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Inventiones mathematicae

Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation

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1. Introduction

Let M be a C^∞ second countable manifold without boundary. We denote by TM the tangent bundle and by $\pi : TM \to M$ the canonical projection. A point in TM will be denoted by (x, v) with $x \in M$ and $v \in T_xM = \pi^{-1}(x)$. In the same way a point of the cotangent space T^*M will be denoted by (x, p) with $x \in M$ and $p \in T^*_xM$, a linear form on the vector space T_xM. We will suppose that g is a *complete* Riemannian metric on M. For $v \in T_xM$, the norm $\|v\|$ is $g(v, v)^{1/2}$. We will denote by $\|\cdot\|$ the dual norm on T^*_xM. We will also use the notations $\|v\|_x$, for $v \in T_xM$, and $\|p\|_x$, for $v \in T^*_xM$.

We will assume in the whole paper that $H : T^*M \to \mathbf{R}$ is a function of class at least C^2, which satisfies the following three conditions

(1) (Uniform superlinearity) for every $K \geq 0$, there exists $C^*(K) \in \mathbf{R}$ such that

$$\forall (x, p) \in T^*M,\ H(x, p) \geq K\|p\| - C^*(K)\ ;$$

(2) (Uniform boundedness in the fibers) for every $R \geq 0$, we have

$$A^*(R) = \sup\{H(x, p) \mid \|p\| \leq R\} < +\infty\ ;$$

(3) (Strict convexity in the fibers) for every $(x, p) \in T^*M$, the second derivative along the fibers $\partial^2 H/\partial p^2(x, p)$ is positive definite.

A locally Lipschitz function $u : U \to \mathbf{R}$, where U is an open subset of M, is said to be a subsolution of $H(x, d_xu) = c$, where $c \in \mathbf{R}$, if we have $H(x, d_xu) \leq c$ for almost every $x \in U$. Recall that, by Rademacher's theorem, a locally Lipschitz function is differentiable almost everywhere.

We say that u is a global subsolution of $H(x, d_x u) = c$ if u is defined on M itself and is a subsolution of $H(x, d_x u) = c$ on the whole of M.

Theorem 1.1. *Under assumptions (1) to (3) above, if there is a global subsolution $u : M \to \mathbf{R}$ of $H(x, d_x u) = c$, then there is a global C^1 subsolution $v : M \to \mathbf{R}$.*

In fact, it is possible to show that there exists $c[0] \in \mathbf{R}$, such that $H(x, d_x u) = c$ admits no subsolution for $c < c[0]$ and has subsolutions for $c \geq c[0]$. The constant $c[0]$ will be called the critical value, or the Mañé critical value. We will say that $u : M \to \mathbf{R}$ is a critical subsolution if it is a subsolution of $H(x, d_x u) = c[0]$.

It can even be shown that the equation $H(x, d_x u) = c[0]$ admits a viscosity solution (see below for definition), see for example [12] or [7]. Moreover it is also well-known that for $c > c[0]$, there exists C^∞ global subsolutions $u : M \to \mathbf{R}$ of $H(x, d_x u) = c$, see [6], for the compact case, or the appendix of [12] for the most general case.

So in fact the new result is the following.

Theorem 1.2. *There exists a C^1 subsolution $u : M \to \mathbf{R}$ of $H(x, d_x u) = c[0]$. In other words, there exists a C^1 global critical subsolution.*

For an introduction to viscosity solutions of the Hamilton-Jacobi equation, see [2] or [3]. We recall the definitions of viscosity subsolution, supersolution and solution. We will say that $u : U \to \mathbf{R}$, defined and continuous on the open subset $U \subset M$, is a viscosity subsolution (resp. supersolution) of $H(x, d_x u) = c$, if for each C^1 function $\phi : U \to \mathbf{R}$ (resp. $\psi : U \to \mathbf{R}$) satisfying $\phi \geq u$ (resp. $\psi \leq u$), and each point $x_0 \in U$ satisfying $\phi(x_0) = u(x_0)$ (resp. $\psi(x_0) = u(x_0)$), we have $H(x_0, d_{x_0}\phi) \leq c$ (resp. $H(x_0, d_{x_0}\psi) \geq c$). A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution. In fact, since $H(x, p)$ is convex and superlinear in p, it is well-known, see [2] or [3], that a function $u : U \to \mathbf{R}$ is a viscosity subsolution of $H(x, d_x u) = c$ if and only if it is locally Lipschitz, and a subsolution in the sense given above (i.e. $H(x, d_x u) \leq c$ almost everywhere).

Even if our statement of Theorems 1.1 and 1.2 do not use viscosity solutions to give their proofs we will need to understand the viscosity solutions of the equation $H(x, d_x u) = c[0]$.

As for subsolutions, we will say that a function u is a critical viscosity solution, if it is a viscosity solution of $H(x, d_x u) = c[0]$. A global critical solution is a function $u : M \to \mathbf{R}$ defined on the whole M and which is a viscosity solution of $H(x, d_x u) = c[0]$ on M itself.

Since we can prove Theorems 1.1 and 1.2 connected component by connected component, there is no loss of generality in assuming M connected. Therefore, *in the remainder of the paper the manifold M is assumed to be connected.*

We can give better versions of Theorem 1.2. We first need to introduce some more tools.

We recall that the Lagrangian $L : TM \to \mathbf{R}$ is defined by

$$\forall (x, v) \in TM, L(x, v) = \max_{p \in T_x^* M} \langle p, v \rangle - H(x, p).$$

Since H is of class C^2 finite everywhere, superlinear, and strictly convex in each fiber $T_x^* M$, it is well known that L is finite everywhere of class C^2, strictly convex and superlinear in each fiber $T_x M$, and satisfies

$$\forall (x, p) \in T^* M, H(x, p) = \max_{v \in T_x M} \langle p, v \rangle - L(x, v).$$

The Legendre transform $\mathcal{L} : TM \to T^* M$ defined by

$$\mathcal{L}(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v) \right)$$

is a diffeomorphism of class C^1. Moreover, we have the equality $\langle p, v \rangle = H(x, p) + L(x, v)$ if and only if $(x, p) = \mathcal{L}(x, v)$.

As in [12], if $c \in \mathbf{R}$, we say that a function $u : M \to \mathbf{R}$ is dominated by $L + c$, and we denote this by $u \prec L + c$, if for every piecewise C^1 curve $\gamma : [a, b] \to M$, with $a \leq b$ we have

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + c(b - a).$$

A function $u : M \to \mathbf{R}$ is dominated by $L + c$ if and only if it is a (viscosity) subsolution of $H(x, d_x u) = c$. See [12], for this and other properties of dominated functions.

The quantity $\mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot\gamma(s))\, ds$ is classically called the action of the curve $\gamma : [a, b] \to M$.

As done by Mather [17], it is convenient to introduce, for $t > 0$, and $x, y \in M$ the following quantity

$$h_t(x, y) = \inf \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

where the infimum is taken over all piecewise C^1 paths $\gamma : [0, t] \to M$, with $\gamma(0) = x, \gamma(t) = y$. It is obvious that $u \prec L + c$ if and only if for each $x, y \in M$, and each $t > 0$, we have $u(y) - u(x) \leq h_t(x, y) + ct$.

The Mañé critical potential $\phi : M \times M \to \mathbf{R}$ and the Peierls barrier $h : M \times M \to]-\infty, +\infty]$ are defined by

$$\phi(x, y) = \inf_{t>0} h_t(x, y) + c[0]t,$$
$$h(x, y) = \liminf_{t \to \infty} h_t(x, y) + c[0]t.$$

We have $h(x, y) \geq \phi(x, y)$ and $\phi(x, x) = 0$. It is clear that the functions ϕ and h satisfy

$$\forall x, y, z \in M, \forall t > 0 \quad \begin{aligned} \phi(x, z) &\leq \phi(x, y) + h_t(y, z) + c[0]t \\ h(x, z) &\leq h(x, y) + h_t(y, z) + c[0]t. \end{aligned}$$

In particular, they even satisfy the triangle inequalities

$$\forall x, y, z \in M, \quad \begin{aligned} \phi(x, z) &\le \phi(x, y) + \phi(y, z), \\ h(x, z) &\le h(x, y) + \phi(y, z), \\ h(x, z) &\le h(x, y) + h(y, z). \end{aligned}$$

It follows that the function h is either identically $+\infty$ or is finite everywhere. If M is compact h is finite everywhere. As is well-known (see Sect. 4 below for proofs) for a given $x \in M$, the function $\phi_x(\cdot) = \phi(x, y)$ is a critical subsolution, and a critical viscosity solution on $M \setminus \{x\}$. On the other hand, if h is finite, then for each $x \in M$, the function $h_x(\cdot) = h(x, \cdot)$ is a global critical viscosity solution.

We recall that the projected Aubry set $\mathcal{A}$ is defined by

$$\mathcal{A} = \{x \in M \mid h(x, x) = 0\}.$$

If $u : M \to \mathbf{R}$ is a critical subsolution then u is differentiable at each $x \in \mathcal{A}$, and $H(x, d_x u) = c[0]$, for such an x, see [9].

The following theorem improves 1.2

Theorem 1.3. *There exists a C^1 function $u : M \to \mathbf{R}$ such that $H(x, d_x u) = c[0]$, for $x \in \mathcal{A}$, and $H(x, d_x u) < c[0]$, for $x \notin \mathcal{A}$.*

We say that a critical subsolution $u : M \to \mathbf{R}$ is strict at some point x if there is an open neighborhood U of x, and a constant $c < c[0]$ such that the restriction $u|_U$ is a subsolution of $H(x, d_x u) = c$ on the open set U.

As a step in the proof of Theorem 1.3, we also obtain the following characterization of the Aubry set $\mathcal{A}$.

Theorem 1.4. *For a point $x \in M$, the following conditions are equivalent*

- *(i) the point x is in $\mathcal{A}$;*
- *(ii) there does not exist a critical subsolution $u : M \to \mathbf{R}$ which is strict at x;*
- *(iii) every critical subsolution is differentiable at x;*
- *(iv) the function $\phi_x : M \to \mathbf{R}$ is a viscosity solution on the whole of M;*
- *(v) the function $\phi_x : M \to \mathbf{R}$ is differentiable at x.*

If $u : M \to \mathbf{R}$ is a critical subsolution, we define $\mathcal{I}(u)$ as the set of points $x \in M$, for which there exists a piecewise C^1 curve $\gamma :]-\infty, +\infty[\to M$ such that $\gamma(0) = x$, and $u(\gamma(t)) - u(\gamma(t')) = \int_{t'}^{t} L(\gamma(s), \dot\gamma(s))\, ds + c[0](t - t')$, for every $t, t' \in \mathbf{R}$ with $t' \le t$. In this case, the curve γ is a minimizer of the action and $\gamma(t) \in \mathcal{I}(u)$, for all $t \in \mathbf{R}$. Moreover, we always have $\mathcal{I}(u) \supset \mathcal{A}$, and the function u is differentiable at each point of $\mathcal{I}(u)$, the derivative $x \to d_x u$ is locally Lipschitz on $\mathcal{I}(u)$, see [9].

If M is compact, we can improve Theorem 1.3

Theorem 1.5. *Suppose that M is compact, and that $u : M \to \mathbf{R}$ is a critical subsolution. For each $\epsilon > 0$, there exists a C^1 function $\tilde u : M \to \mathbf{R}$ such that $\tilde u(x) = u(x)$, $H(x, d_x u) = c[0]$, for $x \in \mathcal{A}$, and $|\tilde u(x) - u(x)| < \epsilon$, $H(x, d_x u) < c[0]$, for $x \notin \mathcal{A}$.*

We now state two consequences of the main theorem. The first one is a slight generalization of a theorem of Mañé, see [15]. In the Riemannian case, this generalization is due to Bangert, see [1].

Theorem 1.6 (Mañé). *Suppose M compact. Call $\mathcal{HM}$ the set of probability measures μ on TM such that $\int_{TM} \|v\|\, d\mu$ is finite and $\int_{TM} d_x\varphi(v)\, d\mu(x, v) = 0$, for each $\varphi : M \to \mathbf{R}$ of class C^∞. Then $-c[0] = \inf\{\int_{TM} L\, d\mu \mid \mu \in \mathcal{HM}\}$. Moreover, each measure $\mu \in \mathcal{HM}$ such that $\int_{TM} L\, d\mu = -c[0]$ is necessarily invariant under the Euler-Lagrange flow of L, and is therefore a minimizing measure.*

The second application to the stable norm of a Riemannian metric is new. See [13] for the definition and properties of the stable norm.

Theorem 1.7. *Suppose M is a compact Riemannian manifold. Each closed 1-form ω on M is cohomologous to a C^0 closed 1-form $\tilde{\omega}$ such that $\|\tilde{\omega}_x\|_x \leq \|[\omega]\|_s$, and such that $\{x \in M \mid \|\tilde{\omega}_x\|_x = \|[\omega]\|_s\}$ is the support of a geodesic lamination.*

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2. Extremals and Euler-Lagrange flow

We need some facts from the classical calculus of variations, see [4] or [9]. If $\gamma : [a, b] \to M$ is a continuous piecewise C^1 curve, its action $\mathbf{L}(\gamma)$ is

$$\mathbf{L}(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

Such a curve γ is called an extremal (for L) (resp. a minimizer) if it is a critical point (resp. a minimum) of the action on the set of curves $\delta : [a, b] \to M$ with $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$.

Extremals satisfy (in local coordinates) the Euler-Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s))\right] = \frac{\partial L}{\partial x}(\gamma(s), \dot{\gamma}(s)).$$

Since $\frac{\partial^2 L}{\partial v^2}$ is everywhere non-degenerate, this equation defines a second order differential equation on M, hence a flow $\varphi_t^L : TM \to TM$. A curve $\gamma : [a, b] \to M$ is an extremal for L, if and only if its *speed curve* $s \mapsto (\gamma(s), \dot{\gamma}(s))$ is an orbit of φ_t^L, i.e. if and only if $\varphi_{t-s}^L(\gamma(s), \dot{\gamma}(s)) = (\gamma(t), \dot{\gamma}(t))$, for $s, t \in [a, b]$.

As is well known, the Euler-Lagrange flow φ_t^L preserves the energy $E : TM \to \mathbf{R}$ defined by

$$E(x, v) = H\left(x, \frac{\partial L}{\partial v}(x, v)\right) = \frac{\partial L}{\partial v}(x, v)(v) - L(x, v).$$

We have to study the behavior of $E(x, v)$ for $\|v\| \to +\infty$.

Replacing the constant $C^*(K)$ expressing the superlinearity of H by a smaller one if necessary we can assume

$$C^*(K) = \sup_{(x,p)\in T^*M} K\|p\| - H(x, p) < +\infty.$$

A proof of the following is contained in [12].

Lemma 2.1. *a)* $A(R) = \sup\{L(x, v) \mid \|v\| \leq R\} = C^*(R) < +\infty$.
b) $C(K) = \sup_{(x,v)\in TM} K\|v\| - L(x, v) = A^*(R) < +\infty$.
c) The function $\theta_1(R) = \sup\{E(x, v) \mid \|v\| \leq R\}$ *is non decreasing and finite everywhere.*
d) The function $\theta_2(R) = \sup\{\|v\| \mid E(x, v) \leq R\}$ *is non decreasing and finite everywhere.*

As a consequence of the last part of this lemma and the conservation of energy the flow φ_t^L is complete. Moreover, see [4] or [9], we can apply Tonelli's theorem to obtain

Theorem 2.2. *For any pair of points* $x, y \in M$*, and any* $a, b \in \mathbf{R}$*, with* $a < b$ *we can find a curve* $\gamma : [a, b] \to M$*, with* $\gamma(a) = x$, $\gamma(b) = y$*, and* $\mathbf{L}(\gamma) \leq \mathbf{L}(\delta)$ *for every curve* $\delta : [a, b] \to M$*, with* $\delta(a) = x$, $\delta(b) = y$*. Such a curve* γ *is called a minimizer, it is necessarily an extremal.*

We draw further consequences of the Lemma 2.1.

Proposition 2.3. *1) There exists a non-decreasing function* $\theta : \mathbf{R} \to \mathbf{R}$ *such that for each extremal* $\gamma : [a, b] \to M$*, we have*

$$\sup_{s\in[a,b]}\|\dot\gamma(s)\| \leq \theta\big(\inf_{s\in[a,b]}\|\dot\gamma(s)\|\big).$$

2) For every continuous piecewise C^1 *curve* $\gamma : [a, b] \to M$*, and each* $K \geq 0$*, we have*

$$\text{length}(\gamma) = \int_a^b \|\dot\gamma(s)\|\, ds \leq \frac{\mathbf{L}(\gamma) + C(K)(b-a)}{K}.$$

In particular

$$\inf_{s\in[a,b]}\|\dot\gamma(s)\| \leq \frac{\mathbf{L}(\gamma)}{K(b-a)} + \frac{C(K)}{K}.$$

3) If $\gamma : [a, b] \to M$ *is an extremal then*

$$\sup_{s\in[a,b]}\|\dot\gamma(s)\| \leq \theta\left[\frac{\mathbf{L}(\gamma)}{K(b-a)} + \frac{C(K)}{K}\right].$$

Proof. 1) We can take $\theta = \theta_2 \circ \theta_1$, where θ_2 and θ_1 are given by Lemma 2.1 above. In fact $\|\dot\gamma(s)\| \leq \theta_2[E(\gamma(s), \dot\gamma(s))]$. Since γ is an extremal $E_\gamma = E(\gamma(s), \dot\gamma(s))$ is constant. Moreover if s_0 is such that $\|\dot\gamma(s_0)\| = \inf_{s\in[a,b]}\|\dot\gamma(s)\|$, we have $E_\gamma = E(\gamma(s_0), \dot\gamma(s_0)) \leq \theta_1(\|\dot\gamma(s_0)\|)$.

2) For $(x, v) \in TM$, we have $K\|v\| \leq L(x, v) + C(K)$. In particular

$$K\|\dot{\gamma}(s)\| \leq L(\gamma(s), \dot{\gamma}(s)) + C(K).$$

Integrating between a and b gives

$$K \int_a^b \|\dot{\gamma}(s)\| \, ds \leq \mathbf{L}(\gamma) + C(K)(b - a).$$

It remains to observe that $\int_a^b \|\dot{\gamma}(s)\| \, ds \geq (b - a) \inf_{s \in [a,b]} \|\dot{\gamma}(s)\|$.

3) This is an easy consequence of 1) and 2). □

3. Dominated functions and calibrated curves

As recalled above, if $c \in \mathbf{R}$, we say that a function $u : U \to \mathbf{R}$, defined on some open set U of M is dominated by $L + c$, and we denote this by $u \prec L + c$, if for every piecewise C^1 curve $\gamma : [a, b] \to M$, with $a \leq b$, and $\gamma([a, b]) \subset U$, we have

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds + c(b - a).$$

We need to understand some of the properties of dominated functions. Most of this is well known by now, see for example [9] or [12].

Parametrizing a curve by arclength, we obtain

Lemma 3.1. *If $u : U \to \mathbf{R}$ is defined on the open subset U of M, and $u \prec L + c$, then for every continuous piecewise C^1 curve $\gamma : [a, b] \to U$, we have*

$$|u(\gamma(b)) - u(\gamma(a))| \leq (A(1) + c) \int_a^b \|\dot{\gamma}(s)\| \, ds.$$

In particular, if $u : M \to \mathbf{R}$ is a function such that $u \prec L + c$, then u is Lipschitzian with Lipschitz constant $\leq A(1) + c$.

Another important result is the characterization of viscosity subsolutions, see [9].

Lemma 3.2. *A function $u : U \to \mathbf{R}$, defined on some open set U of M, is dominated by $L+c$ if and only if it is a viscosity subsolution of $H(x, d_x u) = c$ on U.*

Suppose u is a real valued function defined on some part U of M, and $c \in \mathbf{R}$. A continuous piecewise C^1 curve $\gamma : [a, b] \to U$, $a < b$ is said to be (u, L, c)-calibrated if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds + c(b - a).$$

This notion is interesting only when $u \prec L + c$.

Proposition 3.3. *Let $u : U \to \mathbf{R}$, with U an open set of M, be such that $u \prec L + c$. Suppose that $\gamma : [a, b] \to U$ is (u, L, c)-calibrated, then we have*

1) for each $[a', b'] \subset [a, b]$, the curve $\gamma|[a', b']$ is (u, L, c)-calibrated;
2) the curve γ minimizes action among all curves $\delta : [a, b] \to U$ with $\delta(a) = \gamma(a)$ and $\delta(b) = \gamma(b)$. Hence γ is an extremal.

The proposition above is by now well-known. For a proof, we refer to [9]. Let us prove another property of calibrated curves.

Proposition 3.4. *Given c there exists a constant $K(c)$, such that for every $u : U \to \mathbf{R}$, with U an open set of M, satisfying $u \prec L + c$, and every (u, L, c)-calibrated $\gamma : [a, b] \to U$, $a < b$, we have*

$$\forall s \in [a, b], \|\dot{\gamma}(s)\| \leq K(c).$$

In fact, we can take $K(c) = C[A(1) + c + 1] - c$.

Proof. Since $\gamma|[t, t']$ is also (u, L, c)-calibrated for each $t \leq t'$, $t, t' \in [a, b]$

$$\forall t \leq t', t, t' \in [a, b], u(\gamma(t')) - u(\gamma(t)) = \int_t^{t'} L(\gamma(s), \dot{\gamma}(s))\, ds + c(t' - t),$$

By 3.1, we have $u(\gamma(t')) - u(\gamma(t)) \leq (A(1) + c) \int_t^{t'} \|\dot{\gamma}(s))\|\, ds$, therefore

$$\forall t \leq t', t, t' \in [a, b],$$
$$\int_t^{t'} L(\gamma(s), \dot{\gamma}(s))\, ds + c(t' - t) \leq (A(1) + c) \int_t^{t'} \|\dot{\gamma}(s))\|\, ds.$$

Dividing by $t' - t > 0$ and letting $t' \to t$, we see that

$$\forall t \in [a, b], L(\gamma(t), \dot{\gamma}(t)) + c \leq (A(1) + c)\|\dot{\gamma}(t))\|.$$

Since $L(x, v) \geq (A(1) + c + 1)\|v\| - C[A(1) + c + 1]$. We conclude that $\|\dot{\gamma}(t)\| \leq C[A(1) + c + 1] - c$. □

We will use the following well-known characterization of viscosity solutions. For a proof, one can refer to [12].

Proposition 3.5. *Suppose that $u : U \to \mathbf{R}$ is such that $u \prec L + c$, and for each $y \in U$, there exists a (u, L, c)-calibrated curve $\gamma_y : [-\epsilon_y, 0] \to U$, with $\epsilon_y > 0$ and $\gamma_y(0) = y$, then u is a viscosity solution of $H(x, d_x u) = c$.*

Conversely if $u : M \to \mathbf{R}$ is a global viscosity solution of $H(x, d_x u) = c$, then for each $y \in M$ we can find a (u, L, c)-calibrated curve $\gamma_y :]-\infty, 0] \to M$, with $\gamma_y(0) = y$.

4. Viscosity properties of Mañé potential and the Peierls barrier

The inequalities

$$\begin{aligned} \forall x, y, z \in M, \forall t > 0 \quad & \phi(x, z) \leq \phi(x, y) + h_t(y, z) + c[0]t \\ & h(x, z) \leq h(x, y) + h_t(y, z) + c[0]t \end{aligned}$$

prove that ϕ_x is always a global critical subsolution, and also, if h is finite, that the function h_x is a global critical subsolution. In particular, they are Lipschitzian with Lipschitz constant $\leq A(1)+c[0]$. Since $\phi_x(x) = \phi(x, x) = 0$, we obtain $\phi(x, y) \leq [A(1) + c[0]]d(x, y)$

We would like to first prove that ϕ_x is a viscosity solution on $M \setminus \{x\}$, and that h_x is a global viscosity solution when h is finite.

Proposition 4.1. *Suppose that the Peierls barrier h is finite. For each $x, y \in M$ we can find a curve $\gamma :]-\infty, 0]$ such that $\gamma(0) = y$, and*

$$\forall t \geq 0, h_x(\gamma(0)) - h_x(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t.$$

In particular, when the Peierls barrier is finite, for each $x \in M$, the function h_x is a global critical viscosity solution.

Proof. By Tonelli's Theorem 2.2, and the definition of the Peierls barrier, we can pick a sequence of minimizing extremals $\gamma_n : [-t_n, 0] \to M$ such that $t_n \to +\infty$, $\gamma_n(-t_n) = x$, $\gamma_n(0) = y$, and

$$\int_{-t_n}^{0} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0]t_n \to h(x, y).$$

Discarding the first terms we can also assume that $t_n \geq 1$ and

$$\int_{-t_n}^{0} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0]t_n \leq h(x, y) + 1.$$

By part 3) of Proposition 2.3, we have the estimate

$$\begin{aligned} \sup_{s\in[-t_n,0]} \|\dot{\gamma}_n(s)\| &\leq \theta \left[\frac{h(x, y) + 1 - c[0]t_n}{t_n} + C(1) \right] \\ &\leq \theta(|h(x, y)| + 1 - c[0] + C(1)). \end{aligned}$$

In particular the sequence $(\gamma_n(0), \dot{\gamma}_n(0)) = (y, \dot{\gamma}_n(0))$ remains in a compact subset of TM. Since the speed curves $s \mapsto (\gamma_n(s)\dot{\gamma}_n(s))$ of the extremals γ_n are orbits of the Euler-Lagrange flow φ_t^L, extracting a subsequence if necessary, we can assume that γ_n converges towards a limit γ uniformly in the C^1 topology on each compact subinterval of $]-\infty, 0]$. Fix $t > 0$, set $\delta_n = d(\gamma_n(-t), \gamma(-t))$, we have $\delta_n \to 0$. We can find a smooth curve $\tilde{\gamma}_n : [-t, -t + \delta_n] \to M$ from $\gamma_n(-t)$ to $\gamma(-t)$ whose action is less than $K\delta_n$, with $K = A(1)$. If we piece together the restriction of γ_n to $[-t_n, -t]$ and $\tilde{\gamma}_n$, we see that

$$h_{t_n - t + \delta_n}(x, \gamma(-t)) + c[0](t_n - t + \delta_n) \leq (K + c[0])\delta_n + \int_{-t_n}^{-t} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0](t_n - t).$$

Since γ_n converges uniformly to γ on $[-t, 0]$ in the C^1 topology, we obtain

$$\begin{aligned} h(x, \gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t &\leq \liminf_{n\to\infty} \Bigg[(K + c[0])\delta_n + \\ &\qquad \int_{-t_n}^{-t} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0](t_n - t) \Bigg] + \\ &\qquad \lim_{n\to\infty} \left[\int_{-t}^{0} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0]t \right] \\ = \liminf_{n\to\infty} \left[(K + c[0])\delta_n + \int_{-t_n}^{0} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0]t_n \right] &= h(x, y). \end{aligned}$$

So we obtained the inequality $h(x, \gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t \leq h(x, y)$, but the reverse equality is also true.

The fact that h_x is a global critical viscosity solution now follows from Proposition 3.5. □

Proposition 4.2. *For each $x \in M$, the function ϕ_x is a global critical subsolution.*

For each $x, y \in M$, with $x \neq y$, we can find $\epsilon > 0$ and a curve $\gamma :]-\epsilon, 0]$ such that $\gamma(0) = y$, and

$$\forall t \in [0, \epsilon],\ \phi_x(\gamma(0)) - \phi_x(\gamma(t)) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t.$$

In particular, for each $x \in M$, the function ϕ_x is a critical viscosity solution on $M \setminus \{x\}$.

Proof. We consider first the case where the infimum $\phi(x, y) = \inf_{t>0} h_t(x, y) + c[0]t$ is attained for $t \to \infty$. In that case $h(x, y) = \phi(x, y)$, hence h is finite, and we can find a curve $\gamma :] - \infty, 0]$ such that $\gamma(0) = y$, and

$$\forall t \geq 0,\ h(x, \gamma(0)) - h(x, \gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t.$$

Since we also have the inequalities

$$
\begin{aligned}
h(x, y) &= h(x, \gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t \\
&\geq \phi(x, \gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t \\
&\geq \phi(x, y) = h(x, y)
\end{aligned}
$$

we obtain

$$\forall t \geq 0, \phi_x(\gamma(0)) - \phi_x(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t.$$

If $\phi(x, y) \neq h(x, y)$, then we can find $t_n \to t_\infty \in [0, +\infty[$ such that $h_{t_n}(x, y) + c[0]t_n \to \phi(x, y)$. Discarding the first few terms we will assume that $h_{t_n}(x, y) + c[0]t_n \leq \phi(x, y) + 1$. By Tonelli's Theorem 2.2, we can find a sequence of extremals $\gamma_n : [-t_n, 0] \to M$ with $\gamma_n(-t_n) = x$, $\gamma_n(0) = y$, and $\mathbf{L}(\gamma_n) = h_{t_n}(x, y)$. By part 2) of Proposition 2.3, we have

$$\forall K > 0, d(x, y) \leq \text{length}(\gamma_n) \leq \frac{\phi(x, y) + 1}{K} + \frac{C(K)t_n}{K}.$$

Since $d(x, y) > 0$, we can choose K such that $(\phi(x, y) + 1)/K \leq d(x, y)/2$. Letting then n go to $+\infty$, we obtain

$$0 < \frac{d(x, y)}{2} \leq \frac{C(K)t_\infty}{K}.$$

Therefore $t_\infty > 0$, and we can therefore assume $2t_\infty > t_n > t_\infty/2$ by dropping the first few terms of the sequence γ_n. By part 3) of Proposition 2.3, we get

$$\|\dot\gamma_n(0)\| \leq \theta \left[\frac{2(\phi(x, y) + 1)}{t_\infty} + C(1) \right].$$

Since the γ_n are extremals, by the completeness of the Euler-Lagrange flow, we can extend them to extremals $\gamma_n : \mathbf{R} \to M$. Since the points $(\gamma_n(0), \dot\gamma_n(0)) = (y, \dot\gamma_n(0))$ remain in a compact set of TM, by continuity of the Euler-Lagrange flow, extracting a subsequence if necessary, we can assume that γ_n converge uniformly on compact intervals in the C^1 topology to the extremal $\gamma : \mathbf{R} \to M$. It is clear that $\gamma(-t_\infty) = x$, $\gamma(0) = y$, and $\phi(x, y) = \int_{-t_\infty}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t_\infty$. Since $\phi(x, x) = 0$, this can be rewritten as

$$\phi_x(\gamma(0)) - \phi_x(\gamma(-t_\infty)) = \int_{-t_\infty}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t_\infty.$$

Because $\phi_x \prec L + c[0]$, this implies

$$\forall t \in [0, t_\infty], \phi_x(\gamma(0)) - \phi_x(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t.$$

The fact that ϕ_x is a viscosity solution on $M \setminus \{x\}$ follows from Proposition 3.5. □

Theorem 4.3. *The function ϕ_x is a global viscosity solution if and only if $x \in \mathcal{A}$.*

Proof. If $x \in \mathcal{A}$ then $\phi_x = h_x$, and ϕ_x is therefore a global viscosity solution. Conversely, by a well-known result, see Proposition 3.5, if ϕ_x is a global viscosity solution, then we can find an extremal $\gamma :]-\infty, 0] \to M$ with $\gamma(0) = x$, and such that

$$\forall t \geq 0, \phi_x(\gamma(0)) - \phi_x(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t.$$

Since $\gamma(0) = x$ and $\phi_x(x) = \phi(x, x) = 0$, this can be rewritten as

$$\forall t \geq 0, \phi_x(\gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]t = 0.$$

Fix now $T > 0, \epsilon > 0$, we can find a curve $\gamma_{T,\epsilon} : [0, t_{T,\epsilon}] \to M$ such that $\gamma_{T,\epsilon}(0) = x$, $\gamma_{T,\epsilon}(t_{T,\epsilon}) = \gamma(-T)$, and such that

$$\int_0^{t_{T,\epsilon}} L(\gamma_{T,\epsilon}(s), \dot\gamma_{T,\epsilon}(s))\, ds + c[0]t_{T,\epsilon} \leq \phi(x, \gamma(-T)) + \epsilon.$$

If we piece together the curve $\gamma_{T,\epsilon}$ and $\gamma|[-T, 0]$, we obtain a curve $\tilde\gamma_{T,\epsilon} : [0, t_{T,\epsilon} + T] \to M$ defined by $\tilde\gamma_{T,\epsilon}(s) = \gamma_{T,\epsilon}(s)$, for $s \in [0, t_{T,\epsilon}]$, and $\tilde\gamma_{T,\epsilon}(s) = \gamma(s - T - t_{T,\epsilon})$, for $s \in [t_{T,\epsilon}, t_{T,\epsilon} + T]$. In particular $\tilde\gamma_{T,\epsilon}(0) = \tilde\gamma_{T,\epsilon}(t_{T,\epsilon} + T) = x$, and

$$\begin{aligned}\int_0^{t_{T,\epsilon}+T} & L(\tilde\gamma_{T,\epsilon}(s), \dot{\tilde\gamma}_{T,\epsilon}(s))\, ds + c[0](t_{T,\epsilon} + T) \\ & \leq \phi(x, \gamma(-T)) + \epsilon + \int_{-T}^{0} L(\gamma(s), \dot\gamma(s))\, ds + c[0]T \leq \epsilon.\end{aligned}$$

Since ϵ and T are arbitrary, we conclude $h(x, x) = 0$, which means $x \in \mathcal{A}$. □

5. Differentiability properties for critical subsolutions

For a proof of the following proposition, see [9].

Proposition 5.1. *Let $u : U \to \mathbf{R}$, with U an open set of M, be such that $u \prec L + c$. Suppose that $\gamma : [a, b] \to U$ is (u, L, c)-calibrated, then we have*

1) if $d_{\gamma(t)}u$ exists then it is equal to $\partial L/\partial v(\gamma(t), d_{\gamma(t)}u)$ and $H(\gamma(t), d_{\gamma(t)}u) = c$;
2) if $t \in]a, b[$ then $d_{\gamma(t)}u$ exists.

In particular, if $u \prec L + c$ and $\mathcal{I}(u)$ is the set of $x \in M$ such that there exists $\gamma : \mathbf{R} \to M$, with $\gamma(0) = x$, and γ is (u, L, c)-calibrated, then u is differentiable at each point of $\mathcal{I}(u)$.

In the situation above one can even show that the derivative of u on $\mathcal{I}(u)$ is a locally Lipschitz map.

Proposition 5.2. *If $x \in \mathcal{A}$, then there exists a curve $\gamma : \mathbf{R} \to M$ such that $\gamma(0) = x$ and*

$$\forall t \geq 0, \qquad h(\gamma(t), x) = -\int_0^t L(\gamma(s), \dot\gamma(s))\, ds - c[0]t;$$
$$h(x, \gamma(-t)) = -\int_{-t}^0 L(\gamma(s), \dot\gamma(s))\, ds - c[0]t.$$

In particular, for each critical subsolution u, the curve γ is $(u, L, c[0])$-calibrated. Therefore $\mathcal{A} \subset \mathcal{I}(u)$, and u is differentiable at each point of $\mathcal{A}$.

Proof. If $x \in \mathcal{A}$, we can find a sequence of C^1 curves $\gamma_n : [0, t_n] \to M$, with $\gamma_n(0) = \gamma_n(t_n) = x$, $t_n \to \infty$, and

$$\int_0^{t_n} L(\gamma_n(s), \dot\gamma_n(s))\, ds + c[0]t_n \to 0. \qquad (*)$$

Without loss of generality we can assume that γ_n is an extremal, and $t_n \geq 1$. Therefore by Sect. 3) of Proposition 2.3

$$\sup\{\|\dot\gamma_n(s)\| \mid s \in [0, t_n], n \geq 0\} < +\infty.$$

As the speed curves of the γ_n are orbits of the Euler-Lagrange flow, extracting a sequence if necessary, we can assume that γ_n converges in the C^1 topology to a limit $\gamma : [0, +\infty[\to M$ uniformly on each compact interval contained in $[0, +\infty[$. We fix $t \in [0, +\infty[$, for n large enough to have $t < t_n$, setting $d_n = d(\gamma(t), \gamma_n(t))$, we construct a curve $\tilde\gamma : [t - d_n, t_n] \to M$ from $\gamma(t)$ to x, by piecing together a geodesic from $\gamma(t)$ to $\gamma_n(t)$, parametrized by arc-length, and $\gamma_n|[t, t_n]$. We have

$$\int_{t-d_n}^{t_n} L(\tilde\gamma_n(s), \dot{\tilde\gamma}_n(s))\, ds \leq A(1)d_n + \int_t^{t_n} L(\gamma_n(s), \dot\gamma_n(s))\, ds,$$

moreover, since $t_n - (t - d_n) \geq t_n - t \to +\infty$, and $d_n \to 0$, we obtain

$$h(\gamma(t), x) \leq \liminf_{n\to+\infty} \int_0^{t_n} L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0](t_n - t).$$

It now follows from $(*)$, and the convergence of $\gamma_n \to \gamma$ in the C^1 topology, that

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t + h(\gamma(t), x) \leq 0.$$

But we also have $h(x, \gamma(t)) = h(x, \gamma(t)) - h(x, x) \leq \int_0^t L(\gamma_n(s), \dot{\gamma}_n(s))\, ds + c[0]t$, and $h(x, \gamma(t)) + h(\gamma(t), x) \geq 0$. Therefore we must have

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t + h(\gamma(t), x) = 0.$$

To construct the curve γ on $]-\infty, 0]$, it suffices, in an analogous way, to extract a converging subsequence of the curves $\bar{\gamma}_n : [-t_n, 0] \to M$, with $\bar{\gamma}_n(t) = \gamma_(t + t_n)$.

Suppose now that $u \prec L + c$, for $t \geq 0$, we get

$$u(\gamma(t)) - u(x) \leq \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + c[0],$$

and

$$u(x) - u(\gamma(t)) \leq h(\gamma(t), x) = -\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds - c[0]t.$$

Hence

$$u(\gamma(t)) - u(x) = \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]t.$$

This shows that γ is $(u, L, c[0])$-calibrated on $[0, +\infty[$. An analogous argument shows that is also calibrated on $]-\infty, 0]$, and therefore on the whole of $\mathbf{R}$. □

Theorem 5.3. *The function ϕ_x is differentiable at x if and only if $x \in \mathcal{A}$.*

Proof. If $x \in \mathcal{A}$, the function ϕ_x is differentiable at x by Proposition 5.2. Conversely, assume that ϕ_x is differentiable at x. By Theorem 4.3, we have to prove that ϕ_x is a viscosity solution on the whole of M. Suppose this is false. For each $y \in M \setminus \{x\}$, by Proposition 4.2, we can pick a curve $\gamma_y :]-t_y, 0] \to M \setminus \{x\}$, with $t_y > 0$, such that $\gamma_y(0) = y$, and γ_y is $(\phi_x, L, c[0])$-calibrated. Since γ_y is an extremal, we can extend it to an extremal defined for all time, we will assume that t_y is the largest t such that both $\gamma_y(]-t, 0]) \subset M \setminus \{x\}$ and $\gamma_y\ |]-t, 0]$ is $(\phi_x, L, c[0])$-calibrated. If $t_y < +\infty$ then necessarily $\gamma_y(-t_y) = x$. In fact, if we had $y_- = \gamma_y(-t_y) \neq x$, we can piece together γ_{y_-}

and γ_y to find a curve $\tilde{\gamma}_y :]-t_y - t_{y_-}, 0] \to M \setminus \{x\}$ which is calibrated and satisfies $\tilde{\gamma}_y|[-t_y, 0] = \gamma_y|[-t_y, 0]$. Therefore $\tilde{\gamma}_y$ being also an extremal has to coincide with γ_y everywhere, this gives a contradiction since $t_y + t_{y_-} > t_y$.

We claim that $\limsup_{y\to x} t_y = 0$. In fact if this is not true, we could find a sequence $y_n \to x$ and $\epsilon > 0$ such that $t_{y_n} > \epsilon$. Then each extremal $\gamma_{y_n} : [-\epsilon, 0] \to M$ is $(\phi_x, L, c[0])$-calibrated for every $n \geq 0$. Since $\gamma_n(0) = y_n \to x$, and $\sup_{n\geq 0} \|\dot{\gamma}_{y_n}(0)\| < +\infty$ by Proposition 3.4, extracting a subsequence if necessary, we can assume that $\gamma_{y_n} : [-\epsilon, 0] \to M$ converges uniformly in the C^1 topology to the extremal $\gamma : [-\epsilon, 0] \to M$. It is clear that $\gamma(0) = x$, and that γ is $(\phi_x, L, c[0])$-calibrated; It follows from Propositions 3.5 and 4.2 that ϕ_x is a a critical viscosity solution. This is contrary to our assumption.

We have thus obtained $\limsup_{y\to x} t_y = 0$. We now use the differentiability property at x. First we know that if $d_x\phi_x$ exists then we must have $H(x, d_x\phi_x) \leq c[0]$, because ϕ_x is a global critical subsolution. Moreover, if t_y is finite, we have $\gamma_y(-t_y)) = x$, and since γ_y is $(\phi_x, L, c[0])$-calibrated on $[-t_y, 0]$, by Proposition 3.3, for $\delta > 0$ small enough, we have

$$\phi_x(\gamma_y(-t_y + \delta)) - \phi_x(\gamma_y(-t_y)) = \int_{-t_y}^{-t_y+\delta} L(\gamma_y(s), \dot{\gamma}_y(s))\, ds + c[0]\delta.$$

If we divide by δ and we let $\delta \to 0$, we obtain

$$d_x\phi_x(\dot{\gamma}_y(-t_y)) = L(x, \dot{\gamma}_y(-t_y)) + c[0].$$

Together with the inequality $H(x, d_x\phi_x) \leq c[0]$, this implies $H(x, d_x\phi_x) = c[0]$, and $d_x\phi_x = \partial L/\partial v(x, \dot{\gamma}_y(-t_y))$. By the bijectivity of the Legendre transform, this forces $v_x = \dot{\gamma}_y(-t_y)$ to be independent of the $y \in M \setminus \{x\}$ such that $t_y < +\infty$. If we now call γ the extremal with $\gamma(0) = x$ and $\dot{\gamma}(0) = v_x$, using also that γ_y is an extremal we must have $\gamma_y(s) = \gamma(s + t_y)$ therefore $y = \gamma(t_y)$ if $t_y < +\infty$. Since $\limsup_{y\to x} t_y = 0$, it follows that for each $\delta > 0$, the image $\gamma([0, \delta])$ contains a neighborhood of x in M. Since γ is at least C^1 this is impossible if the dimension of M is ≥ 2. If the dimension of M is 1 then there exists points on $\gamma([0, \delta])$ (locally) to the right and the left of x, therefore we can find $\eta \in]0, \delta]$ with $\gamma(\eta) = x = \gamma(0)$. Since γ is $(u, L, c[0])$-calibrated, we have

$$\int_0^{\eta} L(\gamma(s), \dot{\gamma}(s))\, ds + c[0]\eta = 0,$$

going n times through the loop $\gamma|[0, \eta]$, and letting $n \to \infty$, we obtain $h(x, x) = 0$. □

6. Strict critical subsolutions

As already said a critical subsolution $u : M \to \mathbf{R}$ is said to be strict at $x \in M$, if there exists an open subset $U \subset M$ and $c < c[0]$ such that $x \in U$, and $u|U$ is a subsolution of $H(x, d_x u) = c$.

Proposition 6.1. *There exists a global critical subsolution which is strict at each point of $M \setminus \mathcal{A}$.*

Proof. We first show that given $x \in M \setminus \mathcal{A}$, we can find a global critical subsolution $u_x : M \to \mathbf{R}$ which is strict on an open set U_x containing x, i.e. there exists $c_x < c[0]$ such that the restriction $u_x|U_x$ is a subsolution of $H(y, d_y u_x) = c_x$ on the open subset U_x.

We start with ϕ_x. By Proposition 4.2 we know that ϕ_x is a global critical subsolution and a viscosity solution on $M \setminus \{x\}$. Since by Theorem 4.3, the subsolution ϕ_x is not a viscosity solution, the supersolution viscosity condition must be violated. But the only point where it can be violated is x itself this means that we can find a $\mathbf{C}^1$ function $\theta : V \to \mathbf{R}$ defined on a open neighborhood V of x such that $\theta(y) < \phi_x(y)$, for $y \in V \setminus \{x\}$, $\theta(x) = \phi_x(x)$, and $H(x, d_x\theta) < c[0]$. We can then find $c_x < c[0]$ and an open neighborhood W of x whose closure $\bar{W}$ is compact and contained in V of x such that $H(y, d_y\theta) < c_x$, for each $y \in \bar{W}$. In particular, θ is a strict critical subsolution on W.

We pick $\epsilon > 0$ such that $\phi_x(y) > \theta(y) + \epsilon$, on the compact boundary $\bar{W} \setminus W$, and we define u on $\bar{W}$ by $u_x(y) = \max(\phi_x(y), \theta(y) + \epsilon)$. By the choice of ϵ, the function u_x coincides with ϕ_x on a neighborhood of $\bar{W} \setminus W$, hence we can extend u_x continuously by ϕ_x to M. The function u is obviously locally Lipschitz. Moreover, it is a global critical subsolution. In fact, if $A \subset W$ is a compact set outside which ϕ_x and u_x coincide, then u_x is obviously a critical subsolution outside A. Moreover, it is also a critical subsolution on U, since a max of viscosity subsolutions is itself a viscosity subsolution.

It remains to observe that $\theta(x)+\epsilon > \phi_x(x) = \theta(x)$ to conclude that there exists an open neighborhood U_x of x such that $U_x \subset U$ and $u_x(y) = \theta(y)+\epsilon$. Obviously, for $y \in U_x$, we have $H(y, d_y u_x) = H(y, d_y\theta) < c_x < c[0]$. This of course implies that u_x is dominated by $L + c[0]$ on M, and the restriction $u_x|U_x$ is dominated by $L + c_x$ on U_x.

We now fix some base point y_0 in M. Since u_x is dominated by $L + c[0]$, it is Lipschitz with Lipschitz constant bounded by $A(1) + c[0]$. Hence, for any $y \in M$, we have $|u_x(y) - u_x(y_0)| \leq (A(1) + c[0])d(y, y_0)$. Replacing u_x by $u_x - u_x(y_0)$, we see that we can assume, without loss of generality, that for each compact subset $K \subset M$ we have

$$\sup\{|u_x(y)| \mid x \in M \setminus \mathcal{A}, y \in K\} < +\infty. \tag{$*$}$$

Since the set $M \setminus \mathcal{A}$ is covered by the open sets U_x, $x \in M \setminus \mathcal{A}$, we can extract a countable subcover U_{x_n}, $n \in \mathbf{N}$. We define $u = \sum_{n\in\mathbf{N}} u_{x_n}/2^{n+1}$. From $(*)$ this series is uniformly convergent on each compact subset of M. Observe that $\sum_{n\in\mathbf{N}} 1/2^{n+1} = 1$, hence u is an infinite convex combination of the u_{x_n}. Since each u_{x_n} is dominated by $L+c[0]$ on M, it is not difficult to see that the infinite convex combination u is also dominated by $L+c[0]$ on M. Moreover, since $u_{x_n}|U_{x_n}$ is dominated by $L + c_{x_n}$ on U_{x_n}, the same convexity argument shows that $u|U_{x_n}$ is dominated by $c_{x_n}/2^{n+1} + \sum_{m\neq n} c[0]/2^{m+1} < c[0]$. □

Of course the function u being a critical subsolution is differentiable at each point of $\mathcal{A}$. We would like to find one with the same property of being strict at each point of $M \setminus \mathcal{A}$ and which has the property that its derivative is continuous at each point of $\mathcal{A}$. Since any $T_t^- u$ with $t > 0$ is semi-concave, it suffices to show that it is itself a critical subsolution which is strict at each point of $M \setminus \mathcal{A}$. This is proved in the proposition below.

Proposition 6.2. *If u is a critical subsolution which is strict at each point of $M \setminus \mathcal{I}(u)$, then $T_t^- u$ is also a critical subsolution which is strict at each point of $M \setminus \mathcal{I}(u)$*

We start with a lemma.

Lemma 6.3. *If u is a critical subsolution, then for each $t > 0$, the function $T_t^- u$ is also a critical subsolution. Moreover $\mathcal{I}(T_t^- u) = \mathcal{I}(u)$, and $u = T_t^- u + tc[0]$ on $\mathcal{I}(u) = \mathcal{I}(T_t^- u)$.*

If $t > 0, x \notin \mathcal{I}(u)$, and $\gamma : [0, t] \to M$ are such that $x = \gamma(t)$, and $T_t^- u(x) = u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds$, then $\gamma([0, t]) \subset M \setminus \mathcal{I}(u)$.

Proof. We know that $u \prec L + c[0]$ is equivalent to $u \le T_s^- u + c[0]s$, for all $s \ge 0$, and therefore $T_{s+t}^- u + c[0]s \ge T_t^- u$, for all $s, t \ge 0$, since the semi-group T_t^- preserves the order. Hence $T_t^- u$ is also a critical subsolution.

If $\gamma :]-\infty, +\infty[\to M$ is a $(u, L, c[0])$-calibrated curve, then

$$u(\gamma(s)) = u(\gamma(s-t)) + \int_0^t L(\gamma(\sigma + s - t), \dot\gamma(\sigma + s - t))\, d\sigma + tc[0].$$

It follows that $T_t^- u(\gamma(s)) + tc[0] \le u(\gamma(s))$. The reverse equality is true, as said above, since $u \prec L + c[0]$. This shows that $u = T_t^- u + tc[0]$ on $\gamma(]-\infty, +\infty[)$. Therefore γ is also $(T_t^- u, L, c[0])$-calibrated. This proves $\mathcal{I}(u) \subset \mathcal{I}(T_t^- u)$, and $u = T_t^- u + tc[0]$ on $\mathcal{I}(u)$.

Let us now take a curve $\gamma :]-\infty, +\infty[\to M$ that is $(T_t^- u, L, c[0])$-calibrated. We have $T_t^- u(\gamma(s)) = T_t^- u(\gamma(s-t)) + \int_0^t L(\gamma(\sigma + s - t), \dot\gamma(\sigma + s - t))\, d\sigma + tc[0]$. Since $T_t^- u(\gamma(s-t)) + tc[0] \ge u(\gamma(s-t))$, and $T_t^- u(\gamma(s)) \le u(\gamma(s-t)) + \int_0^t L(\gamma(\sigma + s - t), \dot\gamma(\sigma + s - t))\, dc$, we conclude that $T_t^- u(\gamma(s-t)) + tc[0] = u(\gamma(s-t))$. Therefore $T_t^- u + tc[0] = u$ on $\gamma(]-\infty, +\infty[)$, and γ is also $(u, L, c[0])$-calibrated. This proves $\mathcal{I}(u) \supset \mathcal{I}(T_t^- u)$.

Suppose now that $t > 0, x \notin \mathcal{I}(u)$, and $\gamma : [0, t] \to M$ are such that $x = \gamma(t)$, and $T_t^- u(x) = u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds$. Suppose that $\gamma(0) \in \mathcal{I}(u)$. Then $u(\gamma(0)) = T_t^- u(\gamma(0)) + tc[0]$, and $\gamma(0) \in \mathcal{I}(T_t^- u)$. In particular, the curve γ is $(T_t^- u, L, c[0])$-calibrated, and we can pick an extremal $\delta :]-\infty, +\infty[$ which is $(T_t^- u, L, c[0])$-calibrated and satisfies $\delta(0) = \gamma(0)$. It follows that the curve obtained by piecing up $\delta|]-\infty, 0]$ and γ, is continuous on $]-\infty, t]$ and also $(T_t^- u, L, c[0])$-calibrated. This forces this curve to be an extremal therefore $\gamma = \delta|[0, t]$. Since $x = \gamma(t)$, and the curve δ is contained in $\mathcal{I}(T_t^- u) = \mathcal{I}(u)$, we see that $x \in \mathcal{I}(u)$. This is a contradiction. Therefore $\gamma(0) \notin \mathcal{I}(u)$.

It remains to show that the other points of the curve γ are not in $\mathcal{I}(u)$. But, from the semi-group property of T_t^-, we know that for $t' < t$, we have

$$T_t^- u(x) = T_{t'}^- u(\gamma(t')) + \int_{t'}^{t} L(\gamma(s), \dot\gamma(s))\, ds,$$

therefore from what we just obtained $\gamma(t') \notin \mathcal{I}(T_{t'}^- u) = \mathcal{I}(u)$. □

Proof of Proposition 6.2. Fix $t > 0$, and a compact set K contained in $M \setminus \mathcal{I}(u)$. It suffices to show that there exists $c < c[0]$ such that for each $\gamma : [a, b] \to K$, we have

$$T_t^- u(\gamma(b)) - T_t^- u(\gamma(a)) \le \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + (b - a)c.$$

Consider the set $\mathcal{E}$ of extremals $\gamma : [0, t] \to M$ such that $\gamma(t) \in K$, and

$$T_t^- u(\gamma(t)) = u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds.$$

Since the speed $\dot\gamma(s)$ of such a curve is bounded by a constant independent of γ, the set $\mathcal{E}$ is compact in the C^1 topology, therefore $K' = \{\gamma(s) \mid s \in [0, t], \gamma \in \mathcal{E}\}$ is a compact set disjoint from $\mathcal{I}(u)$. We pick a neighborhood V of this set K' whose closure $\bar V$ is compact and disjoint from $\mathcal{I}(u)$. Since u is strict at each point not in $\mathcal{I}(u)$, by compactness of $\bar V$, we can find $c < c[0]$ such that $u|V$ is a subsolution of $H(x, d_x u) = c$ on V. In particular, for each piecewise C^1 curve $\delta : [\alpha, \beta] \to V$, we have

$$u(\delta(\beta)) - u(\delta(\alpha)) \le \int_\alpha^\beta L(\delta(s), \dot\delta(s))\, ds + (\beta - \alpha)c.$$

Let us now pick a piecewise C^1 curve $\gamma : [a, b] \to K$, we want to show that

$$T_t^- u(\gamma(b)) - T_t^- u(\gamma(a)) \le \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + (b - a)c.$$

It suffices to prove the inequality for $b - a < t$, since we can break the interval $[a, b]$ into a finite family of intervals of length $< t$ and then add up the inequalities obtained.

We pick an extremal $\gamma_- : [0, t] \to M$ such that $\gamma_-(t) = \gamma(a)$ and

$$T_t^- u(\gamma(a)) = u(\gamma_-(0)) + \int_0^t L(\gamma_-(s), \dot\gamma_-(s))\, ds.$$

Of course, the extremal γ_- is in $\mathcal{E}$, and therefore $\gamma_-([0,t]) \subset V$. We define $\delta : [0,t] \to V$ by $\delta(s) = \gamma_-(s+b-a)$, for $0 \le s \le t-(b-a)$, and $\delta(s) = \gamma(s-(t-b))$, for $s \in [t-(b-a), t]$. We have

$$\begin{aligned} T_t^- u(\delta(t)) &\le u(\delta(0)) + \int_0^t L(\delta(s), \dot\delta(s))\, ds \\ &= u(\delta(0)) + \int_{b-a}^t L(\gamma_-(s), \dot\gamma_-(s))\, ds + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds. \end{aligned}$$

We also do have $\delta(0) = \gamma_-(b-a)$ and

$$u(\gamma_-(b-a)) \le u(\gamma_-(0)) + \int_0^{b-a} L(\gamma_-(s), \dot\gamma_-(s))\, ds + (b-a)c.$$

Since $\delta(t) = \gamma(b)$, we obtain

$$\begin{aligned} T_t^- u(\gamma(b)) &\le u(\gamma_-(0)) + \int_0^t L(\gamma_-(s), \dot\gamma_-(s))\, ds \\ &\qquad + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + (b-a)c \\ &= T_t^- u(\gamma(a)) + \int_a^b L(\gamma(s), \dot\gamma(s))\, ds + (b-a)c. \end{aligned}$$

□

Theorem 1.3 is now a consequence of what we just obtained. In fact, we can find a global critical subsolution $u : M \to \mathbf{R}$ which is strict at each point of $M \setminus \mathcal{A}$. Replacing u by $T_t^- u$, which is locally semi-concave for $t > 0$, we can assume that $x \mapsto d_x u$ is continuous on the domain of definition $\mathrm{dom}(du)$ of the derivative. We have $\mathcal{A} \subset \mathrm{dom}(du)$. Since u is a strict critical subsolution on $U = M \setminus \mathcal{A}$, we can find an open cover V_n of U and a sequence $c_n \in \mathbf{R}$, such that $c_n < c[0]$ and $H(x, d_x u) \le c_n$ at each point of $V_n \cap \mathrm{dom}(du)$. We choose a partition of unity φ_n on U subordinated to the cover V_n. If we define the continuous function ψ by $\psi(x) = \sum_n c_n \varphi_n(x)$, we have $\psi(x) < c[0]$, for each $x \in U$ and $H(x, d_x u) \le \psi(x)$ for $x \in U \cap \mathrm{dom}(du)$. We can now apply Theorem 9.2 of Appendix A with $F = \{(x,p) \in T^*U \mid H(x,p) \le \psi(x)\}$ as the closed set and $U = \{(x,p) \in T^*U \mid H(x,p) < (\psi(x) + c[0])/2\}$ as the open set containing F to obtain the required C^1 subsolution. □

7. Approximation

In this section, we will suppose that our manifold M is *compact*, and we will prove Theorem 1.5.

We start by observing that any critical subsolution $u : M \to \mathbf{R}$ is the uniform limit of subsolutions which are strict on $M \setminus \mathcal{A}$. In fact if

$u_0 : M \to \mathbf{R}$ is a critical subsolution which is strict then, for each $\epsilon > 0$, the function $\epsilon u_0 + (1-\epsilon)u$ is also a critical subsolution which is strict on $M \setminus \mathcal{A}$. Obviously, the uniform limit of $\epsilon u_0 + (1-\epsilon)u$ as $\epsilon \to 0$ is u.

If $u : M \to \mathbf{R}$ is a critical subsolution, since M is compact we know from [11] that we can find u_- and u_+, with $u_- = u_+ = u$ on $\mathcal{A}$, such that u_- is a viscosity solution of $H(x, d_x v) = c[0]$, and $-u_+$ is a viscosity solution of $\check{H}(x, d_x v) = c[0]$, where $\check{H}(x, p) = H(x, -p)$. We have $u_+ \le u \le u_-$, and the set $\mathcal{I}(u) \supset \mathcal{A}$ is precisely $\{x \in M \mid u_+(x) = u_-(x)\}$.

For each $x \in M \setminus \mathcal{I}(u)$, we construct a critical subsolution u_x, such that $\|u_x - u\|_\infty \le \epsilon$, $u_x = u$ on $\mathcal{I}(u)$, the critical subsolution u_x is strict at x. Since $x \in M \setminus \mathcal{I}(u)$, we have either $u_-(x) > u(x)$ or $u_+(x) < u(x)$. We will treat the first case, the second is similar. We choose $\delta > 0$ such that $3\delta < u_-(x) - u(x)$, and $3\delta < \epsilon$. We choose $u_1 : M \to \mathbf{R}$ a critical subsolution which is strict on $M \setminus \mathcal{A} \ni x$, and such that $\|u_1 - u\|_\infty \le \delta$. We have $u \le u + \delta \le u_1 + 2\delta \le u + 3\delta$. By the choice of δ, it follows that $u_1(x) + 2\delta < u_-(x)$. Moreover since $u_- = u$ on $\mathcal{I}(u)$, we obtain that $u_1 + 2\delta > u_-$ on a neighborhood of $\mathcal{I}(u)$. If we define the critical subsolution u_x by $u_x = \min(u_-, u_1 + \delta)$. We have $u_x = u_1$ in a neighborhood of x, and therefore u_x is strict at x. Since $u_x = u_-$ on a neighborhood of $\mathcal{I}(u)$, we obtain $u_x = u$ on $\mathcal{I}(u)$. It remains to check that $\|u_x - u\|_\infty \le \epsilon$. Since $u_- \ge u$, and $u_1 \ge u$, we have $u \le u_x$. Of course we also have $u_x \le u_1 + 2\delta \le u + 3\delta$. this gives $\|u_x - u\|_\infty \le 3\delta < \epsilon$.

As in the proof of Theorem 1.3, we can form a convex combination $\bar{u}$ of the family $u_x, x \in M \setminus \mathcal{I}(u)$ to obtain a global critical subsolution such that $\bar{u}$ is a strict subsolution at each $x \in M \setminus \mathcal{I}(u)$, and $\bar{u} = u$ on $\mathcal{I}(u)$. These last two conditions do imply that $\mathcal{I}(\bar{u}) = \mathcal{I}(u)$. Moreover, by convex combination $\|\bar{u} - u\| < \epsilon$. We can now replace $\bar{u}$ by $\bar{\bar{u}} = T_t^- \bar{u}$, with $t > 0$ small enough to have $\|\bar{\bar{u}} - u\|_\infty < \epsilon$. By Proposition 6.2 and Lemma 6.3, the global critical subsolution $\bar{\bar{u}}$ satisfies $\bar{\bar{u}} = u$ on $\mathcal{I}(u) = \mathcal{I}(\bar{\bar{u}})$, and it is also a strict critical subsolution at each point of the complement of $\mathcal{I}(u) = \mathcal{I}(\bar{\bar{u}})$. The derivative of $\bar{\bar{u}}$ is continuous on its domain of definition which contains $\mathcal{I}(u) = \mathcal{I}(\bar{\bar{u}})$.

It is not difficult to adapt the arguments of the end of the proof of Theorem 1.3 to finish the proof of Theorem 1.5. □

8. Applications

We prove first Mañé's Theorem 1.6.

Suppose that M is compact, and that μ is a probability measure on TM with $\int_{TM} \|v\|_x \, d\mu(x, v) < +\infty$. Remark that since M is compact all Riemannian norms on TM are equivalent, therefore the norm used to check the condition $\int_{TM} \|v\|_x \, d\mu(x, v) < +\infty$ is irrelevant. Since the norm $\|d_x g\|_x$ is bounded if $g : M \to \mathbf{R}$ is C^1, the integral $\int_{TM} d_x g(v) \, d\mu(x, v)$ makes sense.

Suppose now that $\int_{TM} d_x f(v)\, d\mu(x, v) = 0$, for each C^∞ function $f : M \to \mathbf{R}$. If $g : M \to \mathbf{R}$ is C^1, we can approximate it in the uniform C^1 topology by a sequence $f_n : M \to \mathbf{R}$ of C^∞ functions. In particular, there is a constant $K < +\infty$ such that $\|d_x f_n\|_x \leq K$, for each $x \in M$, and each $n \in \mathbf{N}$. In particular, we have $|d_x f_n(v)| \leq K\|v\|_x$. Since the right hand side of this last inequality is μ-integrable and $d_x f_n(v) \to d_x g(v)$, by the dominated convergence theorem we obtain that $\int_{TM} d_x g(v)\, d\mu(x, v) = 0$. Therefore if $u : M \to \mathbf{R}$ is a C^1 critical subsolution, integrating the inequality $d_x u(v) \leq L(x, v) + c[0]$, we obtain $-c[0] \leq \int_{TM} L\, d\mu$. Of course we could have used the easier fact that for each $\epsilon > 0$, there exists a C^∞ function $u_\epsilon : M \to \mathbf{R}$ such that $H(x, d_x u_\epsilon) \leq c[0] + \epsilon$, for each $x \in M$, see [6], to obtain a proof of that fact.

Suppose that μ does also satisfy $\int_{TM} L\, d\mu = -c[0]$. We first show that the support of μ is contained in the Aubry set $\tilde{\mathcal{A}} \subset TM$. We choose $u : M \to \mathbf{R}$ a C^1 critical subsolution such that $H(x, d_x u) < c[0]$, for $x \in M \setminus \mathcal{A}$. If we integrate the inequalities

$$d_x u(v) \leq L(x, v) + H(x, d_x u) \leq L(x, v) + c[0],$$

we obtain $0 \leq \int_{TM} L(x, v) H(x, d_x u)\, d\mu = 0$. Therefore, we must have the equalities

$$d_x u(v) = L(x, v) + H(x, d_x u) = L(x, v) + c[0]$$

for (x, v) in the support of μ. The second equality gives $H(x, d_x u) = c[0]$, therefore $x \in \mathcal{A}$. The first shows that $d_x u = \partial L/\partial v(x, v)$, hence (x, v) is precisely the point above $x \in \mathcal{A}$ which is in the Aubry set $\tilde{\mathcal{A}} \subset TM$.

By Mather's graph theorem, the projection $\pi : TM \to M$ induces a bi-Lipschitz homeomorphism from $\tilde{\mathcal{A}}$ onto $\mathcal{A}$. Hence $\tilde{\mathcal{A}} = \{(x, X(x)) | x \in \mathcal{A}\}$, where X is a Lipschitz vector field defined on $\mathcal{A}$. We can extend X to a Lipschitz vector field defined on M. Since the Aubry set $\tilde{\mathcal{A}}$ is invariant by the Euler-Lagrange flow φ_t^L, it is not difficult to see that π induces a conjugacy between $\varphi_t^L|\tilde{\mathcal{A}}$ and $\psi_t^X|\tilde{\mathcal{A}}$, where ψ_t^X is the flow generated by the Lipschitz vector field X. Hence to check the invariance under φ_t^L of μ whose support is contained in $\tilde{\mathcal{A}}$, it suffices to check that the image measure $\pi_*\mu$ is invariant by ψ_t^X. By Proposition 10.3 of Appendix B, we must see that $\int_M d_x f(X(x))\, d\pi_*\mu(x) = 0$, for every $f : M \to \mathbf{R}$ of class C^∞. This follows from $\mu \in \mathcal{HM}$, in fact, since the support of μ is in $\tilde{\mathcal{A}}$, we have $\int_M d_x f(X(x))\, d\pi_*\mu(x) = \int_{TM} d_x f(v)\, d\mu(x, v)$. □

It remains to prove Theorem 1.7. If ω is a smooth 1-form, we can consider the Lagrangian $L(x, v) = \frac{1}{2}\|v\|_x^2 - \omega_x(v)$. Because ω is closed, its Euler-Lagrange flow φ_t^L is the geodesic flow, so its projected Aubry set $\mathcal{A}$ is a geodesic lamination. It is well-known that its associated Hamiltonian is $H(x, p) = \frac{1}{2}\|p + \omega_x\|_x^2$. Moreover, we have $c[0] = \frac{1}{2}\|[\omega]\|_s^2$. It results from Theorem 1.3 that there exists a C^1 function $u : M \to \mathbf{R}$ such that

$H(x, d_x u) = \frac{1}{2}\|d_x u + \omega_x\|_x^2 \leq c[0] = \frac{1}{2}\|[\omega]\|_s^2$, with equality if and only if $x \in \mathcal{A}$. It suffices to set $\tilde{\omega} = \omega + du$. □

9. Appendix A

We will denote by N a smooth metrizable manifold. We will suppose that N is endowed with some auxiliary Riemannian metric (not necessarily complete), we will denote by $\|\cdot\|$ the associated norm on any fiber $T_x N$ or $T_x^* N$. We will denote by $\pi_* : T^* N \to N$ the canonical projection.

If $f : N \to \mathbf{R}$ is a locally Lipschitz function, we will denote by $\mathrm{dom}(df)$ the set of $x \in N$ where the derivative $d_x f$ exists. By Rademacher's theorem $\mathrm{dom}(df)$ is of full (Lebesgue) measure in N.

The following theorem was established in the appendix of [12].

Theorem 9.1. *Let N be a smooth metrizable manifold, and $f : N \to \mathbf{R}$ be a locally Lipschitz function. Suppose that $F \subset O$ are respectively a closed and an open subset of T^*N, such that each $F_x = F \cap T_x^* N$ is convex with $d_x f \in F_x$ for almost every x in $\mathrm{dom}(df)$. If $\epsilon : N \to]0, +\infty[$ is a continuous function, then there exists a C^∞ function $g : N \to \mathbf{R}$ such that $(x, d_x g) \in O$ and $|f(x) - g(x)| < \epsilon(x)$, for each $x \in N$.*

We need an improvement.

Theorem 9.2. *Let N be a smooth metrizable manifold, and $f : N \to \mathbf{R}$ be a locally Lipschitz function. Suppose $A \subset B \subset N$ satisfy*

(1) *A is closed in N,*
(2) *$N \setminus B$ is of (Lebesgue) measure 0,*
(3) *$B \subset \mathrm{dom}(df)$ and the restriction of the derivative $B \to T^*N, x \mapsto (x, d_x f)$ is continuous at each point of A.*

Given any continuous function $\epsilon : N \setminus A \to]0, +\infty[$, and any subsets $F \subset O \subset T^*(N \setminus A)$, which are respectively a closed and an open subset of $T^*(N \setminus A)$, such that

(4) *for each $x \in N \setminus A$, the intersection $F_x = F \cap T_x^* N$ is convex,*
(5) *for almost every x in $(N \setminus A)$, we have $d_x f \in F_x$,*

then there exists a C^1 function $g : N \to \mathbf{R}$ such that $(x, d_x g) \in O$, $|f(x) - g(x)| < \epsilon(x)$, for each $x \in N \setminus A$, and $g(x) = f(x)$, $d_x g = d_x f$, for $x \in A$.

Proof. We set $U = N \setminus A$. We want to apply Theorem 9.1 with U instead of N. The problem is to make sure that we can extend the map obtained on U to a C^1 map on N.

Call d the distance on N obtained from the Riemannian metric. We can assume $\epsilon(x) \leq d(x, A)^2$, for each $x \in U$. If this was not the case, we could replace the continuous positive function ϵ, by $x \mapsto \min(\epsilon(x), d(x, A)^2)$, this function is still continuous > 0 on the open set $U = N \setminus A$.

To simplify the matter, by Tietze-Urysohn theorem, we choose a continuous section $s : N \to T^*N, x \mapsto s(x) \in T^*_x N$, with $s(x) = d_x f$, for $x \in A$. For each $n \in \mathbf{N}$, the set $\mathcal{U}_n = \{(x, p) \in T^*N \mid \|p - s(x)\| < 1/(n+1)\}$ is open and contains $(x, d_x f)$, for each $x \in A$. Since the derivative df restricted to B is continuous at each point of A, we can find an open set $V_n \subset N$, such that $A \subset V_n$, and for each $x \in V_n \cap B$, we have $(x, d_x f) \in \mathcal{U}_n$. Shrinking V_n, if necessary, we can assume $\bar{V}_{n+1} \subset V_n$, and $A = \cap_{n\in\mathbf{N}} V_n$. We define

$$F_1 = \pi_*^{-1}(U \setminus V_0) \cup \bigcup_{n\in\mathbf{N}} \{(x, p) \mid x \in V_n \setminus V_{n+1}, \|p - s(x)\| \le 1/(n+1)\}),$$

$$O_1 = \pi_*^{-1}(U \setminus \bar{V}_1) \cup \bigcup_{n\in\mathbf{N}} (O \cap \{(x, p) \mid x \in V_n \setminus \bar{V}_{n+2}, \|p - s(x)\| < 1/n\}.$$

It is not difficult to check the following properties:

(a) $F_1 \subset O_1$, furthermore F_1 and O_1 are respectively closed and open in T^*U
(b) $(x, d_x f) \in F_1$, for each $x \in U \cap B$
(c) for each $x \in U$, the intersection $F_1 \cap T^*_x U$ is convex.

If we set $\tilde{F} = F_1 \cap F$ and $\tilde{O} = O_1 \cap O$, we evidently have:

(d) $\tilde{F} \subset \tilde{O}$, furthermore $\tilde{F}$ and $\tilde{O}$ are respectively closed and open in T^*U.

Conditions (2), (5), and (b) give:

(e) $(x, d_{xf}) \in \tilde{F}$, for almost every $x \in U$.

Conditions (4), and (c) give:

(f) for each $x \in U$, the intersection $\tilde{F} \cap T^*_x U$ is convex.

Applying Theorem 9.1 with $U, \tilde{F}, \tilde{O}$ instead of N, F, O, we find $g : U \to \mathbf{R}$, of class C^∞, and such that $(x, d_x g) \in \tilde{O}$, $|f(x) - g(x)| < \epsilon(x)$, for each $x \in U$. We extend g by f on A. Since $|f(x) - g(x)| < \epsilon(x) \le d(x, A)^2$, it is easy to see that the extension is also differentiable at each point $x \in A$, with $d_x g = d_x f$. Since $(x, d_x g) \in \tilde{O}$, for $x \in U$, we see that $\|d_x g - s(x)\| < 1/n$, for $x \in V_n$. This implies the continuity of the derivative at each point of A, since $s(x) = d_x f = d_x g$ at such a point. □

10. Appendix B

Suppose that N is a manifold and $f : N \to \mathbf{R}$ is a function. If $(x, v) \in TN$, we will say that f has a derivative at x in the direction of v, if for each C^1 curve $\gamma : [-\eta, \eta] \to N$, such that $\eta > 0$ and $(\gamma(0), \dot{\gamma}(0)) = (x, v)$, the function $t \mapsto f(\gamma(t))$ has a derivative at 0 and the derivative $\frac{d}{dt} f(\gamma(t))|_{t=0}$ is independent of γ. We will then denote by vf this common value. Moreover, if X is a vector field on N, and the derivative of f at each $x \in N$ in the direction of $X(x)$ exists, we will say that f is differentiable in the direction of X, and we will denote by Xf the function $x \mapsto X(x)f$.

The following lemma gives a criterion for verifying directional differentiability when f is Lipschitz (a mild restriction).

Lemma 10.1. *Suppose $f : N \to \mathbf{R}$ is Lipschitz on a neighborhood of $x \in N$. If $v \in T_xN$, and there exists a curve $\gamma_0 : [-\eta_0, \eta_0] \to N$, with $\eta_0 > 0$, $\gamma(0) = x$, admitting a derivative $\dot\gamma_0(0) = v$, and such that both γ and the function $t \mapsto f(\gamma_0(t))$ have a derivative at 0, then f has a derivative at x in the direction of v.*

Proof. This is a local statement, hence we can assume that N is an open subset of $\mathbf{R}^n$, and that $f : N \to \mathbf{R}$ is Lipschitz for the norm $\|\cdot\|$ on $\mathbf{R}^n$, with Lipschitz constant C. If $\gamma : [-\eta, \eta] \to N$ is a curve, differentiable at $t =$, with $(\gamma(0), \dot\gamma(0)) = (x, v)$, then $\gamma(t) - \gamma_0(t) = t\epsilon(t)$, with $\epsilon(t) \to 0$, as $t \to 0$. Therefore, we can write

$$\begin{aligned}\left|\frac{f(\gamma(t)) - f(\gamma(0))}{t} - \frac{f(\gamma_0(t)) - f(\gamma_0(0))}{t}\right| &= \left|\frac{f(\gamma(t)) - f(\gamma_0(t)}{t}\right| \\ &\le \frac{C}{t}\|\gamma(t) - \gamma_0(t)\| \\ &\le \epsilon(t) \to_{t\to 0} 0.\end{aligned}$$

□

Proposition 10.2. *Suppose that N is a metrizable manifold, and that $f : N \to \mathbf{R}$ is locally Lipschitz. We can find a sequence $g_n : N \to \mathbf{R}$ of C^∞ functions converging uniformly to f, and satisfying the following property:*

for each continuous vector field X such that Xf exists and is continuous, the sequence of functions Xg_n converges pointwise to Xf.

Moreover, we can choose the functions g_n with the further property that the subset $\{(x, d_xg_n) \mid x \in K, n \in \mathbf{N}\}$ is relatively compact in TN, for each compact subset $K \subset N$.

Proof. As usual in non-smooth analysis, see [5], we call $\partial f(x)$ the convex hull of the set of accumulation of derivatives d_yf (at points where they exist) when $y \to x$. Because f is locally lipschitz and T_xN is finite dimensional, the set is also compact and non empty.

We first show that if X is a continuous vector field such that Xf exists and is continuous, then $p(X(x)) = Xf(x)$, for each $x \in N$, and each $p \in \partial f(x)$. Suppose first that $p = \lim_{n\to\infty} d_{y_n}f$, where $y_n \to x$, and $y_n \in \mathrm{dom}(df)$, the set of points where f is differentiable, then since necessarily $Xf(y_n) = d_{y_n}f(X(y_n))$, and by continuity of X we have $p(X(x)) = \lim_{n\to\infty} d_{y_n}f(X(y_n))$, we do obtain by continuity of Xf that $Xf(x) = p(X(x))$. Since $\partial f(x)$ is the convex envelope of the accumulation points of d_yf as $y \to x$, this gives the same equality $p(X(x)) = Xf(x)$, for $p \in \partial f(x)$.

In fact, since N is finite-dimensional, say of dimension d, a theorem of Carathéodory states that every convex combination of a set of points is a convex combination of a subset with at most $d + 1$ points. It is then not difficult to show that $C = \cup_{x \in N} \partial f(x)$ is closed in TN. Moreover, for every compact set $K \subset N$, the intersection $C \cap \pi_*^{-1}(K)$ is compact, because the almost everywhere defined derivative is bounded on every compact subset of N. Since TN is a metric space, we can write $C = \bigcap_{n \in \mathbf{N}} U_n$, where U_n is open in TN, we can also assume $U_{n+1} \subset U_n$, and $\bar{U}_0 \cap \pi_*^{-1}(K)$ is compact, for each compact set $K \subset N$, since the same thing is true for C. We can apply Theorem 9.1, to obtain a sequence $g_n : N \to \mathbf{R}$ of C^∞ functions converging uniformly to f and such that $(x, d_x g_n) \in U_n$, for $x \in N$, and $n \in \mathbf{N}$. It remains to show that $d_x g_n(X(x))$ converges to $Xf(x)$. Since $d_x g_n$ is contained in the compact subset $\bar{U}_0 \cap T_x^* N$, it suffices to show that $d_x g_{n_k}(X(x))$ converges to $Xf(x)$ if $d_x g_{n_k}$ converges to p. But this follows from what we proved above, since necessarily $p \in T_x^* N \cap \bigcap_{n \in \mathbf{N}} U_n = \partial f(x)$. □

Proposition 10.3. *Suppose that μ is a probability measure on the compact manifold N. If X is a Lipschitz vector field on N such that $\int_N d_x g(X(x)) d\mu(x) = 0$, for every C^∞ function $g : N \to \mathbf{R}$, then μ is invariant by the flow ψ_t^X generated by X.*

Proof. Using Lebesgue's dominated convergence theorem, it is not difficult to obtain from Proposition 10.2 above, that $\int_N Xf \, d\mu = 0$, for every Lipschitz function which is continuously differentiable in the direction of X. Suppose now that $g : N \to \mathbf{R}$ is a C^∞ function. For each $t \in \mathbf{R}$ the map $x \mapsto \psi_t^X(x)$ is locally Lipschitz on $N \to \mathbf{R}$, therefore the function $g \circ \psi_t^X : N \to \mathbf{R}$ is Lipschitz. Moreover, this function $g \circ \psi_t^X$ is continuously differentiable in the direction of X with

$$X\big(g \circ \psi_t^X\big)(x) = \frac{d}{ds}\big[g \circ \psi_s^X(x)\big]_{|s=t} = (Xg) \circ \psi_t^X(x).$$

It follows that $\int_N \frac{d}{ds}[g \circ \psi_s^X(x)]\, d\mu(x) = 0$, therefore since $g \circ \psi_t^X(x) - g(x) = \int_0^t \frac{d}{ds}[g \circ \psi_s^X(x)]\, ds$, by Fubini theorem we obtain

$$\int_N \big(g \circ \psi_t^X - g\big)\, d\mu = \int_0^t \int_N \frac{d}{ds}\big[g \circ \psi_s^X(x)\big]\, d\mu(x) ds = 0.$$

Since C^∞ functions are dense in $C^0(N, \mathbf{R})$ for the sup-norm, we do obtain $\int_N f \circ \psi_t^X \, d\mu = \int_N f \, d\mu$, for every $f \in C^0(N, \mathbf{R})$. □

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