holds for*

$$m_0 = D, M_0 = B + C + D - F_1,$$

$$m_1 = C, M_1 = B + C - F_1,$$

$$m_2 = B, M_2 = B - F_1, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (2) *has a non-negative (positive), non-decreasing (increasing), convex solution in* $C^4[0,1]$.

**Proof.** Lemma 6 implies

$$m_k \leq y^{(k)}(t) \leq M_k, t \in [0,1], k = 0,1,2,$$

and Lemma 1 yields

$$m_3 \leq y'''(t) \leq M_3, t \in [0,1].$$

Further, as in the proof of Theorem 2, we establish that problem (1), (2) has a solution $y(t) \in C^4[0,1]$. Since $y(t) \geq D \geq 0$ $(y(t) \geq D > 0), y'(t) \geq C \geq 0$ $(y'(t) \geq C > 0)$ and $y''(t) \geq B \geq 0$ for $t \in [0,1]$, this solution has the specified properties. □

**Theorem 4.** *Assume that* $(\mathbf{H}_1)$ *holds, and* $(\mathbf{H}_2)$ *holds for*

$$M_0 = |B| + |D - C| + |C| + \max\{|F_1|, |L_1|\}, m_0 = -M_0,$$

$$M_1 = |B| + |D - C| + \max\{|F_1|, |L_1|\}, m_1 = -M_1,$$

$$M_2 = |B| + \max\{|F_1|, |L_1|\}, m_2 = -M_2, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (3) *has a solution in* $C^4[0,1]$.

**Proof.** It differs from that of Theorem 2 only in that now Lemma 3 guarantees

$$m_k \leq y^{(k)}(t) \leq M_k, t \in [0,1], k = 0,1,2.$$

□

**Theorem 5.** *Assume that $A \le 0, B \le 0, D \ge C \ge 0$ ($D \ge C > 0$). Assume also that* ($\mathbf{H}_1$) *holds for $L_1 \le 0$ and $D - C \ge -(B + F_1)$, and* ($\mathbf{H}_2$) *holds for*

$$m_0 = C, M_0 = D - B - F_1,$$

$$m_1 = B + D - C + F_1, M_1 = D - B - C - F_1,$$

$$m_2 = B + F_1, M_2 = B, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (3) *has a non-negative (positive), non-decreasing, concave solution in* $C^4[0,1]$.

**Proof.** From Lemma 7 for every solution $y \in C^4[0,1]$ to family $(1)_\lambda$, (3), we have

$$\min\{C, D\} \le y(t) \le C + |D - C| - B - F_1 \text{ for } t \in [0,1],$$

i.e.,

$$C \le y(t) \le D - B - F_1 \text{ for } t \in [0,1],$$

since $D \ge C$. Therefore,

$$m_k \le y^{(i)}(t) \le M_k, t \in [0,1], k = 0,1,2,3,$$

by Lemma 7 and Lemma 1. Further, essentially the same reasoning as in Theorem 2 establishes that problem (1), (3) is solvable in $C^4[0,1]$. Since $m_0 = C \ge 0 (m_0 > 0)$, $m_1 = B + D - C + F_1 \ge 0$ and $M_2 = B \le 0$, the solution has the desired properties. □

**Theorem 6.** *Assume that* ($\mathbf{H}_1$) *holds, and* ($\mathbf{H}_2$) *holds for*

$$M_0 = |B| + |C - B| + |D| + \max\{|F_1|, |L_1|\}, m_0 = -M_0,$$

$$M_1 = |B| + |C - B| + \max\{|F_1|, |L_1|\}, m_1 = -M_1,$$

$$M_2 = |C - B| + \max\{|F_1|, |L_1|\}, m_2 = -M_2, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (4) *has a solution in* $C^4[0,1]$.

**Proof.** It follows the proof of Theorem 2. Now Lemma 4 guarantees the estimates

$$m_k \le y^{(k)}(t) \le M_k, t \in [0,1], k = 0,1,2,$$

for every solution $y \in C^4[0,1]$ to $(1)_\lambda$, (4). □

**Theorem 7.** *Assume that $A, D \ge 0 (D > 0), C \le B \le 0 (C \le B < 0)$. Assume also that* ($\mathbf{H}_1$) *holds for $F_1 \ge 0$ and $C - B + L_1 \le 0$, and* ($\mathbf{H}_2$) *holds for*

$$m_0 = D, M_0 = D - C + L_1,$$

$$m_1 = C - L_1, M_1 = B,$$

$$m_2 = C - B - L_1, M_2 = C - B + L_1, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (4) *has a non-negative (positive), non-increasing (decreasing), convex solution in* $C^4[0,1]$.

**Proof.** From Lemma 8 for every solution $y \in C^4[0,1]$ to family $(1)_\lambda$, (4), we know

$$D \le y(t) \le D + |B| + |C - B| + L_1, \ t \in [0,1],$$

$$-(|B| + |C - B| + L_1) \le y'(t) \le \max\{B, C\}, \ t \in [0,1],$$

i.e.,

$$D \le y(t) \le D - C + L_1,\ t \in [0,1],$$
$$C - L_1 \le y'(t) \le B,\ t \in [0,1],$$

because $C \le B \le 0$. Therefore,

$$m_k \le y^{(k)}(t) \le M_k, t \in [0,1], k = 0,1,2,3,$$

by Lemma 8 and Lemma 1. Further, as in the proof of Theorem 2, we establish that (1), (4) is solvable in $C^4[0,1]$. Since $m_0 = D \ge 0(m_0 > 0), M_1 = B \le 0(M_1 < 0)$ and $M_2 = C - B + L_1 \le 0$, the solution has the desired properties. □

**Theorem 8.** *Assume that* $(\mathbf{H}_1)$ *holds, and* $(\mathbf{H}_2)$ *holds for*

$$M_0 = |B| + |C| + |D - C - B| + \max\{|F_1|, |L_1|\}, m_0 = -M_0,$$

$$M_1 = |B| + |D - C - B| + \max\{|F_1|, |L_1|\}, m_1 = -M_1,$$

$$M_2 = |D - C - B| + \max\{|F_1|, |L_1|\}, m_2 = -M_2, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (5) *has a solution in* $C^4[0,1]$.

**Proof.** It does not differ substantially from the proof of Theorem 2. Now Lemma 5 guarantees the bounds

$$m_k \le y^{(k)}(t) \le M_k, t \in [0,1], k = 0,1,2.$$

□

**Theorem 9.** *Assume that* $A \le 0, B, C \ge 0(B, C > 0), D > C$. *Assume also that* $(\mathbf{H}_1)$ *holds for* $L_1 \le 0$ *and* $D - C \ge B - F_1$, *and* $(\mathbf{H}_2)$ *holds for*

$$m_0 = C, M_0 = D - F_1,$$

$$m_1 = B, M_1 = D - C - F_1,$$

$$m_2 = D - C - B + F_1, M_2 = D - C - B - F_1, m_3 = F_1, M_3 = L_1.$$

*Then problem* (1), (5) *has a non-negative (positive), non-decreasing (increasing), concave solution in* $C^4[0,1]$.

**Proof.** According to Lemma 9, for every eventual solution $y \in C^4[0,1]$ to family $(1)_\lambda$, (5), we have

$$C \le y(t) \le D - F_1,$$
$$B \le y'(t) \le D - C - F_1,$$
$$D - C - B + F_1 \le y''(t) \le D - C - B - F_1$$

for $t \in [0,1]$, i.e.,

$$m_k \le y^{(k)}(t) \le M_k, t \in [0,1], k = 0,1,2.$$

In addition, by Lemma 1,

$$m_3 \le y'''(t) \le M_3, t \in [0,1].$$

Further, as in the proof of Theorem 2, we verify that the conditions of Theorem 1 are fulfilled and so (1), (5) is solvable in $C^4[0,1]$. Since $m_0 = C \ge 0(m_0 > 0), m_1 = B \ge 0(m_1 > 0)$ and $m_2 \ge 0$, the solution has the properties from the conclusion of the theorem. □

## 4. Examples

**Example 1.** *Consider the boundary value problems for*

$$y^{(4)} = F(t, y, y', y'')Q_n(y'''), t \in (0, 1),$$

*with BCs either (2)–(4) or (5). Here* $F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous and does not change its sign, and* $Q_n(w), n \geq 2$, *is a polynomial with simple zeros* $w_1$ *and* $w_2$ *such that* $w_1 < A < w_2$.

Clearly, there is a $\tau > 0$ such that $w_1 + \tau \leq A \leq w_2 - \tau$,

$$Q_n(w) \neq 0 \text{ for } w \in \cup_{k=1}^{2}\Big((w_k - \tau, w_k + \tau) \setminus \{w_k\}\Big).$$

Let, for concreteness,

$$Q_n(w) > 0 \text{ on } (w_1 - \tau, w_1) \quad \text{and} \quad Q_n(w) < 0 \text{ on } (w_2, w_2 + \tau);$$

the other cases can be considered in a similar way. It is clear that for the considered case, we have

$$Q_n(w) < 0 \text{ on } (w_1, w_1 + \tau) \quad \text{and} \quad Q_n(w) > 0 \text{ on } (w_2 - \tau, w_2).$$

Now, if $F(t, y, u, v) \geq 0$ on $[0, 1] \times \mathbb{R}^3$, then

$$F(t, y, u, v)Q_n(w) \geq 0 \quad \text{for} \quad (t, y, u, v, w) \in [0, 1] \times \mathbb{R}^3 \times [w_1 - \tau, w_1]$$

and

$$F(t, y, u, v)Q_n(w) \leq 0 \quad \text{for} \quad (t, y, u, v, w) \in [0, 1] \times \mathbb{R}^3 \times [w_2, w_2 + \tau],$$

that is, $(\mathbf{H}_1)$ holds for $F_2 = w_1 - \tau, F_1 = w_1, L_1 = w_2, L_2 = w_2 + \tau$, for example.

On the other hand, if $F(t, y, u, v) \leq 0$ on $[0, 1] \times \mathbb{R}^3$, then

$$F(t, y, u, v)Q_n(w) \geq 0 \quad \text{for} \quad (t, y, u, v, w) \in [0, 1] \times \mathbb{R}^3 \times [w_1, w_1 + \tau]$$

and

$$F(t, y, u, v)Q_n(w) \leq 0 \quad \text{for} \quad (t, y, u, v, w) \in [0, 1] \times \mathbb{R}^3 \times [w_2 - \tau, w_2]$$

and so $(\mathbf{H}_1)$ holds for $F_2 = w_1, F_1 = w_1 + \tau, L_1 = w_2 - \tau, L_2 = w_2$, for example.

Since the right-hand side $F(t, y, u, v)Q_n(w)$ of the equation is a defined and continuous function on $[0, 1] \times \mathbb{R}^4$, i.e., $D_y = D_u = D_v = D_w = \mathbb{R}$, $(\mathbf{H}_2)$ holds for each of the considered BVPs.

Therefore, we can apply Theorems 2, 4, 6 and 8 to BVPs (1),(2), (1),(3), (1),(4) and (1),(5), respectively, to conclude that each of them has at least one solution in $C^4[0, 1]$.

**Example 2.** *Consider the boundary value problem*

$$y^{(4)} = -\frac{t(y''' + 1)\sqrt{100 - y^2}\sqrt{400 - y'^2}}{\sqrt{400 - y''^2}\sqrt{625 - y'''^2}}, t \in (0, 1),$$

$$y'''(0) = -3, y''(1) = 1, y'(0) = 1, y(0) = 2.$$

This problem is of the type (1), (2) with $A = -3, B = 1, C = 1$ and $D = 2$. These values satisfy the condition of Theorem 3, so we will check its applicability.

The nonlinearity

$$f(t, y, u, v, w) = -\frac{t(w + 1)\sqrt{100 - y^2}\sqrt{400 - u^2}}{\sqrt{400 - v^2}\sqrt{625 - w^2}}$$

is defined and continuous for

$$(t, y, u, v, w) \in [0,1] \times [-10,10] \times [-20,20] \times (-20,20) \times (-25,25),$$

that is, $D_y = [-10,10]$, $D_u = [-20,20]$, $D_v = (-20,20)$ and $D_w = (-25,25)$.

It is easily verified that

$$f(t, y, u, v, w) \geq 0 \text{ for } (t, y, u, v, w) \in [0,1] \times [-10,10] \times [-20,20] \times (-20,20) \times [-5,-4]$$

and

$$f(t, y, u, v, w) \leq 0 \text{ for } (t, y, u, v, w) \in [0,1] \times [-10,10] \times [-20,20] \times (-20,20) \times [-1,0],$$

i.e., $(\mathbf{H}_1)$ is satisfied for $F_2 = -5, F_1 = -4, L_1 = -1$ and $L_2 = 0$.

Next, determine the constants $m_k, M_k, k = 0,1,2,3$, from Theorem 3:

$$m_0 = D = 2, M_0 = B + C + D - F_1 = 8,$$

$$m_1 = C = 1, M_1 = B + C - F_1 = 6,$$

$$m_2 = B = 1, M_2 = B - F_1 = 5,$$

$$m_3 = F_1 = -4, M_3 = L_1 = -1.$$

Since $[1.9, 8.1] \subseteq D_y$, $[0.9, 6.1] \subseteq D_u$, $[0.9, 5.1] \subseteq D_v$, $[-4.1, -0.9] \subseteq D_w$, $(\mathbf{H}_2)$ holds for the above constants $m_k, M_k, k = 0,1,2,3$, and $\varepsilon = 0.1$, for example. Therefore, we can apply Theorem 3 to conclude that this problem has a positive, increasing, convex solution in $C^4[0,1]$.

**Example 3.** *Consider the boundary value problem*

$$y^{(4)} = \sqrt{y'' + 20}\sin(y''' - 1), t \in (0,1),$$

$$y'''(0) = -2, y''(0) = -1, y(0) = 1, y(1) = 10.$$

Now, the boundary values satisfy the requirement of Theorem 5 for them. In addition, it is not difficult to verify that

$$f(t, y, u, v, w) = \sqrt{v + 20}\sin(w - 1) \geq 0 \text{ for } (t, y, u, v, w) \in [0,1] \times \mathbb{R}^2 \times [-20, \infty) \times [-5, -4]$$

and

$$f(t, y, u, v, w) = \sqrt{v + 20}\sin(w - 1) \leq 0 \text{ for } (t, y, u, v, w) \in [0,1] \times \mathbb{R}^2 \times [-20, \infty) \times [-1, 0],$$

which means that $(\mathbf{H}_1)$ is satisfied for $F_2 = -5, F_1 = -4, L_1 = -1$ and $L_2 = 0$. Moreover, the condition $D - C \geq -(B + F_1)$ also holds.

We will check that $(\mathbf{H}_2)$ holds for the constants $m_k, M_k, k = 0,1,2,3$, from Theorem 5. Actually specifying the constants $m_k, M_k, k = 0,1$, and $m_3$ and $M_3$ is not necessary because here $D_y = D_u = D_w = \mathbb{R}$ and so the inclusions

$$[m_0 - \varepsilon, M_0 + \varepsilon] \subseteq D_y, \quad [m_1 - \varepsilon, M_1 + \varepsilon] \subseteq D_u \text{ and } [m_3 - \varepsilon, M_3 + \varepsilon] \subseteq D_w$$

are always fulfilled for an arbitrarily fixed $\varepsilon > 0$. Of interest to us are only the constants $m_2$ and $M_2$. Since $B = -1$ and $F_1 = -4$, then $m_2 = B + F_1 = -5$, $M_2 = B = -1$ and obviously

$$[-5 - \varepsilon, -1 + \varepsilon] \subset D_v, \ D_v = [-20, \infty),$$

for sufficiently small $\varepsilon > 0$. Thus, $(\mathbf{H}_2)$ is satisfied because $f(t, y, u, v, w)$ is continuous on $[0,1] \times \mathbb{R}^2 \times [-20, \infty) \times \mathbb{R}$ and in particular on the set $J$.

According to Theorem 5, the considered problem has a positive, non-decreasing, concave solution in $C^4[0,1]$.

**Example 4.** *Consider the problem*

$$y^{(4)} = (5 - y''')^3 - \ln(y''' - 1), t \in (0,1),$$

$$y'''(0) = 3, y'(0) = -1, y'(1) = -7, y(1) = 1.$$

Here, we will check the applicability of Theorem 7. From

$$f(t,y,u,v,w) = (5-w)^3 - \ln(w-1) \geq 0 \quad \text{for} \quad (t,y,u,v,w) \in [0,1] \times \mathbb{R}^3 \times [1.5,2]$$

and

$$f(t,y,u,v,w) = (5-w)^3 - \ln(w-1) \leq 0 \quad \text{for} \quad (t,y,u,v,w) \in [0,1] \times \mathbb{R}^3 \times [5,6]$$

it follows that $(\mathbf{H}_1)$ holds for $F_2 = 1.5, F_1 = 2, L_1 = 5$ and $L_2 = 6$, for example. Since $B = -1$ and $C = -7$, the condition $C - B + L_1 \leq 0$ is also satisfied.

We have $D_y = D_u = D_v = \mathbb{R}$, and $D_w = (1, \infty)$. Therefore, only the constants $m_3 = 2$ and $M_3 = 5$ are interesting to us to see that

$$[1.9, 5.1] \subset D_w$$

and to conclude that $(\mathbf{H}_2)$ holds for $\varepsilon = 0.1$, for example, because $f(t,y,u,v,w)$ is continuous on the set $[0,1] \times \mathbb{R}^3 \times (1,\infty)$.

Consequently, this problem has a positive, decreasing, convex solution in $C^4[0,1]$ by Theorem 7.

**Example 5.** *Consider the boundary value problem*

$$y^{(4)} = \sqrt{y'' + 20}\sin(y''' - 1), t \in (0,1),$$

$$y'''(0) = -2, y'(0) = 1, y(0) = 1, y(1) = 10.$$

This problem is of the type (1), (5) with boundary values $A = -2, B = 1, C = 1$ and $D = 10$, which satisfy the condition of Theorem 9.

As in Example 3, we establish that $(\mathbf{H}_1)$ and the requirement $D - C \leq B - F_1$ of Theorem 9 are satisfied for $F_2 = -5, F_1 = -4, L_1 = -1$ and $L_2 = 0$. Again, we are only interested in the constants $m_2 = D - C - B + F_1 = 4$ and $M_2 = D - C - B - F_1 = 12$ to see that $[4 - \varepsilon, 12 + \varepsilon] \subset D_v$, $D_v = [-20, \infty)$, for a sufficiently small $\varepsilon > 0$. Because of the continuity of $f(t,y,u,v,w)$ on the set $[0,1] \times \mathbb{R}^2 \times [-20,\infty) \times \mathbb{R}$, $(\mathbf{H}_2)$ also is satisfied. Thus, we can apply Theorem 9 to conclude that the considered problem has a positive, increasing, concave solution in $C^4[0,1]$.

## 5. Discussion

The barrier strips technique used in this paper was introduced in 1994 in [46]. This technique does not use the classical tools mentioned in the introduction. It is based on the assumption that the right-hand side of the equation has suitable different signs on suitable subsets of its domain. Subsequently, barrier strips are used by a number of authors investigating the solvability of various boundary value problems for differential, difference and fractional differential equations, as well as of functional boundary value problems for differential equations.

This paper shows how the barrier strips technique (based here on assumption $(\mathbf{H}_1)$) can be used not only to establish the solvability of the boundary value problems under consideration but also to establish the existence of solutions that have important properties, namely, solutions that are monotonous, convex or concave and do not change their sign.

In principle, the barrier strips technique provides an a priori estimate for the $(n-1)$th derivative of initial and boundary value problems for nth-order equations. As a consequence, it provides a priori estimates for both the unknown function and its remaining derivatives if at least one value for all of them is known. Moreover, the type of barrier condition depends on what value of the variable is the known value of the $(n-1)$th derivative—at the end of the set interval or at its interior point. All this makes the barrier strips technique applicable to a wide class of initial and boundary value problems.

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