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Existence of a mild solution for an impulsive nonlocal non-autonomous neutral functional differential equation

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Abstract This paper considers an impulsive neutral differential equation with nonlocal initial conditions in an arbitrary Banach space E. The existence of the mild solution is obtained by using Krasnoselskii's fixed point theorem and approximation techniques without imposing the strong restriction on nonlocal function and impulsive functions. An example is also provided at the end of the paper to illustrate the abstract theory.

Keywords Nonlocal initial conditions · Analytic compact semigroup · Functional evolution equation · Fixed point theorem · Fractional power operator · Impulsive differential equation

Mathematics Subject Classification 34K37 · 34K30 · 35R11 · 47N20

1 Introduction

In recent years, impulsive differential equations have become an active area of research due to their demonstrated applications in widespread fields of science and engineering such as biology, physics, control theory, population dynamics, economics, chemical technology, medicine and many others. Many physical systems which are characterized by the occurrence of an abrupt change in the state of the system can be described by impulsive differential equations. These changes occur at certain time instants over a

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period of negligible duration. Impulsive differential equations are also an appropriate model to hereditary phenomena for which a delay argument arises in the modelling equations. The existence, uniqueness and stability of mild solutions to functional differential equations with impulsive conditions have been considered by many authors in literatures. In [1], authors have considered a class of abstract impulsive mixed-type non-autonomous functional integro-differential equations with finite delay in a Banach space and obtained sufficient conditions for controllability of considered system by virtue of semigroup theory via Mönch fixed point theorem technique and measures of noncompactness. In [2], authors have extended the results of [3,4] and derived a sufficient condition for existence of mild solution by mean of Leray-Schauder alternative theorem. The sufficient condition for controllability of semilinear evolution integro-differential system has been obtained by authors in [5]. The existence of the mild solutions to a class of abstract non-autonomous impulsive functional integro-differential equations have been studied by authors in [6]. The existence and Ulam–Hyers–Rassias stability of mild solution of impulsive non-autonomous differential equations are studied by authors in [7]. In [8], author has shown the controllability of a system of impulsive semilinear non-autonomous differential equations via Rothe's type fixed-point theorem. For more details and study on such differential equation, we refer to the monographs [9,10] and papers [11–20] and reference cited therein.

The qualitative behavior of evolutionary differential and difference equations, whose right-hand side is explicitly time-dependent can be described by nonautonomous dynamics. Over recent years, the theory of such systems has developed into a highly active field related to, yet recognizably distinct from that of classical autonomous dynamical systems. This development was motivated by problems of applied mathematics, in particular in the life sciences where genuinely nonautonomous systems abound. For more details, we refer the monographs [21–23]. On the other hand, the existence of the solution of the differential equations with nonlocal conditions has been investigated widely by many authors as, the nonlocal conditions are more realistic than the classical initial conditions such as in dealing with many physical problems. The differential equation with nonlocal conditions has been firstly considered by Byszewski [24]. In [25], authors have established the existence of mild solutions to a nonlocal impulsive integro-differential equations via measure of noncompactness and new fixed point theorem. By utilizing fixed point theorem for condensing operator and approximation techniques, the existence of the mild solution is obtained by authors in [26]. An impulsive neutral integro-differential equation of Sobolev type with time varying delays has been considered by authors in [27] and the sufficient condition for existence of the mild solution has been provided by using the Monch's fixed point theorem. In [14], authors have studied the existence results for the mild solution of a nonlocal differential equation impulsive conditions. The existence results for mild solutions have been obtained via the techniques of approximate solutions and fixed point theorem. By utilizing approximation techniques and fractional operator, the existence of the mild solution for nonlocal impulsive functional integro-differential equation has been established by authors in [15]. In [16], authors have considered an impulsive stochastic functional integro-differential inclusions with nonlocal conditions in a Hilbert space and provided existence results for mild solution by using approximation technique and BohnenblustKarlins fixed point theorem. In

[17], authors have concerned with the existence of α-mild solutions for a fractional stochastic integro-differential equations with nonlocal initial conditions in a real separable Hilbert space by using an approximation technique. For more details, we refer to papers [11,14–20,24,26,28] and references given therein.

Motivated by above stated work, our main objective of this work is to examine the following nonlocal impulsive non-autonomous neutral functional differential equation in a Banach space E:

$$\frac{d}{dt}\left[y(t)-F(t, y(h_1(t)), \int_0^t a_1(t, s, y(h_2(s)))ds)\right] \in -B(t)y(t)+G(t, y(h_3(t))),$$
$$t \in J = [0, T], t \neq t_i, \tag{1}$$
$$y(0) = u_0 + g(y) \in E, \tag{2}$$
$$\Delta y(t_i) = I_i(y(t_i)), \quad i = 1, \ldots, \delta, \quad \delta \in \mathbb{N}, \tag{3}$$

where, $0 < T < \infty$, $-B(t) : D(B(t)) \subseteq E \to E,\ t \geq 0$ is a closed densely defined linear operator. Here, $h_j : [0, T] \to [0, T]$, $j = 1, 2, 3$ and $0 = t_0 < t_1 < \cdots < t_\delta < t_{\delta+1} = T$ are fixed numbers, $0 < T < \infty$, $\Delta y|_{t=t_i} = y(t_i^+) - y(t_i^-)$ and $y(t_i^-) = \lim_{\varepsilon\to 0-} y(t_i + \varepsilon)$ and $y(t_i^+) = \lim_{\varepsilon\to 0+} y(t_i + \varepsilon)$ denote the left and right limits of $y(t)$ at $t = t_i$, respectively. In (1), $B(t)$ is assumed to be the infinitesimal generator of a compact analytic semigroup of bounded linear operators on a Banach space E. The nonlinear functions F, g, G, a_1 and $I_i : E \to E$ $(i = 1, \ldots, \delta)$ are appropriate functions satisfying some conditions to be mentioned later section. For more details on non-autonomous differential equations, we refer to monograph [21,22], and papers [1–8,11,12,26,27,29–35] and references cited therein.

The organization of the paper is as follows: In Sect. 2, we provide some basic definitions, lemmas and theorems. Section 3 discusses the existence of mild solution to (1)–(3) by utilizing Krasnoselskii's fixed point theorem and limit arguments. Section 4 provides an example to illustrate the obtained abstract theory.

2 Preliminaries

In this section, we state some basic definitions, preliminaries, theorems, lemmas and assumptions which will be used for proving main result.

Throughout the work, we assume that $(E, \|\cdot\|)$ is a Banach space and the notation $C([0, T], E)$ stands for the space of E-valued continuous functions on $[0, T]$ with the norm $\| z\| = \sup\{\| z(\tau)\|, \tau \in [0, T]\}$ and $L^1([0, T], E)$ denotes the space of E-valued Bochner integrable functions on $[0, T]$ endowed with the norm $\| \mathcal{F}\|_{L^1} = \int_0^T \| \mathcal{F}(t)\|dt,\ \mathcal{F} \in L^1([0, T], E)$. We denote by $C^\beta([0, T]; E)$ the space of all uniformly Hӧlder continuous functions from $[0, T]$ into E with exponent $\beta > 0$. It is easy to verify that $C^\beta([0, T]; E)$ is a Banach space with the norm

$$\| y\|_{C^\beta([0,T];E)} = \sup_{0\leq t\leq T} \| y(t)\| + \sup_{0\leq t,s\leq T,\ t\neq s} \frac{\| y(t) - y(s)\|}{|t - s|^\beta}. \tag{4}$$

To consider the mild solution for the impulsive problem, we propose the space $PC([0,T];E) = \{y : [0,T] \to E : y$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and $y(t_i^+)$ exists, for all $i = 1,\ldots,\delta\}$. Clearly, $PC([0,T];E)$ is a Banach space endowed the norm $\| y\|_{PC} = \sup_{t\in[0,T]} \| y(s)\|$. For a function $y \in PC([0,T];E)$ and $i \in \{0,1,\ldots,\delta\}$, we define the function $\widetilde{y_i} \in C([t_i,t_{i+1}],E)$ such that

$$\widetilde{y_i}(t) = \begin{cases} y(t), & \text{for } \; t \in (t_i,t_{i+1}], \\ y(t_i^+), & \text{for } \; t = t_i. \end{cases} \tag{5}$$

For $W \subset PC([0,T];E)$ and $i \in \{0,1,\ldots,\delta\}$, we have $\widetilde{W_i} = \{\widetilde{y_i} : y \in W\}$ and following Accoli–Arzel*à* type criteria.

Lemma 2.1 [16] *A set $W \subset PC([0,T];E)$ is relatively compact in $PC([0,T];E)$ if and only if each set $\widetilde{W}_j (j = 1,2,\ldots,\delta)$ is relatively compact in $C([t_j,t_{j+1}],E)(j = 0,1,\ldots,\delta)$.*

Let $\{B(t) : 0 \leq t \leq T\}, T \in (0,\infty)$ be a family of closed linear operators on the Banach space E. We impose following restrictions {[22]} as:

(P1) The domain $D(B)$ of $\{B(t) : \; t \in [0,T]\}$ is dense in E and $D(B)$ is independent of t.
(P2) For each $0 \leq t \leq T$ and $Re\lambda \leq 0$, the resolvent $R(\lambda; B(t))$ exists and there exists a positive constant K (independent of t and λ) such that

$$\| R(\lambda; B(t))\| \leq K/(|\lambda|+1), \quad Re\lambda \leq 0, \quad t \in [0,T].$$

(P3) For each fixed $\xi \in [0,T]$, there exist constant $K > 0$ and $0 < \mu \leq 1$ such that

$$\| [B(\tau) - B(s)]B^{-1}(\xi)\| \leq K|\tau - s|^{\mu}, \quad \text{for any} \quad \tau, \quad s \in [0,T], \tag{6}$$

where μ and K are independent of τ, s and ξ.
(P4) For every $t \in [0,T]$, the resolvent set of $B(t)$, the resolvent $R(\lambda, B(t))$, is a compact operator for some $\lambda \in \rho(B(t))$.

The assumptions $(P1)$–$(P3)$ permit that there is a unique linear evolution system (linear evolution operator) $\mathcal{S}(t,s)$, $0 \leq s \leq t \leq T$ which is generated by family $\{B(t) : \; t \in [0,T]\}$ and there exists a family of bounded linear operators $\{\Phi(t,s) : 0 \leq s \leq t \leq T\}$ such that $\| \Phi(t,s)\| \leq \frac{K}{|t-s|^{1-\mu}}$. We also have that $\mathcal{S}(t,s)$ can be written as

$$\mathcal{S}(t,s) = e^{-(t-s)B(t)} + \int_s^t e^{-(t-\tau)B(\tau)}\Phi(\tau,s)d\tau. \tag{7}$$

The assumption $(P2)$ guarantees that $-B(s),\; s \in [0,T]$ is the infinitesimal generator of a strongly continuous compact analytic semigroup $\{e^{-tB(s)} : t \geq 0\}$ in $\mathbb{B}(E)$, where the symbol $\mathbb{B}(E)$ stands for the Banach algebra of all bounded linear operators on E.

By the assumptions $(P1)$–$(P4)$ [see, [22]], it follows that there is a unique fundamental solution $\{\mathcal{S}(t,s) : 0 \leq s \leq t \leq T\}$ for the homogenous Cauchy problem such that

(i) $\mathcal{S}(t,s) \in \mathbb{B}(E)$ and the mapping $(t,s) \to \mathcal{S}(t,s)z$ is continuous for $z \in E$, i.e., $\mathcal{S}(t,s)$ is strongly continuous in t, s for all $0 \le s \le t \le T$.
(ii) For each $z \in E$, $\mathcal{S}(t,s)z \in D(B)$, for all $0 \le s \le t \le T$.
(iii) $\mathcal{S}(t,\tau)\mathcal{S}(\tau,s) = \mathcal{S}(t,s)$ for all $0 \le s \le \tau \le t \le T$.
(iv) For each $0 \le s < t \le T$, the derivative $\frac{\partial \mathcal{S}(t,s)}{\partial t}$ exists in the strong operator topology and an element of $\mathbb{B}(E)$, and strongly continuous in t, where $s < t \le T$.
(v) $\mathcal{S}(t,t) = I$.
(vi) $\frac{\partial \mathcal{S}(t,s)}{\partial t} + B(t)\mathcal{S}(t,s) = 0$ for all $0 \le s < t \le T$.

We have also the following inequalities:

$$\| e^{-tB(\tau)} \| \le Ke^{-dt}, \quad t \ge 0; \tag{8}$$

$$\| B(\tau)e^{-tB(\tau)} \| \le \frac{Ke^{-dt}}{t}, \quad t > 0; \tag{9}$$

$$\| B(t)\mathcal{S}(t,\tau) \| \le K|t-\tau|^{-1}, \quad 0 \le s \le t \le T. \tag{10}$$

for all $\tau \in [0,T]$, where d is a positive constant. For $\alpha > 0$, we may define negative fractional powers $B(t)^{-\alpha}$ as

$$B(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-sB(t)} ds. \tag{11}$$

Then, the operator $B(t)^{-\alpha}$ is bounded linear and one to one operator on E and $B^{-\alpha}(t)B^{-\beta}(t) = B^{-(\alpha+\beta)}(t)$. Therefore, it implies that there exists an inverse of the operator $B(t)^{-\alpha}$. We can define $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ which is the positive fractional powers of $B(t)$. The operator $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ is closed densely defined linear operator with domain $D(B(t)^{\alpha}) \subset E$ and for $\alpha < \beta$, we get $D(B(t)^{\beta}) \subset D(B(t)^{\alpha})$. Let $E_\alpha(t_0) = D(B(t_0)^{\alpha})$ be a Banach space with the norm $\|y\|_\alpha = \|B(t_0)^{\alpha}y\|$, $t_0 \in [0,T]$. For $0 < \omega_1 \le \omega_2$, we have that the embedding $E_{\omega_2}(t_0) \hookrightarrow E_{\omega_1}(t_0)$ is continuous and dense. For each $\alpha > 0$, we may define $E_{-\alpha}(t_0) = (E_\alpha)^*(t_0)$, which is the dual space of $E_\alpha(t_0)$. The dual space is a Banach space with natural norm $\|y\|_{-\alpha} = \|B(t_0)^{-\alpha}y\|$. In particular, by the assumption $(P3)$, we conclude a constant $K > 0$, such that

$$\| B(t)B(s)^{-1} \| \le K, \quad \text{for all} \quad 0 \le s, t \le T. \tag{12}$$

We also have following results:

$$\|B^{\alpha}(t)B^{-\beta}(s)\| \le N_{\alpha,\beta}, \tag{13}$$

$$\|B^{\beta}(t)e^{-sB(t)}\| \le \frac{N_\beta}{s^\beta}e^{-ws}, \quad t > 0, \quad \beta \le 0, \quad w > 0, \tag{14}$$

$$\|B^{\beta}(t)\mathcal{S}(t,s)\| \le N_\beta|t-s|^{-\beta}, \quad 0 < \beta < \mu+1, \tag{15}$$

$$\|B^{\beta}(t)\mathcal{S}(t,s)B^{-\beta}(s)\| \le N_\beta^{'}, \quad 0 < \beta < \mu+1, \tag{16}$$

for $s, t \in [0, T]$, $0 \leq \alpha < \beta$ and $t > 0$, where $N_{\alpha,\beta}$ is a constant related to T and μ and $N_{\alpha,\beta}, N_\beta, N_\beta^{'}$ show their dependence on the constants α, β. We also have following results

Lemma 2.2 (Lemmas II.14.1, [29]) *Suppose that $(P1)$–$(P3)$ are satisfied. If $0 \leq \gamma \leq 1$, $0 \leq \beta \leq \alpha < 1 + \mu$, $0 < \alpha - \gamma \leq 1$, then for any $0 \leq \tau < t < t + \Delta t \leq t_0$, $0 \leq \zeta \leq t_0$,*

$$\|B^\gamma(\zeta)[\mathcal{S}(t + \Delta t, \tau) - \mathcal{S}(t, \tau)]B^{-\beta}(\tau)\| \leq N_{\gamma,\beta,\alpha}(\Delta t)^{\alpha-\gamma}|t - \tau|^{\beta-\alpha}. \tag{17}$$

Lemma 2.3 (Lemmas II.14.4, [29]) *Suppose that $(P1)$-$(P3)$ are satisfied and let $0 \leq \gamma < 1$. Then for any $0 \leq \tau \leq t \leq t + \Delta \leq t_0$ and for any continuous function $f(s)$,*

$$\begin{aligned} &\|B^\gamma(\zeta)[\int_t^{t+\Delta t} \mathcal{S}(t + \Delta t, s)f(s)ds - \int_0^t \mathcal{S}(t, s)f(s)ds]\| \\ &\leq N_\gamma(\Delta t)^{1-\gamma}(|\log(\Delta t)| + 1) \max_{\tau \leq s \leq t+\Delta t} \|f(s)\|. \end{aligned} \tag{18}$$

For more details, we refer to monographs [21,22,29].

Our main existence result is based on the Krasnoselskii's fixed point theorem.

Lemma 2.4 [36] *Let E be a Banach space and $\mathbb{D}$ be a non-empty closed bounded and convex subset of E. Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be two operators such that*

1. *$\mathbb{Q}_1 x + \mathbb{Q}_2 y \in \mathbb{D}$, for every pair $x, y \in \mathbb{D}$;*
2. *$\mathbb{Q}_1$ is a contraction mapping;*
3. *$\mathbb{Q}_2$ is completely continuous.*

Then, there exists a fixed point of map $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$.

3 Main result

The main result of this article are provided in this section. We study the existence of mild solutions for (1)–(3) on the Banach subspace $E_\alpha(t_0)$ for some $0 < \alpha < 1$ and $t_0 \in [0, T]$. For this, we make following assumptions:

$(H1)$ The nonlinear function $G : [0, T] \times E_\alpha(t_0) \to E$ satisfies the following conditions.

(1) The function $G : [0, T] \times E_\alpha(t_0) \to E$ fulfills the Carathéodory type conditions, that is, $G(\cdot, y)$ is strongly measurable for each $y \in E_\alpha(t_0)$ and $G(t, \cdot)$ is continuous for almost everywhere $t \in [0, T]$.

(2) For $r > 0$, there exists function $0 < m_r \in L^1([0, T], \mathbb{R}^+)$ such that

$$\begin{aligned} &\sup_{\|y\|_\alpha \leq r} \|G(t, y)\| \leq m_r(t), \quad \forall \quad y \in E_\alpha(t_0) \quad \text{and} \\ &\lim_{r \to +\infty} \inf \frac{\|m_r\|_{L^1}}{r} = \varpi_1 < +\infty, \end{aligned} \tag{19}$$

where $\varpi_1 > 0$ is a constant.

$(H2)$ $F : [0, T] \times E_\alpha(t_0) \times E_\alpha(t_0) \to E$ is a Lipschitz continuous function with $F([0, T] \times E_\alpha(t_0) \times E_\alpha(t_0)) \subset D(B)$. $B(t)F$ is continuous and there exists a positive constant L_F such that

$$\|B(t)F(t, x_1, y_1) - B(t)F(s, x_2, y_2)\| \le L_F[|t - s| + \|x_1 - x_2\|_\alpha + \|y_1 - y_2\|_\alpha], \tag{20}$$

for each $(t, x_1, y_1), (s, x_2, y_2) \in [0, T] \times E_\alpha(t_0) \times E_\alpha(t_0)$ and $\mathcal{C}_4 = \sup_{t\in[0,T]} \|B(t)F(t, 0, 0)\|$.

$(H3)$ $g : PC([0, T], E_\alpha(t_0)) \to D(B)$ is a nonlinear function which satisfies that $B(t)g$ is continuous on $PC([0, T], E_\alpha(t_0))$ and

$$\|B(t)g(z)\| \le L_g\|z\|_{PC([0,T],E_\alpha(t_0))}, \quad \text{for each} \quad z \in PC([0, T], E_\alpha(t_0)),$$

where L_g is a constant. Furthermore, there exists a $\theta \in (0, t_1)$ such that $g(u) = g(v)$ for $u, v \in PC([0, T], E_\alpha(t_0))$ with $u = v$ on $[\theta, T]$.

$(H4)$ (1) The map $a_1 : D \times E_\alpha(t_0) \to E_\alpha(t_0)$, where $D = \{(t, s) \in J \times J : t \ge s\}$ is a continuous mapping and there exists a positive constant L_{a_1} such that

$$\left\| \int_0^t [a_1(t, s, z_1) - a_1(t, s, z_2)]ds \right\|_\alpha \le L_{a_1}\|z_1 - z_2\|_\alpha, \tag{21}$$

for all $z_1, z_2 \in E_\alpha(t_0)$ and $t \in J$ with $\mathcal{C}_3 = \sup_{t\in[0,T]} \int_0^t \|a_1(t, s, 0)\|ds$.

(2) There exists a positive constant L_a such that

$$\left\| \int_0^{t_1} a_1(t_1, s, y)ds - \int_0^{t_2} a_1(t_2, s, y)ds \right\|_\alpha \le L_a|t_1 - t_2|, \quad t_1, t_2 \in [0, T]. \tag{22}$$

$(H5)$ $I_i : E_\alpha(t_0) \to E_\alpha(t_0)$, $(i = 1, \ldots, \delta)$ are continuous and there exist continuous nondecreasing functions $J_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|B(t)I_i(z)\| \le J_i(\|z\|_\alpha), \quad \liminf_{r\to+\infty} \frac{J_i(r)}{r} = \gamma_i < \infty, \tag{23}$$

for each $z \in E_\alpha(t_0)$.

Consider the sets $\mathcal{B}_r = \{y \in E_\alpha(t_0) : \|y\|_\alpha \le r\}$ and $\mathcal{W}_r = \{y \in PC([0, T], E_\alpha(t_0)) : y(t) \in \mathcal{B}_r,$ for all $t \in [0, T]\}$ for each finite constant $r > 0$.

Before expressing and demonstrating the main result, we present the definition of the mild solution to problem (1)–(3).

Definition 3.1 A piecewise continuous function $y(\cdot) : [0, T] \to E$ is called a mild solution for the problem (1)–(3) if $y(0) = y_0 + g(y)$ and $y(\cdot)$ satisfies the integral equation

$$y(t) = \mathcal{S}(t, 0)[y_0 + g(y) - F(0, y(h_1(0)), 0)]$$

$$+F(t, y(h_1(t)), \int_0^t a_1(t, s, y(h_2(s)))ds)$$
$$-\int_0^t \mathcal{S}(t, \tau)B(\tau)F(\tau, y(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y(h_2(\xi)))d\xi)d\tau$$
$$+\int_0^t \mathcal{S}(t, \tau)G(\tau, y(h_3(\tau)))d\xi)d\tau + \sum_{0<t_i<t} \mathcal{S}(t, t_i)\mathcal{S}(\theta_n, 0)I_i(u(t_i^-)), \ t\in[0, T]. \tag{24}$$

Now, we present the existence result for nonlocal impulsive system (1)–(3) by utilizing the fixed point theorem and techniques of approximate solutions under the assumption that nonlocal function is Lipschitz continuous in PC and impulsive function is not completely continuous in PC.

Theorem 3.1 *Let us assume that the hypotheses* $(P1)$-$(P4)$ *and* $(H1)$-$(H5)$ *are satisfied,* $y_0 \in E_\beta(t_0)$ *for some* β, $0 < \alpha < \beta \leq 1$ *and*

$$N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}N_1^{'}L_g + N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}L_F + N_{\alpha,1}L_F(1 + L_{a_1})$$
$$+N_{\alpha,\beta}N_\beta L_F(1 + L_{a_1})\frac{T^{1-\beta}}{(1-\beta)}$$
$$+N_{\alpha,\beta}N_\beta\varpi_1\frac{T^{1-\beta}}{(1-\beta)} + N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}N_1^{'}K\sum_{i=1}^{\delta}\gamma_i < 1. \tag{25}$$

Then, there exists at least one mild solution for the problem (1)–(3) *on* $[0, T]$.

Proof To prove the theorem, we consider the following approximate problem illustrated by

$$\frac{d}{dt}\left[y(t) - F(t, y(h_1(t)), \int_0^t a_1(t, s, y(h_2(s)))ds)\right] = -B(t)y(t) + G(t, y(h_3(t))),$$
$$t \in J = [0, T], t \neq t_i \tag{26}$$
$$y(0) = u_0 + \mathcal{S}(\theta_n, 0)g(y) \in E, \tag{27}$$
$$\Delta y(t_i) = \mathcal{S}(\theta_n, 0)I_i(y(t_i)), \quad i = 1, \ldots, \delta, \tag{28}$$

for each n such that $\theta_n \to 0$ as $n \to \infty$. In order to show the existence of solution for problem (1)–(3), it is sufficient to show that there exists at least one mild solution for the nonlocal problem (26)–(28). To this end, we consider the operator $\mathbb{Q}_n : PC([0, T], E_\alpha(t_0)) \to PC([0, T], E_\alpha(t_0))$ defined by

$$(\mathbb{Q}_n y)(t) = \mathcal{S}(t, 0)[y_0 + \mathcal{S}(\theta_n, 0)g(y) - F(0, y(h_1(0)), 0)] + F(t, y(h_1(t)),$$
$$\times \int_0^t a_1(t, s, y(h_2(s)))ds)$$
$$-\int_0^t \mathcal{S}(t, \tau)B(\tau)F(\tau, y(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y(h_2(\xi)))d\xi)d\tau$$

$$+\int_0^t \mathcal{S}(t,\tau)G(\tau, y(h_3(\tau)))d\xi)d\tau + \sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{S}(\theta_n, 0)I_i(y(t_i^-)),$$
$$t \in [0, T]. \tag{29}$$

Clearly, the fixed point of the map $\mathbb{Q}_n$ is the mild solution of the nonlocal impulsive differential system (26)–(28). Thus, we will show that there exists a fixed point of mapping $\mathbb{Q}_n$ by utilizing Krasnoselskii's fixed point theorem. Now, we decompose $\mathbb{Q}_n$ as $\mathbb{Q}_n = \mathbb{Q}_n^1 + \mathbb{Q}_n^2$, where

$$(\mathbb{Q}_n^1 y)(t) = \begin{cases} \mathcal{S}(t,0)[-F(0, y(h_1(0)), 0))] + F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds) \\ -\int_0^t \mathcal{S}(t,\tau)B(\tau)F(\tau, y(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,y(h_2(\xi)))d\xi)d\tau, \; t\in[0,T], \end{cases} \tag{30}$$

$$(\mathbb{Q}_n^2 y)(t) = \mathcal{S}(t,0)[y_0 + \mathcal{S}(\theta_n, 0)g(y)] + \int_0^t \mathcal{S}(t,\tau)G(\tau, y(h_3(\tau)))d\tau$$
$$+ \sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y(t_i^-)), \; t \in [0,T]. \tag{31}$$

Now, we can show the result in several steps.

Step 1. We show that $(\mathbb{Q}_n^1 + \mathbb{Q}_n^2)(\mathcal{W}_r(PC)) \subset \mathcal{W}_r(PC)$.

Suppose, on the contrary that for each $r \in \mathbb{N}$ there exist $y^r \in \mathcal{W}_r(PC)$ and some $t^r \in [0,T]$ such that $\|(\mathbb{Q}_n y^r)(t^r)\| > r$. Then, we get

$$\begin{aligned}
r &< \|(\mathbb{Q}y^r)(t^r)\|_\alpha \\
&\le \|(\mathbb{Q}_1 y^r)(t^r)\|_\alpha + \|(\mathbb{Q}_2 y^r)(t^r)\|_\alpha \\
&\le \|\mathcal{S}(t,0)[y_0 + \mathcal{S}(\theta_n,0)g(y^r) - F(0, y^r(h_1(0)), 0))]\|_\alpha \\
&\quad + \|F(t^r, y^r(h_1(t^r)), \int_0^{t^r} a_1(t^r, s, y^r(h_2(s)))ds)\|_\alpha \\
&\quad + \|\int_0^{t^r} \mathcal{S}(t^r,\tau)B(\tau)F(\tau, y^r(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,y^r(h_2(\xi)))d\xi)d\tau\|_\alpha \\
&\quad + \|\int_0^{t^r} \mathcal{S}(t^r,\tau)G(\tau, y^r(h_3(\tau)))d\tau\|_\alpha + \|\sum_{0<t_i<t} \mathcal{S}(t^r,t_i)\mathcal{S}(\theta_n,0)I_i(y^r(t_i^-))\|_\alpha, \\
&\le \|B^\alpha(t_0)B^{-\beta}(t^r)\|\|B^\beta(t^r)\mathcal{S}(t^r,0)B^{-\beta}(0)\|[\|B^\beta(0)y_0\| + \|B^\beta(0)B^{-1}(\theta_n)\| \\
&\quad \times \|B(\theta_n)\mathcal{S}(\theta_n,0)B^{-1}(0)\|\|B(0)g(y^r)\| + \|B^\beta(0)B^{-1}(t^r)\| \\
&\quad \times \|B(t^r)F(0, y^r(h_1(0)), 0))\|] \\
&\quad + \|B^\alpha(t_0)B^{-1}(t^r)\| \cdot \|B(t^r)F(t^r, y^r(h_1(t^r)), \int_0^{t^r} a_1(t^r,s,y^r(h_2(s)))ds)\| \\
&\quad + \int_0^{t^r} \|B^\alpha(t_0)B^{-\beta}(t^r)\| \cdot \|B^\beta(t^r)\mathcal{S}(t^r,\tau)\|\|B(\tau)F(\tau, y^r(h_1(\tau)), \\
&\quad \times \int_0^\tau a_1(\tau,\xi,y^r(h_2(\xi)))d\xi)\|d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t^r} \|B^\alpha(t_0)B^{-\beta}(t^r)\|\|B^\beta(t^r)\mathcal{S}(t^r,\tau)\|\cdot\|G(\tau, y^r(h_3(\tau)))\|d\tau\\
&+\sum_{0<t_i<t} \|B^\alpha(t_0)B^{-\beta}(t^r)\|\cdot\|B^\beta(t^r)\|\mathcal{S}(t^r,t_i)B^{-\beta}(t_i)\|\cdot\|B^\beta(t_i)B^{-1}(\theta_n)\|\\
&\times\|B(\theta_n)\mathcal{S}(\theta_n,0)B^{-1}(0)\|\times\|B(0)B^{-1}(t^r)\|\|B(t^r)I_i(y^r(t_i^-))\|,\\
\le{}& N_{\alpha,\beta}N_\beta^{'}\|B^\beta(0)y_0\| + N_{\alpha,\beta}N_\beta^{'}N_1^{'}N_{\beta,1}L_g r + N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}[L_F(T+r)+\mathcal{C}_4]\\
&+N_{\alpha,1}[L_F(r+L_{a_1}r+\mathcal{C}_3)+\mathcal{C}_4] + N_{\alpha,\beta}N_\beta[L_F(r+L_{a_1}r+\mathcal{C}_3)\\
&+\mathcal{C}_4]\int_0^{t^r}(t^r-\tau)^{-\beta}d\tau\\
&+N_{\alpha,\beta}N_\beta\int_0^{t^r}(t^r-\tau)^{-\beta}m_r(\tau)d\tau\\
&+N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}N_1^{'}K\sum_{0<t_i<t^r}^{\delta} J_i(r).
\end{aligned}
\tag{32}
$$

We divide on both sides by r and take the lower limit as $r \to \infty$ and therefore obtain

$$
\begin{aligned}
&N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}N_1^{'}L_g + N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}L_F\\
&\quad+N_{\alpha,1}L_F(1+L_{a_1}) + N_{\alpha,\beta}N_\beta L_F(1+L_{a_1})\frac{T^{1-\alpha}}{(1-\alpha)}\\
&\quad+N_{\alpha,\beta}N_\beta\varpi_1\frac{T^{1-\beta}}{(1-\beta)} + N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}N_1^{'}K\sum_{i=1}^{\delta}\gamma_i > 1,
\end{aligned}
\tag{33}
$$

which contradicts (25). Therefore, we have that there exists a positive integer $r \in \mathbb{N}$ such that $Q(\mathcal{W}_r(PC)) \subset \mathcal{W}_r(PC)$.

Step 2. The mapping $\mathbb{Q}_n^1$ is a contraction map on $PC([0,T], E_\alpha(t_0))$.

Let $y_1,\ y_2 \in PC([0,T], E_\alpha(t_0))$ and $t \in [0,T]$. Then, we obtain

$$
\begin{aligned}
&\|(\mathbb{Q}_n^1 y_1)(t) - (\mathbb{Q}_n^1 y_2)(t)\|_\alpha\\
&\quad\le \|B^\alpha(t_0)B^{-\beta}(t)\|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\|\\
&\qquad\times[\|B^\beta(0)B^{-1}(t)\|\|B(t)F(0,y_1(h_1(0)),0)) - B(t)F(0,y_2(h_1(0)),0))\|]\\
&\qquad+\|B^\alpha t_0B^{-1}(t)\|\times\|B(t)F(t,y_1(h_1(t)),\int_0^t a_1(t,s,y_1(h_2(s)))ds)\\
&\qquad-B(t)F(t,y_2(h_1(t)),\int_0^t a_1(t,s,y_2(h_2(s)))ds)\|\\
&\qquad+\int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\cdot\|B^\beta(t)\mathcal{S}(t,\tau)\|\times\|B(\tau)F(\tau,y_1(h_1(\tau)),\\
&\qquad\quad\times\int_0^\tau a_1(\tau,\xi,y_1(h_2(\xi)))d\xi)\\
&\qquad-B(\tau)F(\tau,y_2(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_2(h_2(\xi)))d\xi)\|d\tau,
\end{aligned}
$$

$$\leq \left[N_{\alpha,\beta} N'_{\beta} N_{\beta,1} L_F + N_{\alpha,1} L_F (1 + L_{a_1}) + N_{\alpha,\beta} N_{\beta} L_F (1 + L_{a_1}) \frac{T^{1-\beta}}{(1-\beta)} \right]$$
$$\times \| y_1 - y_2 \|_{PC([0,T],E_\alpha(t_0))} \tag{34}$$

Since $\Lambda = N_{\alpha,\beta} N'_{\beta} N_{\beta,1} L_F + N_{\alpha,1} L_F (1 + L_{a_1}) + N_{\alpha,\beta} N_{\beta} L_F (1 + L_{a_1}) \frac{T^{1-\beta}}{(1-\beta)} < 1$ by (25). Therefore, we take the supremum of t over $[0, T]$ and deduce

$$\| (\mathbb{Q}_n^1 y_1) - (\mathbb{Q}_n^1 y_2) \|_{PC([0,T],E_\alpha(t_0))} \leq \Lambda \| y_1 - y_2 \|_{PC([0,T],E_\alpha(t_0))} \tag{35}$$

with $\Lambda < 1$. Hence, the operator $\mathbb{Q}_n^1$ is a contraction map on $PC([0, T], E_\alpha(t_0))$. Next, we show that $\mathbb{Q}_n^2$ is completely continuous mapping on $PC([0, T], E_\alpha(t_0))$.

Step 3. We show the continuity of the mapping $\mathbb{Q}_n^2$ on $PC([0, T], E_\alpha(t_0))$.

Let $\{y_p\}_{p=1}^{\infty}$ be a sequence in $PC([0, T], E_\alpha(t_0))$ such that $\lim_{p\to\infty} y_p = y \in PC([0, T], E_\alpha(t_0))$. By the continuity of G and I_i, $i = 1, \ldots, \delta$, we have that for each $t \in [0, T]$

$$G(t, y_p(h_3(t))) \to G(t, y(h_3(t))), \tag{36}$$
$$I_i(y_p(t_i^-)) \to I_i(y(t_i^-)), \tag{37}$$

as $p \to \infty$. Then, by the dominated convergence theorem, we deduce that

$$\begin{aligned}
&\| (\mathbb{Q}_n^2 y_p)(t) - (\mathbb{Q}_n^2 y)(t) \|_\alpha \\
&\quad \leq \| B^\alpha(t_0) \mathcal{S}(t, 0) \mathcal{S}(\theta_n, 0)(g(y_p) - g(y)) \| \\
&\qquad + \int_0^t \| B^\alpha(t_0) B^{-\beta}(t) \| \| B^\beta(t) \mathcal{S}(t, \tau) \| \times \| G(\tau, y_p(h_3(\tau))) - G(\tau, y(h_3(\tau))) \| d\tau \\
&\qquad + \sum_{0<t_i<t} \| B^\alpha(t_0) B^{-\beta}(t) \| \| B^\beta(t) \mathcal{S}(t, t_i) B^{-\beta}(t_i) \| \| B^\beta(t_i) B^{-1}(\theta_n) \| \\
&\qquad \times \| B(\theta_n) \mathcal{S}(\theta_n, 0) B^{-1}(0) \| \\
&\qquad \times \| B(0) B^{-1}(t) \| \| B(t)[I_i(y_p(t_i^-)) - I_i(y(t_i^-))] \|, \\
&\quad \to 0, \quad \text{as} \quad p \to \infty,
\end{aligned} \tag{38}$$

which implies the continuity of the mapping $\mathbb{Q}_n^2$ on $PC([0, T], E_\alpha(t_0))$.

Step 4. In this step, we show that $\mathbb{Q}_n^2$ maps bounded sets into bounded sets in $PC([0, T], E_\alpha(t_0))$.

For this purpose, it is sufficient to prove that there exists a positive constant Π such that one has $\| (\mathbb{Q}_n^2 y)(t) \| \leq \Pi$ for each $y \in \mathcal{W}_r(PC) = \{ y \in PC([0, T], E_\alpha(t_0)) : \| y \|_\alpha \leq r \}$. Thus, for each $t \in [0, T]$ and $y \in \mathcal{W}_k \subset PC([0, T], E_\alpha(t_0))$. Then, we have

$$\begin{aligned}
\| (\mathbb{Q}_n^2 y)(t) \|_\alpha &\leq \| B^\alpha t_0 \mathcal{S}(t, 0)[y_0 + \mathcal{S}(\theta_n, 0) g(y)] \| \\
&\quad + \int_0^t \| B^\alpha(t_0) B^{-\beta}(t) \| \cdot \| B^\beta(t) \mathcal{S}(t, \tau) \| \times \| G(\tau, y(h_3(\tau))) \| d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_i<t}\|B^{\alpha}(t_0)B^{-\beta}(t)\|\cdot\|B^{\beta}(t)\mathcal{S}(t,t_i)B^{-\beta}(t_i)\|\|B^{\beta}(t_i)B^{-1}(\theta_n)\|\\
&\times\|B(\theta_n)\mathcal{S}(\theta_n,0)B^{-1}(0)\|\cdot\|B(0)B^{-1}(t)\|\|B(t)I_i(y(t_i^-))\|\\
\le\ & N_{\alpha,\beta}N_{\beta}^{'}\|B^{\beta}(0)y_0\|+N_{\alpha,\beta}N_{\beta}^{'}N_1^{'}N_{\beta,1}L_g r\\
&+N_{\alpha,\beta}N_{\beta}\int_0^t(t-\tau)^{-\beta}m_r(\tau)d\tau\\
&+N_{\alpha,\beta}N_{\beta}^{'}N_{\beta,1}N_1^{'}K\sum_{i=1}^{\delta}J_i,\\
\le\ & N_{\alpha,\beta}N_{\beta}^{'}\|B^{\beta}(0)y_0\|+N_{\alpha,\beta}N_{\beta}^{'}N_1^{'}N_{\beta,1}L_g r+N_{\alpha,\beta}N_{\beta}\|m_r\|_{L^1}\frac{T^{1-\beta}}{(1-\beta)}\\
&+N_{\alpha,\beta}N_{\beta}^{'}N_{\beta,1}N_1^{'}K\sum_{i=1}^{\delta}J_i,\\
=\ &\Pi.
\end{aligned}
\tag{39}
$$

Thus, for every $y\in\mathcal{W}_r(PC)$, we obtain $\|(\mathbb{Q}_n^2 y)(t)\|_{\alpha}\le\Pi$.

Step 5. $\mathbb{Q}_n^2$ maps bounded sets into equicontinuous sets of $PC([0,T],E_{\alpha}(t_0))$.

Let $t\in J-\{t_1,t_2,\ldots,t_{\delta}\}$ and let $h>0$ be a arbitrary constant with $0\le t\le t+h<T$. Then, for $y\in\mathcal{W}_r(PC)$, we get

$$
\begin{aligned}
&\|(\mathbb{Q}_n^2 y)(t+h)-(\mathbb{Q}_n^2 y)(t)\|_{\alpha}\\
&\le\|[\mathcal{S}(t+h,0)-\mathcal{S}(t,0)][y_0+\mathcal{S}(\theta_n,0)g(y)]\|_{\alpha}\\
&\quad+\|\int_0^{t+h}\mathcal{S}(t+h,\tau)G(\tau,y(h_3(\tau)))d\tau-\int_0^t\mathcal{S}(t,\tau)G(\tau,y(h_3(\tau)))d\tau\|_{\alpha}\\
&\quad+\|\sum_{0<t_i<t}[\mathcal{S}(t+h,t_i)-\mathcal{S}(t,t_i)]\mathcal{S}(\theta_n,0)I_i(y(t_i^-))\|_{\alpha}.
\end{aligned}
\tag{40}
$$

From the Lemma 14.1 [29] and Lemma 14.4 [29], we have following results

$$
\begin{aligned}
&\|[\mathcal{S}(t+h,0)-\mathcal{S}(t,0)][y_0+\mathcal{S}(\theta_n,0)g(y)]\|_{\alpha}\\
&\quad\le N_{\alpha,\beta}h^{\beta-\alpha}\|B^{\beta}(0)[y_0+\mathcal{S}(\theta_n,0)g(y)]\|,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&\|B^{\alpha}(t_0)[\mathcal{S}(t+h,t_i)-\mathcal{S}(t,t_i)]B^{-\beta}(t_i)\|\|B^{\beta}(t_i)B^{-1}(\theta_n)\|\cdot\|B(\theta_n)\mathcal{S}(\theta_n,0)B^{-1}(0)\|\\
&\quad\times\|B(0)B^{-1}(t)\|\|B(t)I_i(y(t_i^-))\|\le N_{\beta,1}N_1^{'}N_1\|B(t)I_i(y(t_i^-))\|\times N_{\alpha,\beta}h^{\beta-\alpha}.
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&\|B^{\alpha}(t_0)\int_0^{t+h}\mathcal{S}(t+h,\tau)G(\tau,y(h_3(\tau)))d\tau-\int_0^t\mathcal{S}(t,\tau)G(\tau,y(h_3(\tau)))d\tau\|\\
&\quad\le N_{\alpha}h^{1-\alpha}(|\log(h)|+1)\|m_r\|_{L^1}.
\end{aligned}
\tag{43}
$$

Since $\mathcal{S}(t,s)$ is a compact. Therefore, using (42) and (43) in (40), we deduce that $\|(\mathbb{Q}_n^2 y)(t+h)-(\mathbb{Q}_n^2 y)(t)\|_\alpha$ tends to zero as $h \to 0$. Hence, $\mathbb{Q}_n^2$ maps $\mathcal{W}_r(PC)$ into a family of equicontinuous functions.

Step 6. Next we show that $\mathbb{Q}_n^2$ maps $\mathcal{W}_r(PC)$ into a precompact set in $E_\alpha(t_0)$ i.e. the set $\mathcal{U}(t)=\{(\mathbb{Q}_n^2 y)(t) : y \in \mathcal{W}_r(PC)\}$ is precompact in $E_\alpha(t_0)$ for each fixed $t \in [0,T]$. For $t=0$, we have that $\mathcal{U}(0)=\{\mathbb{Q}_n^2 y)(0)\}=\{y_0+\mathcal{S}(\theta_n,0)g(y),\ y \in \mathcal{W}_r(PC)\}$. Since $S(t,s)$ is compact, therefore, we deduce that $\mathcal{U}(0)=\{\mathbb{Q}_n^2 y)(0)\}$ is precompact in $E_\alpha(t_0)$. Now, it remains to show that $\mathcal{U}(t)=\{(\mathbb{Q}_n^2 y)(t) : y \in \mathcal{W}_r(PC)\}$ is relatively compact in $E_\alpha(t_0)$ for each $t \in (0,T]$. First, we show that the set $\mathcal{U}_G(t)=\{(\mathbb{Q}_{Gn}^2 y)(t) : y \in \mathcal{W}_r(PC)\}$ is precompact in $E_\alpha(t_0)$, where $\mathbb{Q}_{Gn}^2$ is defined by

$$(\mathbb{Q}_{Gn}^2 y)(t)=\mathcal{S}(t,0)[y_0+\mathcal{S}(\theta_n,0)g(y)]+\int_0^t \mathcal{S}(t,\tau)G(\tau,y(h_3(\tau)))d\tau. \tag{44}$$

Let $\epsilon>0$ be a real number satisfying $\epsilon<t$ for each fixed $0<t\leq T$. For $y \in \mathcal{W}_r(PC)$, we consider

$$\begin{aligned}(\mathbb{Q}_{Gn}^{2,\epsilon} y)(t)&=\mathcal{S}(t,0)[y_0+\mathcal{S}(\theta_n,0)g(y)]\\ &\quad+\mathcal{S}(t,t-\epsilon)\int_0^{t-\epsilon}\mathcal{S}(t-\epsilon,\tau)G(\tau,y(h_3(\tau)))d\tau.\end{aligned} \tag{45}$$

By the compactness of $\mathcal{S}(t,s),\ t-s>0$, we obtain $\mathcal{U}(t)_{G,\epsilon}=\{(\mathbb{Q}_{Gn}^{2,\epsilon} y)(t) :\ y \in \mathcal{W}_r(PC)\}$ is relatively compact in $E_\alpha(t_0)$ for every ϵ, $\epsilon \in (0,t)$. Furthermore, for each $y \in \mathcal{W}_r(PC)$,

$$\begin{aligned}&\|(\mathbb{Q}_{Gn}^2 y)(t)-(\mathbb{Q}_{Gn}^{2,\epsilon} y)(t)\|_\alpha\\ &\quad\leq \|\int_{t-\epsilon}^t \mathcal{S}(t,\tau)G(\tau,y(h_3(\tau)))d\tau\|_\alpha,\\ &\quad\leq \int_{t-\epsilon}^t \|B^\alpha(t_0)B^{-\beta}(t)\|\cdot\|B^\beta(t)\mathcal{S}(t,\tau)\|\times\|G(\tau,y(h_3(\tau)))\|d\tau,\\ &\quad\leq N_{\alpha,\beta}N_\beta\int_{t-\epsilon}^t (t-\tau)^{-\beta}m_r(\tau)d\tau.\end{aligned}$$

Therefore,

$$\|(\mathbb{Q}_{Gn}^2 y)(t)-(\mathbb{Q}_{Gn}^{2,\epsilon} y)(t)\|_\alpha \to 0, \quad \text{as} \quad \epsilon \to 0^+. \tag{46}$$

It implies that there are relatively compact sets which are arbitrarily near to the set $\mathcal{U}_G(t)=\{(\mathbb{Q}_{Gn}^2 y)(t) : y \in \mathcal{W}_r(PC)\}$. Hence, $\mathcal{U}_G(t)=\{(\mathbb{Q}_{Gn}^2 y)(t) : y \in \mathcal{W}_r(PC)\}$ is relatively compact in $E_\alpha(t_0)$. Furthermore, let $J_0=\overline{J_0}=[0,t_1]$, $J_1=(t_1,t_2]$, $\overline{J_1}=[t_1,t_2],\ldots,J_m=\overline{J_m}=[t_m,T]$. Now, we consider the mapping $\mathbb{Q}_{In}^2 : \mathcal{W}_r \to PC([0,T],E_\alpha(t_0))$ defined by

$$\mathbb{Q}^2_{In}y(t) = \begin{cases} 0, & t \in J_0, \\ \mathcal{S}(t,t_1)\mathcal{S}(\theta_n,0)I_1(y(t_1)), & t \in J_1 \\ \dots \\ \sum_{i=1}^{\delta}\mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y(t_i^-)), & t \in J_\delta. \end{cases} \tag{47}$$

By the Lemma 2.1, it sufficient to show that the set

$$\{\mathbb{Q}^2_{In}y : y \in \mathcal{W}_r(PC)\}_{\overline{J_1}} = \{\mathcal{S}(\cdot,t_1)\mathcal{S}(\theta_n,0)I_1(y(t_1^-)) : \cdot \in \overline{J_1},\ y \in \mathcal{W}_r\}$$

is precompact in $C([t_1,t_2],E_\alpha(t_0))$, as the rest cases for $t \in \overline{J_i},\ i = 1,\dots,m$ are the same. Let $v = (\mathbb{Q}^2_{In}y)_{\overline{J_1}}$, $y \in \mathcal{W}_r(PC)$ which means that

$$v(t_1) = (\mathbb{Q}^2_{In}y)(t_1^+) = \mathcal{S}(\theta_n,0)I_1(y(t_1)), \tag{48}$$

$$v(t) = (\mathbb{Q}^2_{In}y)(t) = \mathcal{S}(t,t_1)\mathcal{S}(\theta_n,0)I_1(y(t_1)), \quad t \in J_1. \tag{49}$$

Thus, by the compactness of $\mathcal{S}(\theta_n,0)$ we deduce that $\{\mathbb{Q}^2_{In}y : y \in \mathcal{W}_r(PC)\}_{\overline{J_1}}(t)$ is relatively compact in $PC([0,T],E_\alpha(t_0))$ for every $t \in \overline{J}_1$. Hence, we deduce that $\mathcal{U}(t) = \{(\mathbb{Q}^2_n y)(t) = (\mathbb{Q}^2_{Gn} + \mathbb{Q}^2_{In})y : y \in \mathcal{W}_r(PC)\}$ is also relative compact in $E_\alpha(t_0)$. Thus, $\mathbb{Q}^2_n$ is a completely continuous operator by using ArzelaAscoli theorem. These arguments enable us to conclude that $\mathbb{Q}_n = \mathbb{Q}^1_n + \mathbb{Q}^2_n$ is a condensing map on $\mathcal{W}_r(PC)$ and hence, there exists at least one fixed point y_n for $\mathbb{Q}_n$ in $\mathcal{W}_r(PC)$ which is just a mild solution for the problem (26)–(28).

Now, we prove that there is a subsequence y_n converging to a mild solution of (1)–(3). Let $\sum$ be the set of all the fixed points $y_n(\cdot)$ of operator $\mathbb{Q}_n$ on $\mathcal{W}_r(PC)$ i.e.,

$$\sum = \{y_n(\cdot) \in PC([0,T],E_\alpha(t_0)) :\ y_n = \mathbb{Q}_n y_n,\ \text{for each } n \geq 1\}. \tag{50}$$

Next, we are going to show that $\sum$ is relatively compact in $PC([0,T],E)$. To prove the result, we consider the following decomposition of y_n into y_n^1, y_n^2 and y_n^3 i.e. $y_n = y_n^1 + y_n^2 + y_n^3$ such that

$$\begin{aligned} y_n^1(t) &= \mathcal{S}(t,0)[-F(0,y(h_1(0)),0)] + F(t,y_n(h_1(t)),\int_0^t a_1(t,s,y_n(h_2(s)))ds) \\ &\quad - \int_0^t \mathcal{S}(t,\tau)B(\tau)F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau, \quad t \in [0,T], \end{aligned} \tag{51}$$

$$y_n^2(t) = \mathcal{S}(t,0)[y_0 + \mathcal{S}(\theta_n,0)g(y_n)] + \int_0^t \mathcal{S}(t,\tau)G(\tau,y_n(h_3(\tau)))d\tau, \quad t \in [0,T], \tag{52}$$

$$y_n^3(t) = \sum_{0<t_i<t}\mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y_n(t_i^-)), \quad t \in [0,T]. \tag{53}$$

Now, we show that $\{y_n^1(t) : n \in \mathbb{N}\}$, $\{y_n^2(t) : n \in \mathbb{N}\}$ and $\{y_n^3(t) : n \in \mathbb{N}\}$ are relatively compact in $PC([0,T],E_\alpha(t_0))$.

Claim 1 $\{y_n^1(t) : n \in \mathbb{N}\}$ is equicontinuous on $[0, T]$.

Let $h > 0$ be a arbitrary constant such that $0 < t < t + h \leq T$. Then, for each $y_n \in \mathcal{W}_r(PC)$, we have

$$\begin{aligned}
&\|y_n^1(t+h) - y_n^1(t)\|_\alpha \\
&\leq \|B^\alpha(t_0)[\mathcal{S}(t+h,0) - \mathcal{S}(t,0)]F(0, y(h_1(0)), 0)\| \\
&\quad + \|B^\alpha(t_0)B^{-1}(t)B(t)[F(t+h, y_n(h_1(t+h)), \int_0^{t+h} a_1(t+h, s, y_n(h_2(s)))ds) \\
&\quad - F(t, y_n(h_1(t)), \int_0^t a_1(t, s, y_n(h_2(s)))ds)]\| \\
&\quad + \|B^\alpha(t_0)[\int_0^{t+h} \mathcal{S}(t+h,\tau)F(\tau, y_n(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y_n(h_2(\xi)))d\xi)\|d\tau \\
&\quad - \int_0^t \mathcal{S}(t,\tau)F(\tau, y_n(h_1(\tau)), \quad \int_0^\tau a_1(\tau, \xi, y_n(h_2(\xi)))d\xi)\|d\tau]\|
\end{aligned}$$

By the assumptions $(H2)$, $(H4)$ (2), we get

$$\begin{aligned}
&(1 - N_{\alpha,1}L_F)\|y_n^1(t+h) - y_n^1(t)\|_\alpha \\
&\leq \|B^\alpha(t_0)[\mathcal{S}(t+h,0) - \mathcal{S}(t,0)]B^{-1}(t)\|[\|B(t)F(0, y(h_1(0)), 0)\|] \\
&\quad + N_{\alpha,1}L_F(h + \mathcal{L}_{a_1}h) \\
&\quad + N_\alpha h^{1-\alpha}(|\log(h)| + 1) \max_{\tau \in [0,T]} \|F(\tau, y_n(h_1(\tau)), \quad \int_0^\tau a_1(\tau, \xi, y_n(h_2(\xi)))d\xi)\|
\end{aligned}$$

Since $N_{\alpha,1}L_F < 1$, thus the righthand side of the above inequality tends to zero as $h \to 0$ which proves the equicontinuity of the set $\{y_n^1(t) : n \in \mathbb{N}\}$ on $[0, T]$.

Claim 2 We show that $\{y_n^1(t) : n \in \mathbb{N}\}$ is relatively compact in $E_\alpha(t_0)$.

For $t \in [0, T]$, $y \in \mathcal{W}_r(PC)$ and $0 < \alpha < \beta \leq 1$, we have

$$\begin{aligned}
\|y_n^1(t)\|_\alpha &\leq N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}[L_F(T+r) + \mathcal{C}_4] + N_{\alpha,1}[L_F(r + L_{a_1}r + \mathcal{C}_3) + \mathcal{C}_4] \\
&\quad + N_{\alpha,\beta}N_\beta[L_F(r + L_{a_1}r + \mathcal{C}_3) + \mathcal{C}_4]\int_0^t (t-\tau)^{-\beta}d\tau, \\
&\leq N_{\alpha,\beta}N_\beta^{'}N_{\beta,1}[L_F(T+r) + \mathcal{C}_4] + N_{\alpha,1}[L_F(r + L_{a_1}r + \mathcal{C}_3) + \mathcal{C}_4] \\
&\quad + N_{\alpha,\beta}N_\beta[L_F(r + L_{a_1}r + \mathcal{C}_3) + \mathcal{C}_4]\frac{T^{1-\beta}}{(1-\beta)},
\end{aligned}$$

where $\mathcal{C}_4 = \sup_{t\in[0,T]} \|B(t)F(t, 0, 0)\|$ and $\mathcal{C}_3 = \sup_{t\in[0,T]} \int_0^t \|a_1(t, s, 0)\|ds$. This shows that $\{y_n^1(t) : n \in \mathbb{N}\}$ is uniformly bounded on $E_\alpha(t_0)$. By Arzela-Ascoli theorem, we deduce that $\{y_n^1(t) : n \in \mathbb{N}\}$ is relatively compact in $E_\alpha(t_0)$. Hence, $\{y_n^1(t) : n \in \mathbb{N}\}$ is compact.

Claim 3 $\{y_n^2(t) : n \in \mathbb{N}\}$ is relatively compact in $PC([0, T], E_\alpha(t_0))$.

(i) $\{y_n^2(t) : n \in \mathbb{N}\}$ is equicontinuous on $[0, T]$.

Let us consider $0 < t < T$ and $h > 0$ be very small. Then, we have

$$\begin{aligned}
&\|y_n^2(t+h) - y_n^2(t)\|_\alpha \\
&\quad \leq \|[\mathcal{S}(t+h,0) - \mathcal{S}(t+h,0)][y_0 + \mathcal{S}(\theta_n,0)g(y_n)]\|_\alpha \\
&\qquad + \|\int_0^{t+h} \mathcal{S}(t+h,\tau)G(\tau, y_n(h_3(\tau)))d\tau - \int_0^t \mathcal{S}(t,\tau)G(\tau, y_n(h_3(\tau)))d\tau\|_\alpha,
\end{aligned}$$

Using the similar arguments as showing the equicontinuity for the family $\{\mathbb{Q}_n^2 y : y \in \mathcal{W}_r(PC)\}$ in pervious steps, it is easy to prove the equicontinuity of $\{y_n^2(t) : n \in \mathbb{N}\}$ on $[0, T]$.

(ii) $\{y_n^2(t) : n \in \mathbb{N}\}$ is relatively compact in $E_\alpha(t_0)$. Let us consider $t \in (0, T]$ and $\epsilon > 0$. For $y_n \in \mathcal{W}_r(PC)$, there exists $\varsigma \in (0, t)$ such that

$$\begin{aligned}
\|y_n^2(t) - y_n^{2,\varsigma}(t)\|_\alpha &\leq \|\int_{t-\varsigma}^t \mathcal{S}(t,\tau)G(\tau, y_n(h_3(\tau)))d\tau\|_\alpha, \\
&\leq N_{\alpha,\beta}N_\beta \int_{t-\varsigma}^t (t-\tau)^{-\beta} m_r(\tau)d\tau, \qquad (54)
\end{aligned}$$

where

$$y_n^{2,\varsigma}(t) = \mathcal{S}(t,0)[y_0 + \mathcal{S}(\theta_n,0)g(y)] + \mathcal{S}(t,t-\varsigma)\int_0^{t-\varsigma} \mathcal{S}(t-\varsigma,\tau)G(\tau, y_n(h_3(\tau)))d\tau, \qquad (55)$$

which is compact mapping. By using similar argument as proving relatively compactness of the operator $\mathbb{Q}_n^2$. Thus, we can deduce that $\{y_n^2(t) : n \in \mathbb{N}\}$ is relatively compact in $PC((0, T], E_\alpha(t_0))$.

We set

$$\widehat{y}_n^2(t) = \begin{cases} y_n^2(t), & t \in [\theta, T], \\ y_n^2(\theta), & t \in [0, \theta] \end{cases} \qquad (56)$$

where θ comes from the condition $(H3)$. Thus, by condition $(H3)$, we have $g(y_n^2) = g(\widehat{y}_n^2)$ and $\widehat{y}_n^2(t) = y_n^2(t)$ for $t \in [\theta, T]$. Moreover, one has that $\{\widehat{y}_n^2 : n \in \mathbb{N}\}$ is relatively compact in $PC((0, T], E_\alpha(t_0))$ by applying similar argument as proving relatively compactness Step 6. Therefore, without loss of generality, it can be assumed that $\widehat{y}_n^2 \to y \in PC((0, T], E_\alpha(t_0))$ as $n \to \infty$.

Therefore, by the continuity of $\mathcal{S}(t, s)$ and g, we obtain

$$y_n^2(0) = y_0 + \mathcal{S}(\theta_n,0)g(y_n^2) = y_0 + \mathcal{S}(\theta_n,0)g(\widehat{y}_n^2) \to y_0 + g(y) = y(0) \qquad (57)$$

as $n \to \infty$. Thus, $\{y_n^2(0) : n \in \mathbb{N}\}$ is relatively compact.

Claim 4 $\{y_n^3(t) : n \in \mathbb{N}\}$ is relatively compact in $PC([0, T], E_\alpha(t_0))$.

For $t = t_1$ and $y_n \in PC([0, T], E_\alpha(t_0))$, we obtain that the set $\{y_n^3(t) : n \in \mathbb{N}\}$ is a compact subset of $E_\alpha(t_0)$. To prove the compactness of the set $\{y_n^3(t) : n \in \mathbb{N}\}$, note that

$$y_n^3(t) = \begin{cases} 0, & t \in J_0, \\ \mathcal{S}(t,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1)), & t \in J_1, \\ \sum_{i=1}^{\delta} \mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y_n(t_i)), & t \in J_\delta. \end{cases} \tag{58}$$

we only need to prove that $\{y_n^3(t) : y_n \in \sum\}_{n=1}^{\infty}|_{\overline{J_1}} = \{\mathcal{S}(\cdot,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1)) : \cdot \in \overline{J_1},\ y_n \in \sum,\ n \geq 1\}$ is precompact in $C([t_1,t_2],E_\alpha(t_0))$ as the rest cases for $t \in \overline{J_i}$, $i = 2, 3, \delta$ are the same. We recall that $v_n \in \{y_n^3(t) : y_n \in \sum\}_{n=1}^{\infty}|_{\overline{J_1}}$, which implies that

$$\begin{aligned} v_n(t_1) &= y_n^3(t_1^+) = \mathcal{S}(\theta_n,0)I_1(y_n(t_1)) = y_n(t_1^+) - y_n(t_1), \quad \text{for some} \quad y_n \in \sum, \\ v_n(t) &= y_n^3(t) = \mathcal{S}(t,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1)), \quad \text{for some} \quad y_n \in \sum, \quad t \in J_1. \end{aligned} \tag{59}$$

Since $\mathcal{S}(t,s)$ is compact, therefore, we deduce that $\{y_n^3(t_1) : n \in \mathbb{N}, y_n \in \sum\}$ and $\{y_n^3(t) : y_n \in \sum,\ n \in \mathbb{N}\}$ are relatively compact in X for each $t \in J_1$.

Next, for $t_1 \leq s \leq t \leq t_2$, we get

$$\begin{aligned} &\|\mathcal{S}(t,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1)) - \mathcal{S}(s,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1))\|_\alpha \\ &\quad \leq \|\mathcal{S}(t,s)\mathcal{S}(s,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1)) - \mathcal{S}(s,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1))\|_\alpha, \\ &\quad \|[\mathcal{S}(t,s) - I]\mathcal{S}(s,t_1)\mathcal{S}(\theta_n,0)I_1(y_n(t_1))\|_\alpha \end{aligned} \tag{60}$$

which tends to zero as $t \to s$. Thus, the set $\{y_n^3(t) : n \in \mathbb{N}\}$ is relatively compact for each $y_n \in C([t_1,t_2],E_\alpha(t_0))$ by ArzelaAscoli theorem. The same idea can be used to prove that $\{y_n^3(t) : y_n \in PC([0,T],E_\alpha(t_0)),\ n \in \mathbb{N}\}|_{\overline{J_i}}$ is precompact for each $i = 2, 3, \ldots, \delta$. Therefore, $\{y_n^3(t) : y_n \in PC([0,T],E_\alpha(t_0))\}$ is relatively compact in $PC([0,T],E_\alpha(t_0))$.

Thus, we conclude that $\{y_n(t) : n \in \mathbb{N}\}$ is relatively compact in $PC([0,T],E_\alpha(t_0))$. Hence, $\sum$ is relatively compact in $PC([0,T],E_\alpha(t_0))$. Without loss of generality, let us assume that $y_n \to y^*$ in $PC([0,T],E_\alpha(t_0))$ and

$$\begin{aligned} y_n(t) &= \mathcal{S}(t,0)[y_0 + g(y_n) - F(0,y_n(h_1(0)),0)] + F(t,y_n(h_1(t)), \\ &\quad \times \int_0^t a_1(t,s,y_n(h_2(s)))ds) \\ &\quad - \int_0^t \mathcal{S}(t,\tau)B(\tau)F(\tau,y_n(h_1(\tau)), \quad \int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau \\ &\quad + \int_0^t \mathcal{S}(t,\tau)G(\tau,y_n(h_3(\tau)), \quad \int_0^\tau b_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau \\ &\quad + \sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y_n(t_i^-)), \quad t \in [0,T]. \end{aligned} \tag{61}$$

It is obvious that $y^* \in PC([0,T],E_\alpha(t_0))$ is obtained by taking limits in above equation defined by

$$y^*(t) = \mathcal{S}(t,0)[y_0 + g(y^*) - F(0,y^*(h_1(0)),0)] + F(t,y^*(h_1(t)),$$

$$
\begin{aligned}
&\times \int_0^t a_1(t,s,y^*(h_2(s)))ds) \\
&-\int_0^t \mathcal{S}(t,\tau)B(\tau)F(\tau,y^*(h_1(\tau)), \quad \int_0^\tau a_1(\tau,\xi,y^*(h_2(\xi)))d\xi)d\tau \\
&+\int_0^t \mathcal{S}(t,\tau)G(\tau,y^*(h_3(\tau)), \quad \int_0^\tau b_1(\tau,\xi,y^*(h_2(\xi)))d\xi)d\tau \\
&+\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{S}(\theta_n,0)I_i(y^*(t_i^-)), \quad t\in[0,T]. \qquad (62)
\end{aligned}
$$

This implies that y^* is a mild solution for the nonlocal impulsive system (1)–(3) on $[0,T]$. This finishes the proof of the theorem. □

4 Example

To illustrate the theory, we present an example in this section.

Let us consider the following nonlocal impulsive differential problem illustrated by

$$
\begin{aligned}
&D_t^\alpha\left[w(t,x)-\int_0^\pi a(t,s,\vartheta)w(\sin t,\vartheta)d\theta\right] = \frac{\partial^2}{\partial x^2}w(t,x)+d(t)w(t,x) \\
&\quad +\mathcal{G}(t,\frac{\partial}{\partial x}w(t,x)), \quad 0\le t\le T\le\pi, \qquad (63)\\
&w(t,0)=w(t,\pi)=0, \quad 0\le t\le T, \qquad (64)\\
&w(0,x)=u_0(x)+\int_0^\pi\int_0^1 \cos(x-y)w^{\frac{1}{3}}(s,y)dsdy, \quad 0\le x\le\pi, \qquad (65)\\
&w(t_i^+,x)-w(t_i^-,x)=I_i(w(t_i,x))=\int_0^{t_i}\mathcal{P}_i(x,y)w(t_k,y)dy, \quad i=1,\ldots,\delta, \qquad (66)
\end{aligned}
$$

where $u_0\in E=L^2([0,\pi])$, $d(t)>0$ is a continuous function and is Hölder continuous in t with parameter $0<\mu<1$. The function $\mathcal{P}_i,\ i=1,\ldots,\delta$ are continuous. Consider the operator $B(t)$ defined by

$$B(t)y=-y''-d(t)y \qquad (67)$$

with domain

$$D(B)=\{y(\cdot)\in E:\, y,y' \ \text{ are absolutely continuous and } \ y''\in E, \quad y(0)=y(\pi)=0\}. \qquad (68)$$

It is easy to verify that $B(t)$ generates an evolution operator $\mathcal{S}(t,s)$ satisfying assumptions $(P1)$–$(P4)$ and

$$\mathcal{S}(t,s)=\mathcal{T}(t-s)\exp\left(\int_s^t d(\tau)d\tau\right), \qquad (69)$$

where $\mathcal{T}(t)$ is the compact analytic semigroup generated by the operator $-B$ with $-Ay = -y''$ for $y \in D(B)$. It is not difficult to compute that $-B$ has a discrete spectrum and the eigenvalues are $n^2, \; n \in \mathbb{N}$ with the corresponding normalized eigenvectors $w_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Therefore, for each $y \in D(B)$, we have

$$-B(t)y = \sum_{n=1}^{\infty} (-n^2 + d(t)) < y, w_n > w_n, \tag{70}$$

and clearly the common domain coincides with that of the operator B. Moreover, we define $B^{\alpha}(t_0)$ for $t_0 \in [0, T]$ for self-adjoint operator $B(t_0)$ by the classical spectral theorem and

$$B^{\alpha}(t_0)y = \sum_{n=1}^{\infty} (n^2 - d(t_0))^{\alpha} < y, w_n > w_n, \tag{71}$$

on the domain $D(B^{\alpha}) = \{y(\cdot) \in E, \sum_{n=1}^{\infty} (n^2 - d(t_0))^{\alpha} < y, w_n > w_n \in E\}$. In addition,

$$B^{1/2}(t_0)y = \sum_{n=1}^{\infty} \sqrt{(n^2 - d(t_0))} < y, w_n > w_n. \tag{72}$$

Thus,

$$\mathcal{S}(t,s)y = \sum_{n=1}^{\infty} e^{-n^2(t-s) + \int_s^t d(\tau)d\tau} < y, w_n > w_n, \tag{73}$$

$$B^{\alpha}(t_0)B^{\beta}(t_0)y = \sum_{n=1}^{\infty} (n^2 - d(t_0))^{\alpha - \beta} < y, w_n > w_n, \tag{74}$$

$$B^{\alpha}(t_0)\mathcal{S}(t,s)y = \sum_{n=1}^{\infty} (n^2 - d(t_0))^{\alpha} e^{-n^2(t-s) + \int_s^t d(\tau)d\tau} < y, w_n > w_n. \tag{75}$$

for each $y \in E$. Then, we obtain

$$\|B^{\alpha}(t)B^{-\beta}(s)\| \le (1 + \|d(\cdot)\|)^{\alpha}, \quad \|B^{\beta}(t)\mathcal{S}(t,s)B^{-\beta}(s)\| \le (1 + \|d(\cdot)\|)^{\beta}, \tag{76}$$

for $t, s \in [0, T]$ and $0 < \alpha < \beta$.

Now, we assume that following conditions are fulfilled. The functions $\mathcal{P}_i : [0, \pi] \times [0, \pi] \to \mathbb{R}, \; i = 1, \ldots, \delta$ are continuously differentiable and

$$\gamma_k = \int_0^{\pi} \int_0^{\pi} (\frac{\partial}{\partial x} \mathcal{P}_{\rangle}(x, y))^2 dx dy < \infty. \tag{77}$$

Let us consider the functions $F : [0, T] \times E \to E, G : [0, T] \times E \to E$

$$F(t, w)(\cdot) = \int_0^{\pi} a(t, \cdot, \vartheta) w(\vartheta) d\vartheta, \tag{78}$$

$$G(t, w)(\cdot) = \mathcal{G}(t, w'(\cdot)), \tag{79}$$

$$g(w)(\cdot) = \int_0^\pi \int_\varsigma^1 \cos(x-y)w(s,y)dsdy \quad w \in \mathcal{PC}([0,T], E_{1/2}), \tag{80}$$

$$I_i(z)(x) = \int_0^\pi \mathcal{P}_i(x,y)z(t_k,y)dy, \quad i = 1, \ldots, \delta. \tag{81}$$

Take $h_1(t) = h_3(t) = \sin(t)$. Thus, the system (63)–(66) can be written in the form of system (1)–(3). Furthermore, $B(t)F : [0,T] \times E_{1/2}(t_0) \to E_{1/2}(t_0)$ (we choose $\beta = 1/2$) $G : [0,T] \times E_{1/2}(t_0) \to E$. Hence, there exists a mild solution for (63)–(66) under appropriate functions G, F, g and I_i, $i = 1, \ldots, \delta$ satisfying suitable conditions to verify the assumptions on Theorem 3.1.

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