Existence of Δ_λ-Cycles and Δ_λ-Paths

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ABSTRACT

A cycle C of a graph G is called a D_λ-cycle if every component of $G - V(C)$ has order less than λ. A D_λ-path is defined analogously. D_λ-cycles and D_λ-paths were introduced by Veldman. Here a cycle C of a graph G is called a Δ_λ-cycle if all vertices of G are at distance less than λ from a vertex of C. A Δ_λ-path is defined analogously. In particular, in a connected graph, a D_λ-cycle is a Δ_λ-cycle and a D_λ-path is a Δ_λ-path. Furthermore, a Δ_1-cycle is a Hamilton cycle and a Δ_1-path is a Hamilton path. Necessary conditions and sufficient conditions are derived for graphs to have a Δ_λ-cycle or Δ_λ-path. The results are analogues of theorems on D_λ-cycles and D_λ-paths. In particular, a result of Chvátal and Erdös on Hamilton cycles and Hamilton paths is generalized. A recent conjecture of Bondy and Fan is settled.

1. TERMINOLOGY

We use [2] for basic terminology and notation not introduced here, and consider simple graphs only. Let G be a graph. We will sometimes identify a trail in G with the subgraph induced by its edges. Hence a subgraph T of G is a trail if and only if T is connected and at most two vertices of T have odd degree in T. Let λ be an integer with $\lambda \geq 1$. If T is a trail in G, then $b_\lambda(T)$ denotes the number of vertices of G that are at distance less than λ from a vertex of T. Following Veldman [6], a trail T of G is defined to be a *D_λ-trail* of G if all components of $G - V(T)$ have order less than λ. Here we define T to be a *Δ_λ-trail* of G if all vertices of G are at distance less than λ from a vertex of T. Note that, in a connected graph, a D_λ-trail is a Δ_λ-trail, whereas the converse is only true in general for $\lambda = 1$. A *circuit* is a nontrivial closed trail. Graphs containing a Δ_λ-cycle (D_λ-cycle) will be called *Δ_λ-cyclic* (*D_λ-cyclic*); graphs containing a Δ_λ-path (D_λ-path, Hamilton path) will be called *Δ_λ-traceable* (*D_λ-traceable*, *traceable*). If C is a cycle of G with a fixed orientation and $v \in V(C)$, then v^-

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and v^+ denote the immediate predecessor and immediate successor of v on C, respectively. If H is an oriented path or cycle of G and u and v are vertices of H, then $u\vec{H}v$ and $v\overleftarrow{H}u$ denote, respectively, the segment of H from u to v and the reverse segment from v to u. Two vertices u and v of G are *λ-neighbors* if $d(u,v) < \lambda$. $N_\lambda(v)$ denotes the set of λ-neighbors of a vertex v of G. Note that $N_1(v) = \{v\}$ and $N_2(v) = N(v) \cup \{v\}$. We let $R_\lambda(v) = N_\lambda(v) - \{v\}$. Two vertices u and v of G are *λ-distant* if $d(u,v) \geq \lambda$, i.e., if they are not λ-neighbors. In [6] Veldman introduced $\omega_\lambda(G)$, the number of components of G of order at least λ, and $\alpha_\lambda(G)$, the maximum number of mutually disjoint connected subgraphs of order λ of G such that no edge of G joins two vertices of different subgraphs. If S is a subset of $V(G)$, then $\hat{\omega}_\lambda(G,S)$ denotes the number of components of $G - S$ that contain a vertex that is λ-distant from all vertices of S. Note that, in a connected graph G, $\hat{\omega}_\lambda(G,S) \leqq \omega_\lambda(G - S)$ and $\hat{\omega}_1(G,S) = \omega_1(G - S) = \omega(G - S)$. By $\hat{\alpha}_\lambda(G)$ we denote the maximum cardinality of a set of mutually λ-distant vertices in G. Note that, in a connected graph G, $\hat{\alpha}_{2\lambda}(G) \leq \alpha_\lambda(G)$ and $\hat{\alpha}_2(G) = \alpha_1(G) = \alpha(G)$. If P is a nontrivial path in G with origin v_1 and terminus v_2, then P is called a *λ-bridge* if all edges of P are cut edges of G and, for $i = 1,2$, the component of $G - E(P)$ containing v_i also contains a vertex u_i satisfying $d(u_i, v_{3-i}) \geq \lambda$. Note that a cut edge is a 1-bridge, and a cut edge incident with two vertices of degree at least 2 is a 2-bridge. G is *λ-bridgeless* if G contains no λ-bridge.

2. INTRODUCTION

Our concern will be the existence of Δ_λ-cycles and Δ_λ-paths in graphs. Recognizing Δ_λ-cyclic graphs is an NP-complete problem. This is easily seen, using the NP-completeness of the Hamilton cycle problem. In Sections 3 and 4 necessary conditions and sufficient conditions are derived for the existence of Δ_λ-cycles and Δ_λ-paths. There is a nice analogy with known results and proof techniques concerning the existence of D_λ-cycles and D_λ-paths. D_λ-cycles and D_λ-paths were studied in [4] and [6]; D_2-cycles and D_2-paths in [5]. The conditions for Δ_λ-cyclicity (Δ_λ-traceability) are weaker than the corresponding ones for D_λ-cyclicity (D_λ-traceability), in accordance with the fact that every D_λ-cyclic (D_λ-traceable) connected graph is Δ_λ-cyclic (Δ_λ-traceable), whereas the converse is not true in general.

3. NECESSARY CONDITIONS

The following statement is obvious:

Proposition 1. If a graph G contains a Δ_λ-circuit, then G is λ-bridgeless.

For $\lambda = 1$, Proposition 1 coincides with the statement that a graph containing a spanning circuit is 2-edge-connected.

Theorem 4.2 of [2] states that if a graph G is hamiltonian, then $\omega(G - S) \le |S|$ for every nonempty proper subset S of $V(G)$.

Veldman [6] showed that if a graph G is D_λ-cyclic, then $\omega_\lambda(G - S) \le |S|$ for every nonempty proper subset of S of $V(G)$.

Here we give a similar condition on Δ_λ-cyclic graphs.

Theorem 2. If a graph G is Δ_λ-cyclic, then, for every nonempty proper subset S of $V(G)$, $\hat{\omega}_\lambda(G, S) \le |S|$.

Proof. Let S be a nonempty proper subset of $V(G)$ and C a Δ_λ-cycle of G. Then every vertex of $G - S$ is at distance at most $\lambda - 1$ from a vertex of C in G. If $S \cap V(C) = \emptyset$, $\hat{\omega}_\lambda(G, S) \le 1 \le |S|$. Otherwise, $\hat{\omega}_\lambda(G, S) \le |S \cap V(C)| \le |S|$. ∎

Analogously, one proves a cut set theorem for Δ_λ-traceable graphs.

Theorem 3. If a graph G is Δ_λ-traceable, then, for every nonempty proper subset S of $V(G)$, $\hat{\omega}_\lambda(G, S) \le |S| + 1$.

4. SUFFICIENT CONDITIONS

Chvátal and Erdös [3] showed that a graph G with independence number $\alpha(G)$ and connectivity $\kappa(G)$ is hamiltonian if $\alpha(G) \le \kappa(G)$, while $\alpha(G) \le \kappa(G) + 1$ implies that G is traceable. Veldman [6] proved the following generalization on D_λ-cyclic graphs:

Theorem 4. (Veldman [6]). Let k and λ be positive integers such that either $k \ge 2$ or $k = 1$ and $\lambda \le 2$. If G is a k-connected graph, other than a tree (in case $k = 1$), with $\alpha_\lambda \le k$, then G is D_λ-cyclic.

Here we prove an analogue of Theorem 4. For convenience, we deal with graphs of connectivity 1 separately.

Theorem 5. Let G be a k-connected graph ($k \ge 2$). If $\hat{\alpha}_{2\lambda} \le k$, then G is Δ_λ-cyclic.

Proof. Let G be a non-Δ_λ-cyclic k-connected graph ($k \ge 2$). We will exhibit $k + 1$ mutually 2λ-distant vertices. Let C be a cycle of G such that

(1) $b_\lambda(C)$ is maximum.

Fix an orientation on C. Since G is connected and C is not a Δ_λ-cycle of G, there exists a vertex $u \in A = V(G) - V(C)$ such that

(2) $N_\lambda(u) \cap V(C) = \emptyset$.

Suppose $|V(C)| < k$ and let $xy \in E(C)$. By Menger's theorem there exists a path Q_x from u to x and a path Q_y from u to y such that Q_x and Q_y meet only at

u and no other vertex of C lies on Q_x or Q_y. The cycle C' with $E(C') = E(C) \cup E(Q_x) \cup E(Q_y) - \{xy\}$ satisfies $b_\lambda(C') > b_\lambda(C)$, contradicting (1). Hence $V(C) \geq k$. Since G is k-connected and $|V(C)| \geq k$, a variation on Menger's theorem asserts that u is connected to at least k distinct vertices of C by internally-disjoint paths. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a collection of paths with the following properties:

(3) P_i has origin u and terminus v_i on C $(i = 1, 2, \ldots, k)$.
(4) Two distinct paths of $\mathcal{P}$ have only u in common.
(5) No internal vertex of P_i is on C $(i = 1, 2, \ldots, k)$.

Furthermore, assume that C and $\mathcal{P}$ are chosen such that, subject to conditions (1)–(5),

(6) $|\bigcup_{i=1}^{k} V(P_i)|$ is minimum.

Assume that $v_1, v_2, \ldots, v_k$ occur on C in the order of their indices. From the maximality of $b_\lambda(C)$ it follows that $v_i v_{i+1} \notin E(C)$ $(i = 1, 2, \ldots, k$, indices mod $k)$; otherwise the cycle C' with $E(C') = E(C) \cup E(P_i) \cup E(P_{i+1}) - \{v_i v_{i+1}\}$ contradicts the choice of C. Define a vertex u_i on C by the following requirements:

(7) for all x in $X_i = V(v_i^+ \vec{C} u_i) \cup \{v \in A | v \in N_\lambda(w)$ for some $w \in V(v_i^+ \vec{C} u_i)\}$ there exists a vertex $y \in V(u_i \vec{C} v_i)$ with $d(x, y) < \lambda$, and
(8) $|V(v_i^+ \vec{C} u_i)|$ is maximum $(i = 1, 2, \ldots, k)$.

Since (7) is satisfied for $u_i = v_i^+$, u_i exists $(i = 1, 2, \ldots, k)$. Furthermore, (1) implies that $u_i \in V(v_i^+ \vec{C} v_{i+1}^-)$ $(i = 1, 2, \ldots, k$, indices mod $k)$; otherwise (7) is satisfied for $u_i = v_{i+1}$, and the cycle C' with $E(C') = E(C) \cup E(P_i) \cup E(P_{i+1}) - E(v_i \vec{C} v_{i+1})$ satisfies $b_\lambda(C') > b_\lambda(C)$, since all vertices of $v_i \vec{C} v_{i+1}$ and their λ-neighbors have λ-neighbors on C' and u is on C'. This contradicts (1). Now X_i contains at least one vertex x_i such that all λ-neighbors of x_i on C are in X_i $(i = 1, 2, \ldots, k)$; otherwise (7) is satisfied with u_i replaced by u_i^+, in contradiction to (8). Thus

(9) $N_\lambda(x_i) \cap V(C) \subset X_i$ $(i = 1, 2, \ldots, k)$.

Let H_i denote the component of $G[X_i]$ containing x_i. We make two more observations. Here $i, j \in \{1, 2, \ldots, k\}$.

(10) There exists no path from a vertex of H_i to a vertex of $V(P_j) - \{v_j\}$ that is internally disjoint from C and the paths of $\mathcal{P}$.

This is a consequence of (1) and (6). Suppose to the contrary that there is such a path. Without loss of generality, assume there exists a path Q from a vertex $x \in V(H_i) \cap V(C)$ to a vertex $y \in V(P_j) - \{v_j\}$ that is internally disjoint from C and the paths of $\mathcal{P}$. If $i \neq j$, consider the cycle C' with $E(C') = E(C) \cup E(Q) \cup E(P_i) \cup E(u\vec{P}_j y) - E(v_i \vec{C} x)$ (possibly $y = u$); if $i = j$, consider the cycle C' with $E(C') = E(C) \cup E(Q) \cup E(y\vec{P}_i v_i) - E(v_i \vec{C} x)$. Now $b_\lambda(C') \geq b_\lambda(C)$, since all vertices of $v_i \vec{C} x$ and their λ-neighbors have λ-neighbors on C'. If $b_\lambda(C') = b_\lambda(C)$, then $i = j$, and there exist paths $P_1', P_2', \ldots, P_k'$ with properties (3), (4), and (5) with respect to u and C', and $|\bigcup_{i=1}^{k} V(P_i')| < |\bigcup_{i=1}^{k} V(P_i)|$. Hence C' contradicts the choice of C.

(11) For $i \neq j$, there exists no path from a vertex of H_i to a vertex of H_j that is internally disjoint from C.

This is a consequence of (1) and (10). Suppose to the contrary that there is such a path. Then, by (10), this path is disjoint from the paths of $\mathcal{P}$. Without loss of generality, assume there exists a path Q from a vertex $x \in V(H_i) \cap V(C)$ to a vertex $y \in V(H_j) \cap V(C)$ that is internally disjoint from C and the paths of $\mathcal{P}$ such that $|V(v_i \vec{C} x)|$ is minimum. The choice of x implies that all vertices in $X_i \cup X_j$ have a λ-neighbor in $V(x\vec{C}v_j) \cup V(y\vec{C}v_i)$. Now the cycle C' with $E(C') = E(C) \cup E(Q) \cup E(P_i) \cup E(P_j) - (E(v_i\vec{C}x) \cup E(v_j\vec{C}y))$ contradicts the choice of C.

We complete the proof by showing that $\{u, x_1, \ldots, x_k\}$ is a set of mutually 2λ-distant vertices. If $1 \leq i < j \leq k$, then, by (9) and (11), $d(x_i, x_j) \geq (\lambda - 1) + (\lambda - 1) + 2 = 2\lambda$. For arbitrary $i \in \{1, 2, \ldots, k\}$, consider a shortest path P from u to x_i. By (10), at least one of the internal vertices of P is on C. Let x be the first vertex on P that is on C. By (10), $x \notin X_i$. By (2), $d(u, x) \geq \lambda$, and by (9), $d(x_i, x) \geq (\lambda - 1) + 1$. Hence $d(u, x_i) \geq 2\lambda$. ∎

Note that $\hat{\alpha}_{2\lambda} = 1$ for a graph G if and only if the diameter of G is at most $2\lambda - 1$. Hence the following result can be viewed as the case $k = 1$ of Theorem 5:

Theorem 6. Let G be a connected λ-bridgeless graph other than a tree. If the diameter of G is at most $2\lambda - 1$, then G is Δ_λ-cyclic.

Proof. Let G be a connected non-Δ_λ-cyclic λ-bridgeless graph other than a tree. Let C be a cycle of G such that $b_\lambda(C)$ is maximum. Fix an orientation on C. Since C is not a Δ_λ-cycle of G and since G is connected, there exists a vertex $u \in V(G) - V(C)$ such that $N_\lambda(u) \cap V(C) = \emptyset$ and $d(u, v_1) = \lambda$ for some $v_1 \in V(C)$. Define u_1 and x_1 as in the proof of Theorem 5. If $u_1 \neq v_1$, then, like in the proof of Theorem 5, one can show that $d(u, x_1) \geq 2\lambda$. Now suppose $u_1 = v_1$. Then all vertices of C and their λ-neighbors are λ-neighbors of v_1. Let P be a shortest (u, v_1)-path and let z_1 denote the immediate predecessor of v_1 on P. Now $v_1 z_1$ is a cut edge of G; otherwise there is a cycle C' of G containing v_1 and z_1. Since $d(u, z_1) = \lambda - 1$, $b_\lambda(C') > b_\lambda(C)$, a contradiction.

Since $v_1 z_1$ is not a λ-bridge of G, $N_\lambda(v_1) \subset N_\lambda(z_1)$. Let z be the vertex on P such that all edges of $z\vec{P}v_1$ are cut edges of G and $|V(z\vec{P}v_1)|$ is maximum. Since G is λ-bridgeless, $z \neq u$, implying the existence of a cycle C' containing z. Furthermore, since G is λ-bridgeless, $N_\lambda(v_1) \subset N_\lambda(z)$. This implies that $b_\lambda(C') > b_\lambda(C)$, a contradiction. ∎

Theorem 5 and Theorem 6 generalize the mentioned result of Chvátal and Erdös, and are best possible in the sense that, for any positive integers k and λ, there exist infinitely many k-connected non-Δ_λ-cyclic graphs with

$\hat{\alpha}_{2\lambda} = k + 1$ (which are λ-bridgeless and not trees). Consider, e.g., the graphs $G(k, n_{1,1}, \ldots, n_{1,k+1}, n_{2,1}, \ldots, n_{2,k+1}, \ldots, n_{\lambda,1}, \ldots, n_{\lambda,k+1})$ that are sketched in Figure 1.

They consist of the following mutually disjoint subgraphs: a subgraph $H \cong K_k$, $\lambda(k+1)$ subgraphs $H_{i,j} \cong K_{n_{i,j}}$, where $1 \le i \le \lambda$ and $1 \le j \le k+1$, and the following additional edges:

$$\{xy \mid x \in V(H); y \in \bigcup_{j=1}^{k+1} V(H_{1,j})\}$$
$$\cup \{xy \mid x \in V(H_{i,j}); y \in V(H_{i+1,j}); 1 \le i < \lambda; 1 \le j \le k+1\}.$$

If $n_{i,j} \ge k$, for $1 \le i < \lambda$ and $1 \le j \le k+1$, and $n_{\lambda,j} \ge 1$, for $1 \le j \le k+1$, then $G(k, n_{1,1}, \ldots, n_{\lambda,k+1})$ obviously is k-connected. It is not Δ_λ-cyclic by Theorem 2 (with $S = V(H)$). Obviously, $d(u, v) \ge 2\lambda$ if and only if there exist integers i and j with $1 \le i < j \le k+1$ such that $u \in V(H_{\lambda,i})$ and $v \in V(H_{\lambda,j})$. Hence $\hat{\alpha}_{2\lambda} = k + 1$ (for $k = 1$ we take $n_{1,1} \ge n_{1,2} \ge 2$ to obtain λ-bridgeless graphs other than trees).

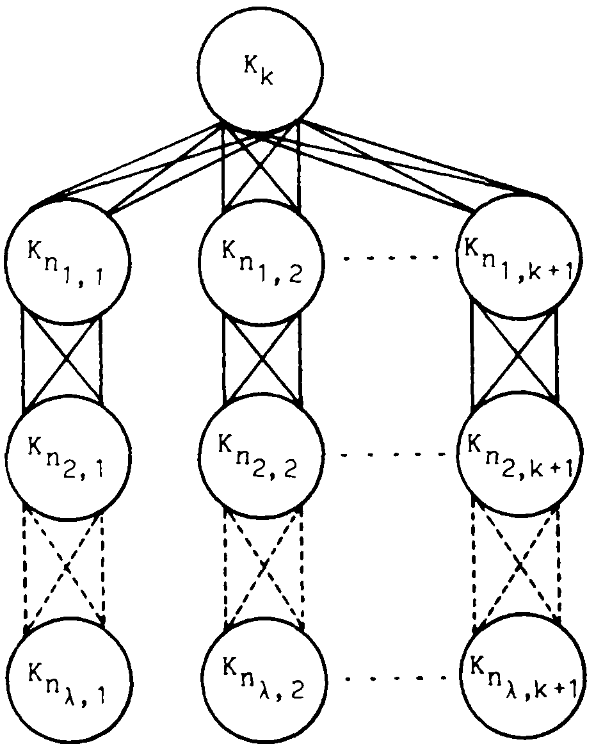

FIGURE 1

Theorem 5 and Theorem 6 have some interesting corollaries. The following one, in terms of the connectivity and independence number, is another generalization of the mentioned result of Chvátal and Erdös.

Corollary 7. Let G be a k-connected λ-bridgeless graph other than a tree ($k \geq 1$). If λ is odd and $\alpha \leq \frac{1}{2}(\lambda k + \lambda + k - 1)$, or λ is even and $\alpha \leq \frac{1}{2}(\lambda k + \lambda)$, then G is Δ_λ-cyclic.

Proof. Let G be a non-Δ_λ-cyclic k-connected λ-bridgeless graph other than a tree ($k \geq 1$). By Theorem 5 and Theorem 6 there exist $k + 1$ vertices $v_0, v_1, \ldots, v_k$ that are mutually 2λ-distant. Now, for $0 \leq i \leq k$, let $x_{i,j}$ be a vertex at distance $2j$ of v_i, with $0 \leq j \leq \lambda/2$. Since, for $0 \leq i < j \leq k$, $d(v_i, v_j) \geq 2\lambda$, the following observations are obvious:

(I) If λ is odd, then $\{x_{i,j} \mid 0 \leq i \leq k; 0 \leq j \leq (\lambda - 1)/2\}$ is a set of mutually independent vertices.
(II) If λ is even, then $\{x_{i,j} \mid 0 \leq i \leq k; 0 \leq j \leq (\lambda - 2)/2\} \cup \{x_{0,\lambda/2}\}$ is a set of mutually independent vertices.

Hence, if λ is odd, $\alpha \geq ((\lambda + 1)/2)(k + 1) = \frac{1}{2}(\lambda k + \lambda + k + 1)$, and, if λ is even, $\alpha \geq (\lambda/2)(k + 1) + 1 = \frac{1}{2}(\lambda k + \lambda) + 1$. ∎

The graphs showing that Theorem 5 and Theorem 6 are best possible also show that Corollary 7 is best possible. Before we state two other corollaries of Theorem 5 and Theorem 6, we prove the following lemma:

Lemma 8. Let G be a k-connected graph ($k \geq 1$) and let $\lambda \geq 2$.
If $\{v_0, v_1, \ldots, v_k\}$ is a set of mutually 2λ-distant vertices, then

$$\sum_{i=0}^{k} d(v_i) < \nu - 2k - (\lambda - 2)k(k + 1) \quad \text{and} \quad \sum_{i=0}^{k} |R_\lambda(v_i)| < \nu - 2k .$$

Proof. Let G be a k-connected graph ($k \geq 1$) and let $\lambda \geq 2$.
Let $\{v_0, v_1, \ldots, v_k\}$ be a set of mutually 2λ-distant vertices.
Define $V_{i,t} = \{v \in V(G) \mid d(v_i, v) = t\}$ ($0 \leq i \leq k$; $1 \leq t \leq \lambda$). Since G is k-connected, $|V_{i,t}| \geq k$ ($0 \leq i \leq k$; $1 \leq t \leq \lambda$). Since, for $0 \leq i < j \leq k$, $d(v_i, v_j) \geq 2\lambda$, the sets in the collection $\mathfrak{R} = \{V_{i,t} \mid 0 \leq i \leq k; 2 \leq t \leq \lambda - 1\} \cup \{V_{0,\lambda}\}$ are mutually disjoint.
Furthermore, no vertex in the union of the sets in $\mathfrak{R}$ is adjacent to any of the vertices $v_0, v_1, \ldots, v_k$. Hence

$$\sum_{i=0}^{k} d(v_i) \leq \nu - (k + 1) - (k + 1)(\lambda - 2)k - k = \nu - 2k - (\lambda - 2)k(k + 1) - 1 .$$

Finally, since $R_\lambda(v_i) = \bigcup_{t=1}^{\lambda-1} V_{i,t}$ and $R_\lambda(v_i) \cap R_\lambda(v_j) = \varnothing$, we get $\sum_{i=0}^{k} |R_\lambda(v_i)| \leq \nu - 2k - 1$. ∎

The next two corollaries are easily obtained by combining Lemma 8 with Theorem 5 and Theorem 6.

Corollary 9. Let G be a k-connected λ-bridgeless graph other than a tree ($k \geq 1$; $\lambda \geq 2$). If the degree-sum of any $k + 1$ mutually $(2\lambda - 1)$-distant vertices is at least $\nu - 2k - (\lambda - 2)k(k + 1)$, then G is Δ_λ-cyclic.

Corollary 10. Let G be a k-connected λ-bridgeless graph other than a tree ($k \geq 1$; $\lambda \geq 2$). If any $k + 1$ mutually $(2\lambda - 1)$-distant vertices $v_0, v_1, \ldots, v_k$ satisfy the inequality $\sum_{i=0}^{k} |R_\lambda(v_i)| \geq \nu - 2k$, then G is Δ_λ-cyclic.

The graphs $G(k, n_{1,1}, \ldots, n_{\lambda,k+1})$, with $n_{i,j} = k$, for $1 \leq i < \lambda$ and $1 \leq j \leq k + 1$, and $n_{\lambda,j} \geq 1$, for $1 \leq j \leq k + 1$, show that Corollaries 9 and 10 are best possible for $k \geq 2$. Corollaries 9 and 10 are more general than the following result of Fraisse [4]:

Corollary 11. (Fraisse [4]). Let G be a k-connected graph ($k \geq 2$). If $\delta(G) > (\nu - 2k - 1)/(k + 1)$, then G is Δ_2-cyclic.

The case $k \geq 2$ of Corollary 10 was recently conjectured by Bondy and Fan [1] and proved for $\lambda = 2$.

Without proof we mention the following analogue of Theorems 5 and 6 on Δ_λ-traceable graphs:

Theorem 12. Let G be a k-connected graph ($k \geq 1$). If $\hat{\alpha}_{2\lambda} \leq k + 1$, then G is Δ_λ-traceable.

As analogues of Corollaries 7, 9, and 10 we find, respectively, the following:

Corollary 13. Let G be a k-connected graph ($k \geq 1$). If λ is odd and $\alpha \leq \frac{1}{2}(\lambda k + 2\lambda + k)$, or λ is even and $\alpha \leq \frac{1}{2}(\lambda k + 2\lambda)$, then G is Δ_λ-traceable.

Corollary 14. Let G be a k-connected graph ($k \geq 1$) and let $\lambda \geq 2$. If the degree-sum of any $k + 2$ mutually $(2\lambda - 1)$-distant vertices is at least $\nu - 2k - 1 - (\lambda - 2)k(k + 2)$, then G is Δ_λ-traceable.

Corollary 15. Let G be a k-connected graph ($k \geq 1$) and let $\lambda \geq 2$. If any $k + 2$ mutually $(2\lambda - 1)$-distant vertices $v_0, v_1, \ldots, v_{k+1}$ satisfy the inequality $\sum_{i=0}^{k+1} |R_\lambda(v_i)| \geq \nu - 2k - 1$, then G is Δ_λ-traceable.

The proofs of the above results on Δ_λ-traceable graphs are similar to the proofs of the corresponding results on Δ_λ-cyclic graphs. Both Theorem 12 and Corollaries 13–15 can be shown to be best possible by considering the graphs sketched in Figure 1 and replacing "$k + 1$" by "$k + 2$."

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