MATH. SCAND. 28 (1971), 257–264

EXISTENCE OF CONDITIONAL PROBABILITIES

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1. Introduction.

Let $(\Omega,\mathfrak{A},P)$ be a probability space, (S,Σ) a measurable space and p a measurable map from $(\Omega,\mathfrak{A})$ into (S,Σ). Given p we then want to construct a regular conditional probability of P, that is, we want to construct a map, R, from $\mathfrak{A}\times S$ into $[0,1]$ satisfying

$$R(\cdot,s) \text{ is a probability measure on } (\Omega,\mathfrak{A}) \ \forall s\in S\,, \tag{1}$$

$$R(A,\cdot) \text{ is a } Q\text{-measurable map } \forall A\in\mathfrak{A}\,, \tag{2}$$

$$P(A\cap p^{-1}(B)) = \int_B R(A,s)\,Q(ds) \quad \forall B\in\Sigma \ \forall A\in\mathfrak{A}\,, \tag{3}$$

where $Q=p\cdot P$, that is, Q is the image measure of P under p. Sometimes it is useful to demand that R has the following additional property

$$R(p^{-1}(s),s) = 1 \ \forall s\in p(\Omega)\,. \tag{4}$$

It is known (see for example [2, p. 370]) that it is not always possible to construct regular conditional probabilities. So one has to put restrictions on the probability space $(\Omega,\mathfrak{A},P)$ in order to derive the desired result. In this connection the notion of regularity of P plays an essential role.

Suppose that Ω is a Hausdorff space, $\mathfrak{A}$ a σ-algebra in Ω, and P a probability measure on $(\Omega,\mathfrak{A})$. Then P is called *regular* if for all A we have

$$P(A) = \sup\{P(K) \mid K \text{ compact},\ K\subseteq A,\ K\in\mathfrak{A}\}\,.$$

It is well known that, if P is regular and $\mathfrak{A}$ is countably generated, then a regular conditional probability exists for an arbitrary measurable map p.

A. and C. Ionescu Tulcea have recently proved that a regular conditional probability exists, if Ω and S are locally compact spaces, P is regular, and $\mathfrak{A}$ and Σ are the Baire σ-algebras (that is the σ-algebras generated by the compact $\mathscr{G}_\delta$-sets in Ω and S, respectively). See [1, Theorem 5 on p. 150].

Received May 27, 1970.

The aim of this paper is to show that a regular conditional probability exists whenever Ω is a Hausdorff space and P is a regular probability measure on $(\Omega,\mathscr{B}(\Omega))$, and to find conditions which assure that (4) can be obtained.

The tool of the proof is the notion of a lifting. Let (S,Σ,μ) be a positive measure space, and let Σ_μ denote the Lebesgue extension of Σ with respect to μ. Then we introduce the following spaces:

$L_\infty(\mu)$ = the space of all μ-measurable, μ-essentially bounded real functions on S. In this space two functions are identified if they coincide μ-a.e.

$B(S,\Sigma_\mu)$ = the space of all Σ_μ-measurable, bounded real functions on S. In this space no identification is made.

In $L_\infty(\mu)$ we introduce an ordering $\leq$, by saying $f \leq g$, if and only if $f(s) \leqq g(s)$ for μ-a.e. $s \in S$, and a norm is introduced by the formula

$$\|f\|_\infty^\mu = \mu-\operatorname{ess\,sup}_{s\in S}|f(s)| \quad \forall f \in L_\infty(\mu) .$$

In $B(S,\Sigma_\mu)$ we introduce an ordering $\leq$, by saying $f \leq g$, if and only if $f(s) \leqq g(s)$ $\forall s \in S$, and a norm is introduced by the formula

$$\|f\|_\infty = \sup_{s\in S}|f(s)| \quad \forall f \in B(S,\Sigma_\mu) .$$

A *lifting* of $L_\infty(\mu)$ is then a map, l, from $L_\infty(\mu)$ into $B(S,\Sigma_\mu)$, which satisfies

$$l(f) \leq l(g) \quad \text{if } f \leq g \ \mu\text{-a.e.} , \tag{5}$$

$$l(h) = al(f)+bl(g) \quad \text{if } h = af+bg \ \mu\text{-a.e.} , \tag{6}$$

$$l(h) = l(f)\cdot l(g) \quad \text{if } h = f\cdot g \ \mu\text{-a.e.} , \tag{7}$$

$$l(h) = 1 \quad \text{if } h = 1 \ \mu\text{-a.e.} , \tag{8}$$

$$l(f) = f \quad \mu\text{-a.e.} . \tag{9}$$

Notice that a lifting becomes an isometric order and algebra isomorphism from $L_\infty(\mu)$ onto a subspace of $B(S,\Sigma_\mu)$. It is well known (see for example [4]) that a lifting for $L_\infty(\mu)$ exists whenever μ is σ-finite.

2. Some properties of liftings.

In this section (S,Σ,Q) will denote a probability space, and l a lifting on $L_\infty(Q)$. The image in $B(S,\Sigma_Q)$ of $L_\infty(Q)$ under l is denoted by $\mathscr{L}$.

It is well known that $(L_\infty(Q), \leq)$ is a complete vector lattice (see for example Corollary 7 of [5, IV.11]). Hence $(\mathscr{L}, \leq)$ becomes a complete vector lattice.

If L_0 is a subset of $\mathscr{L}$ which is bounded above, we define $\mathsf{V}(L_0) = \mathsf{V}\{f \mid f \in L_0\}$ to be the lattice supremum of L_0 in $\mathscr{L}$, and we define

$$\sup(L_0)(s) = \sup\{f(s) \mid f \in L_0\} \quad \forall s \in S .$$

Our first lemma explores the connection between V and sup.

LEMMA 1. *Let L_0 be a subset of $\mathscr{L}$ which is bounded from above. Then*

(a) $\sup(L_0) \leq \mathsf{V}(L_0)$ *everywhere on S,*
(b) $\sup(L_0)$ *is μ-measurable, and*

$$\sup(L_0) = \mathsf{V}(L_0) \quad \mu\text{-a.e.}\,.$$

PROOF. Let $h_0 = \mathsf{V}(L_0)$ and $h_1 = \sup(L_0)$. Then $h_0(s) \geq f(s)$ $\forall s \in S$ $\forall f \in L_0$, hence $h_0 \geq h_1$ everywhere on S, and so (a) is proved.

By Corollary 7 in [5, IV.11], there exists $\{f_n\} \subseteq L_0$, such that

$$h_0 = \mathsf{V}\{f_n \mid n \geqq 1\} .$$

Put $h_2 = \sup\{f_n \mid n \geqq 1\}$. Then obviously $h_2 \leq h_1 \leq h_0$ everywhere on S, and h_2 is μ-measurable. So it suffices to show that $h_2 \geq h_0$ μ-a.e.. Now $h_2 \geq f_n$ $\forall n \geqq 1$, and so $l(h_2) \geq f_n$ everywhere on S $\forall n \geq 1$, which means that $l(h_2) \geq h_0$. But this implies that $h_2 \geq h_0$ μ-a.e., and the lemma is proved.

LEMMA 2. *Let L_0 be a subset of $\mathscr{L}$, which is bounded and filtering to the right, that is*

(a) $\forall f, g \in L_0\, \exists h \in L_0$, *such that* $h \geq \sup(f, g)$.

Then

$$\int_S \sup(L_0)\, dQ = \sup\left\{\int_S f\, dQ \mid f \in L_0\right\}.$$

PROOF. By Corollary 7 in [5, IV.11], we can find $\{f_n\} \subseteq L_0$, such that

$$\mathsf{V}(L_0) = \mathsf{V}\{f_n \mid n \geqq 1\} = h_0 .$$

By assumption (a) it is no loss of generality to assume that $f_1 \leq f_2 \ldots$. By Lemma 1, we have that

$$\sup(L_0) = h_0 = \sup_n f_n = \lim_n f_n \quad \mu\text{-a.e.}$$

So by Lebesgue's dominated convergence theorem we have that

$$\int_S \sup(L_0)\, dQ = \lim_{n\to\infty} \int_S f_n dQ \leq \sup\left\{\int_S f\, dQ \mid f \in L_0\right\},$$

and since the converse inequality is trivially true, the Lemma is proved.

3. Existence of regular conditional probabilities.

We are now ready to prove the main theorem of this paper.

THEOREM 1. *Let Ω be a Hausdorff space, $\mathfrak{A}=\mathscr{B}(\Omega)$, P a regular probability measure on $(\Omega,\mathfrak{A})$, (S,Σ) a measurable space and p a measurable map from $(\Omega,\mathfrak{A})$ into (S,Σ). We put $Q=p\cdot P$. Then there exists a map, R, from $\mathscr{B}(\Omega)\times S$ into $[0,1]$, such that*

(a) *$R(\cdot,s)$ is a regular probability measure on $(\Omega,\mathfrak{A})$ $\forall s\in S$,*

(b) *$R(A,\cdot)$ is Q-measurable $\forall A\in\mathfrak{A}$,*

(c) $\displaystyle\int_B R(A,s)\,Q(ds) = P(A\cap p^{-1}(B)) \quad \forall A\in\mathfrak{A}\;\forall B\in\Sigma.$

PROOF. First we suppose that Ω is compact. If $f\in C(\Omega)$, we define

$$\mu_f(B) = \int_{p^{-1}(B)} f(\omega)\,P(d\omega) \qquad \forall B\in\Sigma\,.$$

Then μ_f is a finite signed measure on (S,Σ), such that μ_f is absolutely continuous with respect to Q, $f \rightsquigarrow \mu_f$ is linear and positive, and

$$|\mu_f|(B) \le \|f\|_\infty Q(B) \qquad \forall f\in C(\Omega)\;\forall B\in\Sigma\,, \tag{10}$$

$$\mu_{1_\Omega} = Q\,, \tag{11}$$

where $|\mu_f|$ denotes the total variation of μ_f. Now let $p(s,f)$ be a Radon–Nikodym derivative of μ_f with respect to Q, and let l be a lifting of $L_\infty(Q)$. By (10) we then see that $p(\cdot,f)$ is μ-essentially bounded by $\|f\|_\infty$. Hence we may define

$$\overline{p}(\cdot,f) = l(p(\cdot,f)) \qquad \forall f\in C(\Omega)\,.$$

From the properties of μ_f it follows that $\overline{p}(s,\cdot)$ is a positive continuous linear functional on $C(\Omega)$ for each $s\in S$, and furthermore

$$\overline{p}(s,1_\Omega) = 1 \qquad \forall s\in S\,,$$
$$|\overline{p}(s,f)| \le \|f\|_\infty \qquad \forall s\in S\;\forall f\in C(\Omega)\,.$$

Hence there exists a regular probability measure $R(\cdot,s)$ on $(\Omega,\mathfrak{A})$, such that

$$\int_\Omega f(\omega)\,R(d\omega,s) = \overline{p}(s,f) \qquad \forall s\in S\;\forall f\in C(\Omega)\,, \tag{12}$$

$$\int_\Omega f(\omega)\,g(p(\omega))\,P(d\omega) = \int_S g(s)\int_\Omega f(\omega)\,R(d\omega,s)\,Q(ds) \qquad \forall f\in C(\Omega)\;\forall g\in B(S,\Sigma)\,. \tag{13}$$

We now show that R has the properties (a), (b) and (c) of the theorem. By the very definition of R, we see that R satisfies (a).

Let $\mathscr{F}$ be the class of all bounded real Borel functions, f, on Ω, satisfying

$$\int_\Omega f(\omega)\, R(d\omega,\cdot) \text{ is } Q\text{-measurable,} \tag{14}$$

$$\int_S g(s) \int_\Omega f(\omega)\, R(d\omega,s)\, Q(ds) = \int_\Omega f\cdot(g\circ p)\, dP \quad \forall g \in B^+(S,\Sigma)\,. \tag{15}$$

First we notice that $C(\Omega)\subseteq\mathscr{F}$. Let U be an open subset of Ω. We shall then show that $1_U \in \mathscr{F}$. Let $g \in B^+(S,\Sigma)$ and put

$$\mathscr{G} = \{f \in C(\Omega) \mid 0 \le f \le 1_U\}\,,$$
$$\mathscr{G}^* = \{\overline{p}(\cdot,f)\, l(g) \mid f \in \mathscr{G}\}\,.$$

Then $\mathscr{G}$ and $\mathscr{G}^*$ are families of functions, which are filtering to the right and bounded above, so by regularity of $R(\cdot,s)$ and $g(p(\omega))P(d\omega)$, we find that

$$R(U,s) = \sup\{\overline{p}(s,f) \mid f \in \mathscr{G}\} \quad \forall s \in S\,, \tag{16}$$

$$\int_U g\circ p\, dP = \sup\left\{\int_\Omega f\cdot(g\circ p)\, dP \mid f \in \mathscr{G}\right\}, \tag{17}$$

since $1_U = \sup(\mathscr{G})$ and $\mathscr{G}$ consists of continuous functions (see for example [3, II, Theorem 35]). Now we notice that

$$\overline{p}(\cdot,f) l(g) \in l(L_\infty(Q)) \quad \text{for all } f \in C(\Omega)\,,$$

so by Lemma 1 and (16), 1_U satisfies (14), and by Lemma 2 we have

$$\int_S g(s)\, R(U,s)\, Q(ds) = \sup\left\{\int_S g(s)\, \overline{p}(s,f)\, Q(ds) \mid f \in \mathscr{G}\right\}.$$

Inserting (13) and (17) in this we find that

$$\int_U g\circ p\, dP = \int_S g(s)\, R(U,s)\, Q(ds)$$

which means that 1_U satisfies (15). That is, $1_U \in \mathscr{F}$.

Now $\mathscr{F}$ is obviously a linear space which is closed under pointwise uniformly bounded sequential convergence, and so from Theorem 20 in [3, I] it follows that $\mathscr{F}$ contains all bounded Borel measurable functions on Ω.

This proves Theorem 1 in the case that Ω is compact. In the general case we choose compact sets $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$, such that $P(\Omega \setminus K_n) \to 0$ as $n \to \infty$. The argument above shows that there exist positive measure kernels R_n, such that for $n \geqq 1$

(18) $\quad R_n(\cdot, s)$ is a positive regular measure on $(\Omega, \mathscr{B}(\Omega))$ $\forall s \in S$,

(19) $$R_n(\Omega, s) = R_n(K_n, s) = P(K_n) \quad \forall s \in S ,$$

(20) $\quad R_n(A, \cdot)$ is Q-measurable $\forall A \in \mathscr{B}(\Omega)$,

(21) $$\int_B R_n(A,s)\, Q(ds) = P(K_n \cap A \cap p^{-1}(B)) \quad \forall A \in \mathscr{B}(\Omega)\ \forall B \in \Sigma ,$$

(22) $$\int_\Omega f(\omega)\, R_n(d\omega, \cdot) \in l(L_\infty(Q)) \quad \forall f \in C(K_n) .$$

Let $n \geqq 1$ and let $f \in B^+(\Omega, \mathscr{B}(\Omega))$, such that $f | K_{n+1}$ is continuous. Then

$$\begin{aligned} \int_B Q(ds) \int_\Omega f(\omega)\, R_n(d\omega, s) &= \int_{K_n \cap p^{-1}(B)} f\, dP \\ &\leqq \int_{K_{n+1} \cap p^{-1}(B)} f\, dP \\ &= \int_B Q(ds) \int_\Omega f(\omega)\, R_{n+1}(d\omega, s) \end{aligned}$$

for all $B \in \Sigma$. This shows that

$$\int_\Omega f(\omega)\, R_n(d\omega, s) \leq \int_\Omega f(\omega)\, R_{n+1}(d\omega, s) \quad Q\text{-a.e.}$$

But from (22) one deduces that this inequality holds everywhere on S. So by (18) and (19) we find that

$$R_n(A,s) \leqq R_{n+1}(A,s) \quad \forall s \in S\ \forall A \in \mathscr{B}(\Omega) .$$

Now put $R(A,s) = \lim_{n\to\infty} R_n(A,s)$ for $s \in S$, $A \in \mathscr{B}(\Omega)$. Then it is easily checked that R has the properties (a), (b) and (c) of Theorem 1.

THEOREM 2. *Let $(\Omega, \mathfrak{A}, P)$ be a probability space, (S, Σ) a measurable space and p a measurable map from $(\Omega, \mathfrak{A})$ into (S, Σ). Suppose that a regular conditional probability, R, of P given p exists, and that the graph of p defined by*

$$G(p) = \{(\omega, p(\omega)) \mid \omega \in \Omega\}$$

belongs to the product σ-algebra $\mathfrak{A} \times \Sigma$. Then we have ($Q$ denotes the image measure: $p \cdot P$)

(a) *$p(\Omega)$ is Q-measurable, and has Q-measure* 1,

(b) $\{s\} \in \Sigma \ \forall s \in p(\Omega)$,

(c) $R(p^{-1}(s),s)=1$ Q-a.e.,

(d) *There exists a regular conditional probability, R_0, of P given p, such that*

$$R_0(p^{-1}(s),s) = 1 \quad \forall s \in p(\Omega) .$$

Proof. If $A \subseteq \Omega \times S$, then we define

$$\begin{aligned} A'(s) &= \{\omega \in \Omega \mid (\omega,s) \in A\} \quad \forall s \in S , \\ A''(\omega) &= \{s \in S \mid (\omega,s) \in A\} \quad \forall \omega \in \Omega . \end{aligned}$$

It is well known that if $A \in \mathfrak{A} \times \Sigma$, then $A'(s) \in \mathfrak{A} \ \forall s \in S$, and $A''(\omega) \in \Sigma \ \forall \omega \in \Omega$, and that

$$\int_S R(A'(s),s)\, Q(ds) = P(\omega \in \Omega \mid (\omega,p(\omega)) \in A) .$$

Now put $A=G(p)$. Then $A''(\omega)=\{p(\omega)\}$, and $A'(s)=p^{-1}(s)$, hence (b) is proved. Furthermore $R(p^{-1}(s),s)$ is Q-measurable, and

$$\int_S R(p^{-1}(s),s)\, Q(ds) = 1 .$$

That is, (c) is proved. Let

$$S_0 = \{s \in S \mid R(p^{-1}(s),s)=0\} .$$

Then S_0 is Q-measurable, $Q(S_0)=0$ and $S \setminus p(\Omega) \subseteq S_0$, which proves (a). Since (d) is a trivial consequence of (a) and (c), the theorem is proved.

Theorem 3. *Let $(\Omega,\mathfrak{A})$ and (S,Σ) be measurable spaces and p a measurable map from $(\Omega,\mathfrak{A})$ into (S,Σ). Put $S_0=p(\Omega)$, and define the graph of p by*

$$G(p) = \{(\omega,p(\omega)) \mid \omega \in \Omega\} .$$

Then the following 5 statements are equivalent:

(a) $\exists \{B_n\} \subseteq \Sigma$, *such that if $s_1 \neq s_2$, $s_1 \in S_0$, then $s_1 \in B_n$, $s_2 \notin B_n$ for some $n \geq 1$.*

(b) $\exists f$ *a measurable map from (S,Σ) into $[0,1]$, such that $f(s_1) \neq f(s_2)$ if $s_1 \in S_0$ and $s_2 \neq s_1$.*

(c) $\exists C \subseteq (S \setminus S_0) \times (S \setminus S_0)$, *such that $\Delta_{S_0} \cup C \in \Sigma \times \Sigma$, where $\Delta_{S_0} = \{(s,s) \mid s \in S_0\}$.*

(d) $G(p) \in \mathfrak{A} \times \Sigma$.

(e) $\exists$ *a sub σ-algebra Σ_0 of Σ, such that Σ_0 is countably generated and $\{s\} \in \Sigma_0 \ \forall s \in S_0$.*

REMARKS. (i) From (b) it follows that the cardinal of S_0 is at most that of the continuum, if $G(p) \in \mathfrak{A} \times \Sigma$.

(ii) If $\Delta_S = \{(s,s) \mid s \in S\} \in \Sigma \times \Sigma$, then (c) and so (a), (b), (d) and (e) holds.

PROOF OF THEOREM 3.

(a) $\Rightarrow$ (b): Let

$$f(s) = \sum_{n=1}^{\infty} 10^{-n} 1_{B_n}(s), \quad s \in S .$$

Then f satisfies the hypothesis of (b).

(b) $\Rightarrow$ (c): Let Δ be the diagonal of the unit square: $[0,1] \times [0,1]$, and put

$$F(s,t) = (f(s),f(t)), \quad s \in S,\ t \in S .$$

Then $F^{-1}(\Delta)$ is of the form $\Delta_{S_0} \cup C$, for some $C \subseteq (S \setminus S_0) \times (S \setminus S_0)$.

(c) $\Rightarrow$ (d): Let

$$q(\omega,s) = (s,p(\omega)), \quad s \in S,\ \omega \in \Omega .$$

Then $q^{-1}(C \cup \Delta_{S_0}) = G(p)$, if C is the set in (c).

(d) $\Rightarrow$ (e): Since $\mathfrak{A} \times \Sigma$ is generated by the sets $A \times B$, with $A \in \mathfrak{A}$ and $B \in \Sigma$, we can find $\{A_n\} \subseteq \mathfrak{A}$ and $\{B_n\} \subseteq \Sigma$, such that

$$G(p) \in \sigma\{A_n \times B_n\} \subseteq \mathfrak{A} \times \sigma\{B_n\} ,$$

where $\sigma\{\mathscr{F}\}$ denotes the least σ-algebra containing $\mathscr{F}$, when $\mathscr{F}$ is a family of subsets of a given set. Put $\Sigma_0 = \sigma\{B_n\}$. Then Σ_0 is countably generated and

$$G(p)''(\omega) = \{p(\omega)\} \in \Sigma_0 \quad \forall\, \omega \in \Omega$$

which proves (e).

(e) $\Rightarrow$ (a): Let $\{B_n\}$ be a countable algebra generating Σ_0. Then it is easily seen that, since $\{s\} \in \sigma\{B_n\}\ \forall\, s \in S_0$, $\{B_n\}$ has the required property in (a).

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