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Existence of entire functions of one variable with prescribed indicator

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Let u be an entire function of one complex variable satisfying

$$\log|u(\zeta)| \leqslant A|\zeta|^{\varrho}+B \quad (\zeta \in \mathbf{C}) \tag{1}$$

for some constants A, B. The number ϱ is positive and fixed throughout the paper. The *indicator* of u is the function

$$p(\zeta)=\overline{\lim_{t\to+\infty}}\frac{\log|u(t\zeta)|}{t^{\varrho}} \quad (\zeta \in \mathbf{C}). \tag{2}$$

It is clear that p is positively homogeneous of order ϱ and that $p(\zeta)\leqslant A|\zeta|^{\varrho}$ if (1) holds. It also follows from standard theorems for subharmonic functions that the *regularized indicator* p^*, defined by

$$p^*(\zeta)=\overline{\lim_{\theta\to\zeta}}\, p(\theta),$$

is subharmonic. However, it is known that we always have $p^*=p$.

The purpose of this note is to provide a proof of the following theorem of V. Bernstein [1, 2] (see also Levin [5] and, for $\varrho=1$, Pólya [8]).

Theorem 1. *A function defined in the complex plane* $\mathbf{C}$ *is the regularized indicator of some entire function satisfying* (1) *for some constants* A *and* B *if* (*and only if*) *it is subharmonic and positively homogeneous of order* ϱ.

As noted above, the theorem can be improved by deleting the word "regularized". We shall not prove this here.

Formulas (1) and (2) have immediate generalizations to functions of several variables; then p^* becomes a plurisubharmonic function. In [6, 7] Martineau has proved that a function in $\mathbf{C}^n$ is the regularized indicator of some entire function satisfying (1) for some constants A and B if and only if it is plurisubharmonic and positively homogeneous of order ϱ. His proof has the form of an induction on the dimension and relies on a more precise version of the same result in one variable given in Levin [5]. It might be a justification for printing the present proof of Theorem 1 that it gives a more unified proof of the characterization of regularized indicators when combined with the induction step in [6, 7]. To be precise, the induction in such a proof could start with a function satisfying the estimate (6) below which could then be extended successively in analogy with Lemma 4 of [6]. The paper also serves to illustrate the fact that the estimates for the $\bar\partial$ operator given by Hörmander [3] are non-trivial even in one variable.

Let F be a given subharmonic function which is positively homogeneous of order ϱ. To prove Theorem 1 we shall construct an entire function u with indicator p_u satisfying

$$p_u(1)=F(1), \qquad p_u(\zeta)\leqslant F(\zeta) \quad (\zeta\in\mathbf{C}). \tag{3}$$

Let us first observe that this implies the desired result, viz. that $p_v^*=F$ for some entire function v. (Using integral transformations one can prove that $p^*=p$ so that the regularization is unnecessary.) This is proved by a category argument which has been carried through by Martineau [6, 7] (cf. also a remark in [4]). In fact, the space of all entire functions satisfying (1) for some constants A, B and with indicator $\leqslant F$ is a Fréchet space with the topology defined by the norms

$$u\mapsto \sup_{\zeta\in\mathbf{C}}|u(\zeta)|e^{-G(\zeta)},$$

where G is an arbitrary continuous function which is positively homogeneous of order ϱ and $>F$ at every point on the unit circle. It is easy to see that F is continuous (the function $\zeta\mapsto F((a\zeta)^{1/\varrho})$ is locally convex) so it suffices to take G of the form $G(\zeta)=F(\zeta)+|\zeta|^\varrho/j\,(j=1,2,\ldots)$. Let E_F be this Fréchet space. Suppose that we have found $u\in E_F$ with $p_u(\theta)=F(\theta)$ for any preassigned $\theta\in\mathbf{C}$ (it is of course enough to do this for $\theta=1$). Then E_G is meager in E_F by the Banach theorem provided $G\leqslant F$, $G\neq F$. Here G, as well as G_j and H below, are assumed to be continuous and positively homogeneous of order ϱ. Hence $\bigcup_{j=1}^\infty E_{G_j}$ is meager in E_F if $G_j\leqslant F$, $G_j\neq F\,(j=1,2,\ldots)$. But it is easy to find a sequence of functions $G_j\leqslant F$, $G_j\neq F$, such that $H\leqslant G_j$ for some j if $H\leqslant F$, $H\neq F$. Therefore all functions in E_F not in $\bigcup E_{G_j}$ must have regularized indicator p^* equal to F.

To find u satisfying (3) we shall use the following adoption to supremum norms of Theorem 4.4.2 in Hörmander [3].

Theorem 2. *Let G be a plurisubharmonic function in $\mathbf{C}^n$. For every form $f\in C^\infty_{(0,1)}(\mathbf{C}^n)$ satisfying*

$$\bar\partial f=0 \quad \textit{and} \quad |f(\zeta)|\leqslant e^{G(\zeta)},$$

there exists a function $u\in C^\infty(\mathbf{C}^n)$ with

$$\bar\partial u=f \quad \textit{and} \quad |u(\zeta)|\leqslant e^{H(\zeta)},$$

where
$$H(\zeta)=\sup_{|\theta|\leqslant 1}G(\zeta+\theta)+a\log(1+|\zeta|^2)+b.$$

Here a may be taken as an arbitrary number $>1+n/2$, and b is a constant which depends only on a and n.

As to the notation in this theorem we only mention that $C^\infty_{(0,1)}(\mathbf{C}^n)$ denotes the space of forms of type (0, 1):

$$f=\sum f_j\,d\bar z_j$$

with C^∞ coefficients f_j; $\bar\partial f$ is defined by

$$\bar\partial f=\sum\frac{\partial f_j}{\partial\bar z_k}\,d\bar z_k\wedge d\bar z_j,$$

whereas $\bar\partial u$ for $u \in C^\infty(\mathbf{C}^n)$ is given by

$$\bar\partial u = \sum \frac{\partial u}{\partial \bar z_j} d\bar z_j,$$

a form of type (0, 1). Note that $\bar\partial f = 0$ is no condition when $n=1$. For other notions we refer to Hörmander [3].

We shall take an entire function u satisfying (3) of the form

$$u = g - hv,$$

where h is the entire function

$$h(\zeta) = \prod_1^\infty (1 - 2^{-j}\zeta)$$

with zeros at 2^j ($j = 1, 2, \ldots$), and g, v are C^∞ functions to be described presently. Let $\varphi \in C_0^\infty(\mathbf{C})$ be a function which is zero when $|z| \geqslant 1$ and equal to one when $|z| \leqslant \frac{1}{2}$, $0 \leqslant \varphi \leqslant 1$. We shall define g by

$$g(\zeta) = \sum_1^\infty \varphi(\zeta - 2^j)\, e^{F(2^j)}.$$

It is then clear that $g \in C^\infty(\mathbf{C})$ and that

$$u(2^j) = g(2^j) = e^{F(2^j)}.$$

Hence, if p_u is the indicator of u, $p_u(1) \geqslant F(1)$. It remains to define $v \in C^\infty(\mathbf{C})$ so that u becomes analytic and $p_u \leqslant F$. That u is analytic means that

$$0 = \bar\partial u = \bar\partial g - h\bar\partial v,$$

i.e.

$$\bar\partial v = f,$$

where

$$f = \frac{1}{h}\bar\partial g \in C^\infty_{(0,1)}(\mathbf{C}).$$

It can easily be proved by estimating the factors $(1 - 2^{-j}\zeta)$ constituting h that for some constant C_1,

$$|h(\zeta)| \geqslant \frac{1}{C_1} > 0,$$

when

$$\tfrac{1}{2} \leqslant |\zeta - 2^j| \leqslant 1 \quad (j = 1, 2, \ldots).$$

Hence, if C_2 is chosen so large that

$$|\bar\partial \varphi| \leqslant C_2,$$

we obtain

$$|f(\zeta)| = \left|\frac{1}{h(\zeta)}\bar\partial g(\zeta)\right| \leqslant C_1 C_2 e^{F(2^j)}, \tag{4}$$

when $|\zeta - 2^j| \leqslant 1$. Define

$$G(\zeta) = \sup_{|\theta| \leqslant 1} F(\zeta + \theta).$$

It is easy to see that G is also continuous and subharmonic. We obtain from (4) that

$$|f(\zeta)| \leqslant C_1 C_2 e^{G(\zeta)}$$

for every $\zeta \in \mathbf{C}$, for either $f(\zeta)=0$ or else we can find a j such that $|\zeta - 2^j| \leqslant 1$ and use (4) for this j. We can therefore apply Theorem 2 to find a $v \in C^\infty(\mathbf{C})$ with

$$\bar{\partial} v = f \quad \text{and} \quad |v(\zeta)| \leqslant C_3 e^{H(\zeta)} \quad (\zeta \in \mathbf{C}),$$

where C_3 is a new constant and

$$H(\zeta) = \sup_{|\theta|\leqslant 1} G(\zeta+\theta) + a \log(1+|\zeta|^2) \leqslant \sup_{|\theta|\leqslant 2} F(\zeta+\theta) + a\log(1+|\zeta|^2). \qquad (5)$$

Now $u = g - hv$ is certainly analytic, and

$$|u(\zeta)| \leqslant g(\zeta) + |h(\zeta)|\,|v(\zeta)| \leqslant e^{G(\zeta)} + |h(\zeta)|\, C_3 e^{H(\zeta)}.$$

But it is well known that h is of order zero, hence for any $\varepsilon > 0$ there are constants A and C_4 such that

$$|h(\zeta)| \leqslant C_4 e^{A|\zeta|^\varepsilon}.$$

($C_4 = 1$ will do.) We finally arrive at the inequality

$$|u(\zeta)| \leqslant C_5 e^{H(\zeta) + A|\zeta|^\varepsilon}. \qquad (6)$$

It now follows in view of (5) and the continuity of F that the indicator p_u of u satisfies $p_u \leqslant F$ provided only $\varepsilon < \varrho$. The proof is complete.

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