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Existence for stationary mean-field games with congestion and quadratic Hamiltonians

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Abstract. Here, we investigate the existence of solutions to a stationary mean-field game model introduced by J.-M. Lasry and P.-L. Lions. This model features a quadratic Hamiltonian and congestion effects. The fundamental difficulty of potential singular behavior is caused by congestion. Thanks to a new class of a priori bounds, combined with the continuation method, we prove the existence of smooth solutions in arbitrary dimensions.

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1. Introduction

The seminal papers [14–19] on mean-field games have inspired research efforts in the field (for instance, see the recent surveys [2,3,11,20] and the references therein). Nevertheless, several fundamental questions remain unanswered. Here, we address one of these questions by proving the existence of smooth solutions for stationary mean-field games with congestion and quadratic Hamiltonians.

Mean-field games model large populations of rational agents that move according to certain stochastic optimal control goals. To simplify the presentation, we will work in the periodic setting, that is in the d dimensional standard torus, $\mathbb{T}^d$, $d \geq 1$. We consider a large stationary population of agents, whose statistical information is encoded in an unknown probability density, $m : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$. Fix $\mu > 0$. Each individual agent seeks to minimize an infinite-horizon discounted cost given by

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$$u(x) = \inf\left\{\mathbb{E}\left[\int_0^{+\infty} e^{-\mu s}\left(\frac{m(X(s),s)^\alpha |v(s)+b(X(s))|^2}{2}\right.\right.\right.$$
$$\left.\left.\left. +V(X(s),m(X(s),s))\right)\,ds\right]\right\}, \tag{1.1}$$

where the infimum is taken over all progressively measurable controls v,

$$dX = vdt + \sqrt{2}dW_t \quad \text{with } X(0) = x, \tag{1.2}$$

where W_t and $\mathbb{E}$ denote a standard d-dimensional Brownian motion and the expected value, respectively. For convenience, we set the discount rate $\mu = 1$. The constant $0 < \alpha < 1$ determines the strength of congestion effects in the term $m^\alpha|v - b(x)|^2$ and makes it costly to move in areas of high density with a drift v substantially different from a reference vector field $b : \mathbb{T}^d \to \mathbb{R}^d$. The function $V : \mathbb{T}^d \times \mathbb{R}^+ \to \mathbb{R}$ accounts for additional spatial preferences of the agents. We assume that b and V are smooth functions.

Suppose the value function u given by (1.1) is $C^2(\mathbb{T}^d)$. Then, it is a solution to the Hamilton–Jacobi equation

$$u - \Delta u + \frac{|Du|^2}{2m^\alpha} + b(x)\cdot Du = V(x,m).$$

Moreover, the optimal control, v, in (1.2) is determined in feedback form as

$$v(t) = -b(X(t)) - \frac{Du(X(t))}{m^\alpha(X(t))}.$$

If the agents are rational, they will use this feedback control. Here, we suppose that the discount rate, μ, in (1.1) is also the death rate of the agents. That is, in (1.1), agents minimize the average lifetime cost. Finally, we suppose also that agents are born into the system randomly at a unit rate. These three assumptions (rationality, death rate and constant birth rate) determine the density m as a solution to the Fokker–Planck equation

$$m - \Delta m - \operatorname{div}\left(m^{1-\alpha}Du\right) - \operatorname{div}(mb) = 1.$$

The assumption that the death and discount rates are equal is not critical for our methods. For instance, the key estimate in Proposition 2.2 holds for distinct discount rates in the Hamilton–Jacobi equation and the Fokker–Planck equation.

According to the previous discussion, for $u, m : \mathbb{T}^d \to \mathbb{R}$, $m > 0$, our model is given by the system

$$\begin{cases} u - \Delta u + \dfrac{|Du|^2}{2m^\alpha} + b(x)\cdot Du = V(x,m) & \text{in } \mathbb{T}^d, \quad (1.3)\\ m - \Delta m - \operatorname{div}\left(m^{1-\alpha}Du\right) - \operatorname{div}(mb) = 1. & \text{in } \mathbb{T}^d, \quad (1.4)\end{cases}$$

Our focus here is to prove the existence of solutions for the stationary problem. We are interested in classical solutions for (1.3)–(1.4), that is, $(u,m) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ with $m > 0$. Previous work by Lions [21] proved the uniqueness of classical solutions. The fundamental difficulty of potential singular behavior is caused by congestion. The dependence on m in the optimal control problem causes the singularity in the Eq. (1.3), for which we had to develop a new class of estimates. Thanks to those, we obtained our main result:

Theorem 1.1. *Assume the following*

(A1) $0 \le \alpha < 1$;
(A2) $V : \mathbb{T}^d \times \mathbb{R}^+ \to \mathbb{R}$, $V(x, m) \in C^\infty(\mathbb{T}^d \times \mathbb{R}^+)$ *is globally bounded with bounded derivatives and nondecreasing with respect to* m;
(A3) $b : \mathbb{T}^d \to \mathbb{R}^d$, $b \in C^\infty(\mathbb{T}^d)$.

Then, there exists a solution $(u, m) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ *to* (1.3)–(1.4) *with* $m > 0$. *Furthermore, if* V *is strictly increasing with respect to* m, *then a solution is unique.*

Numerous a-priori bounds for mean-field games have been proved by various authors (for instance, see [4,6–10,12,16–18,22,23]). However, these bounds were designed to address a different coupling, namely mean-field games, where the local dependence on m is not singular when $m = 0$. A typical example is the following system:

$$\begin{cases} u - \Delta u + \dfrac{|Du|^2}{2} = m^\alpha & \text{in } \mathbb{T}^d \\ m - \Delta m - \operatorname{div}(mDu) = 1 & \text{in } \mathbb{T}^d. \end{cases} \tag{1.5}$$

In (1.5), the main difficulties are caused by the growth of the nonlinearity m, especially for large $\alpha > 0$, rather than singularities caused by m vanishing. Furthermore, (1.5) can be regarded as an Euler–Lagrange equation of a suitable integral functional, whereas (1.3)–(1.4) do not have this structure.

In Sect. 2, we start by exploring the special form of (1.3)–(1.4) to obtain a bound for $\|m^{-1}\|_{L^\infty(\mathbb{T}^n)}$. This estimate, combined with the techniques from [1], yields a priori regularity in $W^{2,p}(\mathbb{T}^d)$ for any $p \ge 1$. From this, a simple argument shows that any solution to (1.3)–(1.4) is bounded a priori in any Sobolev space, $W^{k,p}(\mathbb{T}^d)$. The estimates from Sect. 2 are only a priori estimates, and there is no general existence theory of weak solutions for mean-field games that can be applied immediately. Thus, we need to address the existence question separately. This is done in Sect. 3, where we prove the existence of solutions to (1.3)–(1.4) by using the continuation method together with the aforementioned a priori estimates. For completeness, Appendix A presents the uniqueness proof for solutions to (1.3)–(1.4), following the ideas in [21] (also see [13]).

2. A priori estimates

In this section, we obtain a priori bounds for solutions of (1.3)–(1.4). In particular, we prove an L^∞ bound for m^{-1}. From this, we derive estimates for u, m in $W^{2,p}(\mathbb{T}^d)$ for any $p \ge 1$. Then, by a bootstrapping argument, we establish smoothness of solutions.

Proposition 2.1. *There exists a constant* $C := C(\|V\|_\infty) \geq 0$ *such that for any classical solution* (u, m) *of* (1.3)–(1.4) *we have* $\|u\|_{L^\infty(\mathbb{T}^d)} \leq C$. *Furthermore,* $m \geq 0$ *on* $\mathbb{T}^d$ *and* $\|m\|_{L^1(\mathbb{T}^d)} = 1$.

Proof. The L^∞ bound is obtained by evaluating the equation at points of maximum of u (resp., minimum) and using the fact that at those points $Du = 0$, $\Delta u \leq 0$ (resp., ≥ 0) and V is bounded on $\mathbb{T}^d \times [0, \infty)$. We then observe that m is nonnegative by the maximum principle. Moreover, it has a total mass of 1 by integrating (1.4). □

Proposition 2.2. *There exists a constant* $C := C(\|b\|_\infty, \|V\|_\infty) \geq 0$ *such that for any classical solution* (u, m) *of* (1.3)–(1.4) *we have*

$$\left\|\frac{1}{m}\right\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

We point out that the above a priori estimate is valid for mean-field games without congestion, i.e., $\alpha = 0$ in (1.3).

Proof. Let $r > \alpha$. Subtract Eq. (1.4) divided by $(r+1-\alpha)m^{r+1-\alpha}$ from Eq. (1.3) divided by rm^r. Then,

$$\begin{aligned}
&\int_{\mathbb{T}^d} \left[u - \Delta u + \frac{|Du|^2}{2m^\alpha} + b \cdot Du - V\right] \cdot \frac{1}{rm^r}\, dx \\
&\quad - \int_{\mathbb{T}^d} \left[m - \Delta m - \operatorname{div}\left(m^{1-\alpha} Du\right) - \operatorname{div}(mb)\right] \cdot \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\, dx \\
&= - \int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\, dx. \qquad (2.1)
\end{aligned}$$

Next, observe that

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r}\, dx = \int_{\mathbb{T}^d} \frac{Du \cdot Dm}{m^{r+1}}\, dx,$$

and

$$\int_{\mathbb{T}^d} \frac{\operatorname{div}\left(m^{1-\alpha} Du\right)}{(r+1-\alpha)m^{r+1-\alpha}}\, dx = \int_{\mathbb{T}^d} \frac{Du \cdot Dm}{m^{r+1}}\, dx.$$

Hence

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r}\, dx - \int_{\mathbb{T}^d} \frac{\operatorname{div}\left(m^{1-\alpha} Du\right)}{(r+1-\alpha)m^{r+1-\alpha}}\, dx = 0. \qquad (2.2)$$

Also, note the identity

$$\begin{aligned}
&\int \operatorname{div}(bm) \frac{m^{-r-1+\alpha}}{r+1-\alpha} = \int m^{-r-1+\alpha} b \cdot Dm \\
&= -\int b \cdot D\left(\frac{m^{-r+\alpha}}{r-\alpha}\right) = \frac{1}{r-\alpha} \int \operatorname{div}(b) m^{-r+\alpha}.
\end{aligned}$$

Then (2.1) is reduced to

$$\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{2rm^{r+\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}}\,dx$$
$$= \int_{\mathbb{T}^d} \left[-\frac{V}{rm^r} - \frac{u}{rm^r} + \frac{1}{(r+1-\alpha)m^{r-\alpha}} - \frac{b\cdot Du}{rm^r} - \frac{1}{r-\alpha}\mathrm{div}\,(b)m^{-r+\alpha} \right] dx$$
$$\leq \int_{\mathbb{T}^d} \frac{C}{rm^r}\,dx + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|b|^2}{rm^{r-\alpha}} + \frac{|Du|^2}{4rm^{r+\alpha}}\,dx$$

in view of Proposition 2.1. Consequently,

$$\int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{4rm^{r+\alpha}}\,dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}}\,dx$$
$$\leq \int_{\mathbb{T}^d} \frac{C}{rm^r}\,dx + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}}\,dx.$$

By Young's inequality for $\alpha \in [0,1)$, we have

$$\frac{C}{rm^r} \leq \frac{1}{4(r+1-\alpha)m^{r+1-\alpha}} + C_r^1$$

and

$$\frac{C}{(r-\alpha)m^{r-\alpha}} \leq \frac{1}{4(r-\alpha)m^{r+1-\alpha}} + C_r^2$$

with

$$C_r^1 := \frac{(1-\alpha)4^{\frac{r}{1-\alpha}}C^{\frac{r+1-\alpha}{1-\alpha}}}{r(r+1-\alpha)}, \quad C_r^2 := \frac{4^{r-\alpha}C^{r+1-\alpha}(r-\alpha)^{r-\alpha-1}}{(r+1-\alpha)^{r+1-\alpha}}.$$

Therefore,

$$\frac{1}{r+1-\alpha}\int_{\mathbb{T}^d} \frac{1}{m^{r-\alpha+1}} \leq 2\left(C_r^1 + C_r^2\right).$$

Thus, we get

$$\left\|\frac{1}{m}\right\|_{L^{r+1-\alpha}(\mathbb{T}^d)} \leq \left[2(r+1-\alpha)(C_r^1 + C_r^2)\right]^{\frac{1}{r+1-\alpha}} =: C_\alpha(r).$$

We can easily check that, for any $r_0 > \alpha$ there exists C_α for which

$$C_\alpha(r) \leq C_\alpha \quad \text{for all } r \in [r_0, \infty).$$

□

Proposition 2.3. *For any $p \geq 1$ there exists a constant $C := C_p(\|b\|_\infty, \|V\|_\infty) > 0$ such that, for any classical solution (u,m) of* (1.3)–(1.4)*, we have $\|u\|_{W^{2,p}(\mathbb{T}^d)} + \|m\|_{W^{2,p}(\mathbb{T}^d)} \leq C$.*

Proof. Let (u,m) be a classical solution (u,m) to (1.3)–(1.4). In view of Lemma [1, Lemma4], combined with Proposition 2.2 we conclude that for all $p \in [1,\infty)$ there exists $C = C(\|V\|_\infty, \|b\|_\infty, p)$ such that

$$\|u\|_{W^{2,p}(\mathbb{T}^d)} \leq C.$$

In light of the Sobolev embedding theorem, we get

$$\|u\|_{C^{1,\gamma}(\mathbb{T}^d)} \le C\|u\|_{W^{2,p}(\mathbb{T}^d)} \le C. \tag{2.3}$$

Then, multiplying (1.4) by m^p and using Young's inequality yield

$$\int_{\mathbb{T}^d} m^{p+1}\,dx + p\int_{\mathbb{T}^d} m^{p-1}|Dm|^2\,dx = \int_{\mathbb{T}^d} m^p\,dx - p\int_{\mathbb{T}^d} m^p(g+b)\cdot Dm\,dx$$
$$\le \left[\frac{1}{2}\int_{\mathbb{T}^d} m^{p+1}\,dx + C\right] + \left[\frac{p}{2}\int_{\mathbb{T}^d} m^{p-1}|Dm|^2\,dx + Cp\int_{\mathbb{T}^d}(|g|^2+|b|^2)m^{p+1}\,dx\right],$$

where $g := Du/m^\alpha$. Noting that $|g| \le C$ in view of (2.3) and $m^{p-1}|Dm|^2 = C_p|Dm^{(p+1)/2}|^2$, we get

$$\int_{\mathbb{T}^d} m^{p+1}\,dx + C_p\int_{\mathbb{T}^d}|Dm^{(p+1)/2}|^2\,dx \le C + C_p'\int_{\mathbb{T}^d} m^{p+1}\,dx. \tag{2.4}$$

Using Hölder's inequality, we have

$$\left(\int_{\mathbb{T}^d} m^{p+1}\right)^{1/(p+1)} \le \left(\int_{\mathbb{T}^d} m\right)^{2/(2+dp)} \left(\int_{\mathbb{T}^d} m^{2^*(p+1)/2}\right)^{\frac{dp/(2+dp)}{2^*(p+1)/2}}.$$

Moreover, using the Sobolev embedding theorem, we get

$$\int_{\mathbb{T}^d} m^{p+1} \le \left(\int_{\mathbb{T}^d} m^{2^*(p+1)/2}\right)^{\frac{dp/(2+dp)}{2^*/2}}$$
$$\le C\left(\int_{\mathbb{T}^d} m^{p+1}\,dx + \int_{\mathbb{T}^d}|Dm^{(p+1)/2}|^2\,dx\right)^{dp/(2+dp)}.$$

Then, using the previous estimate and the fact that $dp/(2+dp) < 1$ on the right-hand side of (2.4), we conclude that

$$\int_{\mathbb{T}^d} m^{p+1}\,dx + \int_{\mathbb{T}^d}|Dm^{(p+1)/2}|^2\,dx \le C. \tag{2.5}$$

Note now that if $m \in W^{1,q}(\mathbb{T}^d)$, we have

$$m - \Delta m = m^{1-\alpha}\Delta u + (1-\alpha)m^{-\alpha}Du\cdot Dm + \operatorname{div}(mb) + 1 \in L^q(\mathbb{T}^d). \tag{2.6}$$

Thus, by standard elliptic regularity $m \in W^{2,q}(\mathbb{T}^d)$ and consequently $m \in W^{1,q^*}(\mathbb{T}^d)$. In light of (2.5) for $p = 1$ we have $m \in W^{1,2}(\mathbb{T}^d)$. So we obtain $m \in W^{2,2}(\mathbb{T}^d)$ and $m \in W^{1,2^*}(\mathbb{T}^d)$. By iterating this argument, we finally get $m \in W^{2,q}(\mathbb{T}^d)$ for any $q < \infty$. □

Proposition 2.4. *For any integer $k \ge 0$ there exists a constant $C := C(\|b\|_\infty, \|V\|_\infty, k) > 0$ such that any classical solution (u, m) of (1.3)–(1.4) satisfies $\|u\|_{W^{k,\infty}(\mathbb{T}^d)} + \|m\|_{W^{k,\infty}(\mathbb{T}^d)} \le C$.*

Proof. Note that $D(m^{-\alpha}) = m^{-(1+\alpha)}Dm \in L^p(\mathbb{T}^d)$ for large $p > 1$ in view of Propositions 2.2, 2.3. This implies $m^{-\alpha} \in W^{1,p}(\mathbb{T}^d)$. Hence, according to Morrey's theorem, we have $m^{-\alpha} \in C^\gamma(\mathbb{T}^d)$ for some $\gamma \in (0,1)$. Also, note that $|Du|^2 \in C^\gamma(\mathbb{T}^d)$.

Therefore, going back to Eq. (1.3) we have

$$u - \Delta u = -\frac{|Du|^2}{2m^\alpha} - b \cdot Du + V \in C^\gamma(\mathbb{T}^d). \tag{2.7}$$

Then, in view of the elliptic regularity theory, we get $u \in C^{2,\gamma}(\mathbb{T}^d)$. Note that the norm in $C^\gamma(\mathbb{T}^d)$ of the right-hand side of (2.7) is estimated by a constant, which only depends on $\|b\|_\infty, \|V\|_\infty$. Thus, $u \le C(\|b\|_\infty, \|V\|_\infty)$. Next, returning to Eq. (2.6) for m and noting that the right-hand side is $C^\gamma(\mathbb{T}^d)$, we get $m \in C^{2,\gamma}(\mathbb{T}^d)$ with $m \le C(\|b\|_\infty, \|V\|_\infty)$.

Once we know that $u, m \in C^{2,\gamma}(\mathbb{T}^d)$—(2.7) and (2.6) imply $u, m \in C^{3,\gamma}(\mathbb{T}^d)$. By continuing this so-called "bootstrap" argument, we reach our conclusion. □

3. Existence by continuation method

In this section, we prove the existence of a unique classical solution to (1.3)–(1.4) by using the continuation method. We work under the assumptions of Theorem 1.1. For $0 \le \lambda \le 1$, we consider the problem

$$\begin{cases} u_\lambda - \Delta u_\lambda + \dfrac{|Du_\lambda|^2}{2m_\lambda^\alpha} + \lambda b(x) \cdot Du_\lambda - \lambda V(x,m) - (1-\lambda)V_0(m) = 0 & \text{in } \mathbb{T}^d, \\ m_\lambda - \Delta m_\lambda - \operatorname{div}\left(m_\lambda^{1-\alpha} Du_\lambda\right) - \lambda \operatorname{div}(b m_\lambda) = 1 & \text{in } \mathbb{T}^d, \end{cases} \tag{3.1}$$

where $V_0(m) := \arctan(m)$. Let $E^k := H^k(\mathbb{T}^d) \times H^k(\mathbb{T}^d)$ for $k \in \mathbb{N}$ and $E^0 := L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$.

For any $k_0 \in \mathbb{N}$ with $k_0 > d/2$, we define the map $F : [0,1] \times E^{k_0+2} \to E^{k_0}$ by

$$F(\lambda, u, m) := \begin{pmatrix} u - \Delta u + \dfrac{|Du|^2}{2m^\alpha} + \lambda b(x) \cdot Du - \lambda V(x,m) - (1-\lambda)V_0(m) \\ m - \Delta m - \operatorname{div}\left(m^{1-\alpha} Du\right) - \lambda \operatorname{div}(bm) - 1. \end{pmatrix}.$$

Then, we can rewrite (3.1) as

$$F(\lambda, u_\lambda, m_\lambda) = 0.$$

Note that for any $\gamma > 0$, the map F is C^∞ in the set $\{(u,m) \in E^{k_0+2}(\mathbb{T}^d), m > \gamma\}$. This is because for $k_0 > d/2$, the Sobolev space $H^{k_0}(\mathbb{T}^d)$ is an algebra. Moreover, if k_0 is large enough, then any solution (u_λ, m_λ) in E^{k_0+2} is, in fact, in E^{k+2} for all $k \in \mathbb{N}$, by the a priori bounds in Sect. 2.

We define the set Λ by

$$\Lambda := \left\{\lambda \in [0,1] \mid (3.1) \text{ has a classical solution } (u,m) \in E^{k_0+2}\right\}.$$

When $\lambda = 0$ we have an explicit solution, namely $(u_0, m_0) = (\pi/4, 1)$; therefore, $\Lambda \ne \emptyset$. The main goal of this section is to prove

$$\Lambda = [0,1].$$

To prove this, we show that Λ is relatively closed and open on $[0,1]$.

The closeness of Λ is a straightforward consequence of the estimates in Sect. 2:

Proposition 3.1. *The set Λ is closed.*

Proof. To prove that Λ is closed, we need to check that for any sequence $\lambda_n \in \Lambda$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$, gives $\lambda_0 \in \Lambda$. Next, we must fix such a sequence and corresponding solutions $(u_{\lambda_n}, m_{\lambda_n})$ to (3.1) with $\lambda = \lambda_n$. Since the a-priori bound in Proposition 2.4 is independent of $n \in \mathbb{N}$, by taking a subsequence, if necessary, we may assume that $(u_{\lambda_n}, m_{\lambda_n}) \to (u, m)$ in E^{k_0+2}. Moreover, $m_{\lambda_n}^{-1} \to m^{-1}$ in $C(\mathbb{T}^d)$. Therefore, if we take the limit in (3.1), we get that (u, m) is the solution to (3.1) with $\lambda = \lambda_0$. This implies $\lambda_0 \in \Lambda$. $\square$

To prove that Λ is relatively open in $[0, 1]$, we need to check that for any $\lambda_0 \in \Lambda$ there exists a neighborhood of λ_0 contained in Λ. To do so, we will use the implicit function theorem (for example, see [5], chapter X). For a fixed $\lambda_0 \in \Lambda$, we consider the Fréchet derivative $\mathcal{L}_{\lambda_0} : E^{k_0+2} \to E^{k_0}$ of $(u, m) \mapsto F(\lambda_0, u, m)$ at the point $(u_{\lambda_0}, m_{\lambda_0})$, which is given by

$$\mathcal{L}_{\lambda_0}(v, f) = \begin{pmatrix} v-\Delta v+\dfrac{Du_{\lambda_0}\cdot Dv}{m_{\lambda_0}^{\alpha}}-\dfrac{\alpha|Du_{\lambda_0}|^2 f}{2m_{\lambda_0}^{\alpha+1}}+\lambda_0 b\cdot Dv-(\lambda_0 D_m V+(1-\lambda_0)D_m V_0)\, f \\ f-\Delta f-\mathrm{div}\,\left(m_{\lambda_0}^{1-\alpha}Dv\right)-(1-\alpha)\mathrm{div}\,\left(m_{\lambda_0}^{-\alpha} f Du_{\lambda_0}\right)-\lambda_0\mathrm{div}\,(bf) \end{pmatrix}. \tag{3.2}$$

Because of the a priori bounds for u and m in Sect. 2, we can extend the domain of $\mathcal{L}_{\lambda_0}$ by continuity to E^{k+2} for any $k \leq k_0$. We will prove that $\mathcal{L}_{\lambda_0}$ is an isomorphism from E^{k+2} to E^k for any $k \geq 0$.

Define the bilinear mapping $B_{\lambda_0}[w_1, w_2] : E^1 \to \mathbb{R}$ by

$$\begin{aligned} B_{\lambda_0}[w_1, w_2] := \int_{\mathbb{T}^d} \Bigg[v_1 + &-\frac{\alpha|Du_{\lambda_0}|^2 f_1}{2m_{\lambda_0}^{\alpha+1}} + \lambda_0 b \\ &\cdot Dv_1 - (\lambda_0 D_m V + (1-\lambda_0) D_m V_0)\, f_1 \Bigg] f_2 \\ &+ Dv_1 \cdot Df_2 - m_{\lambda_0}^{1-\alpha} Dv_1 Dv_2 \\ &+ \left[f_1 - (1-\alpha)\mathrm{div}\,\left(m_{\lambda_0}^{-\alpha} f_1 Du_{\lambda_0}\right) - \lambda_0 \mathrm{div}\,(bf_1)\right](-v_2) \\ &- Df_1 \cdot Dv_2\, dx. \end{aligned}$$

We set $Pw := (f, -v)$ for $w = (v, f)$ and observe that if $w_1 \in E^k$ with $k \geq 2$, then

$$B_{\lambda_0}[w_1, w_2] = \int_{\mathbb{T}^d} \mathcal{L}_{\lambda_0}(w_1) \cdot Pw_2\, dx.$$

The boundedness of B_{λ_0} is a straightforward result of Proposition 2.4:

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$|B_{\lambda_0}[w_1, w_2]| \leq C\|w_1\|_{E^1}\|w_2\|_{E^1}$$

for any $w_1, w_2 \in E^1$.

Thus, in view of the Riesz representation theorem for Hilbert spaces, there exists a linear mapping $A : E^1 \to E^1$ such that

$$B_{\lambda_0}[w_1, w_2] = (Aw_1, w_2)_{E^1}.$$

Lemma 3.3. *The operator A is injective.*

Proof. Let $w = (v, f)$. By Young's inequality we have

$$\begin{aligned}
B_{\lambda_0}[w, w] &= \int_{\mathbb{T}^d} \frac{\alpha Du_{\lambda_0} \cdot Dv}{m_{\lambda_0}^{\alpha}} f - \frac{\alpha |Du_{\lambda_0}|^2}{2m_{\lambda_0}^{\alpha+1}} f^2 \\
&\quad - (\lambda_0 D_m V + (1-\lambda_0) D_m V_0)\, f^2 - m_{\lambda_0}^{1-\alpha} |Dv|^2 \, dx \\
&\leq \int_{\mathbb{T}^d} - (\lambda_0 D_m V + (1-\lambda_0) D_m V_0)\, f^2 + \frac{(\alpha-2) m_{\lambda_0}^{1-\alpha} |Dv|^2}{2} \\
&\leq -C_{\lambda_0} (\|Dv\|^2_{L^2(\mathbb{T}^d)} + \|f\|^2_{L^2(\mathbb{T}^d)})
\end{aligned}$$

for a constant C_{λ_0}, which depends on bounds for m_{λ_0} and Du_{λ_0}, but is strictly positive for any solution to (3.1) since $0 \leq \alpha < 1$. We have used Assumption (A1) and the strict monotonicity of V on m. This implies that if $Aw = 0$ then we have $w = (\mu, 0)$ for some constant μ. Next, by computing

$$0 = (Aw, (0, \mu)) = B[(\mu, 0), (0, \mu)] = \mu^2,$$

we conclude that $\mu = 0$. □

Remark 1. Note that the injectivity of the operator A holds for all $0 \leq \alpha < 2$. However, the a priori estimates of the previous section are only valid for $0 \leq \alpha < 1$.

Lemma 3.4. *The range $R(A)$ is closed and $R(A) = E^1$.*

Proof. Take a Cauchy sequence, z_n, in the range of A, that is $z_n = Aw_n$, for some sequence $w_n = (v_n, f_n)$. We claim that w_n is a Cauchy sequence. We have

$$\begin{aligned}
(z_n - z_m, w_n - w_m)_{E^1} &= (A(w_n - w_m), w_n - w_m)_{E^1} \\
&\leq -C_{\lambda_0} (\|D(v_n - v_m)\|^2_{L^2(\mathbb{T}^d)} + \|f_n - f_m\|^2_{L^2(\mathbb{T}^d)}).
\end{aligned}$$

Note that

$$\begin{aligned}
&|(z_n - z_m, w_n - w_m)_{E^1}| \\
&\quad \leq \|z_n - z_m\|_{E^0} \|w_n - w_m\|_{E^0} + \|D(z_n - z_m)\|_{E^0} \|D(w_n - w_m)\|_{E^0} \\
&\quad \leq \|z_n - z_m\|_{E^0} \left(\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|f_n - f_m\|_{L^2(\mathbb{T}^d)}\right) \\
&\qquad + \|D(z_n - z_m)\|_{E^0} \left(\|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)}\right) \\
&\quad \leq \varepsilon \left(\|f_n - f_m\|^2_{L^2(\mathbb{T}^d)} + \|D(v_n - v_m)\|^2_{L^2(\mathbb{T}^d)}\right) + C_\varepsilon \|z_n - z_m\|^2_{E^1} \\
&\qquad + \|z_n - z_m\|_{E^1} \left(\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)}\right)
\end{aligned}$$

for $\varepsilon > 0$.

If we fix a suitable small ε and combine the inequalities above, then we obtain

$$\begin{aligned}
&\|D(v_n - v_m)\|^2_{L^2(\mathbb{T}^d)} + \|f_n - f_m\|^2_{L^2(\mathbb{T}^d)} \\
&\quad \leq C\|z_n - z_m\|^2_{E^1} + C\|z_n - z_m\|_{E^1}(\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2}(\mathbb{T}^d)) \\
&\quad \leq C\|z_n - z_m\|^2_{E^1} + C\|z_n - z_m\|_{E^1}\|w_n - w_m\|_{E^1}.
\end{aligned} \tag{3.3}$$

We have

$$B[w_n - w_m, (-f_n + f_m, v_n - v_m,)] = \|w_n - w_m\|^2_{E^1} + E_{nm},$$

where, using (3.3), E_{nm} satisfies

$$\begin{aligned}
|E_{nm}| &\leq C\|v_n - v_m\|_{L^2(\mathbb{T}^d)}\|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} + C\|f_n - f_m\|_{L^2(\mathbb{T}^d)}\|v_n - v_m\|_{L^2(\mathbb{T}^d)} \\
&\quad + C\|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)}\|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} \\
&\quad + C\|f_n - f_m\|_{L^2(\mathbb{T}^d)}\|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)} \\
&\leq C\|w_n - w_m\|_{E^1}\left(\|z_n - z_m\|^2_{E^1} + \|z_n - z_m\|_{E^1}\|w_n - w_m\|_{E^1}\right)^{1/2}.
\end{aligned} \tag{3.4}$$

On the other hand, by Lemma 3.2 we have

$$B[w_n - w_m, (-f_n + f_m, v_n - v_m,)] \leq C\|z_n - z_m\|_{E^1}\|w_n - w_m\|_{E^1}. \tag{3.5}$$

Combining (3.4) and (3.5) we deduce

$$\begin{aligned}
&\|w_n - w_m\|^2_{E^1} \\
&\quad \leq C\|z_n - z_m\|_{E^1}\|w_n - w_m\|_{E^1} \\
&\qquad + C\|w_n - w_m\|_{E^1}\left(\|z_n - z_m\|^2_{E^1} + \|z_n - z_m\|_{E^1}\|w_n - w_m\|_{E^1}\right)^{1/2}.
\end{aligned}$$

By using Young's inequality we conclude

$$\|w_n - w_m\|^2_{E^1} \leq C\|z_n - z_m\|^2_{E^1}.$$

From this we get convergence in E^1.

Finally, we prove that $R(A) = E^1$. Suppose that $R(A) \neq E^1$. Since $R(A)$ is closed, if $R(A) \neq E^1$ there exists $z \in R(A)^\perp$ with $z \neq 0$ such that $B_{\lambda_0}[z, z] = 0$. The argument in the proof of Lemma 3.3 implies $z = 0$, which is a contradiction. □

Lemma 3.5. *The operator $\mathcal{L}_{\lambda_0} : E^{k+2} \to E^k$ is an isomorphism for all $k \in \mathbb{N}$ with $k \geq 2$.*

Proof. Since $\mathcal{L}_{\lambda_0}$ is injective, it suffices to prove that it is surjective. To do so, fix $w_0 \in E^k$ with $w_0 = (v_0, f_0)$. We claim there exists a solution $w_1 \in E^{k+2}$ to $\mathcal{L}_{\lambda_0} w_1 = w_0$.

Consider the bounded linear functional $w \mapsto (w_0, w)_{E^0}$ in E^1. According to the Riesz representation theorem, $\tilde{w} \in E^1$ exists such that $(w_0, w)_{E^0} =$

$(\tilde{w}, w)_{E^1}$ for any $w \in E^1$. In light of Lemmas 3.3 and 3.4, there exists the inverse of A. We define $w_1 := A^{-1}\tilde{w}$ and write $w_1 = (v, f)$. Set

$$\begin{pmatrix} g_1[v,f] \\ g_2[v,f] \end{pmatrix} := \begin{pmatrix} \dfrac{Du_{\lambda_0} \cdot Dv}{m_{\lambda_0}^{\alpha}} - \dfrac{\alpha |Du_{\lambda_0}|^2 f}{2m_{\lambda_0}^{\alpha+1}} + \lambda_0 b \cdot Dv - (\lambda_0 D_m V + (1-\lambda_0) D_m V_0)\, f \\ -\mathrm{div}\, \left(m_{\lambda_0}^{1-\alpha} Dv\right) - (1-\alpha)\mathrm{div}\, \left(m_{\lambda_0}^{-\alpha} f Du_{\lambda_0}\right) - \lambda_0 \mathrm{div}\,(bf) \end{pmatrix}.$$

Then, the identity

$$(Aw_1, w)_{E^1} = (\tilde{w}, w)_{E^1} = (w_0, w)_{E^0}$$

for any $w \in E^1$, means that v is a weak $H^1(\mathbb{T}^d)$ solution to

$$v - \Delta v = g_1[v, f] + v_0$$

and that $f \in H^1(\mathbb{T}^d)$ is also a weak solution to

$$f - \Delta f = g_2[v, f] + f_0.$$

Observe that if $v, f \in H^{j+1}(\mathbb{T}^d)$ then $g_1, g_2 \in H^j(\mathbb{T}^d)$. Additionally, elliptic regularity yields, from $g_i[v, f] \in H^j(\mathbb{T}^d)$, that $v, j \in H^{j+2}(\mathbb{T}^d)$. Since we have $v, f \in H^1(\mathbb{T}^d)$, we conclude by induction that $v, f \in H^{j+2}(\mathbb{T}^d)$ holds for all $j \leq k$. □

A straightforward result from Lemma 3.5 and the implicit function theorem in Banach space is

Proposition 3.6. *The set* Λ *is relatively open in* $[0, 1]$.

Finally, we address the existence of solutions to (1.3)–(1.4), and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. If V is strictly increasing on m, the existence of a classical solution to (1.3)–(1.4) is a straightforward result of Proposition 3.6. Uniqueness of the solution is discussed in the Appendix, Proposition A.1.

If we only assume V to be nondecreasing on m, existence can be obtained by using a perturbation argument similar to the one in (3.1). More precisely, we add a small perturbation $\varepsilon \arctan(m)$ to V so that we make the potential term strictly monotone. This problem admits a unique classical solution (u_ϵ, m_ϵ). Because the a priori bounds in the previous section do not depend on the strict monotonicity of V, (u_ϵ, m_ϵ) satisfy uniform bounds in any Sobolev space. Thus, by compactness, we can extract a convergent subsequence to a limit (u, m), which solves (1.3)–(1.4). □

Remark 2. In this paper, because our main focus is to achieve a lower bound on m, we investigate the case where the potential V is bounded. In principle, unbounded potentials can be studied by adapting the techniques in [4,7,9,18], for instance.

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Appendix A. Uniqueness

Uniqueness of solutions of (1.3)–(1.4) is well understood (see [13,17] for a related problem). However, to make this paper self-contained, we give a proof based on Lions ideas in [21].

Proposition A.1. *The system* (1.3)–(1.4) *admits at most one classical solution* (u, m).

Proof. Let (u_0, m_0) and (u_1, m_1) be classical solutions to (1.3)–(1.4). Subtract (1.3) for (u_1, m_1) from (1.3) for (u_0, m_0) and (1.4) for (u_1, m_1) from (1.4) for (u_0, m_0), respectively, and then

$$u_0 - u_1 = \Delta(u_0 - u_1) + \frac{|Du_1|^2}{2m_1^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} + b \cdot D(u_1 - u_0) + V(x, m_0) - V(x, m_1), \tag{A.1}$$

$$m_0 - m_1 = \Delta(m_0 - m_1) + \operatorname{div}(m_0^{1-\alpha} Du_0) - \operatorname{div}(m_1^{1-\alpha} Du_1) + \operatorname{div}(b(m_0 - m_1)). \tag{A.2}$$

Subtract (A.2) multiplied by $u_0 - u_1$ from (A.1) multiplied by $m_0 - m_1$: then

$$\int_{\mathbb{T}^d} \left(\frac{|Du_1|^2}{2m_1^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} \right)(m_0 - m_1)\, dx + \int_{\mathbb{T}^d} \left(m_0^{1-\alpha} Du_0 - m_1^{1-\alpha} Du_1 \right) \cdot D(u_0 - u_1)\, dx = \int_{\mathbb{T}^d} (V(x, m_1) - V(x, m_0))(m_0 - m_1)\, dx. \tag{A.3}$$

We prove that the left-hand side of (A.3) is nonnegative if $\alpha \in [0, 2]$, following the technique in [13]. Set $u_\theta := u_0 + \theta(u_1 - u_0)$ and $m_\theta := m_0 + \theta(m_1 - m_0)$ for $\theta \in [0, 1]$. Define

$$I(\theta) := \left[-\int_{\mathbb{T}^d} \left(\frac{|Du_\theta|^2}{2m_\theta^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} \right)(m_1 - m_0) + \int_{\mathbb{T}^d} \left(m_\theta^{1-\alpha} Du_\theta - m_0^{1-\alpha} Du_0 \right) \cdot D(u_1 - u_0)\, dx \right].$$

Then

$$\frac{d}{dt} I(\theta) = -\alpha \int_{\mathbb{T}^d} \frac{Du_\theta \cdot D(u_1 - u_0)(m_1 - m_0)}{m_\theta^\alpha}\, dx + \frac{\alpha}{2} \int_{\mathbb{T}^d} \frac{|Du_\theta|^2 (m_1 - m_0)^2}{m_\theta^{1+\alpha}}\, dx + \int_{\mathbb{T}^d} m_\theta^{1-\alpha} |D(u_1 - u_0)|^2\, dx$$
$$\geq \left(1 - \frac{\alpha}{2}\right) \int_{\mathbb{T}^d} m_\theta^{1-\alpha} |D(u_1 - u_0)|^2\, dx \geq 0$$

for $\alpha \in [0, 2]$. Noting that $I(0) = 0$, we conclude that $I(1) \geq 0$, which proves that the left-hand side of (A.3) is nonnegative as claimed. The proposition follows using the assumption that V is strictly increasing on m. □

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