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EXISTENCE OF MONOMORPHISMS OF PARTIAL UNARY ALGEBRAS

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1. MONOMORPHISMS

1.0. Notation. We denote by Ord the class of all ordinals and by N the set of all finite ordinals.

Let $\infty_1, \infty_2 \notin \mathrm{Ord}$. We suppose that $\alpha < \infty_1 < \infty_2$ for each $\alpha \in \mathrm{Ord}$.

If $\alpha \in \mathrm{Ord}$ then we put $W(\alpha) = \{\beta \in \mathrm{Ord};\ \beta < \alpha\}$. Further, $W(\infty_1) = N$, $W(\infty_2) = N \cup \{\infty_1\}$.

If A is a set we denote by $|A|$ the cardinal number of A. Let φ be a partial map from A into a set B. We put $\mathrm{dom}\ \varphi = \{x \in A;$ there exists $y \in B$ such that $(x, y) \in \varphi\}$. If $\mathrm{dom}\ \varphi = A$ then we write $\varphi : A \to B$. Further, if $C \subseteq A$, $D \subseteq B$ then we put $\varphi C = \{\varphi x;\ x \in C\}$, $\varphi^{-1} D = \{x \in A;\ \varphi x \in D\}$, $\varphi \mid C = \varphi \cap (C \times B)$ (the restriction of φ on C).

Let A_i be an ordered set with an order $\leqq_i$ $(i = 1, 2)$. We put $B = \bigcup_{a \in A_1} (A_1, a) \cup \bigcup_{a' \in A_2} (A_2, a')$ and define an order $\leqq$ on B in the following way: if $(t, a), (u, b) \in B$ then we put

$$(t, a) \leqq (u, b) \quad \text{iff} \quad t = u = A_1\,, \quad a \leqq_1 b \quad \text{or} \quad t = u = A_2\,, \quad a \leqq_2 b \quad \text{or} \quad t = A_1\,, \quad u = A_2\,.$$

Then we write $B = A_1 \oplus A_2$ and the ordered set B is called the *ordinal sum of* A_1 *and* A_2.

1.1. Definition. Let A, B be sets, $S_1 : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$, $S_2 : B \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ arbitrary maps. Let $F : A \to B$ be a map such that $S_1 x \leqq S_2 F x$ for each $x \in A$. Then F is called a *degree map* (abbreviation a *d-map*) with respect to S_1, S_2. Further, if F is an injection then we speak about a *d-injection*.

1.2. Theorem. *Let A, B be finite sets, $S_1 : A \to N \cup \{\infty_1, \infty_2\}$, $S_2 : B \to N \cup \{\infty_1, \infty_2\}$ maps. We put $A^i = S_1^{-1} i$, $B^i = S_2^{-1} i$ for each $i \in N \cup \{\infty_1, \infty_2\}$. Then there is a d-injection $F : A \to B$ with respect to S_1, S_2 if and only if $|A - \bigcup_{i \in W(n)} A^i| \leqq |B - \bigcup_{i \in W(n)} B^i|$ for each $n \in N \cup \{\infty_1, \infty_2\}$.*

Proof. *Necessity.* Let there be a d-injection $F : A \to B$. Let $n \in N \cup \{\infty_1, \infty_2\}$ be arbitrary. Then, for each $x \in A - \bigcup_{i \in W(n)} A^i$, $S_2 Fx \geqq S_1 x \notin W(n)$ holds and thus, $Fx \in B - \bigcup_{i \in W(n)} B^i$. Hence $F(A - \bigcup_{i \in W(n)} A^i) \subseteq B - \bigcup_{i \in W(n)} B^i$ and we obtain $|A - \bigcup_{i \in W(n)} A^i| = |F(A - \bigcup_{i \in W(n)} A^i)| \leqq |B - \bigcup_{i \in W(n)} B^i|$.

Sufficiency. Since $S_1 A$ is finite there is $n_0 \in N \cup \{\infty_1, \infty_2\}$ such that $n_0 = \max S_1 A$. Then $|A^{n_0}| = |A - \bigcup_{i \in W(n_0)} A^i| \leqq |B - \bigcup_{i \in W(n_0)} B^i|$. Thus, there is a d-injection $F_0 : A^{n_0} \to B$.

Let now $p \in N$, let $n_0 > n_1 > \ldots > n_p$ be the last $p+1$ elements in $S_1 A$ and let there be a d-injection $F_p : \bigcup_{i \in W(p+1)} A^{n_i} \to B$.

If $A = \bigcup_{i \in W(p+1)} A^{n_i}$ then the proof is complete. Therefore, let $A \neq \bigcup_{i \in W(p+1)} A^{n_i}$, i.e. $S_1 A \cap W(n_p) \neq \emptyset$. Then we put $n_{p+1} = \max (S_1 A \cap W(n_p))$. Hence $|A^{n_{p+1}}| + |\bigcup_{i \in W(p+1)} A^{n_i}| = |A - \bigcup_{i \in W(n_{p+1})} A^i| \leqq |B - \bigcup_{i \in W(n_{p+1})} B^i|$. Further, $\bigcup_{i \in W(p+1)} A^{n_i} = A - \bigcup_{i \in W(n_p)} A^i$ and since F_p is a d-map we have $F_p \bigcup_{i \in W(p+1)} A^{n_i} = F_p(A - \bigcup_{i \in W(n_p)} A^i) \subseteq B - \bigcup_{i \in W(n_p)} B^i \subseteq B - \bigcup_{i \in W(n_{p+1})} B^i$. Thus, since F_p is an injection we obtain $|A^{n_{p+1}}| \leqq |B - \bigcup_{i \in W(n_{p+1})} B^i| - |\bigcup_{i \in W(p+1)} A^{n_i}| = |B - \bigcup_{i \in W(n_{p+1})} B^i| - |F_p \bigcup_{i \in W(p+1)} A^{n_i}| \leqq |(B - \bigcup_{i \in W(n_{p+1})} B^i) - F_p \bigcup_{i \in W(p+1)} A^i| = |B - (\bigcup_{i \in W(p+1)} B^i \cup F_p \bigcup_{i \in W(p+1)} A^{n_i})|$. It follows that there is an injection $G : A^{n_{p+1}} \to B - (\bigcup_{i \in W(n_{p+1})} B^i \cup F_p \bigcup_{i \in W(p+1)} A^{n_i})$. Further, G is a d-injection and since $F_p \bigcup_{i \in W(p+1)} A^{n_i}$ and $GA^{n_{p+1}}$ are disjoint we obtain that $F_{p+1} = F_p \cup G$ is a d-injection $F_{p+1} : \bigcup_{i \in W(p+2)} A^{n_i} \to B$.

1.3. Lemma. *Let A_1, A_2 be finite sets, $S_i : A_i \to N \cup \{\infty_1, \infty_2\}$ maps $(i = 1, 2)$. We put $A_i^n = S_i^{-1} n$ for each $n \in N \cup \{\infty_1, \infty_2\}$ $(i = 1, 2)$. Let $x_i \in A_i$ $(i = 1, 2)$ be such that $S_1 x_1 = S_2 x_2$. We put $\bar{A}_i = A_i - \{x_i\}$, $\bar{A}_i^n = (S_i \mid \bar{A}_i)^{-1} n$ for each $n \in N \cup \{\infty_1, \infty_2\}$ $(i = 1, 2)$. Let $n \in N \cup \{\infty_1, \infty_2\}$ be arbitrary. If $|A_1 - \bigcup_{i \in W(n)} A_1^i| \leqq |A_2 - \bigcup_{i \in W(n)} A_2^i|$ then $|\bar{A}_1 - \bigcup_{i \in W(n)} \bar{A}_1^i| \leqq |\bar{A}_2 - \bigcup_{i \in W(n)} \bar{A}_2^i|$.*

Proof. We put $S_1 x_1 = S_2 x_2 = p$. Then

$$\bar{A}_i - \bigcup_{j \in W(n)} \bar{A}_i^j = \begin{cases} (A_i - \bigcup_{j \in W(n)} A_i^j) - \{x_i\} & \text{if} \quad n \leqq p \\ A_i - \bigcup_{j \in W(n)} A_i^j & \text{if} \quad n > p \end{cases} \quad \text{where} \quad i = 1, 2\,.$$

If $n \in N \cup \{\infty_1, \infty_2\}$, $n \leqq p$ then $|\bar{A}_1 - \bigcup_{i \in W(n)} \bar{A}_1^i| = |A_1 - \bigcup_{i \in W(n)} A_1^i| - 1 \leqq$
$\leqq |A_2 - \bigcup_{i \in W(n)} A_2^i| - 1 = |\bar{A}_2 - \bigcup_{i \in W(n)} \bar{A}_2^i|$.

If $n \in N \cup \{\infty_1, \infty_2\}$, $n > p$ then $|\bar{A}_1 - \bigcup_{i \in W(n)} \bar{A}_1^i| = |A_1 - \bigcup_{i \in W(n)} A_1^i| \leqq |A_2 -$
$- \bigcup_{i \in W(n)} A_2^i| = |\bar{A}_2 - \bigcup_{i \in W(n)} \bar{A}_2^i|$.

1.4. Definition. (a) Let A be a non-empty set, f a partial map from the set A into A. Then the ordered pair $\boldsymbol{A} = (A, f)$ is called a *partial unary algebra*.

(b) Let $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ be partial unary algebras, let $F : A \to B$ be a map. Then F is called a *homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$ if $x \in \operatorname{dom} f$ implies $Fx \in \operatorname{dom} g$ and $Ffx = gFx$ for each $x \in A$; we write $F : \boldsymbol{A} \to \boldsymbol{B}$. If F is a homomorphism and an injective map then F is called a *monomorphism*.

(c) The category of all partial unary algebras where morphisms are homomorphisms is denoted by $\mathscr{U}$.

1.5. Definition. Let $\boldsymbol{A} = (A, f) \in \mathscr{U}$.

(a) We put $D\boldsymbol{A} = A - \operatorname{dom} f$.

(b) If $A' \subseteq A$ then we put $\langle A' \rangle_{\boldsymbol{A}} = f^{-1}A'$; especially, we put $\langle x \rangle_{\boldsymbol{A}} = f^{-1}x$ for every $x \in A$.

(c) We put $f^0 = \mathrm{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \operatorname{dom} f^{n-1}$ and $f^{n-1}x \in \operatorname{dom} f$ then we put $f^n x = ff^{n-1}x$.

(d) Let $x \in A$ be arbitrary. Then we define $[x)_{\boldsymbol{A}} = \{y \in A$; there is $n \in N$ with $x \in \operatorname{dom} f^n$ and $y = f^n x\}$.

(e) Let $x, y \in A$ be arbitrary. Then we put $\varrho\boldsymbol{A} = \{(x, y) \in A^2;\ [x)_{\boldsymbol{A}} \cap [y)_{\boldsymbol{A}} \neq \emptyset\}$. If $\varrho\boldsymbol{A} = A^2$ then $\boldsymbol{A}$ is called a *connected partial unary algebra* (abbreviation a *c-algebra*). The category of all c-algebras where morphisms are homomorphism defined in 1.4(b) is denoted by $\mathscr{U}^c$.

(f) Let $\boldsymbol{A} \in \mathscr{U}^c$, $x \in A$ be arbitrary. Then we put $[x)^0_{\boldsymbol{A}} = [x)_{\boldsymbol{A}}$, $[x)^{-1}_{\boldsymbol{A}} = \langle [x)_{\boldsymbol{A}} \rangle_{\boldsymbol{A}} -$
$- [x)_{\boldsymbol{A}}$. Suppose that we have defined the set $[x)^{-n}_{\boldsymbol{A}}$ for $n \in N - \{0\}$. Then we put $[x)^{-(n+1)}_{\boldsymbol{A}} = \langle [x)^{-n}_{\boldsymbol{A}} \rangle_{\boldsymbol{A}}$.

(g) Let $\boldsymbol{A} \in \mathscr{U}^c$, $x, y \in A$ be arbitrary. Then we put $\langle y \rangle_{\boldsymbol{A},x} = \langle y \rangle_{\boldsymbol{A}} - [x)_{\boldsymbol{A}}$.

(i) Let $\boldsymbol{A} = (A, f) \in \mathscr{U}^c$, $x \in A$ be arbitrary. Then the following assertions hold:

(a) $[x)^{-(n+1)}_{\boldsymbol{A}} = \bigcup_{y \in [x)^{-n}_{\boldsymbol{A}}} \langle y \rangle_{\boldsymbol{A},x}$ with disjoint terms for each $n \in N$,

(b) $A = \bigcup_{n \in N} [x)^{-n}_{\boldsymbol{A}}$ with disjoint terms.

(See [7], 2.1 and [4], 3.9 (c).)

1.6. Definition. Let $A = (A, f) \in \mathscr{U}^c$. Then we define $ZA = \{x \in A;$ there is $n \in N - \{0\}$ such that $f^n x = x\}$, $RA = |ZA|$.

(ii) Let $A = (A, f) \in \mathscr{U}^c$. If $ZA \neq \emptyset$, $x \in A$ then there is $n \in N$ such that $f^n x \in ZA$. (See [3], 1.17 (a).)

1.7. Lemma. *Let $A = (A, f) \in \mathscr{U}^c$, $x, y \in A$ be arbitrary. We put $p_0 = \min \{n \in N; f^n x \in ZA\}$. Then the following assertions hold:*

(a) *If $y \in \bigcup_{n \in N - \{0\}} [x]_A^{-n}$ then $\langle y \rangle_A = \langle y \rangle_{A,x}$.*

(b) $\langle x \rangle_A = \begin{cases} \langle x \rangle_{A,x} & \text{if } x \notin ZA, \\ \langle x \rangle_{A,x} \cup \{f^{RA-1}x\} & \text{if } x \in ZA. \end{cases}$

(c) *If $y = f^p x$, $p \in N - \{0, p_0\}$ then $\langle y \rangle_A = \langle y \rangle_{A,x} \cup \{f^{p-1}x\}$.*

(d) *If $y = f^{p_0}x$ where $p_0 \in N - \{0\}$ then $\langle y \rangle_A = \langle y \rangle_{A,x} \cup \{f^{p_0-1}x, f^{RA+p_0-1}x\}$.*

(e) *If $x \in ZA$ then* $\langle y \rangle_A = \begin{cases} \langle y \rangle_{A,x} & \text{if } y \in \bigcup_{n \in N - \{0\}} [x]_A^{-n} \\ \langle y \rangle_{A,x} \cup \{f^{p-1}x\} & \text{if } y = f^p x,\ p \in N - \{0\}. \end{cases}$

(f) *If $x \notin ZA$ then*

$$\langle y \rangle_A = \begin{cases} \langle y \rangle_{A,x} & \text{if } y \in \{x\} \cup \bigcup_{n \in N - \{0\}} [x]_A^{-n}, \\ \langle y \rangle_{A,x} \cup \{f^{p-1}x\} & \text{if } y = f^p x,\ p \in N - \{0, p_0\}, \\ \langle y \rangle_{A,x} \cup \{f^{p_0-1}x, f^{RA+p_0-1}x\} & \text{if } y = f^{p_0}x. \end{cases}$$

Proof of (a). Let $y \in [x]_A^{-n}$ where $n \in N - \{0\}$. Then $\langle y \rangle_A \subseteq [x]_A^{-(n+1)}$ by 1.5 (f). Hence $\langle y \rangle_A \cap [x]_A = \emptyset$ by (i) (b) which implies $\langle y \rangle_{A,x} = \langle y \rangle_A - [x]_A = \langle y \rangle_A$.

Proof of (b). If $x \notin ZA$ then $\langle x \rangle_A \cap [x]_A = \emptyset$ by 1.5 (d) and 1.6 and thus $\langle x \rangle_{A,x} = \langle x \rangle_A - [x]_A = \langle x \rangle_A$.

If $x \in ZA$ then $\langle x \rangle_A \cap [x]_A = \{f^{RA-1}x\}$ by 1.5 (d) and 1.6 which implies $\langle x \rangle_{A,x} = \langle x \rangle_A - \{f^{RA-1}x\}$.

Proof of (c). Let $y = f^p x$, $p \in N - \{0, p_0\}$. Then $\langle y \rangle_A \cap [x]_A = \{f^{p-1}x\}$ and therefore, $\langle y \rangle_{A,x} = \langle y \rangle_A - \{f^{p-1}x\}$.

Proof of (d). Let $y = f^{p_0}x$ where $p_0 \in N - \{0\}$. Then $\langle y \rangle_A \cap [x]_A = \{f^{p_0-1}x, f^{RA+p_0-1}x\}$ and we have $\langle y \rangle_{A,x} = \langle y \rangle_A - \{f^{p_0-1}x, f^{RA+p_0-1}x\}$.

(e) and (f) are consequences of (a), (b), (c), (d) and (i) (b).

1.8. Definition. Let $A = (A, f) \in \mathscr{U}^c$. Then we define

(a) the set $KA = \{x \in A - ZA;$ there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f x_{i+1} = x_i$ for each $i \in N\}$;

(b) $A^{\infty_2} = ZA$, $A^{\infty_1} = KA$, $A^0 = \{x \in A; \langle x \rangle_A = \emptyset\}$; if $\alpha \in \mathrm{Ord} - \{0\}$ is arbitrary and if the sets $A^\varkappa$ have been defined for each $\varkappa \in W(\alpha)$ then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W(\alpha)} A^\varkappa; \langle x \rangle_A \subseteq \bigcup_{\varkappa \in W(\alpha)} A^\varkappa\}$;

(c) $\vartheta A = \min\{\varkappa \in \mathrm{Ord};\ A^\varkappa = \emptyset\}$;

(d) a map $SA : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $SAx = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W(\vartheta A) \cup \{\infty_1, \infty_2\}$; SAx is called the *degree of* x.

(iii) Let $A = (A, f) \in \mathscr{U}^c$. Then

(a) $(KA \cup ZA, f \mid KA \cup ZA)$, $(ZA, f \mid ZA)$ are subalgebras of A,

(b) $|DA| \leqq 1$,

(c) $DA \neq \emptyset$ iff the following conditions hold: $RA = 0$ and there is $x \in A$ such that $|[x)_A| < \aleph_0$.

(See [4], 2.10, 2.15 (a), 2.1 and 2.9.)

1.9. Definition. Let $A \in \mathscr{U}^c$.

(a) The set $DA \cup KA \cup ZA$ is called the *kernel of* A. If $DA \cup KA \cup ZA \neq \emptyset$ then we say that A has the kernel.

(b) If $DA \neq \emptyset$ then we denote by dA the only point with the property $\{dA\} = DA$.

(iv) Let $A = (A, f) \in \mathscr{U}^c$, $x \in A$ be arbitrary. Then

(a) $SAx = \infty_1$ implies $\langle x \rangle_A \cap A^{\infty_1} \neq \emptyset$,

(b) $SAx = \infty_2$ implies $\langle x \rangle_A \cap A^{\infty_2} \neq \emptyset$,

(c) $SAx = \alpha \in \mathrm{Ord}$ implies that $W(\alpha)$ is cofinal with $SA\langle x \rangle_A$,

(d) $SAx \in \mathrm{Ord}$, $x \in \mathrm{dom}\, f^n$, where $n \in N$, implies $SAf^n x \geqq SAx + n$.

(See [4], 2.17 (a), (b), 2.25 (b) and 2.26 (a).)

1.10. Lemma. *Let $A = (A, f) \in \mathscr{U}^c$, $x \in A$ be such that $SAx \in \mathrm{Ord}$. Let $\alpha \in \mathrm{Ord}$. Then the following assertions hold*:

(a) *$SAx > \alpha$ iff there is $y \in \langle x \rangle_A$ such that $SAy \geqq \alpha$.*

(b) *$\langle x \rangle_A - \bigcup_{\lambda \in W(\alpha)} A^\lambda \neq \emptyset$ iff $x \in A - \bigcup_{\lambda \in W(\alpha+1)} A^\lambda$.*

Proof of (a). Let $y \in \langle x \rangle_A$ be such that $SAy \geqq \alpha$. Then $SAx = SAy > SAy \geqq \alpha$ by (iv) (d).

Let, on the other hand, $SAx > \alpha$. Then there is $y \in \langle x \rangle_A$ such that $SAy \geqq \alpha$ by (iv) (c).

(b) is a consequence of (a).

1.11. Convention. Let $A = (A, f) \in \mathscr{U}^c$, $B = (B, g) \in \mathscr{U}^c$. If we speak about a d-map (d-injection) $F : A \to B$ we already mean a d-map (d-injection) with respect to SA, SB (see 1.1).

1.12. Definition. Let $\boldsymbol{A} = (A, f) \in \mathscr{U}^c$, $\boldsymbol{B} = (B, g) \in \mathscr{U}^c$. Then

(a) we put $H(\boldsymbol{A}, \boldsymbol{B}) = \{(x, x') \in A \times B;$ for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ and $S\boldsymbol{A}f^n x \leqq S\boldsymbol{B}g^n x'\}$,

(b) we define

$$(\boldsymbol{A}, \boldsymbol{B}) \in h\text{-Ad} \quad \text{iff} \quad (1)\ R\boldsymbol{B} \neq 0 \quad \text{implies} \quad R\boldsymbol{B} \mid R\boldsymbol{A}\ ^{*)}, \quad (2)\ R\boldsymbol{B} = 0 \quad \text{implies} \quad H(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset\,;$$

(c) we put $M_0(\boldsymbol{A}, \boldsymbol{B}) = \{(x, x') \in H(\boldsymbol{A}, \boldsymbol{B}); R\boldsymbol{B} \neq 0$ implies $x \in Z\boldsymbol{A} \cup \bigcup_{i \in W(R\boldsymbol{B})} [\mathrm{d}\boldsymbol{A})_{\boldsymbol{A}}^{-i}\}$,

(d) supposing that $D\boldsymbol{A} = \emptyset$ implies $R\boldsymbol{A} = R\boldsymbol{B}$ we define $(x, x') \in M(\boldsymbol{A}, \boldsymbol{B})$ iff the following conditions hold: $(x, x') \in M_0(\boldsymbol{A}, \boldsymbol{B})$ and if we define a map $F_0 : [x)_{\boldsymbol{A}} \to [x')_{\boldsymbol{B}}$ such that $F_0 f^n x = g^n x'$ for each $f^n x \in [x)_{\boldsymbol{A}}$ then there is a sequence of maps $\{F_n : [x)_{\boldsymbol{A}}^{-n} \to [x')_{\boldsymbol{B}}^{-n}; n \in N\}$ such that, for each $n \in N$ and each $y \in [x)_{\boldsymbol{A}}^{-n}$, $F_{n+1} \mid \langle y \rangle_{\boldsymbol{A},x}$ is a d-injection and $F_{n+1}\langle y \rangle_{\boldsymbol{A},x} \subseteq \langle F_n y \rangle_{\boldsymbol{B},x'}$;

(e) we define

$$(A, B) \in m\text{-Ad} \quad \text{iff} \quad (1)\ D\boldsymbol{A} = \emptyset \quad \text{implies} \quad R\boldsymbol{A} = R\boldsymbol{B}, \quad (2)\ M(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset\,.$$

(v) m-Ad $\subseteq h$-Ad. (See [7], 3.17.)

(vi) Let $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{U}^c$. Then there is a monomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ if and only if $(\boldsymbol{A}, \boldsymbol{B}) \in m$-Ad. (See [7], 3.21.)

2. EXISTENCE OF MONOMORPHISMS

The characteristic condition of the existence of monomorphisms contained in (vi) is rather complicated because of the definition of $M(\boldsymbol{A}, \boldsymbol{B})$ (1.12(d)). In the following special cases, it can be simplified.

2.1. Definition. We put $\mathscr{U}_0^c = \{\boldsymbol{A} = (A, f) \in \mathscr{U}^c; |\langle x \rangle_{\boldsymbol{A}}| < \aleph_0$ for each $x \in A\}$.

2.2. Lemma. *Let $\boldsymbol{A} = (A, f) \in \mathscr{U}_0^c$. Then*

(a) $\vartheta\boldsymbol{A} \leqq \omega_0$,

(b) $S\boldsymbol{A}A \subseteq N \cup \{\infty_1, \infty_2\}$.

Proof of (a). Let, on the contrary, $\vartheta\boldsymbol{A} > \omega_0$. Then there is $x \in A$ such that $S\boldsymbol{A}x = \omega_0$ by 1.8(c). Further, $W(\omega_0)$ is cofinal with $S\boldsymbol{A}\langle x \rangle_{\boldsymbol{A}}$ by (iv) (c) and thus $|\langle x \rangle_{\boldsymbol{A}}| \geqq |S\boldsymbol{A}\langle x \rangle_{\boldsymbol{A}}| = \aleph_0$ which is a contradiction to $\boldsymbol{A} \in \mathscr{U}_0^c$.

(b) follows immediately from (a).

2.3. Definition. Let $d, \bar{d} \notin (N - \{0\}) \cup \{\omega_0\}$. We put $N_1 = (N - \{0\}) \cup \{\omega_0, d, \bar{d}\}$, $N_2 = N - \{0\}$.

*) $p \mid q$ for $p, q \in N$ means that p is a divisor of q.

(a) We define a relation

(α) $\leqq_1$ on N_1 in this way: if $a, b \in N_1$ then we put $a \leqq_1 b$ iff (1) $a, b \in N - \{0\}$, $a \leqq b$ or (2) $a \in (N - \{0\}) \cup \{\omega_0\}$, $b = \omega_0$ or (3) $a \in (N - \{0\}) \cup \{d\}$, $b = d$ or (4) $b = \bar{d}$;

(β) $\leqq_2$ on N_2 in this way: if $a, b \in N_2$ then we put $a \leqq_2 b$ iff $b \mid a$.
(N_1, N_2 are ordered sets. Compare [5], 1.16, 1.17.)

(b) We put $C_0 = N_1 \oplus N_2$ (see 1.0) and we denote the order on C_0 by $\lesssim_0$. Further, if $(N_1, a), (N_2, b) \in C_0$ (where $a \in N_1$, $b \in N_2$) are arbitrary then we put (for brevity) $a = (N_1, a)$, $b = (N_2, b)$.

(c) If $a, b \in C_0$ are arbitrary then we put $a \lesssim b$ iff (1) $a \lesssim_0 b$ and (2) $b \in N_2$ implies $a \in N_1 - \{\omega_0, \bar{d}\}$ or $a = b$.

(d) Let $A \in \mathscr{U}_0^c$. Then we define

$$\chi A = \begin{cases} RA \in N_2 & \text{if} \quad RA \neq 0\,, \\ \bar{d} \in N_1 & \text{if} \quad RA = 0\,, \quad KA \neq \emptyset\,, \quad DA = \emptyset\,, \\ d \in N_1 & \text{if} \quad RA = 0\,, \quad KA \neq \emptyset\,, \quad DA \neq \emptyset\,, \\ \vartheta A \in N_1 & \text{if} \quad RA = 0\,, \quad KA = \emptyset\,. \end{cases}$$

(vii) Let $A, B \in \mathscr{U}_0^c$. Then

(a) $DA = \emptyset$ iff $\chi A \in \{\omega_0, \bar{d}\} \cup N_2$,

(b) $(A, B) \in h$-Ad implies $\chi A \lesssim_0 \chi B$.

(See [5], 1.12 and 1.23.)

2.4. Lemma. *Let $A, B \in \mathscr{U}_0^c$, Then*

(a) *$\chi A \lesssim \chi B$ iff (1) $\chi A \lesssim_0 \chi B$ and (2) $DA = \emptyset$ implies $RA = RB$,*

(b) *$(A, B) \in m$-Ad implies $\chi A \lesssim \chi B$.*

Proof of (a). Let $\chi A \lesssim \chi B$. Then $\chi A \lesssim_0 \chi B$. Further, let $DA = \emptyset$. Then $\chi A \in \{\omega_0, \bar{d}\} \cup N_2$ by (vii) (a).

Now, if $RB = 0$ then $\chi B \in N_1$ which implies $\chi A \in N_1$, i.e. $RA = 0$ and we have $RA = RB$.

If $RB \neq 0$ then $\chi B = RB \in N_2$ which implies $\chi A \in N_1 - \{\omega_0, \bar{d}\}$ or $\chi B = \chi A$ by 2.3 (c). Since $\chi A \notin N_1 - \{\omega_0, \bar{d}\}$ we have $RB = \chi B = \chi A = RA$.

Let, on the other hand, $\chi A \lesssim_0 \chi B$ and let $DA = \emptyset$ implies $RA = RB$.

Let $\chi B \in N_2$. Then $RB \neq 0$. If $DA \neq \emptyset$ then $\chi A \in N_1 - \{\omega_0, \bar{d}\}$ by (vii) (a) and if $DA = \emptyset$ then $RA = RB \neq 0$ which implies $\chi A = RA = RB = \chi B$. It follows that $\chi A \lesssim \chi B$.

Proof of (b). Let $(A, B) \in m$-Ad. Then $\chi A \lesssim_0 \chi B$ by (v) and (vii) (b). Hence $\chi A \lesssim \chi B$ by 1.12 (e) and by (a) of this lemma.

2.5. Definition. Let $n \in N \cup \{\infty_1, \infty_2\}$ be arbitrary. Then we put

$$W^*(n) = \begin{cases} W(n+1) & \text{if} \quad n \in N \\ W(n) & \text{if} \quad n \in \{\infty_1, \infty_2\} \quad (\text{see } 1.0) . \end{cases}$$

2.6. Lemma. *Let* $A = (A, f) \in \mathscr{U}_0^c$, $x \in A$, $n \in N \cup \{\infty_1, \infty_2\}$ *be arbitrary. Then* $\langle x \rangle_A - \bigcup_{i \in W(n)} A^i \neq \emptyset$ *iff* $x \in A - \bigcup_{i \in W^*(n)} A^i$.

Proof. Let $n \in N$. If $SAx \in W(\vartheta A)$ then the assertion holds by 1.10 (b). If $SAx \in \{\infty_1, \infty_2\}$ then $x \in A^{\infty_1} \cup A^{\infty_2} \subseteq A - \bigcup_{i \in W^*(n)} A^i$ and $\langle x \rangle_A - \bigcup_{i \in W(n)} A^i \supseteq \langle x \rangle_A \cap (A^{\infty_1} \cup A^{\infty_2}) \neq \emptyset$ by (iv) (a), (b).

Let $n = \infty_1$. If $y \in \langle x \rangle_A - \bigcup_{i \in W(n)} A^i$ then $y \in A^{\infty_1} \cup A^{\infty_2}$ which implies $x = fy \in A^{\infty_1} \cup A^{\infty_2}$ by (iii) (a) and 1.8 (b). If, on the other hand, $x \in A - \bigcup_{i \in W^*(n)} A^i = A^{\infty_1} \cup A^{\infty_2}$ then $\langle x \rangle_A - \bigcup_{i \in W(n)} A^i \supseteq \langle x \rangle_A \cap (A^{\infty_1} \cup A^{\infty_2}) \neq \emptyset$ by (iv) (a), (b).

Similarly, the assertion holds for $n = \infty_2$ by (iii) (a) and (iv) (b).

2.7. Definition. Let $A = (A, f) \in \mathscr{U}_0^c$. If there is a sequence $\{c_n;\ n \in N \cup \{\infty_1, \infty_2\}\}$ such that for each $n \in N \cup \{\infty_1, \infty_2\}$ and each $x \in A - \bigcup_{i \in W^*(n)} A^i$, $|\langle x \rangle_A - \bigcup_{i \in W(n)} A^i| = c_n$ holds then A is called *homogeneous* (with the sequence $\{c_n\}$).

2.8. Definition. Let $A = (A, f) \in \mathscr{U}_0^c$, $B = (B, g) \in \mathscr{U}_0^c$, let B be homogeneous with the sequence $\{c_n\}$. Then B is said to be a *majorant* of A if $\chi A \lesssim \chi B$ and if, for arbitrary $n \in N \cup \{\infty_1, \infty_2\}$, $|\langle x \rangle_A - \bigcup_{i \in W(n)} A^i| \leq c_n$ holds for each $x \in A$.

2.9. Theorem. *Let* $A, B \in \mathscr{U}_0^c$, B *be homogeneous. If* $(A, B) \in m\text{-Ad}$ *then* B *is a majorant of* A.

Proof. $\chi A \lesssim \chi B$ by 2.4 (b).

Further, let B be homogeneous with the sequence $\{c_n\}$. Let $(x, x') \in M(A, B)$. Then $(x, x') \in M_0(A, B)$ and there is a sequence of maps $\{F_n : [x]_A^{-n} \to [x']_B^{-n};\ n \in N\}$ such that for each $f^n x \in [x)_A$, $F_0 f^n x = g^n x'$ and for each $n \in N$ and each $y \in [x]_A^{-n}$, $F_{n+1} \mid \langle y \rangle_{A,x}$ is a d-injection and $F_{n+1} \langle y \rangle_{A,x} \subseteq \langle F_n y \rangle_{B,x'}$.

1. Now let $y \in A$ be arbitrary. Then is $n \in N$ such that $y \in [x]_A^{-n}$ by (i) (b). It will be shown that there is a d-injection of $\langle y \rangle_A$ into $\langle F_n y \rangle_B$.

1a. Let $x \in ZA$. Then $x' \in ZB$ because $(x\ x') \in H(A, B)$.

If $\langle y \rangle_A = \langle y \rangle_{A,x}$ then $n \in N - \{0\}$ by 1.7 (e). Hence $F_n y \in [x']_B^{-n}$ and $\langle F_n y \rangle_B = \langle F_n y \rangle_{B,x'}$ by 1.7 (e). It follows that $F_{n+1} \mid \langle y \rangle_{A,x}$ is a d-injection of $\langle y \rangle_A$ into $\langle F_n y \rangle_B$.

If, by 1.7 (e), $\langle y\rangle_A = \langle y\rangle_{A,x} \cup \{f^{p-1}x\}$ then $n = 0$, $y = f^p x$; hence $F_0 y = F_0 f^p x = g^p x'$ and thus $\langle F_0 y\rangle_B = \langle F_0 y\rangle_{B,x'} \cup \{g^{p-1}x'\}$ by 1.7 (e). Further, $F_0 f^{p-1}x = g^{p-1}x'$ and we obtain that $F_1 \mid \langle y\rangle_{A,x} \cup F_0 \mid \{f^{p-1}x\}$ is a d-injection of $\langle y\rangle_A$ into $\langle F_0 y\rangle_B$.

1b. Let, on the other hand, $x \notin ZA$. Then $ZA = \emptyset$ because if we had $ZA \neq \emptyset$ then we should have $DA = \emptyset$ by (iii) (c) and $RB = RA = 0$ by 1.12 (e) which is a contradiction to $(x, x') \in M_0(A, B)$ by 1.12 (c). Thus,

$$\langle y\rangle_A = \begin{cases} \langle y\rangle_{A,x} & \text{if } \ y \in \{x\} \cup \bigcup_{n\in N-\{0\}} [x]_A^{-n} \\ \langle y\rangle_{A,x} \cup \{f^{p-1}x\} & \text{if } \ y = f^p x\,, \quad p \neq 0 \end{cases} \quad \text{by 1.7 (f)}\,.$$

1ba. Let now $\langle y\rangle_A = \langle y\rangle_{A,x}$. Then $F_{n+1}\langle y\rangle_A = F_{n+1}\langle y\rangle_{A,x} \subseteq \langle F_n y\rangle_{B,x'} \subseteq \langle F_n y\rangle_B$ and thus $F_{n+1} \mid \langle y\rangle_{A,x}$ is a d-injection of $\langle y\rangle_A$ into $\langle F_n y\rangle_B$.

1bb. Let $\langle y\rangle_A = \langle y\rangle_{A,x} \cup \{f^{p-1}x\}$ where $y = f^p x$, $p \neq 0$. Then $F_0 y = F_0 f^p x = g^p x'$.

If $x' \in ZB$ then $\langle F_0 y\rangle_B = \langle F_0 y\rangle_{B,x} \cup \{g^{p-1}x'\}$ by 1.7 (e).

If $x' \notin ZB$ and if we put $p_0 = \min\{n \in N;\ g^n x' \in ZB\}$ then

$$\langle F_0 y\rangle_B = \begin{cases} \langle F_0 y\rangle_{B,x'} \cup \{g^{p-1}x'\} & \text{if } \ p \neq p_0 \\ \langle F_0 y\rangle_{B,x'} \cup \{g^{p-1}x', g^{RB+p-1}x'\} & \text{if } \ p = p_0 \end{cases} \quad \text{by 1.7 (f)}\,.$$

Since $(F_0 \cup F_1)\langle y\rangle_A = F_1\langle y\rangle_{A,x} \cup \{F_0 f^{p-1}x\} \subseteq \langle F_0 y\rangle_{B,x'} \cup \{g^{p-1}x'\} \subseteq \langle F_0 y\rangle_B$ we obtain that $F_1 \mid \langle y\rangle_{A,x} \cup F_0 \mid \{f^{p-1}x\}$ is a d-injection of $\langle y\rangle_A$ into $\langle F_0 y\rangle_B$.

2. Let $y \in A$, $m \in N \cup \{\infty_1, \infty_2\}$ be arbitrary.

Let $y \in [x]_A^{-n}$. Then by 1 there exists a d-injection of $\langle y\rangle_A$ into $\langle F_n y\rangle_B$ which implies $\left|\langle y\rangle_A - \bigcup_{i\in W(m)} A^i\right| \leqq \left|\langle F_n y\rangle_B - \bigcup_{i\in W(m)} B^i\right|$ by 1.2.

If $y \in A - \bigcup_{i\in W^*(m)} A^i$ then $SAy \notin W^*(m)$ and we have $SBF_n y \notin W^*(m)$ because $SBF_n y \geqq SAy$. Thus $F_n y \in B - \bigcup_{i\in W^*(m)} B^i$ and we obtain $\left|\langle F_n y\rangle_B - \bigcup_{i\in W(m)} B^i\right| = c_m$ by 2.7. It follows that $\left|\langle y\rangle_A - \bigcup_{i\in W(m)} A^i\right| \leqq c_m$.

Now if $y \in \bigcup_{i\in W^*(m)} A^i$ then $\left|\langle y\rangle_A - \bigcup_{i\in W(m)} A^i\right| = 0$ by 2.6; therefore $\left|\langle y\rangle_A - \bigcup_{i\in W(m)} A^i\right| \leqq c_m$.

Consequently, $\boldsymbol{B}$ is a majorant of $\boldsymbol{A}$.

2.10. Lemma. *Let $A = (A, f) \in \mathscr{U}_0^c$, $B = (B, g) \in \mathscr{U}_0^c$ be such that $RA = RB$. Let $x \in DA \cup KA \cup ZA$, $(x, x') \in M_0(A, B)$ be such that $DA \neq \emptyset$ implies $x = dA$. Let $n \in N$, $y \in [x]_A^{-n}$, $y' \in [x']_B^{-n}$ be such that $n = 0$ implies $y = f^p x$, $y' = g^p x'$ for some $p \in N$. If $\left|\langle y\rangle_A - \bigcup_{i\in W(m)} A^i\right| \leqq \left|\langle y'\rangle_B - \bigcup_{i\in W(m)} B^i\right|$ for each $m \in N \cup \{\infty_1, \infty_2\}$ then there is a d-injection of $\langle y\rangle_{A,x}$ into $\langle y'\rangle_{B,x'}$.*

Proof. 1. Let $RA \neq 0$. Then $RB \neq 0$ and $(x, x') \in ZA \times ZB$ by 1.12 (c).

If $n = 0$ then $y = f^p x \in ZA$, $y' = g^p x' \in ZB$ where $p \in N$. We can suppose that $p \neq 0$. Thus $\langle y \rangle_A = \langle y \rangle_{A,x} \cup \{f^{p-1}x\}$, $\langle y' \rangle_B = \langle y' \rangle_{B,x'} \cup \{g^{p-1}x'\}$ by 1.7 (e) and since $SAf^{p-1}x = \infty_2 = SBg^{p-1}x'$ it follows that there is a d-injection of $\langle y \rangle_{A,x}$ into $\langle y' \rangle_{B,x'}$ by 1.3 and 1.2.

If $n > 0$ then $\langle y \rangle_A = \langle y \rangle_{A,x}$, $\langle y' \rangle_B = \langle y' \rangle_{B,x'}$ by 1.7 (e) and therefore there is a d-injection of $\langle y \rangle_{A,x}$ into $\langle y \rangle_{B,x'}$ by 1.2.

2. Let $RA = 0$. Then $RB = 0$.

2a. Let $DA = \emptyset$. Then $x = dA$.

In this case, $\langle y \rangle_A = \langle y \rangle_{A,x}$ by 1.7 (f) because $y \in A = \{dA\} \cup \bigcup_{i \in N - \{0\}} [dA)_A^{-i}$ by (i) (b). Further, $y' \in \{x'\} \cup \bigcup_{i \in N - \{0\}} [x')_B^{-i}$ because $y \in [dA)_A$, $y' \in [x')_B$ imply $y = f^0 dA$, $y' = g^0 x' = x'$ and since $x' \notin ZB$ we have $\langle y' \rangle_B = \langle y' \rangle_{B,x'}$ by 1.7 (f). Therefore, there is a d-injection of $\langle y \rangle_{A,x}$ into $\langle y' \rangle_{B,x'}$ by 1.2.

2b. Let $DA = \emptyset$. Then $x \in KA$ which implies $[x)_A \subseteq KA$, $[x')_B \subseteq KB$ because $(x, x') \in H(A, B)$, $ZA = ZB = \emptyset$.

If $y \in \{x\} \cup \bigcup_{i \in N - \{0\}} [x)_A^{-i}$ then $\langle y \rangle_A = \langle y \rangle_{A,x}$ by 1.7 (f). Further, $y' \in \{x'\} \cup \bigcup_{i \in N - \{0\}} [x')_B^{-i}$ because $y \in [x)_A$, $y' \in [x')_B$ and we complete the proof again by 1.2.

If $y = f^p x$ where $p > 0$ then $y' = g^p x'$ and thus $\langle y \rangle_A = \langle y \rangle_{A,x} \cup \{f^{p-1}x\}$, $\langle y' \rangle_B = \langle y' \rangle_{B,x'} \cup \{g^{p-1}x'\}$ by 1.7 (f) because $ZA = ZB = \emptyset$. Since $SAf^{p-1}x = \infty_1 = SBg^{p-1}x'$ there is a d-injection of $\langle y \rangle_{A,x}$ into $\langle y' \rangle_{B,x'}$ by 1.3 and 1.2.

2.11. Theorem. *Let $A, B \in \mathscr{U}_0^c$ be such that A has the kernel, B is homogeneous and $DA \neq \emptyset$ implies $RB = 0$. If B is a majorant of A then $(A, B) \in m$-*Ad.

Proof. Let $A = (A, f)$, $B = (B, g)$ and let B be homogeneous with the sequence $\{c_n;\ n \in N \cup \{\infty_1, \infty_2\}\}$. Since $\chi A \lesssim \chi B$ we obtain that $DA = \emptyset$ implies $RA = RB$ by 2.4 (a). Further, we take $x \in DA \cup KA \cup ZA$ such that $RA \neq 0$ implies $x \in ZA$ and $DA \neq \emptyset$ implies $x = dA$.

If $RA \neq 0$ then $RB = RA \neq 0$ by (iii) (c) and 1.12 (e) and we take $x' \in ZB$. Then $(x, x') \in H(A, B)$ and further $(x, x') \in M_0(A, B)$ by 1.12 (c).

Let now $RA = 0$. Then $RB = 0$ because $RB \neq 0$ implies $DA = \emptyset$ and $RA = RB$ which is a contradiction to $RA = 0$. Since $\chi A \lesssim \chi B$ there is $x' \in B$ such that $SAx \leqq SBx'$ by 2.3 (d), (c) and (iv) (e). If $x = dA$ then directly $(x, x') \in H(A, B) = M_0(A, B)$ by 1.12 (a), (c) because $ZA = ZB = \emptyset$ and it follows that $(x, x') \in H(A, B) = M_0(A, B)$.

We define a d-map $F_0 : [x)_A \to [x')_B$ such that $F_0 f^n x = g^n x'$ for each $f^n x \in [x)_A$ (see 1.12 (a)). Let $n \in N$ be arbitrary. Suppose that we have defined a d-map $F_n : [x)_A^{-n} \to [x')_B^{-n}$. It will be shown that there is a map $F_{n+1} : [x)_A^{-(n+1)} \to [x')_B^{-(n+1)}$ such that for each $y \in [x)_A^{-n}$, $F_{n+1} \mid \langle y \rangle_{A,x}$ is a d-injection and $F_{n+1}\langle y \rangle_{A,x} \subseteq \langle F_n y \rangle_{B,x'}$.

Let $y \in [x)_A^{-n}$ be arbitrary. Since $\boldsymbol{B}$ is a majorant of $\boldsymbol{A}$ we have $|\langle y \rangle_A - \bigcup_{i \in W(m)} A^i| \leqq$ $\leqq c_m$ for each $m \in N \cup \{\infty_1, \infty_2\}$. Further, $SAy \leqq SBF_n y$ because F_n is a d-map.

Let $m \in N \cup \{\infty_1, \infty_2\}$ be arbitrary.

If $F_n y \in B - \bigcup_{i \in W^*(m)} B^i$ then $|\langle F_n y \rangle_B - \bigcup_{i \in W(m)} B^i| = c_m$ because $\boldsymbol{B}$ is homogeneous. Thus $|\langle y \rangle_A - \bigcup_{i \in W(m)} A^i| \leqq |\langle F_n y \rangle_B - \bigcup_{i \in W(m)} B^i|$.

If $F_n y \in \bigcup_{i \in W^*(m)} B^i$ then $SBF_n y \in W^*(m)$ which implies $SAy \in W^*(m)$. It follows that $|\langle y \rangle_A - \bigcup_{i \in W(m)} A^i| = 0$ by 2.6. Hence $|\langle y \rangle_A - \bigcup_{i \in W(m)} A^i| = 0 = |\langle F_n y \rangle_B - \bigcup_{i \in W(m)} B^i|$.

Thus for each $m \in N \cup \{\infty_1, \infty_2\}$, $|\langle y \rangle_A - \bigcup_{i \in W(m)} A^i| \leqq |\langle F_n y \rangle_B - \bigcup_{i \in W(m)} B^i|$ and we obtain that there is a d-injection $F_{n+1}^y : \langle y \rangle_{A,x} \to \langle F_n y \rangle_{B,x'}$ by 2.10.

If we put $F_{n+1} = \bigcup_{y \in [x)_A^{-n}} F_{n+1}^y$ then F_{n+1} is the map sought for.

Consequently, $(x, x') \in M(\boldsymbol{A}, \boldsymbol{B})$ and finally, $(\boldsymbol{A}, \boldsymbol{B}) \in m$-Ad.

2.12. Theorem. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{U}_0^c$ be such that $\boldsymbol{A}$ has the kernel, $\boldsymbol{B}$ is homogeneous and $DA \neq \emptyset$ implies $RB = 0$. Then the following assertions are equivalent:*

(α) *There is a monomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

(β) $(\boldsymbol{A}, \boldsymbol{B}) \in m$-Ad.

(γ) *$\boldsymbol{B}$ is a majorant of $\boldsymbol{A}$.*

The assertion is a consequence of (vi), 2.9 and 2.11.

2.13. Corollary. *Let $\boldsymbol{A} \in \mathscr{U}^c$ be finite and complete, $\boldsymbol{B} \in \mathscr{U}_0^c$ homogeneous. Then the following assertions are equivalent:*

(α) *There is a monomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$.*

(β) $(\boldsymbol{A}, \boldsymbol{B}) \in m$-Ad.

(γ) *$\boldsymbol{B}$ is a majorant of $\boldsymbol{A}$.*

Indeed, if $\boldsymbol{A}$ is finite then $\boldsymbol{A} \in \mathscr{U}_0^c$ and it has the kernel.

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