Existence of (Dirac-)harmonic Maps from Degenerating (Spin) Surfaces

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Abstract

We study the existence of harmonic maps and Dirac-harmonic maps from degenerating surfaces to a nonpositive curved manifold via the scheme of Sacks and Uhlenbeck. By choosing a suitable sequence of α-(Dirac-)harmonic maps from a sequence of suitable closed surfaces degenerating to a hyperbolic surface, we get the convergence and a cleaner energy identity under the uniformly bounded energy assumption. In this energy identity, there is no energy loss near the punctures. As an application, we obtain an existence result about (Dirac-)harmonic maps from degenerating (spin) surfaces. If the energies of the map parts also stay away from zero, which is a necessary condition, both the limiting harmonic map and Dirac-harmonic map are nontrivial.

Keywords Harmonic map · Dirac-harmonic map · α-harmonic map · α-Dirac-harmonic map · Degenerating surfaces · Energy identity · Existence

Mathematics Subject Classification 53C43 · 58E20

1 Introduction

The fundamental paper [15] by Sacks and Uhlenbeck approached the theory of harmonic maps from a Riemann surface M into a Riemannian manifold N, that is critical points $u : M \to N$ of the energy functional

$$E(u) = \int_M |du|^2 dvol_g, \tag{1.1}$$

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by perturbing the energy functional E to the functional

$$E_\alpha(u) = \int_M (1 + |du|^2)^\alpha dvol_g, \tag{1.2}$$

for $\alpha > 1$ whose critical points are called α-harmonic maps. In contrast to (1.1), (1.2) satisfies a Palais-Smale condition so that standard methods apply. The difficult part then consists in controlling the limit $\alpha \to 1$. By studying this limit behavior of a sequence of α-harmonic maps as $\alpha \searrow 1$, they obtained the existence of harmonic maps and insight into the formation of bubbles.

Motivated by the supersymmetric nonlinear sigma model from quantum field theory, see [7], Dirac-harmonic maps from spin Riemann surfaces into Riemannian manifolds were introduced in [4]. They are generalizations of the classical harmonic maps and harmonic spinors. From the variational point of view, they are critical points of a conformal invariant action functional whose Euler–Lagrange equation is a coupled elliptic system consisting of a second-order equation and a Dirac equation. Being generalizations of harmonic maps, it looks natural to extend the scheme of (1.2) to them. However, new difficulties arise.

In fact, it turns out that the existence of Dirac-harmonic maps from closed surfaces is a very difficult problem because the kernel of the Dirac operator is a linear space. Moreover, different from the Dirichlet problem, even if there is no bubble, the non-triviality of the limit is also an issue. Here, a solution is considered trivial if the spinor part ψ vanishes identically. So far, there are only a few results in this direction. See [1] and [5] for uncoupled Dirac-harmonic maps (here uncoupled means that the map part is harmonic) based on index theory and the Riemann-Roch theorem, respectively. The problem has also been approached by linking theory, see [10]. Recently, we are succeed in get an existence result by the heat flow method [11]. Precisely, we use α-Dirac-harmonic map flow starting from an initial map with nontrivial α-genus to get nontrivial α-Dirac-harmonic maps. Then a nontrivial Dirac-harmonic map comes from a sequence of nontrivial α-Dirac-harmonic maps by blow-up analysis. This procedure can be viewed as an extension of Sacks–Uhlenbeck scheme to Dirac-harmonic maps.

Given these existence results about (Dirac-)harmonic maps on closed surfaces, it is natural to consider the compactness. When the domain surfaces are fixed, the compactness problem is well understood, see [15] for harmonic maps and [3,16] for Dirac-harmonic maps. When the domain surfaces vary, it is necessary to consider the degeneration of conformal structures on a Riemann surface. Topologically, the limit surface is obtained by collapsing finitely many simple closed geodesics. There are two types of collapsing curves: one is homotopically trivial, which corresponds to the bubbling near isolated singularities, and the other one is homotopically nontrivial, which corresponds to the degeneration of complex structure. By the bubbling procedure in [17,18], two generalized energy identities were proved. These identities tell us that energy may get lost from the formation of some necks that fail to converge to points. Therefore, given a sequence of harmonic maps with bounded energy, we cannot assure the nontriviality of the limit harmonic maps from degenerating surfaces even if the energy uniformly stays away from zero. For the limit Dirac-harmonic map, the situa-

tion is worse. We cannot give a geometric condition to assure the nontriviality of the spinor because, so far, there is no condition to get rid of nontrivial Dirac-harmonic spheres.

In this paper, motivated by the existence of nontrivial α-(Dirac-)harmonic maps from closed (spin) surfaces, we consider the (Dirac-)harmonic maps on degenerating surfaces. We first study the compactness of a sequence of α-Dirac-harmonic maps from closed hyperbolic surfaces degenerating to a hyperbolic Riemann surface and get a nice energy identity. In this new energy identity, there is no energy loss on the degenerating cylinder, which turns out to be very important in proving the nontriviality of limit (Dirac-)harmonic maps.

Theorem 1.1 (Compactness and energy identity) *Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface $(M, h, c, \mathfrak{S})$ by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. We choose a sequence of constants, $\{\alpha_n \in (1, 2)\}$, such that*

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{1.3}$$

For each n, suppose that (u_n, ψ_n) is an α_n-Dirac-harmonic map from $(M_n, h_n, c_n, \mathfrak{S}_n)$ into a fixed compact manifold N with nonpositive curvature and that it satisfies

$$E_{\alpha_n}(u_n) + E(\psi_n) \leq \Lambda, \tag{1.4}$$

for some positive constant Λ, where $E(\psi_n) := E(\psi_n, h_n, M_n) = \int_{M_n} |\psi|^4 dvol_{h_n}$. Moreover, we assume that the first positive eigenvalue $\lambda_1(h_n)$ of the usual Dirac operator $\eth_{h_n}$ stays away from zero, i.e.,

$$\lambda_1(h_n) \geq c_0 \tag{1.5}$$

for some positive constant $c_0 > 0$. Then there exists a Dirac-harmonic map $(u, \psi) : (M, h, c, \mathfrak{S}) \to N$ such that, after a selection of a subsequence,

$$(u_n, \psi_n) \to (u, \psi) \text{ in } C_{loc}^\infty(M) \times C_{loc}^\infty(M) \tag{1.6}$$

and

$$\lim_{n \to \infty} E(u_n, h_n, M_n) = E(u, h, M), \tag{1.7}$$

$$\lim_{n \to \infty} E(\psi_n, h_n, M_n) = E(\psi, h, M). \tag{1.8}$$

In the special case of harmonic maps, the result becomes

Theorem 1.2 *Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface (Σ, h, c) by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. We choose a sequence of constants, $\{\alpha_n \in (1, 2)\}$, such that*

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0.. \tag{1.9}$$

For each n, suppose that u_n is an α_n-harmonic map from (Σ_n, h_n, c_n) into a fixed manifold N with nonpositive curvature which satisfies

$$E_{\alpha_n}(u_n) \leq \Lambda, \tag{1.10}$$

for some positive constant Λ. Then there exists a harmonic map $u : (\Sigma, h, c) \to N$ such that, after a selection of a subsequence,

$$u_n \to u \text{ in } C_{loc}^{\infty}(\Sigma) \tag{1.11}$$

and

$$\lim_{n \to \infty} E(u_n, h_n, \Sigma_n) = E(u, h, \Sigma). \tag{1.12}$$

Actually, such compactness and energy identities are also true for $\alpha(> 1)$-(Dirac-)harmonic maps, see Theorems 4.4 and 4.3 .

Moreover, combining these results with the existence of α-(Dirac-)harmonic maps in [1,15], we get an existence result about (Dirac-)harmonic maps from degenerating (spin) surfaces. Moreover, if the energy of the map parts also stays away from zero, both the limiting harmonic map and Dirac-harmonic map are nontrivial.

Theorem 1.3 (Existence of Dirac-harmonic maps from degenerating surfaces) *Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface $(M, h, c, \mathfrak{S})$ by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that u_n is a map from M_n into a fixed manifold N with nonpositive curvature which satisfies*

$$E_{\alpha_n}(u_n) \leq \Lambda, \tag{1.13}$$

$$\operatorname{ind}_{u_n^* TN}(M_n) \neq 0, \tag{1.14}$$

where Λ is a positive constant, and $\{\alpha_n \in (1, 2)\}$ satisfies

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{1.15}$$

Moreover, we assume that the first positive eigenvalue $\lambda_1(h_n)$ of the usual Dirac operator $\displaystyle{\not}\partial_{h_n}$ stays away from zero, i.e.,

$$\lambda_1(h_n) \geq c_0 \tag{1.16}$$

for some positive constant $c_0 > 0$. Then there exists a Dirac-harmonic map (u, ψ) : $(M, h, c, \mathfrak{S}) \rightarrow N$ such that

$$\psi \neq 0. \tag{1.17}$$

In addition, let the minimizing harmonic map $u_n^{\min}$ in $[u_n]$ satisfy

$$\lim_{n \to \infty} E(u_n^{\min}) > 0. \tag{1.18}$$

Then u is not a constant.

Theorem 1.4 *Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surfaces (Σ, h, c) by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that u_n is a map from Σ_n into a fixed manifold N with nonpositive curvature and satisfies*

$$E_{\alpha_n}(u_n) \leq \Lambda, \tag{1.19}$$

where Λ is a positive constant, and $\{\alpha_n \in (1, 2)\}$ satisfies

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{1.20}$$

Then there exists a harmonic map $u_0 : (\Sigma, h, c) \rightarrow N$, which can be extended to a smooth harmonic map on $(\bar{\Sigma}, \bar{c})$.

In addition, let the minimizing harmonic map $u_n^{\min}$ in $[u_n]$ satisfy

$$\lim_{n \to \infty} E(u_n^{\min}) > 0. \tag{1.21}$$

Then u_0 is not a constant.

Here is a remark on Theorem 1.3 above. First, unlike [17], we do not make an assumption on the type of the punctures of M. Therefore, our result gives an existence result for Dirac-harmonic map on those Riemann surfaces with Ramond type punctures. Even in the Neveu-Schwarz case, in which the Dirac-harmonic map (u, ψ) can be extended to a smooth Dirac-harmonic map $(\bar{u}, \bar{\psi})$ on $(\bar{M}, \bar{c}, \bar{\mathfrak{S}})$, the existence of a Dirac-harmonic map on $(\bar{M}, \bar{c}, \bar{\mathfrak{S}})$ does not directly follow from the result in [1] because we do not know whether $\mathrm{ind}_{\bar{u}^*TN}(\bar{M}, \bar{\mathfrak{S}}) \neq 0$. Second, the uniformly

bounded energy assumption (1.13) is possible. One can take a simple example where u_n is constant in a fixed neighborhood of the degenerating curves. Third, the index assumption (1.14) is used to assure the nontriviality of the spinor, i.e., (1.17). In [1], the authors discussed some situations to realize the assumption (1.14). See for example Corollary 10.3 and Theorem 10.6 in [1]. Last, if the limit surface M has a discrete Dirac spectrum, then Pfäffle [14] proved that the usual Dirac operators $\partial\!\!\!/_{h_n}$ are (Υ, ϵ)-spectral close to $\partial\!\!\!/_h$. In this case, if we also assume that the dimension of the kernel of $\partial\!\!\!/_{h_n}$ converges to the that of $\partial\!\!\!/_h$, then the assumption (1.16) follows. This leads to the question when the limit surface M has a discrete spectrum. An answer was provided by Bär in [2].

Theorem 1.5 [2] *Let M be a hyperbolic manifold of finite volume equipped with a spin structure. If the spin structure is nontrivial along all cusps, i.e., all the punctures of M are of Neveu-Schwarz type, then the spectrum is discrete. In particular, if M is a 2- or 3-manifold and has only one cusp, then the spectrum is always discrete.*

Here the cusps of M are approximated by degenerating tubes around short closed geodesics in M_n of length l_n^j, $j \in J$, where $|J|$ is the number of cusps.

The finite volume assumption follows from the assumption (1.13) (see the proof of Theorem 4.1). Combining all these facts, we have the following corollary.

Corollary 1.6 *Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface $(M, h, c, \mathfrak{S})$ with only Neveu-Schwarz type punctures by collapsing finitely many pairwisely disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. We assume*

(i) *the dimension of the kernel of $\partial\!\!\!/_{h_n}$ converges to that of $\partial\!\!\!/_h$,*

(ii) *$\{u_n\}$ is a sequence of maps from M_n into a closed even-dimensional orientable nonpositive curved manifold N with nontrivial pull-back bundle $u_n^* T N \to M_n$ and uniformly bounded α_n-energy for α_n satisfying (1.15).*

Then there exists a smooth Dirac-harmonic map $(u, \psi) : (M, h, c, \mathfrak{S}) \to N$, which can be extended to a smooth Dirac-harmonic map $(\bar{u}, \bar{\psi})$ on $(\bar{M}, \bar{c}, \tilde{\mathfrak{S}})$, such that

$$\bar{\psi}, \psi \neq 0. \tag{1.22}$$

In addition, let the minimizing harmonic map $u_n^{\min}$ in $[u_n]$ satisfy

$$\lim_{n \to \infty} E(u_n^{\min}) > 0. \tag{1.23}$$

Then both $\bar{u}$ and u are nontrivial.

Remark 1.7 When the target manifold N is odd-dimensional, one can choose spin structures $\mathfrak{S}_n$ on M_n and $\mathfrak{S}$ on M such that the conclusion in the corollary above still holds.

The rest of paper is organized as follows: In Sect. 2, we collect some well-known facts about the hyperbolic Riemann surface theory and some lemmas about α-Dirac-harmonic maps. In Sect. 3, we show the generalized energy identity for a sequence of

α-Dirac-harmonic maps from nondegenerating spin surfaces. In Sect. 4, we will prove all the compactness and existence results stated in this introduction.

2 Preliminaries

In order to study the (Dirac-)harmonic maps from degenerating (spin) surfaces, we recall some basic knowledge about degenerating surfaces and refer to [17,18] for more details.

2.1 Degenerating (Spin) Surfaces

A Riemann surface is an orientable surface with a complex structure. A hyperbolic surface is an orientable surface with a complete Riemannian metric of constant curvature -1 having finite area. The topological type of a surface is determined by its signature (g, k), where k is the number of punctures and g is the genus of the surface obtained by adding a point at each puncture. The type (g, k) is called general if $2g + k > 2$. Two surfaces of type (g, k) are called equivalent if there exists a conformal diffeomorphism between them preserving the punctures (if there are any). The space of equivalence classes is called the moduli space $\mathcal{M}_{g,k}$ of Riemann surfaces of type (g, k).

Now we consider closed Riemann surfaces of genus $g > 1$. Any such surface is of general type and it acquires a complete hyperbolic metric. Therefore, they all stay in $\mathcal{M}_g$. This moduli space is noncompact because the conformal structure can degenerate. The only process by which the conformal structure can degenerate is the shrinking of simple closed geodesics. Hence, when we vary these in $\mathcal{M}_g$, they can degenerate into a surface with punctures. Generally, a sequence of surfaces of the same type can degenerate into another surface of different type. This inspires us to study the existence of (Dirac-)harmonic maps from degenerating surfaces based on the known existence results on closed surfaces.

The natural way to compactify $\mathcal{M}_g$ is to allow the lengths of the geodesics to become zero and thus admit surfaces with nodes as singularities. Topologically, one cuts a given surface along a collection of finitely many homotopically independent pairwise disjoint simple closed curves and pinches the cut curves to points. More precisely, let Σ_0 be a topological model surface and $\Gamma^J = \{\gamma^j, j \in J\}$ a possibly empty collection of finitely many pairwise disjoint, homotopically nontrivial, simple closed curves on Σ_0. Let $\tilde{\Sigma}$ be the surface obtained from Σ_0 by pinching all curves γ^j to points $\mathcal{P}^j$. Next, we remove all $\mathcal{P}^j$ from $\tilde{\Sigma}$ and place a complete hyperbolic metric h on the resulting surface $\Sigma = \tilde{\Sigma} \setminus \cup_{j \in J} \mathcal{P}^j$. For $j \in J$, we denote by $(\mathcal{P}^{j,1}, \mathcal{P}^{j,2})$ a pair of punctures on (Σ, h) corresponding to $\mathcal{P}^j$. Denote by $\bar{\Sigma}$ the surface obtained by adding a point at each puncture of Σ. Then the complex structure c on Σ that is compatible with the hyperbolic metric h extends to a complex structure $\bar{c}$ on $\bar{\Sigma}$. We call $(\tilde{\Sigma}, h, \bar{c})$ a nodal surface. $(\bar{\Sigma}, \bar{c})$ is called the normalization of $(\tilde{\Sigma}, h, \bar{c})$ or (Σ, h, c). $\bar{\Sigma}$ is of lower topological type.

Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic Riemann surfaces of genus $g > 1$. We say that (Σ_n, h_n, c_n) converges to a nodal surface $(\tilde{\Sigma}, h, \bar{c})$ or (Σ, h, c) if there

exist possibly empty collections $\Gamma_n^J = \{\gamma_n^j, j \in J\}$ of finitely many pairwise disjoint simple closed geodesics on each (Σ_n, h_n, c_n) and continuous maps $\tau_n : \Sigma_n \to \tilde{\Sigma}$ with $\tau_n(\gamma_n^j) = \mathcal{P}^j$ for $j \in J$ and each n, such that

(1) For any $j \in J$, the length $l(\gamma_n^j) = l_n^j \to 0$ as $n \to \infty$,
(2) $\tau_n : \Sigma_n \setminus \cup_{j \in J} \gamma_n^j \to \Sigma$ is a diffeomorphism for each n,
(3) $\bar{h}_n := (\tau_n)_* h_n \to h$ in C_{loc}^∞ on Σ,
(4) $\bar{c}_n := (\tau_n)_* c_n \to c$ in C_{loc}^∞ on Σ.

If $|J| > 0$, we say that (Σ_n, h_n, c_n) *degenerates to a nodal surface* $(\tilde{\Sigma}, h, \bar{c})$ or (Σ, h, c). Thus, the analysis of the degeneration of hyperbolic surfaces is reduced to the local behavior of pinched geodesics. The following collar lemma is a fundamental tool to analyze this localization.

Lemma 2.1 [18] *Let γ be a simple closed geodesic of length $l(\gamma) = l$ in a closed Riemann surface Σ of genus $g > 1$. Then there is a collar of area $\frac{2l}{\sinh(l/2)}$ around γ which is isometric to*

$$Z = \left\{ re^{iw} \in \mathbb{H} : 1 \le r \le e^l, \arctan\left(\sinh\left(\frac{l}{2}\right)\right) < w < \pi - \arctan\left(\sinh\left(\frac{l}{2}\right)\right) \right\}, \tag{2.1}$$

where γ corresponds to $\{re^{i\pi/2} \in \mathbb{H} : 1 \le r \le e^l\}$, and the lines $\{r = 1\}, \{r = e^l\}$ are identified via $z \to e^l z$.

Topologically, this collar neighborhood is a cylinder. Under the conformal transformation

$$re^{iw} \to (t, \theta) = \left(\frac{2\pi}{l} w, \frac{2\pi}{l} \log r \right), \tag{2.2}$$

the collar Z is isometric to following cylinder

$$P = \left\{ (t, \theta) : \frac{2\pi}{l} \arctan\left(\sinh\left(\frac{l}{2}\right)\right) < t < \frac{2\pi}{l}\left(\pi - \arctan\left(\sinh\left(\frac{l}{2}\right)\right)\right), \right.$$
$$\left. 0 \le \theta \le 2\pi \right\} \tag{2.3}$$

with metric

$$ds^2 = \left(\frac{l}{2\pi \sin \frac{lt}{2\pi}} \right)^2 (dt^2 + d\theta^2), \tag{2.4}$$

where $\gamma \subset Z$ corresponds to $\{t = \pi^2/l\} \subset P$, and the lines $\{\theta = 0\}, \{\theta = 2\pi\}$ are identified. In these coordinates, we have

$$\sinh(\text{inj}(t, \theta)) \sin\left(\frac{lt}{2\pi}\right) = \sinh\left(\frac{l}{2}\right), \tag{2.5}$$

where $\mathrm{inj}(t, \theta)$ is the injectivity radius at the point $(t, \theta) \in P$. Sometimes, we also denote by $\mathrm{inj}(x; h)$ the injectivity radius at the point at x with respect to the metric h.

To better understand the degeneration near the punctures, we recall the thick-thin decomposition (see [6]). Let (Σ, h) be a hyperbolic surface of type (g, k). For $0 < \delta < \mathrm{arcsinh}\, 1$, define the δ-thin part of (Σ, h) as the set of points at which the injectivity radius is less than δ, and the δ-thick part as its complement. The following results show what the components of δ-thin part of (Σ, h) look like.

Lemma 2.2 *Let Σ be a hyperbolic surface of type (g, k). Then the simple closed geodesics in Σ of length smaller then $2\,\mathrm{arcsinh}\, 1$ are pairwise disjoint and there are at most $3g - 3 + k$ of them.*

Proposition 2.3 *Let (Σ, h) be a hyperbolic surface of type (g, k) and $U \subset \Sigma$ a component of $\{z \in \Sigma | \mathrm{inj}(z; h) < \mathrm{arcsinh}\, 1\}$. Then either*
(1) U contains a simple closed geodesic γ of length $l = l(\gamma) < 2\,\mathrm{arcsinh}\, 1$ and is isometric to

$$\left\{ re^{iw} \in \mathbb{H} : 1 \le r \le e^l, \ \arcsin\left(\sinh\left(\frac{l}{2}\right)\right) < w < \pi - \arcsin\left(\sinh\left(\frac{l}{2}\right)\right) \right\},$$

where γ corresponds to $\{re^{i\pi/2} \in \mathbb{H} : 1 \le r \le e^l\}$, and the lines $\{r = 1\}$, $\{r = e^l\}$ are identified via $z \to e^l z$.
   or
(2) the closure of U in Σ is a standard puncture and hence isometric to

$$\left\{ z = x + iy \in \mathbb{H} : 0 \le x \le 1, \ y \ge \frac{1}{2} \right\},$$

where the lines $\{x = 0\}$, $\{x = 1\}$ are identified via $z \to z + 1$.

For the degeneration of spin surfaces, we also need to consider the spin structure. Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic Riemann surfaces of genus $g > 1$ with spin structures $\mathfrak{S}_n$. We assume that (M_n, h_n, c_n) degenerates to a hyperbolic Riemann surface (M, h, c) by collapsing $|J|$ $(1 \le |J| \le 3g - 3)$ pairwise disjoint simple closed geodesics on M_n. Let $(\bar{M}, \bar{c})$ be the normalization of (M, h, c). For each n, the diffeomorphism τ_n and the spin structure $\mathfrak{S}_n$ together determine a pull-forward spin structure on M. Since there are finitely many spin structures on a surface with punctures (c.f. [17]), by taking a subsequence, we can assume that the pull-forward of $\mathfrak{S}_n$ is a fixed spin structure on M. Let us denote it by $\mathfrak{S}$. For each $j \in J$, the unit tangent vector field of γ_n^j together with the corresponding unit normal vector field forms a section of the oriented orthonormal frame bundle $P_{SO(2)}$ of M_n. The spin structure $\mathfrak{S}_n$ is called trivial along γ_n^j if this section lifts to a closed curve in $P_{\mathrm{spin}(2)}$; otherwise, it is nontrivial along γ_n^j. Therefore, $\mathfrak{S}$ is nontrivial or trivial along the pair of punctures $(\mathcal{P}^{j,1}, \mathcal{P}^{j,2})$ if and only if $\mathfrak{S}_n$ is nontrivial or trivial along the geodesics γ_n^j for all n. If the spin structure $\mathfrak{S}$ is nontrivial along all punctures on M, we say that all the punctures on $(M, \mathfrak{S})$ are of *Neveu-Schwarz* type. In this case, the spin structure

$\mathfrak{S}$ on M extends to some spin structure $\bar{\mathfrak{S}}$ on $\bar{M}$. Furthermore, by the removable singularity theorem (c.f. [17]), any smooth Dirac-harmonic map from $(M, \mathfrak{S})$ to N with finite energy extends to a smooth Dirac-harmonic map from $(\bar{M}, \bar{\mathfrak{S}})$ to N.

In the rest of this section, we recall the definition of Dirac-harmonic maps and collect some lemmas about α-Dirac-harmonic maps which will be used later.

2.2 Dirac-harmonic Maps

Let (M, g) be a compact surface with a fixed spin structure. On the spinor bundle ΣM, we denote the Hermitian inner product by $\langle \cdot, \cdot \rangle_{\Sigma M}$. For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\Sigma M)$, the Clifford multiplication is skew-adjoint in the sense that

$$\langle X \cdot \xi, \eta \rangle_{\Sigma M} = -\langle \xi, X \cdot \eta \rangle_{\Sigma M}. \tag{2.6}$$

Let ∇ be the Levi-Civita connection on (M, g). There is a connection (also denoted by ∇) on ΣM compatible with $\langle \cdot, \cdot \rangle_{\Sigma M}$. Choosing a local orthonormal basis $\{e_\beta\}_{\beta=1,2}$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\beta \cdot \nabla_\beta$, where $\beta = 1, 2$. Here and in the sequel, we use the Einstein summation convention. One can find more about spin geometry in [12].

Let u be a smooth map from M to another compact Riemannian manifold (N, h) of dimension $n \geq 2$. Let u^*TN be the pull-back bundle of TN by u and consider the twisted bundle $\Sigma M \otimes u^*TN$. On this bundle, there is a metric $\langle \cdot, \cdot \rangle_{\Sigma M \otimes u^*TN}$ induced from the metric on ΣM and u^*TN. Also, we have a connection $\tilde{\nabla}$ on this twisted bundle naturally induced from those on ΣM and u^*TN. In local coordinates $\{y^i\}_{i=1,\dots,n}$, the section ψ of $\Sigma M \otimes u^*TN$ is written as

$$\psi = \psi_i \otimes \partial_{y^i}(u),$$

where each ψ^i is a usual spinor on M. We also have the following local expression of $\tilde{\nabla}$

$$\tilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(u) + \Gamma^i_{jk}(u)\nabla u^j \psi^k \otimes \partial_{y^i}(u),$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map u is defined as

$$D\!\!\!\!/ := e_\alpha \cdot \tilde{\nabla}_{e_\alpha}\psi = \partial\!\!\!/\psi^i \otimes \partial_{y^i}(u) + \Gamma^i_{jk}(u)\nabla_{e_\alpha}u^j(e_\alpha \cdot \psi^k) \otimes \partial_{y^i}(u), \tag{2.7}$$

which is self-adjoint [8]. Sometimes, we use $D\!\!\!\!/_u$ to distinguish the Dirac operators defined on different maps. In [4], the authors introduced the functional

$$
\begin{aligned}
L(u, \psi) &:= \frac{1}{2}\int_M (|du|^2 + \langle \psi, D\!\!\!\!/\psi \rangle_{\Sigma M \otimes u^*TN}) \\
&= \frac{1}{2}\int_M h_{ij}(u)g^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta} + h_{ij}(u)\langle \psi^i, D\!\!\!\!/\psi^j \rangle_{\Sigma M}.
\end{aligned}
\tag{2.8}
$$

They computed the Euler–Lagrange equations of L:

$$\tau^m(u) - \frac{1}{2} R^m_{lij} \langle \psi^i, \nabla u^l \cdot \psi^j \rangle_{\Sigma M} = 0, \tag{2.9}$$

$$\slashed{D}\psi^i = \slashed{\partial}\psi^i + \Gamma^i_{jk}(u)\nabla_{e_\alpha} u^j (e_\alpha \cdot \psi^k) = 0, \tag{2.10}$$

where $\tau^m(u)$ is the m-th component of the tension field [8] of the map u with respect to the coordinates on N, $\nabla u^l \cdot \psi^j$ denotes the Clifford multiplication of the vector field ∇u^l with the spinor ψ^j, and R^m_{lij} stands for the component of the Riemann curvature tensor of the target manifold N. Denote

$$\mathcal{R}(u, \psi) := \frac{1}{2} R^m_{lij} \langle \psi^i, \nabla u^l \cdot \psi^j \rangle_{\Sigma M} \partial_{y^m}.$$

We can write (2.9) and (2.10) in the following global form:

$$\begin{cases} \tau(u) = \mathcal{R}(u, \psi), & (2.11) \\ \slashed{D}\psi = 0, & (2.12) \end{cases}$$

and call the solutions (u, ψ) Dirac-harmonic maps from M to N.

By [13], we can isometrically embed N into $\mathbb{R}^q$. Then (2.11)-(2.12) is equivalent to the system

$$\begin{cases} \Delta_g u = II(du, du) + Re(P(\mathcal{S}(du(e_\beta), e_\beta \cdot \psi); \psi)), & (2.13) \\ \slashed{\partial}\psi = \mathcal{S}(du(e_\beta), e_\beta \cdot \psi), & (2.14) \end{cases}$$

where II is the second fundamental form of N in $\mathbb{R}^q$, and

$$\mathcal{S}(du(e_\beta), e_\beta \cdot \psi) := (\nabla u^A \cdot \psi^B) \otimes II(\partial_{z^A}, \partial_{z^B}), \tag{2.15}$$

$$Re(P(\mathcal{S}(du(e_\beta), e_\beta \cdot \psi); \psi)) := P(\mathcal{S}(\partial_{z^C}, \partial_{z^B}); \partial_{z^A})Re(\langle \psi^A, du^C \cdot \psi^B \rangle). \tag{2.16}$$

Here $A, B, C = 1, \ldots, q$, $P(\xi; \cdot)$ denotes the shape operator, defined by $\langle P(\xi; X), Y \rangle = \langle A(X, Y), \xi \rangle$ for $X, Y \in \Gamma(TN)$ and $Re(z)$ denotes the real part of $z \in \mathbb{C}$.

The existence of nontrivial Dirac-harmonic maps is a natural and interesting problem. The following is known. When the surface M has boundary, the nontriviality directly follows from that of the boundary values. In [9], the authors used the heat flow for α-Dirac-harmonic maps to get the existence of α-Dirac-harmonic maps which are the critical points of the functional

$$L^\alpha(u, \psi) = \frac{1}{2} \int_M (1 + |du|^2)^\alpha + \frac{1}{2} \int_M \langle \psi, \slashed{D}^u \psi \rangle_{\Sigma M \otimes u^* TN} \tag{2.17}$$

with a fixed boundary value. To get the existence of Dirac-harmonic maps, Jost–Liu–Zhu considered the limit behavior of a sequence of α-Dirac-harmonic maps. In the

context of harmonic maps, this was the method of Sacks–Uhlenbeck in [15]. It is well known that bubbles generally arise as $\alpha \searrow 1$. We also know that bubbles come from the rescaling. Therefore, it is necessary to know how the α-Dirac-harmonic equations change under rescaling. Precisely, in isothermal coordinates around a point $x_0 \in M$, suppose the metric is given by

$$h = e^{\varphi_0(x)}((dx^1)^2 + (dx^2)^2) \tag{2.18}$$

with $\varphi_0(x_0) = 0$; we define

$$(\tilde{u}_\alpha(x), \tilde{v}_\alpha(x)) := (u_\alpha(x_0 + \lambda_\alpha x), \sqrt{\lambda_\alpha}\psi_\alpha(x_0 + \lambda_\alpha x)) \tag{2.19}$$

for a small number $\lambda_\alpha > 0$. Then $(\tilde{u}_\alpha, \tilde{v}_\alpha)$ satisfies

$$\begin{cases} \Delta_{h_\alpha}\tilde{u}_\alpha = -(\alpha - 1)\dfrac{\nabla_{h_\alpha}|\nabla_{h_\alpha}\tilde{u}_\alpha|^2 \nabla_{h_\alpha}\tilde{u}_\alpha}{\sigma_\alpha + |\nabla_{h_\alpha}\tilde{u}_\alpha|^2} + II(d\tilde{u}_\alpha, d\tilde{u}_\alpha) \\[2mm] \qquad + \dfrac{Re(P(\mathcal{S}(d\tilde{u}_\alpha(e_\beta), e_\beta \cdot \tilde{v}_\alpha); \tilde{v}_\alpha))}{\alpha(1 + \sigma_\alpha^{-1}|\nabla_{h_\alpha}\tilde{u}_\alpha|^2)^{\alpha-1}}, \end{cases} \tag{2.20}$$

$$\slashed{\partial}_{h_\alpha}\tilde{v}_\alpha = \mathcal{S}(d\tilde{u}_\alpha(e_\beta), e_\beta \cdot \tilde{v}_\alpha), \tag{2.21}$$

where $h_\alpha = e^{\varphi_0(x_0+\lambda x)}((dx^1)^2 + (dx^2)^2)$ and $\sigma_\alpha = \lambda_\alpha^2 > 0$. In order to get a useful bubbling equation, it is convenient to add another factor $\lambda_\alpha^{\alpha-1}$ in the rescaling, i.e.,

$$(u_\alpha(x), v_\alpha(x)) := (u_\alpha(x_0 + \lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_0 + \lambda_\alpha x)). \tag{2.22}$$

Then one can check that $(u_\alpha(x), v_\alpha(x))$ satisfies the system

$$\begin{cases} \Delta_{h_\alpha}u_\alpha = -(\alpha - 1)\dfrac{\nabla_{h_\alpha}|\nabla_{h_\alpha}u_\alpha|^2 \nabla_{h_\alpha}u_\alpha}{\sigma_\alpha + |\nabla_{h_\alpha}u_\alpha|^2} + II(du_\alpha, du_\alpha) \\[2mm] \qquad + \dfrac{Re(P(\mathcal{S}(du_\alpha(e_\beta), e_\beta \cdot v_\alpha); v_\alpha))}{\alpha(\sigma_\alpha + |\nabla_{h_\alpha}u_\alpha|^2)^{\alpha-1}}, \end{cases} \tag{2.23}$$

$$\slashed{\partial}_{h_\alpha}v_\alpha = \mathcal{S}(du_\alpha(e_\beta), e_\beta \cdot v_\alpha). \tag{2.24}$$

Once having these equations, it is natural to consider the small energy regularity lemma, an important ingredient in the Sacks–Uhlenbeck scheme. In the case of Dirac-harmonic maps, we need a small energy regularity lemma for both systems above.

Lemma 2.4 [9] *Let $D_1 = D_1(0) \subset \mathbb{R}^2$ be the unit disk with a family of metrics as follows:*

$$g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2), \tag{2.25}$$

where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0$, $g_\alpha \to g_0$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Suppose that $(u_\alpha, \psi_\alpha) : (D_1, g_\alpha) \to N$ satisfies the system (2.20)–(2.21) *or* (2.23)–(2.24) *with*

$$0 < \beta_0 < \liminf_{\alpha \searrow 1} \sigma_\alpha^{\alpha-1} \le 1 \qquad (2.26)$$

for some $\beta_0 > 0$. For any $1 < p < \infty$, there exist two positive constants ϵ_0 and $\alpha_0 > 1$ depending only on g_0, N, such that if $E_\alpha(u_\alpha) + E(\psi_\alpha) \le \Lambda$ and

$$E(u_\alpha) \le \epsilon_0, \ 1 < \alpha \le \alpha_0, \qquad (2.27)$$

then there hold

$$\|\nabla u_\alpha\|_{W^{1,p}(D_{1/2})} \le C(p, g_0, \Lambda, N) \|\nabla u_\alpha\|_{L^2(D_1)}, \qquad (2.28)$$

$$\|\psi_\alpha\|_{W^{1,p}(D_{1/2})} \le C(p, g_0, \Lambda, N) \|\psi_\alpha\|_{L^4(D_1)}. \qquad (2.29)$$

3 α-Dirac-harmonic Maps from Nondegenerating Spin Surfaces

In this section, we will show the generalized energy identity for a sequence of α-Dirac-harmonic maps from nondegenerating spin surfaces. This will be used later at the blow-up points away from the punctures of the degenerating surfaces.

For a sequence of α-Dirac-harmonic maps from a fixed closed surface, the following generalized energy identity was proved in [9].

Theorem 3.1 [9] *Let (u_α, ψ_α) be a sequence of smooth α-Dirac-harmonic maps from a fixed closed spin surface $(M, h, \mathfrak{S})$ to a compact manifold N. If (u_α, ψ_α) satisfies the uniformly bounded energy condition*

$$E_\alpha(u_\alpha) + E(\psi_\alpha) \le \Lambda, \qquad (3.1)$$

then there exist a finite set $S = \{x_1, \ldots, x_I\}$, finitely many Dirac-harmonic spheres $(\sigma^{i,l}, \xi^{i,l}) : S^2 \to N$, $i = 1, \ldots, I$, $l = 1, \ldots, L_i$ and a Dirac-harmonic map $(u, \psi) : (M, h, \mathfrak{S}) \to N$ such that, after selection of a subsequence,

$$(u_\alpha, \psi_\alpha) \to (u, \psi) \ in \ C_{loc}^\infty(M \setminus S) \times C_{loc}^\infty(M \setminus S) \qquad (3.2)$$

and

$$\lim_{\alpha \searrow 1} E_\alpha(u_\alpha) = E(u) + |M| + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\sigma^{i,l}), \qquad (3.3)$$

$$\lim_{\alpha \searrow 1} E(\psi_\alpha) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\xi^{i,l}), \qquad (3.4)$$

where $\mu_{il} \in [1, \frac{\Lambda}{\epsilon_1}]$ and $\epsilon_1 > 0$ is the constant such that, if (σ, ξ) is a smooth Dirac-harmonic sphere satisfying $\int_{S^2} |d\sigma|^2 < \epsilon_1$, then both σ and ξ are trivial.

In addition, if $\|\psi_\alpha\|_{L^q}$ is also uniformly bounded for some $q > 4$, then bubbles are just harmonic spheres.

Theorem 3.1 directly follows from Lemma 2.4 and the following local model case of a single interior blow-up point.

Lemma 3.2 *Let $D_1 = D_1(0) \subset \mathbb{R}^2$ be the unit disk with a family of metrics as follows:*

$$g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2), \tag{3.5}$$

where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi_0$, $g_\alpha \to g_0$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Suppose that a sequence of solutions $(u_\alpha, \psi_\alpha) \in C^\infty(D_1, N)$ to the system (2.20)–(2.21) satisfies:

$$\sup_\alpha (E_{\alpha, \sigma_\alpha}(u_\alpha) + E(\psi_\alpha)) \le \Lambda, \tag{3.6}$$

$$0 < \beta_0 < \liminf_{\alpha \searrow 1} \sigma_\alpha^{\alpha-1} \le 1 \tag{3.7}$$

and

$$(u_\alpha, \psi_\alpha) \to (u, \psi) \text{ in } C^\infty_{loc}(D_1 \setminus \{0\}) \text{ as } \alpha \searrow 1. \tag{3.8}$$

Then there exist a subsequence of (u_α, ψ_α) (still denoted by (u_α, ψ_α)) and a nonnegative integer L_1 such that for any $i = 1, \ldots, L_1$, there exist sequences of positive numbers λ_α^i and nontrivial Dirac-harmonic spheres (σ^i, ξ^i) such that

$$\lim_{\delta \to 0} \lim_{\alpha \searrow 1} E_{\alpha, \sigma_\alpha}(u_\alpha, D_\delta) = \sum_{i=1}^{L_1} \mu_i^2 E(\sigma^i), \tag{3.9}$$

$$\lim_{\delta \to 0} \lim_{\alpha \searrow 1} E(\psi_\alpha, D_\delta) = \sum_{i=1}^{L_1} \mu_i^2 E(\xi^i), \tag{3.10}$$

where $\mu_i = \lim_{\alpha \searrow 1} (\lambda_\alpha^i)^{2-2\alpha}$.

It is then natural to ask what happens when the domain of α-Dirac-harmonic maps (u_α, ψ_α) varies. In this section, we consider the simplest case where (M_n, h_n, c_n) converges to a closed hyperbolic Riemann surface (M, h, c). By the pull-forward discussed in the previous section, we view $(u_n, \psi_n) := (u_{\alpha_n}, \psi_{\alpha_n})$ as a sequence of α_n-Dirac-harmonic maps defined on $(M, \bar{h}_n, \bar{c}_n, \mathfrak{S})$ with respect to $(\bar{c}_n, \nabla_n)$, where ∇_n is the connection on the spinor bundle ΣM coming from $\bar{h}_n$ and $\alpha_n \searrow 1$. Then we have the following energy identity:

Theorem 3.3 *With the notations above, we assume that (u_n, ψ_n) satisfies the uniformly bounded energy assumption*

$$E_{\alpha_n}(u_n, \bar{h}_n) + E(\psi_n, \bar{h}_n) \leq \Lambda. \tag{3.11}$$

Then there exist a finite set $S = \{x_1, \ldots, x_I\}$, finitely many Dirac-harmonic spheres $(\sigma^{i,l}, \xi^{i,l}) : S^2 \to N$, $i = 1, \ldots, I$, $l = 1, \ldots, L_i$ and a Dirac-harmonic map $(u, \psi) : (M, h, \mathfrak{S}) \to N$ such that, after selection of a subsequence,

$$(u_n, \psi_n) \to (u, \psi) \text{ in } C^\infty_{loc}(M \setminus S) \times C^\infty_{loc}(M \setminus S) \tag{3.12}$$

and

$$\lim_{n \to \infty} E_{\alpha_n}(u_n, \bar{h}_n) = E(u, h) + |M|_h + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\sigma^{i,l}), \tag{3.13}$$

$$\lim_{n \to \infty} E(\psi_n, \bar{h}_n) = E(\psi, h) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\xi^{i,l}), \tag{3.14}$$

where $\mu_{il} \in [1, \frac{\Lambda}{\epsilon_1}]$ and $\epsilon_1 > 0$ is the constant such that, if (σ, ξ) is a smooth Dirac-harmonic sphere satisfying $\int_{S^2} |d\sigma|^2 < \epsilon_1$, then both σ and ξ are trivial.

Proof Since $(\bar{c}_n, \nabla_n) \to (h, c)$ in $C^\infty(M)$ as $n \to \infty$, all the geometric data converges in $C^\infty(M)$. In particular, $\nabla_n - \nabla \to 0$ in $C^\infty(M)$, where ∇ is the connection on ΣM coming from h. Therefore, by the uniformly bounded energy assumption, we can assume (u_n, ψ_n) weakly converges to some (u, ψ) in $W^{1,2}(M, N) \times L^4(\Sigma M \otimes \mathbb{R}^K)$ with respect to (c, ∇), where we have isometrically embedded N into $\mathbb{R}^K$. Note that all the constants in the small energy regularity lemma (Lemma 2.4) and the local version of generalized energy identity (Lemma 3.2) are uniform with respect to $\bar{h}_n$ and $\bar{c}_n$. Hence, we have the generalized energy identity. $\square$

Moreover, it follows from the following lemma that $\|\psi_\alpha\|_{L^q}$ is also uniformly bounded for some $q > 4$. Therefore, the bubbles are just harmonic spheres.

Lemma 3.4 *Let M be a compact spin Riemann surface with boundary ∂M, N be a compact Riemann manifold. Let $u \in W^{1,2\alpha}(M, N)$ for some $\alpha > 1$ and $\psi \in W^{1,p}(M, \Sigma M \otimes u^*TN)$ for $1 < p < 2$, then there exists a positive constant $C = C(p, M, N, \|\nabla u\|_{L^{2\alpha}})$ such that*

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|\not{D}\psi\|_{L^p(M)} + \|\psi\|_{L^p(M)}). \tag{3.15}$$

Proof Applying the following lemma to $\eta\psi$ for some cut-off η, we complete the proof. $\square$

Lemma 3.5 [9] *Let M be a compact spin Riemann surface with boundary ∂M, N be a compact Riemann manifold. Let $u \in W^{1,2\alpha}(M, N)$ for some $\alpha > 1$ and $\psi \in$*

$W^{1,p}(M, \Sigma M \otimes u^* TN)$ *for* $1 < p < 2$, *then there exists a positive constant* $C = C(p, M, N, \|\nabla u\|_{L^{2\alpha}})$ *such that*

$$\|\psi\|_{W^{1,p}(M)} \leq C(\|\not{D}\psi\|_{L^p(M)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(\partial M)}), \tag{3.16}$$

where $\mathcal{B}$ *is the Chiral boundary operator for spinors along a map.*

4 Dirac-harmonic Maps from Degenerating Spin Surfaces

We divide this section into two parts. In the first part, we discuss the compactness of a sequence of α-(Dirac-)harmonic maps from closed Riemann (spin) surfaces degenerating to a hyperbolic Riemann (spin) surface. In the second part, based on the compactness result in the first part, we prove an existence result about (Dirac-)harmonic maps from degenerating surfaces.

4.1 Compactness and Energy Identity

The following theorem is the main result of this subsection.

Theorem 4.1 *Let* $(M_n, h_n, c_n, \mathfrak{S}_n)$ *be a sequence of closed hyperbolic surfaces of genus* $g > 1$ *degenerating to a hyperbolic Riemann surface* $(M, h, c, \mathfrak{S})$ *by collapsing finitely many pairwise disjoint simple closed geodesics* $\{\gamma_n^j, j \in J\}$. *Denote by* l_n^j *the length of* γ_n^j *and* $l_n := \max_{j \in J}\{l_n^j\}$. *We choose a sequence of constants,* $\{\alpha_n \in (1, 2)\}$, *such that*

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{4.1}$$

For each n, suppose that (u_n, ψ_n) *is an* α_n-*Dirac-harmonic map from* $(M_n, h_n, c_n, \mathfrak{S})$ *into a fixed compact manifold* N *with nonpositive curvature which satisfies*

$$E_{\alpha_n}(u_n) + E(\psi_n) \leq \Lambda, \tag{4.2}$$

for some positive constant Λ. *Moreover, we assume that the first positive eigenvalue* $\lambda_1(h_n)$ *of the usual Dirac operator* $\not{\partial}_{h_n}$ *stays away from zero, i.e.,*

$$\lambda_1(h_n) \geq c_0 \tag{4.3}$$

for some positive constant $c_0 > 0$. *Then there exists a Dirac-harmonic map* $(u, \psi) : (M, h, c, \mathfrak{S}) \to N$ *such that, after selection of a subsequence,*

$$(u_n, \psi_n) \to (u, \psi) \text{ in } C^\infty_{loc}(M) \times C^\infty_{loc}(M) \tag{4.4}$$

and

$$\lim_{n\to\infty} E(u_n, h_n, M_n) = E(u, h, M), \tag{4.5}$$

$$\lim_{n\to\infty} E(\psi_n, h_n, M_n) = E(\psi, h, M). \tag{4.6}$$

Proof We first consider the case $|J| = 1$. For $0 < \delta < \operatorname{arcsinh} 1$, let

$$M^\delta = \{x \in M, \operatorname{inj}(z; h) \geq \delta\} \tag{4.7}$$

be the δ-thick part of the hyperbolic surface (M, h). As explained in Sect. 2, there are diffeomorphisms $\tau_n : M_n \setminus \gamma_n \to M$ such that $((\tau_n)_* h_n, (\tau_n)_* c_n) \to (h, c)$ in $C^\infty_{loc}(M)$. We set

$$\bar{u}_n := (\tau_n)_* u_n, \ \bar{v}_n := (\tau_n)_* \psi_n, \ \bar{h}_n := (\tau_n)_* h_n, \ \bar{c}_n := (\tau_n)_* c_n \tag{4.8}$$

and consider the following sequence of α_n-Dirac-harmonic maps

$$(\bar{u}_n, \bar{v}_n) : (\Sigma, \bar{h}_n, \bar{c}_n, \mathfrak{S}) \to N. \tag{4.9}$$

Then, for each fixed $\delta > 0$, we have

$$(\bar{h}_n, \bar{c}_n) \to (h, c) \text{ in } C^\infty(M^\delta). \tag{4.10}$$

We choose a sequence $\delta_n \searrow 0$ such that M^{δ_n} exhaust M. By Theorem 3.3 and a standard diagonal argument, there exists a Dirac-harmonic map $(u, \psi) : (M, h, c) \to N$ such that the following hold

$$\lim_{n\to\infty} E_{\alpha_n}(\bar{u}_n, \bar{h}_n, M^{\delta_n}) = E(u, h, M) + |M|_h + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\sigma^{i,l}), \tag{4.11}$$

$$\lim_{n\to\infty} E(\bar{v}_n, \bar{h}_n, M^{\delta_n}) = E(\psi, h, M) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} \mu_{il}^2 E(\xi^{i,l}). \tag{4.12}$$

Moreover, by Lemma 3.4 and our assumption on the target manifold N, we get (4.4) and

$$\lim_{n\to\infty} E(u_n, h_n, \tau_n^{-1}(M^{\delta_n})) = \lim_{n\to\infty} E(\bar{u}_n, \bar{h}_n, M^{\delta_n}) = E(u, h, M), \tag{4.13}$$

$$\lim_{n\to\infty} E(\psi_n, h_n, \tau_n^{-1}(M^{\delta_n})) = \lim_{n\to\infty} E(\bar{v}_n, \bar{h}_n, M^{\delta_n}) = E(\psi, h, M). \tag{4.14}$$

To recover the energy concentration at the punctures $(\mathcal{P}^1, \mathcal{P}^2)$, we need to study $(\bar{u}_n, \bar{v}_n)$ on $M \setminus M^{\delta_n}$, or equivalently (u_n, ψ_n) on $M_n \setminus \tau_n^{-1}(M^{\delta_n})$. For each n, δ, $M_n \setminus \tau_n^{-1}(M^\delta)$ is not the δ-thin part of (M_n, h_n). However, for fixed $\delta > 0$ and sufficiently large n, $M_n \setminus \tau_n^{-1}(M^\delta)$ is almost the δ-thin part of (M_n, h_n).

To see this, fix $\delta > 0$ small and let $x \in M$ be a point with $\mathrm{inj}(x; h) = \delta$. Since $\bar{h}_n \to h$ in C^∞_{loc} on M, for any $\delta_1, \delta_2 > 0$ such that $\delta_1 < \delta < \delta_2$, we have

$$\delta_1 < \mathrm{inj}(x; \bar{h}_n) < \delta_2 \text{ for large } n. \tag{4.15}$$

Recall that for $0 < \delta < \mathrm{arcsinh}\, 1$, the δ-thin part of a hyperbolic surface is either an annulus or a cusp. For $n \geq 1$ and $\delta \in [\frac{l_n}{2}, \mathrm{arcsinh}\, 1]$, let us see what the δ-thin part of (M_n, h_n) looks like. Recall that P_n is the cylinder collar about γ_n. Now, we define the following δ-subcollar of P_n

$$P_n^\delta := [T_n^{1,\delta}, T_n^{2,\delta}] \times S^1 \subset P_n, \tag{4.16}$$

where

$$T_n^{1,\delta} = \frac{2\pi}{l_n} \arcsin\left(\frac{\sinh(\frac{l_n}{2})}{\sinh \delta}\right), \quad T_n^{2,\delta} = \frac{2\pi^2}{l_n} - \frac{2\pi}{l_n} \arcsin\left(\frac{\sinh(\frac{l_n}{2})}{\sinh \delta}\right). \tag{4.17}$$

By (2.5), P_n^δ is exactly the δ-thin part of (M_n, h_n), namely

$$P_n^\delta = \{x \in M_n : \mathrm{inj}(x; h_n) \leq \delta\}. \tag{4.18}$$

Thus, it follows from (4.15) and (4.18) that

$$P_n^{\delta_1} \subset M_n \setminus \tau_n^{-1}(M^\delta) \subset P_n^{\delta_2} \text{ for all } n \text{ large enough.} \tag{4.19}$$

If we choose δ_1, δ_2 in (4.19) sufficiently close to δ, then for n large enough, $M_n \setminus \tau_n^{-1}(M^\delta)$ is almost the δ-thin part P_n^δ of (M_n, h_n).

Now, for $\delta > 0$ small and n large enough, we define

$$\Omega_n^\delta := \{(M_n \setminus \tau_n^{-1}(M^\delta)) \setminus P_n^\delta\} \cup \{P_n^\delta \setminus (M_n \setminus \tau_n^{-1}(M^\delta))\}. \tag{4.20}$$

Then the image of Ω_n^δ under τ_n is uniformly away from the punctures of M and actually converges to ∂M^δ. Therefore, we have

$$\lim_{n \to \infty} E(u_n, \Omega_n^\delta) = 0, \tag{4.21}$$

$$\lim_{n \to \infty} E(\psi_n, \Omega_n^\delta) = 0. \tag{4.22}$$

Thus, after passing to a subsequence, we conclude that

$$\lim_{n \to \infty} E(u_n, (M_n \setminus \tau_n^{-1}(M^{\delta_n})) = \lim_{n \to \infty} E(u_n, P_n^{\delta_n}), \tag{4.23}$$

$$\lim_{n \to \infty} E(\psi_n, (M_n \setminus \tau_n^{-1}(M^{\delta_n})) = \lim_{n \to \infty} E(\psi_n, P_n^{\delta_n}). \tag{4.24}$$

For the right-hand side of (4.23)–(4.24), by the Hölder inequality, we get

$$
\begin{aligned}
\lim_{n\to\infty} E(u_n, P_n^{\delta_n}) &= \lim_{n\to\infty} \int_{P_n^{\delta_n}} |du_n|^2 \\
&\leq \lim_{n\to\infty} \left(\int_{P_n^{\delta_n}} |du_n|^2 \right)^{\frac{1}{\alpha_n}} (\mathrm{Area}(P_n^{\delta_n}))^{1-\frac{1}{\alpha_n}} \qquad (4.25) \\
&\leq \Lambda \lim_{n\to\infty} (\mathrm{Area}(P_n^{\delta_n}))^{1-\frac{1}{\alpha_n}}
\end{aligned}
$$

and

$$
\lim_{n\to\infty} E(\psi_n, P_n^{\delta_n}) = \lim_{n\to\infty} \int_{P_n^{\delta_n}} |\psi_n|^4 \leq \lim_{n\to\infty} \|\psi_n\|_{L^4(M_n)}^4 (\mathrm{Area}(P_n^{\delta_n}))^{1-\frac{4}{q}} \quad (4.26)
$$

for some $q > 4$. By Lemma 3.4, we have

$$
\|\psi_n\|_{L^q(M_n)} \leq C(q, M_n, N, \Lambda). \qquad (4.27)
$$

Note that when (M_n, h_n) varies, the constant above actually depends on the lower bound of $\lambda_1(h_n)$. Therefore, by our assumption (4.3) on the first eigenvalue of $\partial\!\!\!/_{h_n}$, we have a uniform bound for $\|\psi_n\|_{L^q(M_n)}$, and (4.26) becomes

$$
\lim_{n\to\infty} E(\psi_n, P_n^{\delta_n}) \leq C_0 \lim_{n\to\infty} (\mathrm{Area}(P_n^{\delta_n}))^{1-\frac{4}{q}} \qquad (4.28)
$$

for some $C_0 > 0$.

It remains to consider $\lim_{n\to\infty} (\mathrm{Area}(P_n^{\delta_n}))^{1-\frac{1}{\alpha_n}}$. By the definition of $P_n^{\delta_n}$, the area of $P_n^{\delta_n}$ can be computed as

$$
\begin{aligned}
\mathrm{Area}(P_n^{\delta_n}) &= \int_{T_n^{1,\delta_n}}^{T_n^{2,\delta_n}} \int_0^{2\pi} \left(\frac{l_n}{2\pi \sin(\frac{l_n t}{2\pi})} \right)^2 dt d\theta \\
&= 2\pi \int_{T_n^{1,\delta_n}}^{T_n^{2,\delta_n}} \frac{l_n^2}{4\pi^2} \frac{1}{\sin^2(\frac{l_n t}{2\pi})} dt \\
&= 2\pi \int_{\frac{l_n}{2\pi} T_n^{1,\delta_n}}^{\frac{l_n}{2\pi} T_n^{2,\delta_n}} \frac{l_n^2}{4\pi^2} \frac{1}{\sin^2 s} \frac{2\pi}{l_n} ds \\
&= l_n \int_{\frac{l_n}{2\pi} T_n^{1,\delta_n}}^{\frac{l_n}{2\pi} T_n^{2,\delta_n}} \frac{1}{\sin^2 s} ds \\
&= -l_n(\cot(\pi - \arcsin\varphi) - \cot(\arcsin\varphi)) \\
&= 2l_n \sqrt{\frac{\sinh \delta_n}{\sinh \frac{l_n}{2}} - 1}
\end{aligned}
$$

$$\leq \frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}}, \tag{4.29}$$

where $s = \frac{l_n t}{2\pi}$ and $\varphi = \frac{\sinh \frac{l_n}{2}}{\sinh \delta_n}$. Thus,

$$\begin{aligned}
\lim_{n\to\infty} (\text{Area}(P_n^{\delta_n}))^{1-\frac{4}{q}} &= (\lim_{n\to\infty} \text{Area}(P_n^{\delta_n}))^{1-\frac{4}{q}} \\
&\leq (\lim_{n\to\infty} \frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}})^{1-\frac{4}{q}} \\
&= (\lim_{n\to\infty} 8\sqrt{\sinh \frac{l_n}{2}})^{1-\frac{4}{q}} = 0
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
\lim_{n\to\infty} (\text{Area}(P_n^{\delta_n}))^{1-\frac{1}{\alpha_n}} &= \lim_{n\to\infty} (\text{Area}(P_n^{\delta_n}))^{\alpha_n-1} \\
&\leq \lim_{n\to\infty} (\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}})^{\alpha_n-1} \\
&= 0,
\end{aligned} \tag{4.31}$$

where we have used the assumption (4.1) in the last equality. Plugging (4.30) and (4.31) into (4.28) and (4.25), we get

$$\lim_{n\to\infty} E(u_n, P_n^{\delta_n}) = 0 \tag{4.32}$$

and

$$\lim_{n\to\infty} E(\psi_n, P_n^{\delta_n}) = 0. \tag{4.33}$$

Last, by combining (4.13)–(4.14), (4.23)–(4.24) and (4.32)–(4.33), we have (4.5)–(4.6) in the case of $|J| = 1$. By the thick-thin decomposition of hyperbolic surfaces in Sect. 2, both the short simple closed geodesics of lengths less than 2 arcsinh 1 and the corresponding (arcsinh 1)-thin parts of the collars around them are pairwisely disjoint, Hence we can deal with the corresponding subcollars separately, and the case just studied applies. This completes the proof.

□

The preceding proof directly yields a similar theorem for harmonic maps.

Theorem 4.2 *Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface (Σ, h, c) by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length*

of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. We choose a sequence of constants, $\{\alpha_n \in (1, 2)\}$, such that

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{4.34}$$

For each n, suppose that u_n is an α_n-harmonic map from (Σ_n, h_n, c_n) into a fixed compact manifold N with nonpositive curvature which satisfies

$$E_{\alpha_n}(u_n) \leq \Lambda, \tag{4.35}$$

for some positive constant Λ. Then there exists a harmonic map $u : (\Sigma, h, c) \to N$ such that, after selection of a subsequence,

$$u_n \to u \text{ in } C_{loc}^\infty(\Sigma) \tag{4.36}$$

and

$$\lim_{n \to \infty} E(u_n, h_n, \Sigma_n) = E(u, h, \Sigma). \tag{4.37}$$

We conclude this subsection with a remark on the two theorems above. First, different from [17], we do not need to restrict the type of degeneration in Theorem 4.1. Second, it follows from our cleaner energy identity that the limit map u (or u) is nontrivial under the necessary condition $\lim_{n \to \infty} E(u_n) \neq 0$ (or $\lim_{n \to \infty} E(u_n) \neq 0$). Last, for fixed $\alpha > 1$, a sequence of α-Dirac-harmonic maps always has a convergent subsequence with no bubbles (see [10]), and one can similarly prove the following theorems for α-(Dirac-)harmonic maps from degenerating surfaces into an arbitrary compact target manifold N.

Theorem 4.3 Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface (Σ, h, c) by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that u_n is a $\alpha(> 1)$-harmonic map from (Σ_n, h_n, c_n) into a fixed compact manifold N which satisfies

$$E_\alpha(u_n) \leq \Lambda, \tag{4.38}$$

for some positive constant Λ. Then there exists an α-harmonic map $u : (\Sigma, h, c) \to N$ such that, after selection of a subsequence,

$$u_n \to u \text{ in } C_{loc}^\infty(\Sigma) \tag{4.39}$$

and

$$\lim_{n \to \infty} E(u_n, h_n, \Sigma_n) = E(u, h, \Sigma). \tag{4.40}$$

Theorem 4.4 *Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface $(M, h, c, \mathfrak{S})$ by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that (u_n, ψ_n) is a $\alpha(> 1)$-Dirac-harmonic map from $(M_n, h_n, c_n, \mathfrak{S}_n)$ into a fixed compact manifold N which satisfies*

$$E_\alpha(u_n) + E(\psi_n) \le \Lambda, \tag{4.41}$$

for some positive constant Λ. Then there exists an α-Dirac-harmonic map $(u, \psi) : (M, h, c, \mathfrak{S}) \to N$ such that, after selection of a subsequence,

$$(u_n, \psi_n) \to (u, \psi) \text{ in } C_{loc}^\infty(M) \times C_{loc}^\infty(M) \tag{4.42}$$

and

$$\lim_{n\to\infty} E(u_n, h_n M_n) = E(u, h, M), \tag{4.43}$$

$$\lim_{n\to\infty} E(\psi_n, h_n, M_n) = E(\psi, h, M). \tag{4.44}$$

4.2 Existence

Theorems 4.1 and 4.2 yield an existence result for (Dirac-)harmonic maps from degenerating surfaces if the corresponding assumptions are satisfied. In this subsection, we will realize those assumptions based on the existence results in [1] and [15].

For α-Dirac-harmonic maps from a closed surface, we have the following existence result:

Theorem 4.5 *Let M be a closed spin surface and N a compact manifold. Suppose there exists a map $u_0 \in C^{2+\mu}(M, N)$ for some $\mu \in (0, 1)$ such that*

$$\mathrm{ind}_{u_0^* TN}(M) = [\dim_{\mathbb{H}} \ker \mathcal{D}^{u_0}]_{\mathbb{Z}_2} \ne 0. \tag{4.45}$$

Then for any $\alpha \ge 1$ and any α-harmonic map u_α in the homotopy class $[u_0]$, there exists a nontrivial smooth α-Dirac-harmonic map (u_α, ψ_α) such that $\|\psi_\alpha\|_{L^2} = 1$.

The proof is similar to the one in [1] for Dirac-harmonic maps, see also [11].

Proof If $\mathcal{D}^{u_\alpha}$ has minimal kernel, that is, $\dim_{\mathbb{H}} \ker \mathcal{D}^{u_0} = 1$, then for any $\psi \in \ker \mathcal{D}^{u_\alpha}$, (u_α, ψ) is an α-Dirac-harmonic map by Proposition 8.2 in [1] (see also [11] for a proof by the heat flow). If $\mathcal{D}^{u_\alpha}$ has nonminimal kernel, we use the decomposition of the twisted spinor bundle through the $\mathbb{Z}_2$-grading $G \otimes \mathrm{id}$ (see [1]). More precisely, for any smooth variation $(u_s)_{s\in(-\epsilon,\epsilon)}$ of u_0, we split the bundle $\Sigma M \otimes u_s^* TN$ into $\Sigma M \otimes u_s^* TN = \Sigma^+ M \otimes u_s^* TN \oplus \Sigma^- M \otimes u_s^* TN$, which is orthogonal in the complex sense and parallel. Consequently, for any $\psi_0 \in \ker \mathcal{D}^{u_0}$, we have

$$(\mathcal{D}^{u_0}\psi_0^+, \psi_0^+)_{L^2} = (\mathcal{D}^{u_0}\psi_0^-, \psi_0^-)_{L^2} = 0 \tag{4.46}$$

for $\psi_0 = \psi_0^+ + \psi_0^-$, where $\psi_0^\pm = \psi_\pm \otimes u_0^* TN$ and $\psi_\pm \in \Sigma^\pm$. Therefore, $\psi_s^\pm :=$
$\psi_\pm \otimes u_s^* TN$ are smooth variations of $\psi_0^\pm$, respectively, such that

$$\frac{d}{dt}\bigg|_{t=0} (\mathcal{D}^{u_s} \psi_s^\pm, \psi_s^\pm)_{L^2} = 0. \tag{4.47}$$

By taking $u_0 = u_\alpha$ and $\psi_0 = \psi_\alpha \in \ker \mathcal{D}^{u_\alpha}$, the first variation formula of L^α implies that $(u_\alpha, \psi_\alpha^\pm)$ are α-Dirac-harmonic maps (see Corollary 5.2 in [1]). In particular, we can choose ψ_α such that $\|\psi_\alpha^+\|_{L^2} = 1$ or $\|\psi_\alpha^-\| = 1$. $\qquad\square$

Note that $\mathrm{ind}_{u_0^* TN}(M)$ is independent of the choice of the Riemannian metrics on M and N. It is also invariant in the homotopy class $[u_0]$. Combining these facts and the results in the previous subsection, we directly get the following existence results.

Theorem 4.6 *Let $(M_n, h_n, c_n, \mathfrak{S}_n)$ be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface $(M, h, c, \mathfrak{S})$ by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that u_n is a map from M_n into a fixed compact manifold N with nonpositive curvature and satisfies*

$$E_{\alpha_n}(u_n) \le \Lambda, \tag{4.48}$$
$$\mathrm{ind}_{u_n^* TN}(M_n) \ne 0, \tag{4.49}$$

where Λ is a positive constant, and $\{\alpha_n \in (1, 2)\}$ satisfies

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{4.50}$$

Moreover, we assume that the first positive eigenvalue $\lambda_1(h_n)$ of the usual Dirac operator $\mathcal{D}_{h_n}$ stays away from zero, i.e.,

$$\lambda_1(h_n) \ge c_0 \tag{4.51}$$

for some positive constant $c_0 > 0$. Then there exists a Dirac-harmonic map $(u, \psi) : (M, h, c, \mathfrak{S}) \to N$ such that

$$\psi \ne 0. \tag{4.52}$$

In addition, let the minimizing harmonic map $u_n^{\min}$ in $[u_n]$ satisfy

$$\lim_{n \to \infty} E(u_n^{\min}) > 0. \tag{4.53}$$

Then u is not a constant.

Proof By our assumptions, we get a sequence α-Dirac-harmonic maps satisfying the assumption in Theorem 4.1. Therefore, there exists a Dirac-harmonic map (u, ψ) from M to N. Combining the energy identity (4.5)–(4.6) and Theorem 4.5, we have the nontriviality of u and ψ.   □

Similarly, based on the existence of α-harmonic maps in [15], we obtain an analogous theorem for harmonic maps.

Theorem 4.7 *Let (Σ_n, h_n, c_n) be a sequence of closed hyperbolic surfaces of genus $g > 1$ degenerating to a hyperbolic Riemann surface (Σ, h, c) by collapsing finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$. Denote by l_n^j the length of γ_n^j and $l_n := \max_{j \in J}\{l_n^j\}$. For each n, suppose that u_n is a map from Σ_n into a fixed compact manifold N with nonpositive curvature which satisfies*

$$E_{\alpha_n}(u_n) \leq \Lambda, \tag{4.54}$$

where Λ is a positive constant, and $\{\alpha_n \in (1, 2)\}$ satisfies

$$\lim_{n \to \infty} \left(\frac{2l_n}{\sqrt{\sinh \frac{l_n}{2}}} \right)^{\alpha_n - 1} = 0. \tag{4.55}$$

Then there exists a harmonic map $u_0 : (\Sigma, h, c) \to N$, which can be extended to a smooth harmonic map on $(\bar{\Sigma}, \bar{c})$.

In addition, let $u_n^{\min}$ be the minimizing harmonic map in $[u_n]$ and let it satisfy

$$\lim_{n \to \infty} E(u_n^{\min}) > 0. \tag{4.56}$$

Then u_0 is not a constant.

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