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Existence, global nonexistence, and asymptotic behavior of solutions for the Cauchy problem of a multidimensional generalized damped Boussinesq-type equation

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Abstract: We consider the existence, both locally and globally in time, the global nonexistence, and the asymptotic behavior of solutions for the Cauchy problem of a multidimensional generalized Boussinesq-type equation with a damping term.

Key words: Existence, global nonexistence, asymptotic behavior, Boussinesq equations, damping term

1. Introduction

In this paper, we study the Cauchy problem of the generalized multidimensional Boussinesq-type equation with a damping term

$$u_{tt} - \triangle u - a \triangle u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - k \triangle u_t = \triangle f(u), \quad (x,t) \in R^n \times (0,+\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in R^n, \tag{1.2}$$

where $u(x,t)$ denotes the unknown function, $f(s)$ is the given nonlinear function, $u_0(x)$ and $u_1(x)$ are the given initial value functions, k is a constant, the subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x, and $\triangle$ denotes the Laplace operator in R^n.

The effects of small nonlinearity and dispersion are taken into consideration in the derivation of Boussinesq equations, but in many real situations, damping effects are compared in strength to the nonlinear and dispersive ones. Therefore, the damped Boussinesq equation is considered as well:

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta\left(u^2\right)_{xx}, \tag{1.3}$$

where u_{txx} is the damping term, $a,\ b = const > 0$, and $\beta = const \in R$ (see [6] and references therein).

Varlamov [12] investigated the long-time behavior of solutions to initial value, spatially periodic, and initial-boundary value problems for equation (1.3) in 2 space dimensions. Polat et al. [8] established the blow up of the solution for the initial boundary value problem of the damped Boussinesq equation

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = f(u)_{xx}.$$

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The asymptotic behavior and the the blow up of the solution for a nonlinear evolution equation of fourth order

$$u_{tt} - a_1 u_{xx} - a_2 u_{xxt} - a_3 u_{xxtt} = f\left(u_x\right)_x$$

were established in [1].

Polat and Kaya [7] studied the existence, both locally and globally in time, the asymptotic behavior, and the blow up of the solution for a class of nonlinear wave equations with dissipative and dispersive terms

$$u_{tt} - u_{xx} - u_{xxtt} - \lambda u_{xxt} + u = f\left(u_x\right)_x .$$

Wang and Chen [14] studied the global existence and the blow up of the solution for the Cauchy problem of a generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = f\left(u\right)_{xx} . \tag{1.4}$$

Polat and Ertaş [6] extended the result of [14] to the multidimensional version of equation (1.4).

Recently, higher order Boussinesq equations have been investigated. Schneider and Eugene [9] considered a class of Boussinesq equations that models the water wave problem with surface tension as follows:

$$u_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxx} + u_{xxxxtt} = \left(u^2\right)_{xx} ,$$

where $x, t, \mu \in R$ and $u\left(x, t\right) \in R$.

Wang and Mu [15] obtained the global existence and the blow up of the solution for the Cauchy problem of equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} + u_{xxxxtt} = f\left(u\right)_{xx} \tag{1.5}$$

in multidimensional form. Wang and Guo [16] obtained the local existence and the blow up of the solution for the initial boundary value problem of equation (1.5) in the absence of u_{xx} and u_{xxtt} terms; in order to prove the local existence the Galerkin method was used, and the blow up was obtained by using the concavity method of Glassey. Wang and Xue [17] obtained the global existence and nonexistence of the solution for the Cauchy problem of equation (1.5) when $f\left(u\right) = \beta \left|u\right|^p$, $\beta \neq 0$, and $p > 1$ are constants, by the potential well method. Wang and Xu [18] obtained the global existence and nonexistence of the solution for the Cauchy problem of equation (1.5) in the absence of u_{xx} and u_{xxtt} terms. When $f\left(u\right) = -\beta \left|u\right|^p u$, $\beta > 0$, and $p > 1$ are constants, the global existence and nonexistence are proved with the aid of the potential well method. Duruk et al. [2] established the global well-posedness of the Cauchy problem of equation (1.5) in the absence of u_{xxxx} term.

Throughout this paper, we use the following notations and lemmas.

L^p $(1 \leq p \leq \infty)$ denotes the usual space of all L^p functions on R^n with norm $\left\|f\right\|_{L^p} = \left\|f\right\|_p$ and the abbreviations $\left\|f\right\|_{L^2} = \left\|f\right\|$ will be used. H^s denotes the usual Sobolev space on R^n with norm $\left\|f\right\|_{H^s} = \left\|\left(I - \triangle\right)^{\frac{s}{2}} f\right\|_2$, where $s \in R$. $u * v$ is the convolution defined by

$$u * v\left(x\right) = \int_R u\left(y\right) v\left(x - y\right) dy.$$

Lemma 1.1 *(see [13]). Assume that* $f(u) \in C^k(R)$, $f(0) = 0$, $u \in H^s \cap L^\infty$, *and* $k = [s] + 1$, *where* $s \geq 0$. *Then we have*

$$\|f(u)\|_s \leq C_0(\|u\|_\infty)\|u\|_s,$$

where $C_0(\|u\|_\infty)$ *is a constant dependent on* $\|u\|_\infty$.

Lemma 1.2 *(see [13]). If* $s > 0$, *then* $H^s \cap L^\infty$ *is an algebra. Moreover,*

$$\|uv\|_s \leq C(\|u\|_\infty\|v\|_s + \|u\|_s\|v\|_\infty)$$

for $u, v \in H^s \cap L^\infty$.

Lemma 1.3 *(Sobolev imbedding theorem) (see [11]). (1) If* $s > \frac{n}{2} + k$, *where* k *is a nonnegative integer, then*

$$H^s \subset C^k(R^n) \cap L^\infty,$$

(2) If $s = \frac{n}{2}$, *then for* $p \in [2, \infty)$

$$H^s \subset L^p,$$

(3) If $s < \frac{n}{2}$, *we have*

$$H^s \subset L^{\frac{2n}{n-2}}.$$

Let $G(x)$ *be the fundamental solution of the partial differential equation*

$$u(x) - \triangle u(x) = 0.$$

We use the Fourier transform to obtain

$$G(x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty e^{-|x|^2/4\delta} e^{-\delta} \delta^{-n/2} d\delta, \quad x \in R^n.$$

From [10], we can prove that the fundamental solution $G(x)$ *satisfies the following properties.*

Lemma 1.4 *(see [10]).*

(1) The fundamental solution $G(x)$ *is defined and continuous on* R^n, *and* $G(x) > 0$.

(2) $G(x) \in L^q(R^n)$ *and* $\|G(x)\|_1 = 1$, *where* $1 \leq q \leq \infty$ *if* $n = 1$, $1 \leq q < \infty$ *if* $n = 2$, $1 \leq q \leq n/(n-2)$ *if* $n \geq 3$.

(3) $G(x)$ *satisfies the equation*

$$G(x) - \triangle G(x) = \delta(x),$$

where $\delta(x)$ *is the Dirac delta function.*

(4)

$$\|G * u\|_s = \|u\|_{s-2},$$

where $u * v$ *denotes the convolution of* u *and* v.

The plan of this paper is as follows. In section 2, for the special case $a = 2$, we prove the existence and the uniqueness of the local solution for problem (1.1), (1.2). The existence and the uniqueness of the global solution of the problem are proved in section 3. The proof of the global nonexistence of the solution of the problem is given in Section 4. In Section 5, the asymptotic behavior of the global solution for the problem is discussed.

2. Existence and uniqueness of local solution

In this section, we prove the existence and the uniqueness of the local solution for problem (1.1), (1.2) by contraction mapping principle. For this, we construct the solution of the problem as a fixed point of the solution operator associated with a related family of the Cauchy problem for a linear wave equation.

We can rewrite equation (1.1) as follows:

$$[u_{tt} - \triangle u - \triangle u_{tt} + f(u)] - \triangle [u_{tt} - \triangle u - \triangle u_{tt} + f(u)] = f(u) + k \triangle u_t. \tag{2.6}$$

Using the fundamental solution $G(x)$, equation (2.1) is equivalent to

$$u_{tt} - \triangle u - \triangle u_{tt} = -f(u) + G * [f(u) + k \triangle u_t]. \tag{2.7}$$

Now, we proceed with the following linear wave equation

$$u_{tt} - \triangle u - \triangle u_{tt} = h(x,t), \quad x \in R^n, \quad t > 0, \tag{2.8}$$

with the initial value conditions (1.2). By means of the Galerkin method and integral estimations we can prove the following lemma.

Lemma 2.1 *(see[5]). Assume that $u_0 \in H^{s+1}$, $u_1 \in H^{s+1}$, for any $T > 0$, $h \in L^2([0,T]; H^s) \cap C([0,T]; H^{s-1})$, then problem (2.3), (1.2) has a unique solution $u \in C^1([0,T]; H^{s+1})$ and there exists the estimation*

$$\begin{aligned} & \|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 + \|u_{tt}(.,t)\|_{s+1}^2 \\ \leq \; & 6e^{3T}\left(\|u_0\|_{s+1}^2 + \|u_1\|_{s+1}^2 + \int_0^t \|h(.,\tau)\|_s^2 \, d\tau\right) \\ & +2\max_{t\in[0,T]} \|h(.,t)\|_{s-1}^2, \quad 0 \leq t \leq T, \end{aligned} \tag{2.9}$$

where $s \geq 1$ is an arbitrary integer.

Let us define the function space

$$B(T) = C^2([0,T]; H^{s+1}),$$

which is endowed with the norm defined by

$$\|u(t)\|_{B(T)}^2 = \max_{t\in[0,T]}\left[\|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 + \|u_{tt}(.,t)\|_{s+1}^2\right], \quad \forall u \in B(T).$$

It is easy to see that $B(T)$ is a Banach space.

For M, $T > 0$, $u_0 \in H^{s+1}$, $u_1 \in H^{s+1}$, set

$$X(M,T) = \left\{u \mid u \in B(T), \|u(t)\|_{B(T)} \leq M\right\}. \tag{2.10}$$

Obviously, $X(M,T)$ is a nonempty complete metric space for any $M, T > 0$. The Sobolev imbedding theorem and Lemma 1.4 imply that if $u \in X(M,T)$ and $s > \frac{n}{2} + 1$ is a positive integer, then $u \in C([0,T] \times R^n)$. Moreover, it follows from (2.5) and Sobolev imbedding theorem that

$$\max_{(x,t)\in R^n \times [0,T]} \|\triangle u\| \leq M, \quad \forall u \in X(M,T). \tag{2.11}$$

For $\forall w \in X(M,T)$, *we consider the linear equation*

$$u_{tt} - \triangle u - \triangle u_{tt} = -f(w) + G * [f(w) + k \triangle w_t], \tag{2.12}$$

and let S *denote the map that carries* w *into the unique solution. Our goal is to show that* S *has a unique fixed point in* $X(M,T)$ *for appropriately chosen* M *and* T*. For this purpose, we shall employ the contraction mapping principle and Lemma 2.1. Firstly, we prove the following lemma.*

Lemma 2.2 *Assume that* $s > \frac{n}{2} + 1$, $u_0 \in H^{s+1}$, $u_1 \in H^{s+1}$, *and* $f, g \in C^{[s]+1}$; *then* S *maps* $X(M,T)$ *into* $X(M,T)$ *for* M *sufficiently large and* T *sufficiently small relative to* M.

Proof Let $M, T > 0$ and $w \in X(M,T)$ is given. Define $h(x,t)$ by

$$h(x,t) = -f(w(x,t)) + G * [f(w(x,t)) + k \triangle w_t(x,t)]. \tag{2.13}$$

Using Lemmas 1.1 and 1.4, it easily follows that

$$\begin{aligned} \|h(x,t)\|_s &\le \|f(w)\|_s + \|G * [f(w) + k \triangle w_t]\|_s \\ &\le (2C_0 + k) M, \end{aligned}$$

and

$$\int_0^t \|h(.,\tau)\|_s^2 \, d\tau \le (2C_0 + k)^2 M^2 T. \tag{2.14}$$

Moreover, from Lemmas 1.1, 1.2, and 1.3, we have

$$\begin{aligned} h_t(x,t) &= -f'(w(x,t)) w_t(x,t) \\ &\quad + G * [f'(w(x,t)) w_t(x,t) + k \triangle w_{tt}(x,t)], \end{aligned} \tag{2.15}$$

which yields

$$\begin{aligned} \|h_t(x,t)\|_{s-1} &\le C \left(\|f'(w(x,t))\|_\infty \|w_t\|_{s-1} + \|f'(w(x,t))\|_{s-1} \|w_t\|_\infty \right) \\ &\quad + C \left(\|f'(w(x,t))\|_\infty \|w_t\|_{s-2} + \|f'(w(x,t))\|_{s-2} \|w_t\|_\infty \right) \\ &\quad + k \|\triangle w_{tt}\|_{s-1} \\ &\le 4CC_0 M^2 + kM \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \max_{t\in[0,T]} \|h(.,t)\|_{s-1} &= \max_{t\in[0,T]} \left\| h(.,0) + \int_0^t h_\tau(.,\tau) \, d\tau \right\|_{s-1} \\ &\le \|-f(u_0) + G * [f(u_0) + k \triangle u_1]\|_{s-1} \\ &\quad + (4CC_0 M + k) MT. \end{aligned} \tag{2.17}$$

From (2.9) and (2.12) we conclude that $h(x,t) \in L^2([0,T]; H^s) \cap C([0,T]; H^{s-1})$.

From Lemma 2.1 we have

$$\begin{aligned} & \|u(.,t)\|_{s+1}^2+\|u_t(.,t)\|_{s+1}^2+\|u_{tt}(.,t)\|_{s+1}^2 \\ \leq\ & 6e^{3T}\left(\|u_0\|_{s+1}^2+\|u_1\|_{s+1}^2\right)+6e^{3T}\left(2C_0+k\right)^2M^2T \\ & +4\left\|-f(u_0)+G*\left[f(u_0)+k\triangle u_1\right]\right\|_{s-1}^2 \\ & +4\left(4CC_0M+k\right)^2M^2T^2. \end{aligned} \tag{2.18}$$

If M and T satisfy

$$\begin{aligned} M^2 \ \geq\ & 12e^3\left(\|u_0\|_{s+1}^2+\|u_1\|_{s+1}^2\right) \\ & +8\left\|-f(u_0)+G*\left[f(u_0)+k\triangle u_1\right]\right\|_{s-1}^2, \end{aligned} \tag{2.19}$$

$$T\leq \min\left\{1,\left[12e^3\left(2C_0+k\right)^2+8\left(4CC_0M+k\right)^2\right]^{-1}\right\}, \tag{2.20}$$

then the right-hand side of (2.13) is dominated by M^2 and consequently

$$\|(Sw)(.,t)\|_{s+1}^2+\|(Sw)_t(.,t)\|_{s+1}^2+\|(Sw)_{tt}(.,t)\|_{s+1}^2\leq M^2,\qquad \forall w\in X(M,T).$$

This completes the proof of Lemma 2.2. □

Lemma 2.3 *$S:X(M,T)\longrightarrow X(M,T)$ is strictly contractive if M is sufficiently large and T is sufficiently small relative to M.*

Proof Let $M,T>0$ and $w,\overline{w}\in X(M,T)$ be given. For w and $\overline{w}$ there are the corresponding solutions $u=Sw,\ \overline{u}=S\overline{w}$, for problem (2.3), (1.2). Set $U=u-\overline{u},\ W=w-\overline{w}$, and note that

$$U_{tt}-\triangle U-\triangle U_{tt}=H(x,t),\quad (x,t)\in R^n\times[0,T], \tag{2.21}$$

$$U(x,0)=0,\ U_t(x,0)=0,\quad x\in R^n, \tag{2.22}$$

where $H(x,t)$ is defined by

$$H(x,t)=-f(w)+f(\overline{w})+G*\left[f(w)-f(\overline{w})+k\triangle W_t\right]. \tag{2.23}$$

It is observed that H has the smoothness required to apply Lemma 1.1 to (2.16), (2.17). By aid of Lemma 2.1 we estimate U in terms of W.

A simple computation shows that

$$\begin{aligned} H(x,t) \ =\ & -\int_0^1 f'\left(\theta\overline{w}+(1-\theta)w\right)d\theta W \\ & +G*\left[\int_0^1 f'\left(\theta\overline{w}+(1-\theta)w\right)d\theta W+k\triangle W_t\right], \end{aligned} \tag{2.24}$$

$$\begin{aligned}H_t(x,t) &= f'(\overline{w})W_t - \int_0^1 f''(\theta\overline{w}+(1-\theta)w)\,d\theta W w_t \\ &+G*\left[\int_0^1 f''(\theta\overline{w}+(1-\theta)w)\,d\theta W w_t - f'(\overline{w})W_t + k\triangle W_{tt}\right], \qquad (2.25)\end{aligned}$$

where $0<\theta<1$ is a constant. Making use of Lemmas 1.1–1.4, we deduce from (2.19) that

$$\begin{aligned}\|H(.,t)\|_s &\le C\left(\int_0^1 \|f'(\theta\overline{w}+(1-\theta)w)\|_\infty d\theta\|W\|_s + \int_0^1 \|f'(\theta\overline{w}+(1-\theta)w)\|_s d\theta\|W\|_\infty\right) \\ &+C\left(\int_0^1 \|f'(\theta\overline{w}+(1-\theta)w)\|_\infty d\theta\|W\|_{s-2} + \int_0^1 \|f'(\theta\overline{w}+(1-\theta)w)\|_{s-2} d\theta\|W\|_\infty\right) \\ &+k\|\triangle W_t\|_s \\ &\le 4CC_0M\|W\|_{s+1} + k\|W_t\|_{s+1}. \qquad (2.26)\end{aligned}$$

Similarly, from (2.20) we have

$$\|H_t(.,t)\|_{s-1} \le 4CC_0M\|W\|_{s+1} + 4CC_0M\|W_t\|_{s+1} + k\|W_{tt}\|_{s+1}. \qquad (2.27)$$

Using the fact that $H(x,0)=0$, we have

$$\int_0^t \|H(.,\tau)\|_s^2 d\tau \le 2\left[(4CC_0M)^2 \max_{t\in[0,T]}\|W\|_{s+1}^2 + k^2\|W_t\|_{s+1}^2\right]T, \qquad (2.28)$$

$$\begin{aligned}\max_{t\in[0,T]}\|H_t(.,t)\|_{s-1}^2 &\le 3\Big[(4CC_0M)^2\max_{t\in[0,T]}\|W\|_{s+1}^2 + \left(4CC_0M^2\right)^2\max_{t\in[0,T]}\|W_t\|_{s+1}^2 \\ &+k^2\max_{t\in[0,T]}\|W_{tt}\|_{s+1}^2\Big]T^2. \qquad (2.29)\end{aligned}$$

Therefore, by Lemma 2.1 we have

$$\begin{aligned}&\max_{t\in[0,T]}\left[\|U(.,t)\|_{s+1}^2 + \|U_t(.,t)\|_{s+1}^2 + \|U_{tt}(.,t)\|_{s+1}^2\right] \\ \le\ &12e^{3T}\left[(4CC_0M)^2\max_{t\in[0,T]}\|W\|_{s+1}^2 + k^2\|W_t\|_{s+1}^2\right]T \\ &+6\Big[(4CC_0M)^2\max_{t\in[0,T]}\|W\|_{s+1}^2 + \left(4CC_0M^2\right)^2\max_{t\in[0,T]}\|W_t\|_{s+1}^2 \\ &+k^2\max_{t\in[0,T]}\|W_{tt}\|_{s+1}^2\Big]T^2 \\ \le\ &\left[12e^3(4CC_0M)^2 + 96(CC_0M)^2 + k^2\right] \\ &\left(\max_{t\in[0,T]}\|W\|_{s+1}^2 + \max_{t\in[0,T]}\|W_t\|_{s+1}^2 + \max_{t\in[0,T]}\|W_{tt}\|_{s+1}^2\right)\left(T^2+T\right).\end{aligned}$$

If M, T satisfy (2.14) and (2.15), respectively and

$$T \leq \min\left\{1, \left[12e^3\left(4CC_0M\right)^2 + 96\left(CC_0M\right)^2 + 6k^2\right]^{-1}\right\}, \tag{2.30}$$

then S is strictly contractive. This completes the proof of Lemma 2.3. □

Theorem 2.1 *Assume that* $s > \frac{n}{2}+1$, $n \geq 1$, $u_0 \in H^{s+1}$, $u_1 \in H^{s+1}$, *and* $f, g \in C^{[s]+1}$; *then problem (1.1), (1.2) has a unique local solution* $u(x,t)$ *defined on a maximal time interval* $[0, T_0)$, $T_0 > 0$ *with*

$$u \in C^2\left([0,T_0); H^{s+1}\right).$$

Moreover, if

$$\sup_{t\in[0,T_0)}\left[\left\|u(.,t)\right\|_{s+1}^2 + \left\|u_t(.,t)\right\|_{s+1}^2 + \left\|u_{tt}(.,t)\right\|_{s+1}^2\right] < \infty, \tag{2.31}$$

then $T_0 = \infty$.

Proof From Lemmas 2.2, 2.3 and the contraction mapping principle, it follows that for appropriately chosen $T > 0$, S has a unique fixed point $u(x,t) \in X(M,T)$, which is a strong solution of problem (1.1), (1.2). It is not difficult to prove the uniqueness of the solution that belongs to $B\left(T^{'}\right)$ for each $T^{'} > 0$.

In fact, let u_1, $u_2 \in B\left(T^{'}\right)$ be 2 solutions of problem (1.1), (1.2). Let $u = u_1 - u_2$, then

$$u_{tt} - \triangle u - 2\triangle u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - k\triangle u_t = \triangle\left[f(u_1) - f(u_2)\right].$$

Multiplying the above equation by $(-\triangle)^{-1}u_t$ and integrating the product with respect to x, we obtain that

$$\frac{1}{2}\frac{d}{dt}\left[\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\bigtriangledown u\|^2 + \|\bigtriangledown u_t\|^2\right] + k\|u_t\|^2 = \int_{R^n}\left[f(u_2) - f(u_1)\right]u_t dx.$$

From the definition of $B\left(T^{'}\right)$, $s > \frac{n}{2}+1$, and Sobolev imbedding theorem, we have $\|u_i(t)\|_\infty \leq C_1\left(T^{'}\right)$ for $i = 1, 2$ and $0 \leq t \leq T^{'} < T$, where $C_1\left(T^{'}\right)$ is a constant dependent on $T^{'}$. Thus, we get from the Cauchy inequality that

$$\begin{aligned}\left|\int_{R^n}\left[f(u_1) - f(u_2)\right]u_t dx\right| &\leq \|f(u_1) - f(u_2)\|\,\|u_t\| \\ &\leq C_2\left(T^{'}\right)\|u\|\,\|u_t\|,\end{aligned}$$

where $C_2\left(T^{'}\right)$ is a constant dependent on $C_1\left(T^{'}\right)$. From the Young inequality it follows that

$$\left[\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\bigtriangledown u\|^2 + \|\bigtriangledown u_t\|^2\right] + k\int_0^t\|u_\tau\|^2 d\tau \leq C_2\left(T^{'}\right)\int_0^t\left[\|u\|^2 + \|u_\tau\|^2\right]d\tau.$$

From the above inequality we have

$$\|u\|^2 + \|u_t\|^2 \leq \left[C_2\left(T^{'}\right) + 2|k|\right] \int\limits_0^t \left[\|u\|^2 + \|u_\tau\|^2\right] d\tau. \tag{2.32}$$

By using Gronwall's inequality in (2.27), we get $\|u\|^2 + \|u_t\|^2 \equiv 0$ for $0 \leq t \leq T^{'}$. Hence $u \equiv 0$ for $0 \leq t \leq T^{'}$, i.e. problem (1.1), (1.2) has at most one solution that belongs to $B\left(T^{'}\right)$.

Now, let $[0, T_0)$ be the maximal time interval of existence for $u \in B(T_0)$. We want to show that if (2.26) is satisfied, then $T_0 = \infty$.

Suppose that (2.26) holds and $T_0 < \infty$. For any $T^{'} \in [0, T_0)$, we consider the integral equation

$$v_{tt} - \triangle v - \triangle v_{tt} = -f(v) + G * [f(v) + k \triangle v_t] \tag{2.33}$$

$$v(x,0) = u\left(x, T^{'}\right), \quad v_t(x,0) = u_t\left(x, T^{'}\right). \tag{2.34}$$

By virtue of (2.26), $\sup\limits_{t \in [0,T_0)} \left[\|u\|^2_{s+1} + \|u_t\|^2_{s+1} + \|u_{tt}\|^2_{s+1}\right] < \infty$ is uniformly bounded about $T^{'} \in [0, T_0)$, which allows us to choose $T^* \in (0, T_0)$ such that for each $T^{'} \in [0, T_0)$, the integral equation (2.28), (2.29) has a unique solution $v(x,t) \in B(T^*)$. The existence of such a T^* follows from Lemmas 2.2 and 2.3 and the contraction mapping principle. In particular, (2.15) and (2.25) reveal that T^* can be selected independent of $T^{'} \in [0, T_0)$. Set $T^{'} = T_0 - T^*/2$, let v denote the corresponding solution of the integral equation (2.28), (2.29) and define $\overline{u}(x,t)$ by

$$\overline{u}(x,t) = \begin{cases} u(x,t), & t \in [0, T'] \\ v\left(x, t - T^{'}\right), & t \in [T', T_0 + T^*/2] \end{cases} \tag{2.35}$$

By construction, $\overline{u}(x,t)$ is the solution of problem (1.1), (1.2) on $[0, T_0 + T^*/2]$, and by the local uniqueness, $\overline{u}(x,t)$ extends $u(x,t)$. This violates the maximality to $[0, T_0)$. Hence, if (2.26) holds, then $T_0 = \infty$. This completes the proof of the theorem. □

3. Existence and uniqueness of global solution

In this section we prove the existence and the uniqueness of the global solution for problem (1.1), (1.2). For this purpose we are going to make a priori estimates of the local solutions for the problem.

Lemma 3.1 *Suppose that $f(u) \in C(R)$, $F(u) = \int\limits_0^u f(s)\,ds$, $u_0 \in H^1$, $u_1 \in H^1$ and $F(u_0) \in L^1$. Then for the solution $u(x,t)$ of problem (1.1), (1.2), we have the energy identity*

$$\begin{aligned} E(t) &= \left\|(-\triangle)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\bigtriangledown u\|^2 + \|\bigtriangledown u_t\|^2 + 2k\int\limits_0^t \|u_\tau\|^2 d\tau + 2\int\limits_{R^n} F(u)\,dx \\ &= E(0). \end{aligned} \tag{3.1}$$

Here and in the sequel $(-\triangle)^{-\alpha} u(x) = \mathcal{F}^{-1}\left[|x|^{-2\alpha} \mathcal{F}u(x)\right]$, $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *denote respectively Fourier transformation and inverse Fourier transformation in* R^n *(see [10]).*

Proof Multiplying Eq. (1.1) by $(-\triangle)^{-1} u_t$ and integrating the product with respect to x, we obtain that

$$\left(u_{tt} - \triangle u - 2\triangle u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - k\triangle u_t - \triangle f(u), (-\triangle)^{-1} u_t\right) = 0,$$

$$\left((-\triangle)^{-1} u_{tt} + u + 2u_{tt} - \triangle u - \triangle u_{tt} + ku_t + f(u), u_t\right) = 0,$$

$$\left((-\triangle)^{-\frac{1}{2}} u_{tt}, (-\triangle)^{-\frac{1}{2}} u_t\right) + (u, u_t) + 2(u_{tt}, u_t) - (\triangle u, u_t) - (\triangle u_{tt}, u_t) + k(u_t, u_t) + (f(u), u_t) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\left[\left\|(-\triangle)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 + 2\int_{R^n} F(u)\,dx\right] + k\|u_t\|^2 = 0,$$

where $(.,.)$ denotes the inner product of L^2 space. Integrating the above equality with respect to t over $[0,t]$, we get (3.1). The lemma is proved. □

Lemma 3.2 *Suppose that the assumptions of Lemma 3.1 hold and* $F(u) \geq 0$ *or* $f'(u)$ *is bounded below, i.e. there is a constant* A_0 *such that* $f'(u) \geq A_0$ *for any* $u \in R$, *then the solution* $u(x,t)$ *of problem (1.1), (1.2) has the estimation*

$$E_1(t) = \left\|(-\triangle)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2 \leq M_1(T), \ \forall t \in [0,T]. \tag{3.2}$$

Here and in the sequel $M_i(T)$ $(i = 1, 2, ...)$ *are constants dependent on* T.

Proof i) If $F(u) \geq 0$, then from energy identity (3.1) we get

$$E_1(t) \leq E(0) + 2|k| \int_0^t \|u_\tau\|^2\, d\tau.$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t) \leq E(0)\, e^{2|k|T}. \tag{3.3}$$

ii) If $f'(u)$ is bounded below, let $f_0(u) = f(u) - k_0 u$, where $k_0 = \min\{A_0, 0\} (\leq 0)$, then $f_0(0) = 0$, $f_0'(u) = f'(u) - k_0 \geq 0$ and $f_0(u)$ is a monotonically increasing function. Then $F_0(u) = \int_0^u f_0(s)\, ds \geq 0$ and

$F(u)=\int_0^u f(s)\,ds=\int_0^u (f_0(s)+k_0 s)\,ds=F_0(u)+\frac{k_0}{2}u^2$. From (3.1) we have

$$\begin{aligned} E_1(t)+2\int_{R^n} F_0(u)\,dx &= E(0)-2k\int_0^t \|u_\tau\|^2\,d\tau-k_0\|u\|^2 \\ &\le E(0)-2k\int_0^t \|u_\tau\|^2\,d\tau-k_0\|u_0\|^2+\int_0^t\left(k_0^2\|u\|^2+\|u_\tau\|^2\right)d\tau \\ &\le E(0)-k_0\|u_0\|^2+\left(2|k|+1+k_0^2\right)\int_0^t\left(\|u\|^2+\|u_\tau\|^2\right)d\tau. \end{aligned}$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t)\le\left(E(0)-k_0\|u_0\|^2\right)e^{\left(2|k|+1+k_0^2\right)T}. \tag{3.4}$$

We get (3.2) from inequalities (3.3) and (3.4). The lemma is proved. □

Lemma 3.3 *Under the conditions of Lemma 3.2 assume that* $1\le n\le 3$ *and* $|f'(s)|\le A|s|^\rho+B$, *for* $n=2$ *or* 3, *where* ρ *satisfies*

$$\begin{aligned} \frac{1}{2} &\le \rho<\infty \text{ if } n=2, \\ \frac{1}{2} &\le \rho\le\frac{3}{2} \text{ if } n=3. \end{aligned}$$

Then the solution $u(x,t)$ *of problem (1.1), (1.2) has the estimation*

$$E_2(t)=\|\nabla u_t\|^2+\|\triangle u\|^2+2\|\triangle u_t\|^2+\left\|\triangledown^3 u\right\|^2+\left\|\triangledown^3 u_t\right\|^2\le M_2(T),\ \forall t\in[0,T]. \tag{3.5}$$

Proof Multiplying equation (1.1) by $\triangle u_t$ and integrating the product over R^n, we obtain that

$$\frac{d}{dt}E_2(t)+2k\|\triangle u_t\|^2=2\int_{R^n}\triangle f(u)\triangle u_t dx. \tag{3.6}$$

When $n=1,$ we conclude from Lemma 3.2 and Sobolev imbedding theorem that $u\in L^{\infty},\ \bigtriangledown u\in L^{\infty}$. Therefore, from (3.6), Hölder inequality, Cauchy inequality, Lemma 1.1, and (3.2), we get

$$\begin{aligned}
\frac{d}{dt}E_2(t) &= -2k\left\|\triangle u_t\right\|^2+2\left(\triangle f(u),\triangle u_t\right)\\
&\leq 2|k|\left\|\triangle u_t\right\|^2+2\left\|\triangle f(u)\right\|\left\|\triangle u_t\right\|\\
&\leq 2|k|\left\|\triangle u_t\right\|^2+2\left\|f(u)\right\|_{H^2}\left\|\triangle u_t\right\|\\
&\leq 2|k|\left\|\triangle u_t\right\|^2+2K_1\left(\left\|u\right\|_{\infty}\right)\left\|u\right\|_{H^2}\left\|\triangle u_t\right\|\\
&\leq 2|k|\left\|\triangle u_t\right\|^2+K_1\left(\left\|u\right\|_{\infty}\right)\left\|u\right\|_{H^2}^2+\left\|\triangle u_t\right\|^2\\
&\leq 2|k|\left\|\triangle u_t\right\|^2+2K_1\left(\left\|u\right\|_{\infty}\right)\left(\left\|u\right\|^2+\left\|\nabla u\right\|^2+\left\|\triangle u\right\|^2\right)+\left\|\triangle u_t\right\|^2\\
&\leq C_1\left(M_1(T)\right)\left(\left\|\triangle u\right\|^2+\left\|\triangle u_t\right\|^2+1\right). \qquad (3.7)
\end{aligned}$$

Here and in the sequel $C_i\left(M_j(T)\right)$ $(i=1,2,...,\quad j=1,2,...)$ are constants dependent on $M_j(T)$. Integrating (3.7) with respect to t and using the Gronwall's inequality we obtain (3.5).

When $n=2$ or $3,$ from Hölder inequality, Cauchy inequality, Lemma 1.3 and (3.2) we have

$$\begin{aligned}
\int_{R^n}\triangle f(u)\triangle u_t dx &= -\int_{R^n}\nabla f(u)\nabla^3 u_t dx\\
&= -\int_{R^n}f'(u)\nabla u\nabla^3 u_t dx\\
&\leq \left\|f'(u)\right\|_4\left\|\nabla u\right\|_4\left\|\nabla^3 u_t\right\|\\
&\leq \left\|f'(u)\right\|_4\left\|\nabla u\right\|_{H^1}\left\|\nabla^3 u_t\right\|\\
&\leq \left\|f'(u)\right\|_4\left\|u\right\|_{H^2}\left\|\nabla^3 u_t\right\|\\
&\leq \left\|f'(u)\right\|_4\left(\left\|u\right\|_{H^2}^2+\left\|\nabla^3 u_t\right\|^2\right)\\
&\leq \left\|f'(u)\right\|_4\left(\left\|u\right\|^2+\left\|\nabla u\right\|^2+\left\|\triangle u\right\|^2+\left\|\nabla^3 u_t\right\|^2\right).
\end{aligned}$$

When $n=2,$ we have $H^1\hookrightarrow L^q$ for $2\leq q<\infty$. From $\frac{1}{2}\leq\rho<\infty,$ we have $2\leq 4\rho<\infty$. Hence by (3.2) we have $\left\|f'(u)\right\|_4\leq C(\rho)$. Similarly, when $n=3,$ we have $H^1\hookrightarrow L^q$ for $2\leq q\leq 6$. From $\frac{1}{2}\leq\rho\leq\frac{3}{2}$ we have $2\leq 4\rho\leq 6$. Hence by (3.2) we have $\left\|f'(u)\right\|_4\leq C(\rho)$.

Substitute the above inequality into (3.6) to get

$$\frac{d}{dt}E_2(t)\leq 2|k|\left\|\triangle u_t\right\|^2+2\left(\triangle f(u),\triangle u_t\right)\leq C_3\left(M_1(T)\right)\left(\left\|u\right\|^2+\left\|\nabla u\right\|^2+\left\|\triangle u\right\|^2+\left\|\nabla^3 u_t\right\|^2\right). \qquad (3.8)$$

Integrating (3.8) with respect to t and using the Gronwall's inequality, we obtain (3.5). The lemma is proved. □

Lemma 3.4 *Under the conditions of Lemma 3.3 assume that* $1 \le n \le 3,\ f(u) \in C^{[s]}(R),\ u_0 \in H^{s+1},$ *and* $u_1 \in H^{s+1}$; *then the solution* $u(x,t)$ *of problem (1.1), (1.2) has the estimation*

$$E_3(t) = \left\|\nabla^{s-1}u_t\right\|^2 + \|\nabla^s u\|^2 + 2\|\nabla^s u_t\|^2 + \left\|\bigtriangledown^{s+1}u\right\|^2 + \left\|\bigtriangledown^{s+1}u_t\right\|^2 \le M_3(T),\ \forall t \in [0,T]. \tag{3.9}$$

Proof Multiplying Eq. (1.1) by $\triangle^{s-1}u_t$ and integrating the product over R^n, we obtain that

$$\frac{d}{dt}E_3(t) + 2k\|\nabla^s u_t\|^2 = 2\int_{R^n} \nabla^s f(u)\,\nabla^s u_t dx. \tag{3.10}$$

From Lemmas 3.2 and 3.3 and Sobolev imbedding theorem, we know that $\nabla u \in L^\infty,\ u_t \in L^\infty,\ u \in L^\infty$. We get from Hölder inequality, Cauchy inequality, Lemma 1.1, and (3.2) that

$$\begin{aligned}\frac{d}{dt}E_3(t) &= -2k\|\nabla^s u_t\|^2 + 2\int_{R^n} \nabla^s f(u)\,\nabla^s u_t dx \\ &\le 2|k|\,\|\nabla^s u_t\|^2 + 2K_2(\|u\|_\infty)(\|u\| + \|\nabla^s u\|)\|\nabla^s u_t\| \\ &\le C_4(M_1(T))(\|\nabla^s u\| + \|\nabla^s u_t\|).\end{aligned}$$

Integrating the above inequality with respect to t and using the Gronwall's inequality, we obtain (3.9). The lemma is proved. □

Theorem 3.1 *Assume that* $1 \le n \le 3,\ s > \frac{n}{2}+1,\ u_0 \in H^{s+1},\ u_1 \in H^{s+1},\ f(u) \in C^{[s]+1}(R),\ f'(u)$ *is bounded below, i.e. there is a constant* A_0 *such that* $f'(u) \ge A_0$ *for any* $u \in R$. *Moreover, assume that* $1 \le n \le 3$, *and* $|f'(s)| \le A|s|^\rho + B$, *for* $n = 2$ *or* 3, *where* ρ *satisfies*

$$\begin{aligned}\frac{1}{2} &\le \rho < \infty \text{ if } n = 2, \\ \frac{1}{2} &\le \rho \le \frac{3}{2} \text{ if } n = 3.\end{aligned}$$

Then problem (1.1), (1.2) admits a unique global solution $u(x,t) \in C^2\left([0,\infty); H^{s+1}\right)$.

Proof By virtue of Theorem 2.1, it is enough to show that

$$\sup_{t\in[0,T_0)}\left[\|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 + \|u_{tt}(.,t)\|_{s+1}^2\right] < \infty. \tag{3.11}$$

From Lemmas 3.2–3.4, we know that

$$\|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 < M_4(T),\ \forall t \in [0,T), \tag{3.12}$$

where $M_4(T)$ is a constant dependent on T. From equation (2.2) we obtain

$$u_{tt} = \triangle u + \triangle u_{tt} - f(u) + G * [f(u) + k \triangle u_t]$$

Using Lemmas 1.1 and 1.4 and (3.12), we get

$$\begin{aligned}\|u_{tt}\|_{s+1} &\leq \|\triangle u\|_{s+1} + \|-f(u)\|_{s+1} + \|G * [f(u) + k \triangle u_t]\|_{s+1} \\ &\leq \|\triangle u\|_{s+1} + C_0(M_1(T))\|u\|_{s+1} + \|f(u) + k \triangle u_t\|_{s-1}.\end{aligned}$$

And hence by Lemma 3.4 we have

$$\|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 + \|u_{tt}(.,t)\|_{s+1}^2 < M_4(T), \ \forall t \in [0,T),$$

i.e.

$$\sup_{t\in[0,T_0)} \left[\|u(.,t)\|_{s+1}^2 + \|u_t(.,t)\|_{s+1}^2 + \|u_{tt}(.,t)\|_{s+1}^2\right] < \infty,$$

by Theorem 2.1, we get $T \to \infty$, namely, the Cauchy problem (1.1), (1.2) admits a unique global solution $u(x,t) \in C^2\left([0,\infty); H^{s+1}\right)$. The theorem is proved. □

4. Nonexistence of global solution

In this section, we are going to consider the nonexistence of the solution for problem (1.1), (1.2) by the concavity method. For this purpose, we give the following lemma [3] which is a generalization of Levine's result [4].

Lemma 4.1 *(see [3]). Suppose that a positive, twice differentiable function $F(t)$ satisfies on $t \geq 0$ the inequality*

$$F(t)F''(t) - (1+\upsilon)(F'(t))^2 \geq -2M_1 F(t) F'(t) - M_2(F(t))^2,$$

where $\upsilon > 0$ and $M_1, \ M_2 \geq 0$ are constants.

(i) If $M_1 = M_2 = 0, \ F(0) > 0$ and $F'(0) > 0,$ then there is a $t_1 \leq t_2 = \frac{F(0)}{\upsilon F'(0)}$ such that $F(t) \longrightarrow \infty$ as $t \longrightarrow t_1$.

(ii) If $M_1 + M_2 > 0, \ F(0) > 0$ and $F'(0) > -\gamma_2 \upsilon^{-1} F(0),$ then there is a $t_1 \leq t_2$ such that $F(t) \longrightarrow \infty$ as $t \longrightarrow t_1,$ where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \upsilon M_2}$ and

$$t_2 = \frac{1}{2\sqrt{M_1^2 + \upsilon M_2}} \ln \frac{\gamma_1 F(0) + \upsilon F'(0)}{\gamma_2 F(0) + \upsilon F'(0)}.$$

Theorem 4.1 *Assume that $k \geq 0, \ f(u) \in C(R), \ u_0 \in H^1, \ u_1 \in H^1, \ (-\triangle)^{-\frac{1}{2}} u_0, (-\triangle)^{-\frac{1}{2}} u_1 \in L^2$, $F(u) = \int_0^u f(s)\,ds, \ F(u_0) \in L^1,$ and there exists a constant $\alpha > 0$ such that*

$$uf(u) \leq (2+\alpha+k) F(u) + \frac{\alpha}{2} u^2, \ \forall u \in R. \tag{4.1}$$

Then the solution $u(x,t)$ of problem (1.1), (1.2) blows up in finite time if one of the following conditions is valid:

(i) $E(0) = \left\|(-\triangle)^{-\frac{1}{2}} u_1\right\|^2 + \|u_0\|^2 + 2\|u_1\|^2 + \|\triangledown u_0\|^2 + \|\triangledown u_1\|^2 + 2\int_{R^n} F(u_0)\,dx < 0,$

(ii) $E(0)=0$ *and* $\left((-\triangle)^{-\frac{1}{2}}u_0,(-\triangle)^{-\frac{1}{2}}u_1\right)+(\nabla u_0,\nabla u_1)+2(u_0,u_1)>0,$

(iii) $E(0)>0$ *and* $\left((-\triangle)^{-\frac{1}{2}}u_0,(-\triangle)^{-\frac{1}{2}}u_1\right)+(\nabla u_0,\nabla u_1)+2(u_0,u_1)$

$$>\sqrt{2\frac{4\alpha+k+2}{\alpha+2}E(0)\left(\left\|(-\triangle)^{-\frac{1}{2}}u_0\right\|^2+\|\nabla u_0\|^2+2\|u_0\|^2\right)}.$$

Proof Suppose that the maximal time of existence of the solution for problem (1.1), (1.2) is infinite. A contradiction will be obtained by Lemma 4.1. Let

$$F(t)=\left\|(-\triangle)^{-\frac{1}{2}}u\right\|^2+\|\nabla u\|^2+2\|u\|^2+\beta(t+\tau)^2, \tag{4.2}$$

where β and τ are nonnegative constants to be specified later. Obviously we have

$$F'(t)=2\left[\left((-\triangle)^{-\frac{1}{2}}u,(-\triangle)^{-\frac{1}{2}}u_t\right)+(\nabla u,\nabla u_t)+2(u,u_t)+\beta(t+\tau)\right]. \tag{4.3}$$

Using the Schwartz inequality and the inequality

$$(a_1b_1+a_2b_2+...+a_nb_n)^2\leq\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right),$$

where $a_i,b_i\geq 0,\ i=1,2,...,n,$ we have

$$\begin{aligned}(F'(t))^2 &\leq 4\left[\left\|(-\triangle)^{-\frac{1}{2}}u\right\|^2+\|\nabla u\|^2+2\|u\|^2+\beta(t+\tau)^2\right]\\ &\quad .\left[\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+\|\nabla u_t\|^2+2\|u_t\|^2+\beta\right]\\ &= 4F(t)\left[\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+\|\nabla u_t\|^2+2\|u_t\|^2+\beta\right]. \end{aligned} \tag{4.4}$$

We get from equation (1.1)

$$\begin{aligned}F''(t) &= 2\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+2\|\nabla u_t\|^2+4\|u_t\|^2+2\left((-\triangle)^{-\frac{1}{2}}u,(-\triangle)^{-\frac{1}{2}}u_{tt}\right)\\ &\quad +2(\nabla u,\nabla u_{tt})+4(u,u_{tt})+2\beta\\ &= 2\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+2\|\nabla u_t\|^2+4\|u_t\|^2+2\beta+2\left(u,(-\triangle)^{-1}u_{tt}-\triangle u_{tt}+2u_{tt}\right)\\ &= 2\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+2\|\nabla u_t\|^2+4\|u_t\|^2+2\beta-2(u,u-\triangle u+ku_t+f(u))\\ &= 2\left\|(-\triangle)^{-\frac{1}{2}}u_t\right\|^2+2\|\nabla u_t\|^2+4\|u_t\|^2+2\beta-2\|u\|^2-2\|\nabla u\|^2\\ &\quad -2k(u,u_t)-2\int_{R^n}uf(u)\,dx. \end{aligned} \tag{4.5}$$

By the aid of the Cauchy inequality and equality (3.1), we have

$$\begin{aligned} 2k\left(u,u_t\right) &\leq k\left(\|u\|^2+\|u_t\|^2\right) \\ &\leq k\left(\|u\|^2+2\|u_t\|^2\right) \\ &= k\Bigg[E\left(0\right)-\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_t\right\|^2-\|\nabla u\|^2 \\ &\quad -\|\nabla u_t\|^2-2k\int_0^t\|u_\tau\|^2\,d\tau-2\int_{R^n}F\left(u\right)dx\Bigg]. \end{aligned} \tag{4.6}$$

From (4.2)–(4.6) we obtain that

$$\begin{aligned} & F\left(t\right)F''\left(t\right)-\left(1+\frac{\alpha}{4}\right)\left(F'\left(t\right)\right)^2 \\ \geq\ & F\left(t\right)F''\left(t\right)-\left(1+\frac{\alpha}{4}\right)4F\left(t\right)\left[\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_t\right\|^2+\|\nabla u_t\|^2+2\|u_t\|^2+\beta\right] \\ \geq\ & F\left(t\right)\Bigg[\left(k-\alpha-2\right)\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_t\right\|^2+\left(k-\alpha-2\right)\|\nabla u_t\|^2 \\ & +\left(-2\alpha-4\right)\|u_t\|^2+\left(-\alpha-2\right)\beta+\left(k-2\right)\|\nabla u\|^2+ \\ & \int_{R^n}\left[2kF\left(u\right)-2uf\left(u\right)-2u^2\right]dx+2k^2\int_0^t\|u_\tau\|^2\,d\tau-kE\left(0\right)\Bigg]. \end{aligned} \tag{4.7}$$

From equality (3.1) we have

$$\begin{aligned} & \left(k-\alpha-2\right)\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_t\right\|^2+\left(k-\alpha-2\right)\|\nabla u_t\|^2+\left(k-2\right)\|\nabla u\|^2+2\left(-\alpha-2\right)\|u_t\|^2 \\ \geq\ & \left(-\alpha-2\right)\left(\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_t\right\|^2+\|\nabla u_t\|^2+\|\nabla u\|^2+2\|u_t\|^2\right) \\ \geq\ & \left(\alpha+2\right)\left(\|u\|^2+2k\int_0^t\|u_\tau\|^2\,d\tau+2\int_{R^n}F\left(u\right)dx-E\left(0\right)\right). \end{aligned}$$

Thus, from the above inequality and inequalities (4.7) and (4.1), we get

$$\begin{aligned} & F\left(t\right)F''\left(t\right)-\left(1+\frac{\alpha}{4}\right)\left(F'\left(t\right)\right)^2 \\ \geq\ & F\left(t\right)\Bigg[-\left(2+\alpha\right)\beta-\left(2+\alpha+k\right)E\left(0\right)+\left(2k\left(2+\alpha\right)+2k^2\right)\int_0^t\|u_\tau\|^2\,d\tau \\ & +\int_{R^n}\left[2\left(2+\alpha+k\right)F\left(u\right)+\alpha u^2-2uf\left(u\right)\right]dx\Bigg] \\ \geq\ & -\left[\left(2+\alpha\right)\beta+\left(2+\alpha+k\right)E\left(0\right)\right]F\left(t\right). \end{aligned} \tag{4.8}$$

If $E(0) < 0,$ taking $\beta = -\frac{(2+\alpha+k)}{(2+\alpha)} E(0) > 0,$ then

$$F(t) F''(t) - \left(1 + \frac{\alpha}{4}\right) (F'(t))^2 \geq 0.$$

We may now choose τ so large that $F'(0) > 0$. From Lemma 4.1 we know that $F(t)$ becomes infinite at a time T_1 at most equal to

$$T_2 = \frac{4F(0)}{\alpha F'(0)} < \infty.$$

If $E(0) = 0,$ taking $\beta = 0,$ then we get from (4.8) that

$$F(t) F''(t) - \left(1 + \frac{\alpha}{4}\right) (F'(t))^2 \geq 0.$$

Also $F'(0) > 0$ by assumption (ii). Thus, we obtain from Lemma 4.1 that $F(t)$ becomes infinite at a time T_1 at most equal to

$$T_2 = \frac{4F(0)}{\alpha F'(0)} < \infty.$$

If $E(0) > 0,$ then taking $\beta = 0,$ inequality (4.8) becomes

$$F(t) F''(t) - \left(1 + \frac{\alpha}{4}\right) (F'(t))^2 \geq -(2 + \alpha + k) E(0) F(t). \tag{4.9}$$

Define $J(t) = (F(t))^{-\upsilon}$, where $\upsilon = \frac{\alpha}{4}$. Then

$$J'(t) = -\upsilon (F(t))^{-\upsilon-1} F'(t),$$

$$\begin{aligned} J''(t) &= -\upsilon (F(t))^{-\upsilon-2} \left[F(t) F''(t) - (1+\upsilon)(F'(t))^2\right] \\ &\leq \upsilon (2 + \alpha + 4\upsilon) E(0) (F(t))^{-\upsilon-1}, \end{aligned} \tag{4.10}$$

where inequality (4.9) is used. Assumption (iii) implies $J'(0) < 0$. Let

$$t^* = \sup \{t \,|\, J'(\tau) < 0, \ \tau \in (0,t)\}. \tag{4.11}$$

By the continuity of $J'(t)$, t^* is positive. Multiplying (4.10) by $2J'(t)$ yields

$$\begin{aligned} \left[(J'(t))^2\right]' &\geq -2\upsilon^2 (2 + \alpha + 4\upsilon) E(0) (F(t))^{-2\upsilon-2} F'(t) \\ &= 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0) \left[(F(t))^{-2\upsilon-1}\right]', \ \forall t \in [0, t^*). \end{aligned} \tag{4.12}$$

Integrating (4.12) with respect to t over $[0,t)$ gives

$$\begin{aligned} (J'(t))^2 &\geq (J'(0))^2 + 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0) (F(t))^{-2\upsilon-1} \\ &\quad -2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0) (F(0))^{-2\upsilon-1} \\ &\geq (J'(0))^2 - 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0) (F(0))^{-2\upsilon-1}. \end{aligned}$$

By assumption (iii)

$$\left(J'(0)\right)^2 - 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0)\left(F(0)\right)^{-2\upsilon-1} > 0.$$

Hence by continuity of $J'(t)$, we obtain

$$J'(t) \leq -\left[\left(J'(0)\right)^2 - 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0)\left(F(0)\right)^{-2\upsilon-1}\right]^{\frac{1}{2}} \tag{4.13}$$

for $0 \leq t < t^*$. By the definition of t^*, it follows that inequality (4.13) holds for all $t \geq 0$. Therefore,

$$J(t) \leq J(0) - \left[\left(J'(0)\right)^2 - 2\upsilon^2 \frac{(2+\alpha+4\upsilon)}{2\upsilon+1} E(0)\left(F(0)\right)^{-2\upsilon-1}\right]^{\frac{1}{2}} t, \ \forall t > 0.$$

So, $J(T_1) = 0$ for some T_1 and

$$0 < T_1 \leq T_2 = \frac{J(0)}{\left[\left(J'(0)\right)^2 - 2\alpha^2 \frac{(2+\alpha+k)}{4\alpha+8} E(0)\left(F(0)\right)^{-\frac{\alpha+2}{2}}\right]^{\frac{1}{2}}}.$$

Thus, $F(t)$ becomes infinite at a time T_1.

Therefore, $F(t)$ becomes infinite at a time T_1 under either assumptions (i), (ii), or (iii). We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof. □

5. Asymptotic behavior of solution

In this section, we discuss the asymptotic behavior of the solution for problem (1.1), (1.2).

Theorem 5.1 *Let $k > 0$ and assume that*

$$0 \leq F(u) \leq f(u)\, u, \quad \forall u \in R, \quad F(u) = \int_0^u f(s)\, ds.$$

Then for the global solution of problem (1.1), (1.2) there exist positive constants c and λ such that

$$E(t) \leq cE(0)\, e^{-\lambda t}, \quad 0 \leq t < \infty, \tag{5.1}$$

where

$$E(t) = \frac{1}{2}\left(\left\|(-\triangle)^{-\frac{1}{2}} u_t\right\|^2 + \|u\|^2 + 2\|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u_t\|^2\right) + \int_{R^n} F(u)\, dx.$$

Proof Let $u(x,t)$ be a global solution of problem (1.1), (1.2). Multiplying (1.1) by $(-\triangle)^{-1} u_t$ and integrating on R^n, it follows that

$$\frac{d}{dt} E(t) + k\|u_t\|^2 = 0. \tag{5.2}$$

Multiplying (5.2) by $e^{\delta t}$ we get

$$\frac{d}{dt}\left(e^{\delta t}E\left(t\right)\right)+ke^{\delta t}\left\|u_{t}\right\|^{2}=\delta e^{\delta t}E\left(t\right). \tag{5.3}$$

Integrating (5.3) over $(0,t)$ we get

$$\begin{aligned} e^{\delta t}E\left(t\right)+k\int_{0}^{t}e^{\delta\tau}\left\|u_{\tau}\right\|^{2}d\tau &= E\left(0\right)+\delta\int_{0}^{t}e^{\delta\tau}E\left(\tau\right)d\tau \\ &= E\left(0\right)+\frac{\delta}{2}\int_{0}^{t}e^{\delta\tau}\left(\left\|\left(-\triangle\right)^{-\frac{1}{2}}u_{\tau}\right\|^{2}+\left\|u\right\|^{2}+2\left\|u_{\tau}\right\|^{2}+\left\|\nabla u_{\tau}\right\|^{2}\right)d\tau \\ &\quad +\delta\int_{0}^{t}e^{\delta\tau}\left(\frac{1}{2}\left\|\nabla u\right\|^{2}+\int_{R^{n}}F\left(u\right)dx\right)d\tau. \end{aligned} \tag{5.4}$$

From $0\leq F\left(u\right)\leq f\left(u\right)u$ and Eq. (1.1) we have

$$\begin{aligned} \frac{1}{2}\left\|\nabla u\right\|^{2}+\int_{R^{n}}F\left(u\right)dx &\leq \frac{1}{2}\left\|\nabla u\right\|^{2}+\int_{R^{n}}f\left(u\right)udx \\ &= \frac{1}{2}\left\|\nabla u\right\|^{2}-\left(\left(-\triangle\right)^{-1}u_{tt}+u+2u_{tt}-\triangle u-\triangle u_{tt}+ku_{t},u\right) \\ &= \frac{1}{2}\left\|\nabla u\right\|^{2}-\left(\left(-\triangle\right)^{-1}u_{tt},u\right)+\left(\triangle u_{tt},u\right)-2\left(u_{tt},u\right) \\ &\quad -\left\|u\right\|^{2}-\left\|\nabla u\right\|^{2}-\frac{k}{2}\frac{d}{dt}\left\|u\right\|^{2}. \end{aligned}$$

Hence we have

$$\begin{aligned} \delta\int_{0}^{t}e^{\delta\tau}\left(\frac{1}{2}\left\|\nabla u\right\|^{2}+\int_{R^{n}}F\left(u\right)dx\right)d\tau &\leq \delta\int_{0}^{t}e^{\delta\tau}[-\frac{1}{2}\left\|\nabla u\right\|^{2}-\left(\left(-\triangle\right)^{-1}u_{\tau\tau},u\right)-\left(\nabla u_{\tau\tau},\nabla u\right) \\ &\quad -2\left(u_{\tau\tau},u\right)-\left\|u\right\|^{2}-\frac{k}{2}\frac{d}{d\tau}\left\|u\right\|^{2}]d\tau. \end{aligned} \tag{5.5}$$

We will estimate the terms on the right-hand side of (5.5) separately. For the second, third, and fourth terms, by using integration by parts and the Cauchy inequality, we have

$$\begin{aligned}
-\int_0^t e^{\delta\tau}\left((-\triangle)^{-1} u_{\tau\tau}, u\right) d\tau &= -e^{\delta t}\left((-\triangle)^{-\frac{1}{2}} u_t, (-\triangle)^{-\frac{1}{2}} u\right) + \left((-\triangle)^{-\frac{1}{2}} u_1, (-\triangle)^{-\frac{1}{2}} u_0\right) \\
&\quad + \delta \int_0^t e^{\delta\tau}\left((-\triangle)^{-\frac{1}{2}} u_\tau, (-\triangle)^{-\frac{1}{2}} u\right) d\tau + \int_0^t e^{\delta\tau}\left\|(-\triangle)^{-\frac{1}{2}} u_\tau\right\|^2 d\tau \\
&\leq \frac{1}{2} e^{\delta t}\left(\left\|(-\triangle)^{-\frac{1}{2}} u_t\right\|^2 + \left\|(-\triangle)^{-\frac{1}{2}} u\right\|^2\right) \\
&\quad + \frac{1}{2}\left(\left\|(-\triangle)^{-\frac{1}{2}} u_1\right\|^2 + \left\|(-\triangle)^{-\frac{1}{2}} u_0\right\|^2\right) \\
&\quad + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\left\|(-\triangle)^{-\frac{1}{2}} u_\tau\right\|^2 + \left\|(-\triangle)^{-\frac{1}{2}} u\right\|^2\right) d\tau \\
&\quad + \int_0^t e^{\delta\tau}\left\|(-\triangle)^{-\frac{1}{2}} u_\tau\right\|^2 d\tau, \qquad (5.6)
\end{aligned}$$

$$\begin{aligned}
-\int_0^t e^{\delta\tau}\left(\nabla u_{\tau\tau}, \nabla u\right) d\tau &= -e^{\delta t}\left(\nabla u_t, \nabla u\right) + \left(\nabla u_1, \nabla u_0\right) + \delta\int_0^t e^{\delta\tau}\left(\nabla u_\tau, \nabla u\right) d\tau + \int_0^t e^{\delta\tau}\left\|\nabla u_\tau\right\|^2 d\tau \\
&\leq \frac{1}{2} e^{\delta t}\left(\left\|\nabla u_t\right\|^2 + \left\|\nabla u\right\|^2\right) + \frac{1}{2}\left(\left\|\nabla u_1\right\|^2 + \left\|\nabla u_0\right\|^2\right) \\
&\quad + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\left\|\nabla u_\tau\right\|^2 + \left\|\nabla u\right\|^2\right) d\tau + \int_0^t e^{\delta\tau}\left\|\nabla u_\tau\right\|^2 d\tau, \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
-2\int_0^t e^{\delta\tau}\left(u_{\tau\tau}, u\right) d\tau &= -2e^{\delta t}\left(u_t, u\right) + 2\left(u_1, u_0\right) + 2\delta\int_0^t e^{\delta\tau}\left(u_\tau, u\right) d\tau + 2\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau \\
&\leq e^{\delta t}\left(\left\|u_t\right\|^2 + \left\|u\right\|^2\right) + \left(\left\|u_1\right\|^2 + \left\|u_0\right\|^2\right) \\
&\quad + \delta\int_0^t e^{\delta\tau}\left(\left\|u_\tau\right\|^2 + \left\|u\right\|^2\right) d\tau + 2\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau. \qquad (5.8)
\end{aligned}$$

For the last term, by using the integrating by parts, we have

$$-\frac{k}{2}\int_0^t e^{\delta\tau}\frac{d}{d\tau}\left\|u\right\|^2 d\tau = -\frac{k}{2} e^{\delta t}\left\|u\right\|^2 + \frac{k}{2}\left\|u_0\right\|^2 + \frac{k}{2}\delta\int_0^t e^{\delta\tau}\left\|u\right\|^2 d\tau. \qquad (5.9)$$

Substituting (5.6)–(5.9) into (5.4) and (5.5) it follows that there exist positive constants c_0, c_1, and c_2 such that

$$\begin{aligned} e^{\delta t}E(t)+k\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau &\leq c_0E(0)+c_1\delta e^{\delta t}E(t) \\ &+c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d+\frac{3}{2}\delta\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau. \end{aligned} \tag{5.10}$$

Taking δ satisfying $0<\delta<\min\left\{\frac{1}{2c_1},\frac{2k}{3}\right\}$ we get from (5.10) that

$$e^{\delta t}E(t)\leq 2c_0E(0)+2c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau,$$

which together with the Gronwall inequality gives

$$e^{\delta t}E(t)\leq 2c_0E(0)\,e^{2c_2\delta^2 t},\quad 0\leq t<\infty$$

and

$$E(t)\leq 2c_0E(0)\,e^{-\left(\delta-2c_2\delta^2\right)t},\quad 0\leq t<\infty.$$

Again taking δ satisfying $0<\delta<\min\left\{\frac{1}{2c_1},\frac{2k}{3},\frac{1}{2c_2}\right\}$ we can obtain (5.1), where $\lambda=\delta-2c_2\delta^2>0$, $c=2c_0$. Thus, the theorem is proved. □

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