Existence of Multiple Periodic Orbits on Star-Shaped Hamiltonian Surfaces

HENRI BERESTYCKI
Université Paris XIII

JEAN-MICHEL LASRY
Université Paris IX

GIOVANNI MANCINI
Università di Trieste and Courant Institute

AND

BERNHARD RUF
Universität Zürich and Courant Institute

Abstract

Consider the Hamiltonian system

$$\text{(HS)} \qquad \frac{dx_i}{dt} = -\frac{\partial H}{\partial p_i}(x, p), \qquad \frac{dp_i}{dt} = \frac{\partial H}{\partial x_i}(x, p), \qquad i = 1, \cdots, N.$$

Here, $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$. In this paper, we investigate the existence of periodic orbits of (HS) on a given energy surface $\Sigma = \{z \in \mathbb{R}^{2N}; H(z) = c\}$ ($c > 0$ is a constant). The surface Σ is required to verify certain geometric assumptions: Σ bounds a star-shaped compact region $\mathcal{R}$ and $\alpha\mathcal{E} \subset \mathcal{R} \subset \beta\mathcal{E}$ for some ellipsoid $\mathcal{E} \subset \mathbb{R}^{2N}$, $0 < \alpha < \beta$. We exhibit a constant $\delta > 0$ (depending in an explicit fashion on the lengths of the main axes of $\mathcal{E}$ and one other geometrical parameter of Σ) such that if furthermore $\beta^2/\alpha^2 < 1 + \delta$, then (HS) has at least N distinct geometric orbits on Σ. This result is shown to extend and unify several earlier works on this subject (among them works by Weinstein, Rabinowitz, Ekeland-Lasry and Ekeland). In proving this result we construct index theories for an S^1-action, from which we derive abstract critical point theorems for S^1-invariant functionals. We also derive an estimate for the minimal period of solutions to differential equations.

1. Introduction

This paper is concerned with the existence of periodic orbits on a given energy surface for a Hamiltonian system

$$\dot{z} = JH'(z). \tag{1.1}$$

Here, $z = z(t): \mathbb{R} \to \mathbb{R}^{2N}$, $\dot{z} = dx/dt$, $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ is the Hamiltonian and J is the standard skew symmetric matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I stands for the identity in $\mathbb{R}^N$. Trajectories of (1.1) remain on energy surfaces $H = \text{constant}$.

Communications on Pure and Applied Mathematics, Vol. XXXVIII, 253–289 (1985)

CCC 0010-3640/85/030253-37$04.00

1.1. The main result. Let us first explain our main result. Let Ω be the matrix defined by

$$\Omega=\begin{pmatrix}\Omega' & 0\\ 0 & \Omega'\end{pmatrix},$$

Ω' being the diagonal $N\times N$ matrix

$$\Omega'=\begin{pmatrix}\omega_1 & & 0\\ & \ddots & \\ 0 & & \omega_N\end{pmatrix},$$

where $\omega_1, \cdots, \omega_N$ are positive reals. The set

$$\mathscr{E}=\left\{\frac{1}{2}(\Omega x, x)\equiv\sum_{i=1}^{N}\tfrac{1}{2}\omega_i\,(x_i^2+x_{i+N}^2)\leqq 1\right\}$$

defines an ellipsoid in $\mathbb{R}^{2N}$ ($(\ ,\ )$ denotes the scalar product in $\mathbb{R}^{2N}$ and $x=(x_1, \cdots, x_{2N})$.) Without loss of generality we may assume that the energy surface on which we are looking for periodic solutions of (1.1) is defined by

$$\Sigma=\{x\in\mathbb{R}^{2N};\ H(x)=1\}. \tag{1.2}$$

We assume that $H'(z)\neq 0$ for all $z\in\Sigma$ and that

(1.3) Σ is a C^2-manifold which is strictly star shaped[1] with respect to the origin and bounds $\mathscr{R}=\{x\in\mathbb{R}^{2N};\ H(x)\leqq 1\}$, which is compact,

$$\alpha\mathscr{E}\subset\mathscr{R}\subset\beta\mathscr{E} \text{ for some } 0<\alpha<\beta. \tag{1.4}$$

By assumption (1.3), the tangent plane $T_x\Sigma$ to Σ at a point $x\in\Sigma$ never hits the origin. We may therefore define $\rho>0$ to be the largest positive real such that

$$T_x\Sigma\cap\mathring{B}_\rho=\varnothing \quad\text{for all}\quad x\in\Sigma, \tag{1.5}$$

where $\mathring{B}_\rho=\{x\in\mathbb{R}^{2N}\,|\,|x|<\rho\}$.

Our main result is the following

THEOREM 1.1. *Given $\mathscr{E}$, there exists a constant $\delta=\delta(\rho^2/\alpha^2, \omega_1, \cdots, \omega_N)>0$ such that* (1.1) *possesses at least N distinct periodic orbits on any surface Σ satisfying* (1.3)-(1.5) *with $\beta^2/\alpha^2<1+\delta$.*

The explicit dependence of δ on ρ^2/α^2 and the frequencies $\omega_1, \cdots, \omega_N$ are given in subsection 1.2 below.

Remark 1.2. Actually, and more precisely (see Section 7), we prove that given p, $1\leqq p\leqq N$, there is a constant δ_p with $0<\delta=\delta_N\leqq\delta_{N-1}\leqq\cdots\leqq\delta_1=+\infty$

[1] That is, $x\to x/|x|$ is a diffeomorphism from Σ onto S^{2N-1}.

such that, if $\beta^2/\alpha^2 < 1+\delta_p$, then (1.1) has at least p periodic orbits on Σ. In general, the constant δ_p that we obtain increases as p decreases.

1.2. Estimate of δ. Let $l \in \mathbb{N}$ be the number of equivalence classes of the set $\{\omega_1, \cdots, \omega_N\}$ in $\mathbb{R}^*/\mathbb{Q}^*$. That is, relabeling the ω_i, we assume that

$$\{\omega_1, \cdots, \omega_N\} = \{\omega_1^1, \cdots, \omega_{p_1}^1, \omega_1^2, \cdots, \omega_{p_l}^l\}$$

with $\omega_j^i \in \mathbb{R}_+^*$, $p_1 + \cdots + p_l = N$, and

$$\omega_j^i = n_j^i \omega^i, \qquad n_j^i \in \mathbb{N}, \qquad j = 1, \cdots, p_i, \tag{1.6}$$

where ω^i is defined to be the largest positive real satisfying (1.6) (i.e., $\omega^i > 0$ is the largest common integral divider of the ω_j^i). Note that $\omega^i/\omega^j \notin \mathbb{Q}$ for all $i \neq j$. We define $\delta_1 > 0$ by setting

$$\delta_1 = \min_{i=1,\cdots,l} \left\{ \frac{\omega^i \rho^2}{2\alpha^2} \right\}. \tag{1.7}$$

Furthermore, we set

$$\delta_2 = \min \left\{ \frac{n}{m} \frac{\omega^i}{\omega^j} - 1;\ 1 \leqq n,\ m < 1 + 1/\delta_1,\ 1 \leqq i \neq j \leqq l,\ n\omega^i > m\omega^j \right\}. \tag{1.8}$$

Observe that $\delta_2 > 0$, since $\omega^i/\omega^j \notin \mathbb{Q}$, $i \neq j$. Now, we set

$$\delta = \min\{\delta_1, \delta_2\}. \tag{1.9}$$

We shall show that this δ is a possible choice in Theorem 1.1.

1.3. Remarks. The above results extend and unify most of the known results dealing with the existence of periodic solutions on an energy surface.

Indeed, it is easily seen that the local results of A. Weinstein [19], [20] are obtained from Theorem 1.1 by observing that if $H(0) = 0$, $H'(0) = 0$ and $H''(0)$ is positive definite,[2] then, for $\varepsilon > 0$ small enough, the surface $\{H = \varepsilon\}$ (near 0) satisfies the hypothesis of the theorem. The global results of A. Weinstein [21] or P. H. Rabinowitz [18] who show the existence of at least *one* periodic orbit under the assumption (1.3) (in [18]), or less generally that $\mathcal{R}$ is convex (in [21]) just correspond to the case $p = 1$ in Remark 1.2. Note that in this case (since $\delta_1 = +\infty$) the assumptions (1.4) and (1.5) may as well be dropped.

The global multiplicity result of I. Ekeland and J. M. Lasry [11] corresponds to the particular case when $\mathscr{E}$ is a ball ($\Omega' = I$) and $\mathcal{R}$ is convex. In fact, taking $B_\rho = \alpha\mathscr{E}$, the explicit formula (1.7) for δ yields $1 + \delta = 2$, which is precisely the result of Ekeland and Lasry. Hence, in this particular case Theorem 1.1 covers the results of [11] and the constants δ_p (cf. Section 7) also allow us to recover

[2] The extension of this result due to J. Moser [15], where $H''(0)$ is not necessarily positive, does not seem to follow from the above results.

the extension for this case due to Ambrosetti and Mancini [1]. Finally, Theorem 1.1 can be seen to contain a recent perturbation result of I. Ekeland [10], Theorem 18, for Hamiltonian systems.

We present two proofs of Theorem 1.1, both of which rely on critical point theory via S^1-action index theories. In Section 2, we construct an index related to some S^1-action. As a generalization of this index we then define a relative index which allows us to obtain critical points for unbounded indefinite functionals. In Section 3, we derive general and abstract critical point theorems for functionals which are invariant under the S^1-action. A crucial feature in these two sections is that the S^1-action is allowed to have a nontrivial fixed-point space. In Section 4, we give estimates for the minimal period of solutions for some differential equations. (The main result in Section 4 extends a theorem contained in the work of Croke and Weinstein [9].) Sections 2 to 4 are more general than the framework of Theorem 1.1 and, we believe, are of independent interest.

In Section 5, we use the relative index defined in subsection 2.3 to give a proof of Theorem 1.1 by working with an indefinite functional. An alternate proof of Theorem 1.1 by the methods of convex analysis (and using the index of subsection 2.2) is given in Section 6. Finally, in Section 7, we outline some extensions and make a few comments.

Remark 1.3. It will be seen in the course of the proof (see Sections 5 and 6) that we actually also obtain fairly precise estimates on the minimal periods of the orbits.

Remark 1.4. Theorem 1.1 has been announced in our note [7].

Remark 1.5. Existence of periodic orbits for conservative systems which are not necessarily Hamiltonian, on a given energy surface, is investigated in [4], [5], [6].

2. Index Theory for the S^1-Action and a Relative Index Theory

With the aim of constructing critical values by a Ljusternik-Schnirelman type minimax principle, we first require an "index theory" with respect to an S^1-action which has a *nontrivial fixed-point space.* This is the purpose of the present section. Then we shall define an index theory relative to an invariant subspace X of E. Abstract critical point theorems for invariant functionals will be derived in the next section by using the index (for functionals which are "essentially" bounded below) and the relative index (for indefinite functionals). The applications to Hamiltonian systems will be detailed afterwards.

2.1. Notations and basic definitions. To begin with let us specify some notations and definitions. In all the sequel E is a *complex* separable Hilbert space. The generic element of S^1 will be denoted either by θ ($\in \mathbb{R}/2\pi\mathbb{Z}$) or by $e^{i\theta}$. Let T be a unitary representation of S^1 in E, that is, $T_\theta \in \text{Isom}(E)$ is defined for all θ, $\|T_\theta u\| = \|u\|$ for all u, $T_{\theta+\theta'} = T_\theta T_{\theta'}$ (hence $T_0 = \text{Id}$, $T_{-\theta} = T_\theta^{-1} = T_\theta^*$) and $\theta \to T_\theta$ is continuous.

A representation R of S^1 in $\mathbb{C}^k$ (or an S^1-action on $\mathbb{C}^k$) will be termed *regular* if it only has a trivial fixed-point space (i.e., $R_\theta u = u$ for all θ, $\Rightarrow u = 0$). Given $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$, an example of an S^1-action R^α is defined as follows: for $\xi = (\xi_1, \cdots, \xi_k)$, $R_\theta^\alpha \xi = (\exp\{i\alpha_1\theta\}\xi_1, \cdots, \exp\{i\alpha_k\theta\}\xi_k)$. In this example R is regular if and only if $\alpha_1, \cdots, \alpha_k$ are all non-zero. We recall that due to the Peter–Weyl representation theorem, *any* S^1-action on $\mathbb{C}^k$ is of the form R^α for some k-tuple in some orthonormal basis. A set $A \subset E$ is said to be *invariant* (under T) if $T_\theta A = A$ for all θ. Note that $S^{2k-1} = \{\xi \in \mathbb{C}^k;\ |\xi| = 1\}$ is invariant under any S^1-action on $\mathbb{C}^k$. A functional $F: E \to \mathbb{R}$ is *invariant* if $F(T_\theta u) = F(u)$ for all $u \in E$ and for all θ. Finally, a mapping $\phi: E \to \mathbb{C}^k$ is said to be *equivariant* with respect to (T, R) if $\phi \circ T_\theta = R_\theta \circ \phi$ for all θ. We denote by $M_k(A; R) = C_{\text{eq}}(A, \mathbb{C}_R^k \backslash \{0\})$ the space of all continuous maps $\phi: A \to \mathbb{C}^k \backslash \{0\}$ which are equivariant with respect to (T, R).

We denote by E^0 the space of fixed points of T: $E^0 = \{u \in E;\ T_\theta u = u$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. Henceforth we assume that

(2.1) $$E^0 \text{ is finite-dimensional.}$$

Let us define two classes of subsets of E:

$$\mathscr{E} = \{A \subset E;\ A \cap E^0 = \emptyset,\ A \text{ is closed and invariant under } T\},$$

$$\mathscr{E}_c = \{A \in \mathscr{E};\ A \text{ is compact}\}.$$

Our first step will be to recall the geometrical S^1-index theory of V. Benci [2] for these classes of subsets.

DEFINITION 2.1. For $A \in \mathscr{E}$ we define:

(2.2) $$\gamma(A) = \inf\{k \in \mathbb{N};\ \text{there exists a regular } S^1\text{-action } R \text{ on } \mathbb{C}^k \text{ with } M_k(A; R) \neq \emptyset\}.$$

As usual, $\gamma(A) = +\infty$ if $A \neq \emptyset$ and no such $k \in \mathbb{N}$ can be found, and $\gamma(\emptyset) = 0$.

For the properties of this index we refer to Benci [2]. At any rate, this index will be seen to be a special case of the index defined in subsection 2.2 below.

2.2. An index for S^1-actions with fixed points. Let us now consider the broader classes

$$\mathscr{F} = \{A \subset E \backslash \{0\};\ A \text{ is closed and invariant under } T\},$$

$$\mathscr{F}_c = \{A \in \mathscr{F};\ A \text{ is compact}\}.$$

Let us first observe that if $A \in \mathscr{F}$ were any set such that $A \cap E^0 \neq \varnothing$, then the definition (2.1) would lead to $\gamma(A) = +\infty$. We therefore require a more discriminating definition of the index.

DEFINITION 2.2. Let $A \in \mathscr{F}$ and let R be a regular S^1-representation on $\mathbb{C}^k$. We denote by $M_k^0(A, R)$ the set of all continuous mappings $h: A \to E^0 \times \mathbb{C}^k$ having the properties:

$$(0,0) \notin h(A), \tag{2.3}$$

$$\begin{gathered} h \text{ is equivariant with respect to } (T, \hat{R}), \text{ where} \\ \hat{R}_\theta(x_0, \xi) = (x_0, R_\theta \xi) \quad \text{for all} \quad x_0 \in E^0 \quad \text{and} \quad \xi \in \mathbb{C}^k, \end{gathered} \tag{2.4}$$

$$h(u) = (u, 0) \quad \text{for all} \quad u \in A \cap E^0. \tag{2.5}$$

DEFINITION 2.3. Let $A \in \mathscr{F}$. The index of A, $\gamma_0(A)$, is defined to be

$$\begin{aligned} \gamma_0(A) = \inf\{k \in \mathbb{N};\ &\text{there exists a regular } S^1\text{-action} \\ &R \text{ on } \mathbb{C}^k \text{ with } M_k^0(A, R) \neq \varnothing\}. \end{aligned} \tag{2.6}$$

We now list the basic properties of this index.

1. MONOTONICITY. Let $A, B \in \mathscr{F}$. Assume there exists an equivariant mapping $g: A \to B$ and a continuous map $\phi: E^0 \to E^0$ such that $\phi \neq 0$ on $E^0 \backslash \{0\}$ and $\phi[g(u)] = u$ for all $u \in E^0 \cap A$. Then $\gamma_0(A) \leqq \gamma_0(B)$.

2. SUBADDITIVITY. Let $A \in \mathscr{F}$ and $B \in \mathscr{E}$. Then

$$\gamma_0(A \cup B) \leqq \gamma_0(A) + \gamma(B).$$

3. Let $A \in \mathscr{F}$, $B \in \mathscr{E}$ with $\gamma(B) < +\infty$. Then $A \backslash B \in \mathscr{F}$ and $\gamma_0(\overline{A \backslash B}) \geqq \gamma_0(A) - \gamma(B)$.

4. Let $A \in \mathscr{F}$ have an index $\gamma_0(A) \geqq k$. Suppose $(B^0)^\perp = F_1 + F_2$, with F_1, F_2 invariant and orthogonal and $\dim_{\mathbb{C}} F_1 < k$. Then $A \cap F_2 \neq \varnothing$.

5. Let $G \subset (E^0)^\perp$ be an invariant subspace of finite dimension. Let $S = \{x \in E^0 + G, \|x\| = \rho\}$ for some $\rho > 0$. Then $S \in \mathscr{F}_c$ and

$$\gamma_0(S) = \dim G.\ [3]$$

For the proof of these properties we refer to subsection 2.3 for the relative index defined there. (Indeed, it will be seen that γ_0 can be obtained as a particular case of this index.)

[3] Here and thereafter, the dimension of a subspace always refers to the *complex* dimension (unless otherwise specified).

2.3. A relative index. The classes of sets $\mathcal{F}$ and $\mathcal{F}_c$ are defined as before. Let $X \subset E$ be a closed linear subspace invariant under the action such that $E^0 \subset X^\perp$. We shall also write $Y = (E^0 \oplus X)^\perp$, and we shall denote by P_Y and P_0 the orthogonal projections onto Y, E^0, respectively, and $P_1 = P_Y + P_0$.

DEFINITION 2.4. Assume $A \in \mathscr{E}$ and let R be a regular S^1-representation on $\mathbb{C}^k$. We denote by $D_k(A, R)$ the set of all continuous mappings

$$h: A \to X^\perp \times \mathbb{C}^k, \qquad h(u) = (h_1(u), h_2(u)),$$

having the following properties:

$$(0,0) \notin h(A), \tag{2.7}$$

(2.8) h is equivariant with respect to (T, R) in the following sense:

$$h(T_\theta u) = (T_\theta h_1(u), R_\theta h_2(u)),$$

$$h(u) = (u, 0) \quad \text{for all} \quad u \in A \cap E^0, \tag{2.9}$$

$$P_Y h_1 = P_Y + K, \tag{2.10}$$

with $K: A \to Y$ compact (i.e., K continuous, and B bounded$\Rightarrow \overline{K(B)}$ compact).

Remark 2.5. If $X^\perp = E^0$, then (2.10) is automatically satisfied, and hence the relative index reduces to the index introduced in subsection 2.2.

DEFINITION 2.6. Let $A \in \mathcal{F}$. The relative index of A with respect to X, $\gamma_r(A|X) = \gamma_r(A)$, is defined to be

$$\begin{aligned} &\gamma_r(A) = \inf\{k \in \mathbb{N}; \text{there exists a regular } S^1\text{-action} \\ &\qquad R \text{ on } \mathbb{C}^k \text{ with } D_k(A, R) \neq \emptyset\}, \\ &\gamma_r(A) = +\infty \text{ if } A \neq \emptyset \text{ and no such } k \in \mathbb{N} \text{ can be found, and } \gamma_r(\emptyset) = 0. \end{aligned} \tag{2.11}$$

Remark 2.7. The preceding constructions of γ, γ_0, and γ_r are somewhat reminiscent of the notions of index and relative index introduced by E. Fadell, S. Husseini and P. H. Rabinowitz [13]. These authors build, using algebraic topology tools, a general theory of relative indices for spaces in the presence of some G-action, where G is a compact Lie group.

Here we are giving a geometrical construction which reduces algebraic topology to the use of the S^1-version of the Borsuk-Ulam theorem.

The papers [2], [3] of V. Benci have the same purpose of a "geometrical" construction. The index and pseudo-indices of V. Benci have the same type of properties as our index and relative index: compare for example

$$\gamma_r(A \cup B) \leqq \gamma_r(A) + \gamma(B)$$

with V. Benci's

$$i^*(A \cup B) \leqq i^*(A) + i(B).$$

However, the two constructions are in some sense "dual". V. Benci measures the topological complexity of the space by using the existence of groups of deformations, while we are measuring it by obstructions: nonexistence of equivariant maps into spheres of too small dimension.

In the next propositions, we list the basic properties of this relative index.

PROPOSITION 2.8. *Let $A \in \mathscr{F}$. Suppose that $X = F_1 + F_2$, with F_1 and F_2 invariant orthogonal subspaces. If $A \cap F_2 = \varnothing$, then $\gamma_r(A) \leqq \dim F_1$.*

Proof: Let $Q: E \to F_1$ denote the orthogonal projection onto F_1. Define $h(u) = (P_1 u, Qu)$. Then it is easily seen that $h: A \to X^{\perp} \times F_1$ satisfies conditions (2.7)-(2.10) (identify F_1 with $\mathbb{C}^k$, $k = \dim F_1$, which we may assume to be finite). Note that the S^1-action on F_1, the trace of T on F_1, is regular. Hence $h \in D_k(A, T)$. This shows that $\gamma_r(A) \leqq \dim F_1$.

COROLLARY 2.9. *Let $A \in F$ have an index $\gamma_r(A) \geqq k$. Suppose $X = F_1 + F_2$, with F_1, F_2 invariant and orthogonal and* dim *$F_1 < k$. Then $A \cap F_2 \neq \varnothing$.*

PROPOSITION 2.10. *Let $G \subset X$ be an invariant subspace of finite dimension. Let $S = \{x \in X^{\perp} \oplus G \mid \|x\| = \rho\}$ for some $\rho > 0$. Then $S \in \mathscr{F}$ and $\gamma_r(S) = \dim G$ (if G is infinite-dimensional, then $\gamma_r(S) = +\infty$).*

Proof: First, $\gamma_r(S) \leqq \dim G$, by Proposition 2.8. Let $k = \dim G$. If $\gamma_r(S) = j < k$, there exists a regular S^1-action R on $\mathbb{C}^j$ and a mapping $h: S \to X^{\perp} \times \mathbb{C}^j$ satisfying (2.7)-(2.10). We show that h has necessarily a zero, contradicting (2.7).

Let $Y = \oplus_{i \in \mathbb{N}} Y^i$ be the decomposition of Y into its irreducible subspaces with respect to the S^1-action. Let $P_n: E \to \oplus_{i=1}^n Y^i = Y_n$ denote the orthogonal projection onto Y_n. Thus, $Y_n \subset Y$ forms a sequence of orthogonal invariant subspaces such that $\dim Y_n = n$, $Y_n \subset Y_{n+1}$, and, for all $x \in Y$, $P_n x \to x$ as $n \to \infty$ ($P_n x \to P_Y x$ for all $x \in E$). Lastly, let $Q: E \to G$ be the orthogonal projection onto G. Then define h^n: $(Y_n \oplus E^0 \oplus G) \cap S \to (Y_n \oplus E^0) \times \mathbb{C}^j$ by letting $h^n = (h_1^n, h_2^n)$ with

$$h_1^n(u) = P_n h_1(u) + P_0 h_1(u),$$

$$h_2^n(u) = h_2(u),$$

for all n. On the space $(Y_n \oplus E^0) \times \mathbb{C}^j$ define an S^1-action U by setting

$$U_\theta(\xi, \eta) = (T_\theta \xi, R_\theta \eta)$$

with $\xi \in Y_n \oplus E^0$, $\eta \in \mathbb{C}^j$. The mapping h^n satisfies

$$h^n(u) = (u, 0) \quad \text{for all} \quad u \in E^0 \cap S, \tag{2.12}$$

$$h^n \circ T_\theta = U_\theta \circ h^n. \tag{2.13}$$

Identify (after a choice of suitable bases) $Y_n \oplus E^0$ and $Y_n \oplus E^0 \oplus G$ with $\mathbb{C}^n \times \mathbb{C}^l$ and $\mathbb{C}^n \times \mathbb{C}^l \times \mathbb{C}^k$, respectively, where $l = \dim E^0$. We have thus obtained a map from the sphere in $\mathbb{C}^n \times \mathbb{C}^l \times \mathbb{C}^k$ into $\mathbb{C}^n \times \mathbb{C}^l \times \mathbb{C}^j$ with $j < k$ which is equivariant according to the above actions and leaves $\mathbb{C}^l$, the fixed-point set, invariant. Therefore, h^n has a zero by the S^1-version of the Borsuk-Ulam theorem due to Fadell, Husseini and Rabinowitz [13]. An elegant proof of this result (modulo the use of the Peter-Weyl theorem) can be found in L. Nirenberg [16], Theorem 3.

We thus have a sequence $(u_n) \subset S$ such that $0 = h^n(u_n) = (u_n + P_n K(u_n) + P_0 h_1^n(u_n), h_2^n(u_n))$. Using the compactness of K, we therefore find a convergent subsequence $u_n \to u$, $h(u) = 0$, $u \in S$.

Finally, if G is infinite-dimensional, one defines as above invariant spaces $G_n \subset G$, $\dim G_n = n$, and obtains for $S_n = \{x \in X^\perp \oplus G_n \mid \|x\| = \rho\}$ that $\gamma_r(S_n) = n$ for all $n \in \mathbb{N}$. Hence $\gamma_r(S) = +\infty$.

PROPOSITION 2.11, MONOTONICITY. *Let A, $B \in \mathcal{F}$. Suppose that there exists a continuous equivariant map $g: A \to B$ such that*

(i) *$P_Y g = P_Y + K$, with $K: A \to Y$ compact,*

(ii) *there exists $\phi: E^0 \to E^0$ continuous such that $\phi \neq 0$ on $E^0 \setminus \{0\}$ and $\phi(g(u)) = u$ for all $u \in E^0 \cap A$.*

Then $\gamma_r(A) \leqq \gamma_r(B)$.

Proof: It suffices to assume that $k = \gamma_r(B) < \infty$. Let $f \in D_k(B, R)$ and define

$$\tilde{\phi}: Y \times E^0 \times \mathbb{C}^k \to Y \times E^0 \times \mathbb{C}^k$$

by

$$\tilde{\phi}(\eta, x_0, \xi) = (\eta, \phi(x_0), \xi).$$

Identifying $Y \times E^0$ with $X^\perp$, we have a mapping $h = \tilde{\phi} \circ f \circ g: A \to X^\perp \times \mathbb{C}^k$. It is easy to see that $0 \notin h(A)$ and that h is $(T, \hat{R})$ equivariant. Since g is equivariant, it maps fixed points into fixed points. Hence if $u \in E^0$, then $h(u) = (u, 0)$. Finally, $P_Y h = P_Y(\tilde{\phi} \circ f \circ g) = P_Y + K$, $K: A \to Y$ compact. This shows that $h \in D_k(A, R)$, and hence $\gamma_r(A) \leqq k$.

PROPOSITION 2.12. *Let $L: E \to E$ be an equivariant isomorphism with $LX^\perp = X^\perp$. Let $A \in \mathcal{F}$. Then $\gamma_r(LA) = \gamma_r(A)$.*

Proof: Assume $\gamma_r(LA) = k$. Let $f \in D_k(LA, R)$ and define $g: X^\perp \times \mathbb{C}^k \to X^\perp \times \mathbb{C}^k$ by $g(u, \xi) = (L^{-1}u, \xi)$. Let $h = g \circ f: LA \to X^\perp \times \mathbb{C}^k$. It is easy to see that

h has no zero. Now let $\tilde{h} = h \circ L: A \to X^{\perp} \times \mathbb{C}^k$. It is readily verified that $\tilde{h}$ satisfies (2.7)-(2.10). Hence $\gamma_r(A) \leqq k$. Replacing L by L^{-1}, one obtains $\gamma_r(LA) \leqq \gamma_r(A)$.

PROPOSITION 2.13. *Let $\hat{\rho}$ be an invariant functional $\hat{\rho}: E \backslash \{0\} \to [c_1, c_2]$, $0 < c_1 < c_2 < \infty$, and set $\rho(u) = \hat{\rho}(u) \cdot u$. Let $A \in \mathscr{E}$. Then*

$$\gamma_r(\rho(A)) \geqq \gamma_r(A).$$

Proof: Assume $\gamma_r(\rho(A)) = k$. Let $f \in D_k(\rho(A), R)$ and define $g: A \to X^{\perp} \times \mathbb{C}^k$ by

$$g(u) = \frac{1}{\hat{\rho}(u)} f \circ \rho(u).$$

Clearly, $g \in D_k(A, R)$ and therefore $\gamma_r(\rho(A)) \geqq \gamma_r(A)$.

Remark: The same result holds with $A \in \mathscr{F}$ provided one assumes, for instance, that $\rho(u) = u$ for all $u \in E^0$.

PROPOSITION 2.14. SUBADDITIVITY. *Let $A \in \mathscr{F}$ and $B \in \mathscr{E}$; then*

$$\gamma_r(A \cup B) \leqq \gamma_r(A) + \gamma(B).$$

Proof: It suffices to consider the case $k = \gamma_r(A) < \infty$, $m = \gamma(B) < \infty$. Let $f \in D_k(A, R)$ and $g \in M_m(B, S)$ for some regular S^1-actions R and S on $\mathbb{C}^k$ and $\mathbb{C}^m$, respectively. It is straightforward to show that there are continuous and equivariant (under (T, R) and (T, S), respectively) extensions of f, g, denoted by $\bar{f}$, $\bar{g}$:

$$\bar{f}: E \to X^{\perp} \times \mathbb{C}^k,$$
$$\bar{g}: E \to \mathbb{C}^m.$$

Indeed, by Tietze's theorem, g, for instance, has a continuous extension $\hat{g}: E \to \mathbb{C}^m$. Then define

$$\bar{g}(x) = \frac{1}{2\pi} \int_0^{2\pi} S_{-\theta} \hat{g}(T_\theta x)\, d\theta.$$

It is clear that $\bar{g}$ is continuous and equivariant. The proof is similar for $\bar{f}$. By the above argument, and since $B \cap E^0 = \varnothing$, one can choose $\bar{g}$ such that

$$\bar{g}(u) = 0 \quad \text{for all} \quad u \in E^0. \tag{2.14}$$

Now define $h: A \cup B \to X^{\perp} \times \mathbb{C}^{k+m} \simeq X^{\perp} \times \mathbb{C}^k \times \mathbb{C}^m$ by setting (with an obvious identification)

$$h(u) = (\bar{f}(u), \bar{g}(u)).$$

We claim that $h \in D_{k+m}(A \cup B, U)$, where the S^1-action U on $\mathbb{C}^{k+m}$ is defined by $U_\theta x = (R_\theta x_1, S_\theta x_2)$ for $x_1 \in \mathbb{C}^k$, $x_2 \in \mathbb{C}^m$ and $x = (x_1, x_2) \in \mathbb{C}^{k+m}$.

Indeed, h is continuous. By the above construction it is seen that $h_Y = P_Y + K$, and for $u \in A \cup B$ either $\bar{f}(u)$ or $\bar{g}(u)$ is non-zero. It is readily checked that h is equivariant with respect to $(T, \hat{U})$, since it is obvious that h satisfies (2.9). The proof is thereby complete.

COROLLARY 2.15. *Let $A \in \mathscr{E}$, $B \in \mathscr{F}$ with $\gamma(B) < \infty$. Then $\overline{A \backslash B} \in \mathscr{E}$, and*

$$\gamma_r(\overline{A \backslash B}) \geqq \gamma_r(A) - \gamma(B).$$

Proof: $\overline{A \backslash B} \in \mathscr{F}$ (recall that $\varnothing \in \mathscr{F}$ and by definition $\gamma_r(\varnothing) = 0$). Now $A \subset \overline{A \backslash B} \cup B$ and Propositions 2.14 and 2.11 yield

$$\gamma_r(A) \leqq \gamma_r(\overline{A \backslash B}) + \gamma(B).$$

3. Critical Points of S^1-Invariant Functionals

We present in this section some abstract critical point theorems based on the indices introduced above. The hypotheses and notations are the same as in the previous section. In particular, E is an infinite-dimensional complex Hilbert space, T is a unitary representation of S^1 in E, and E^0 denotes the space of fixed points of T, which is assumed to be finite-dimensional.

We shall work with real-valued functionals $f \in C^1(E, \mathbb{R})$ satisfying

(3.1) f is invariant under T:

$$f(T_\theta x) = f(x) \quad \text{for all} \quad \theta \in [0, 2\pi] \quad \text{and} \quad \text{for all} \quad x \in E, \tag{3.1}$$

$$f(0) = 0. \tag{3.2}$$

Furthermore, f is required to satisfy the following classical compactness condition of Palais–Smale.

(PS) For all $\alpha < \beta$, and for any sequence $(x_n) \subset E$ such that $\alpha \leqq f(u_n) \leqq \beta$ and $f'(u_n) \to 0$ strongly in E', there is a convergent subsequence of (x_n).

f is said to satisfy $(\mathrm{PS})^-$ if f satisfies (PS) for all α, β with $\alpha < \beta < 0$.

3.1. Functionals which are essentially bounded from below. Let us now construct critical values of functionals satisfying the above conditions by a minimax principle relying on the index of subsection 2.2. We let

$$\Gamma_k = \{A \in \mathscr{F};\ \gamma_0(A) \geqq k\}, \tag{3.3}$$

$$c_k = \inf_{A \in \Gamma_k} \max_{x \in A} f(x). \tag{3.4}$$

Then we have the following

PROPOSITION 3.1. *Let $f \in C^1(E, \mathbb{R})$ satisfy conditions* (3.1), (3.2) *and* $(\mathrm{PS})^-$. *Assume furthermore that, for some k, $m \in \mathbb{N}^*$, $\Gamma_{k+m-1} \neq \varnothing$ and that*

$$-\infty < c = c_k = c_{k+1} = \cdots = c_{k+m-1} < 0.$$

Let $K_c = \{x \in E\,;\, f(x) = c, f'(x) = 0\}$. Then, if $K_c \cap E^0 = \varnothing$,

$$\gamma(K_c) \geqq m.$$

In particular, c is a critical value of f.

Proof: First, since $K_c \cap E^0 = \varnothing$, and K_c is invariant and compact by $(\mathrm{PS})^-$, K_c belongs to $\mathscr{E}_c$, and $\gamma(K_c)$ is well defined and finite. Furthermore, there exists $\delta > 0$ such that $N_\delta(K_c) = \{x \in E\,;\, \mathrm{dist}\,(x, K_c) < \delta\}$ satisfies $\gamma(N_\delta(K_c)) = \gamma(K_c)$ (cf. Benci [2]). By the deformation lemma of Morse (see e.g. P. H. Rabinowitz [17], or more precisely, for the "equivalent deformation lemma", V. Benci [2]), for any $0 < \varepsilon < \bar{\varepsilon}$ small enough ($0 < \varepsilon < \bar{\varepsilon}$ are fixed such that $c + \bar{\varepsilon} < 0$), there exists $\eta : E \to E$ with the following properties:

(3.5) $\quad\eta$ is a homeomorphism: $E \to E$;

(3.6) $\quad\eta$ is equivariant under T;

(3.7) $\quad\eta(x) = x$ for all x such that $|f(x) - c| \geqq \bar{\varepsilon}$;

(3.8) $\quad\eta[\{f \leqq c + \varepsilon\} \backslash N_\delta(K_c)] \subset \{f \leqq c - \varepsilon\}$.

To prove Proposition 3.1 we argue by contradiction and suppose that $\gamma(K_c) \leqq m - 1$. By the definition of $c_{k+m-1} = c$ there exists $A \in \Gamma_{k+m-1}$ such that $\max_{x \in A} f(x) \leqq c + \varepsilon$, that is $A \subset \{f \leqq c + \varepsilon\}$. Now let $B = \eta(\overline{A \backslash N_\delta(K_c)})$. By (3.7), we know that $\eta(0) = 0$ and, since η is an equivariant homeomorphism, it is clear that $B \in \mathscr{F}$. By the subadditivity property of γ_0 (property 3 of γ_0), we know that

$$\gamma_0(B) \geqq \gamma_0(A) - \gamma(N_\delta(K_c)). \tag{3.9}$$

Hence, since $\gamma(N_\delta(K_c)) = \gamma(K_c) \leqq m - 1$ and $A \in \Gamma_{k+m-1}$, (3.9) yields

$$\gamma_0(B) \geqq k. \tag{3.10}$$

On the other hand, from (3.8) we infer that $B \subset \{f \leqq c - \varepsilon\}$ which is a contradiction to the fact that $c = c_k$ as $B \in \Gamma_k$. The proof is thereby complete.

From the viewpoint of applications it is crucial to have conditions which will *a priori* guarantee that the c_k constructed by (3.3), (3.4) satisfy $-\infty < c_k < 0$. We now give some results in this direction.

PROPOSITION 3.2. *Let $f \in C^1(E, \mathbb{R})$ satisfy* (3.1), (3.2) *and* $(\mathrm{PS})^-$. *Assume that $(E^0)^\perp = F_1 + F_2$, F_1 and F_2 being invariant and orthogonal subspaces. Assume* *that f is bounded from below on F_2 and that* $\dim F_1 = p < \infty$. *Then, for any $k > p$, c_k is finite* $(c_k > -\infty)$.

Proof: This is but a consequence of property 4 of γ_0: for any $A \in \Gamma_k$ and $k > p$, $A \cap F_2 \neq \varnothing$. Hence,

$$c_k \geqq \inf_{F_2} f > -\infty.$$

PROPOSITION 3.3. *Let $f \in C^1(E, \mathbb{R})$ satisfy* (3.1), (3.2) *and* $(\mathrm{PS})^-$. *Suppose that there exists an invariant subspace G of E such that $G \subset (E^0)^\perp$ and $\dim G = m$. Suppose moreover that, for some $\rho > 0$, f is strictly negative on $S = \{x \in E^0 + G;\ \|x\| = \rho\}$. Then, $c_k < 0$ for all $k \leqq m$.*

Proof: By property 5 of γ_0, $S_\rho \in \mathscr{F}_c$ and $\gamma_0(S_\rho) = m$. Hence, $S_\rho \in \Gamma_k$ for all $k \leqq m$, which shows that, for all $k \leqq m$,

$$c_k \leqq \max_{S_\rho} f < 0.$$

We sum up the results of this subsection in the following statement:

THEOREM 3.4. *Let E be an infinite-dimensional complex Hilbert space, T an S^1-action of E with a finite-dimensional fixed point space E^0. Let $f \in C^1(E, \mathbb{R})$ be an invariant functional such that $f(0) = 0$ and f satisfies* $(\mathrm{PS})^-$. *We assume that there are two invariant subspaces V and W of E such that*

(3.11) $$V \subset (E^0)^\perp;$$

(3.12) f *is bounded from below on* V;

(3.13) $W \supset E^0$ *and, for some* $\rho > 0, f(u) < 0$ *for all* $u \in W$ *with* $\|u\| = \rho$;

(3.14) $$m = \dim W < \infty, \qquad p = \operatorname{codim} V < \infty;$$

(3.15) $$E^0 \cap K_c = \{x \in E^0 | f(x) = c, f'(x) = 0\} = \varnothing \quad \textit{for all} \quad c < 0.$$

Assume $m \geqq p$. Then f has at least $m - p$ distinct critical orbits corresponding to negative critical levels.

Remark 3.5. If x is a critical point of f, then, due to the invariance of f, all the points in the orbit of $x, \{T_\theta x\}$, are critical points. Thus, we speak of critical orbits. Observe that distinct orbits have empty intersection.

Proof: Let c_k be defined by (3.4). We know that $-\infty \leqq c_1 \leqq c_2 \leqq \cdots \leqq c_k \leqq \cdots$. By Proposition 3.1, c_k is a critical value provided $-\infty < c_k < 0$. By Propositions 3.2 and 3.3, we have

(3.16) $$-\infty < c_k < 0 \quad \textit{for all} \quad k, \qquad p < k \leqq m.$$

If $c_{p+1} < c_{p+2} < \cdots < c_{m-1} < c_m$ are all distinct, the theorem is proved. Suppose on the contrary that

$$-\infty < c_j = c_{j+1} < 0$$

for some j, $p < j < m$. Then, let $c = c_j = c_{j+1}$; by (3.15), $K_c \cap E^0 = \emptyset$. Hence, by Proposition 3.1, $\gamma(K_c) \geqq 2$. This implies that K_c contains infinitely many distinct critical orbits, and hence the theorem is proved also in this case.

Results in a similar spirit to Theorem 3.4 have been obtained previously by V. Benci [3], Theorem 4.1, by using his pseudo-index theories.

3.2. Indefinite functionals. In this subsection we consider critical points of quadratic functionals restricted to manifolds which are radially diffeomorphic to a sphere.

Let Z be a complex Hilbert space with scalar product $(,)$ and norm $|\cdot|$ and with an S^1-action T_θ. Let $\mathscr{L}$: dom $(\mathscr{L}) \subset Z \to Z$ be a densely defined, selfadjoint, linear, and equivariant (with respect to T_θ) operator with closed range. We let $N = \ker(\mathscr{L})$ be the kernel of $\mathscr{L}$, and we observe that $\operatorname{Im}(\mathscr{L}) = N^\perp$, which implies that $\mathscr{L}^{-1} = [\mathscr{L} | \operatorname{dom}(\mathscr{L}) \cap N^\perp]^{-1} : N^\perp \to N^\perp$ is a well-defined, continuous, linear operator. We assume that $\mathscr{L}^{-1}$ is compact and $\dim N < \infty$.

It is a consequence of these hypotheses that $\sigma(\mathscr{L})$, the spectrum of $\mathscr{L}$, is a pure point spectrum. More precisely, every $\lambda \in \sigma(\mathscr{L}) \backslash \{0\}$ is an eigenvalue of finite multiplicity, and $\sigma(\mathscr{L}) \backslash \{0\}$ has no finite cluster point. Hence, $\sigma(\mathscr{L})$ is at most countable, and we can enumerate it as follows:

$$- \cdots \leqq \lambda_{-2} \leqq \lambda_{-1} \leqq \lambda_0 = 0 < \lambda_1 \leqq \lambda_2 \leqq \cdots.$$

Let ϕ_k be the orthonormal base of eigenfunctions corresponding to the eigenvalues λ_k, and define the space $E = \{u \in Z | \sum_{k \in \mathbb{Z}} |\lambda_k| (u, \phi_k)^2 < \infty\}$. E is endowed with the scalar product $\langle u, v \rangle = \sum_{\mathbb{Z}} (|\lambda_k| + 1) u_k \cdot \bar{v}_k$, where $u_k = (u, \phi_k)$, and the norm $\|u\| = \langle u, u \rangle^{1/2}$. E is a Hilbert space which is compactly embedded in Z. We now define the operator L on E by

$$\langle Lu, v \rangle = \sum_{k \in \mathbb{Z}} \lambda_k u_k \bar{v}_k \quad \text{for all} \quad u, v \in E.$$

Then L is a bounded selfadjoint operator in E, whose spectrum is $\sigma(L) = \{\lambda_k / (|\lambda_k| + 1),\ k \in \mathbb{Z}\}$.

Furthermore, let $A : Z \to Z$ be a selfadjoint, positive, equivariant isomorphism such that $LA = AL$. Note that this implies that $\ker(\mathscr{L} - \lambda_j)$ is an invariant subspace for A, and thus we can assume that ϕ_j is an eigenfunction of A. Hence the ϕ_j can be assumed to form a complete system of eigenfunctions for the eigenvalue problem

$$\mathscr{L} u = \mu A u. \tag{3.17}$$

For later purposes we normalize the ϕ_j as follows: $(A\phi_i, \phi_j) = \delta_{ij}$, $i, j \in \mathbb{Z}$. Let us denote by

$$\cdots \leqq \mu_{-2} \leqq \mu_{-1} \leqq \mu_0 = 0 < \mu_1 \leqq \mu_2 \leqq \cdots$$

the sequence of eigenvalues of (3.17), each repeated according to multiplicity.

We are interested in finding a minimax characterization of the positive eigenvalues of (3.17). A suitable tool for this is given by the relative index $\gamma_r(\cdot|E^+)$ introduced in subsection 2.3, where E^+ denotes the positive eigenspace of L (here we assume for simplicity that $E^0 \subset (E^+)^\perp$). Denoting by $p: E\backslash\{0\} \to S = \{u \in E;\ \|u\| = 1\}$ the radial projection, and setting $\gamma_r(\cdot) = \gamma_r(\cdot|E^+)$, let

$$\Gamma_k(G_1) = \{B \in \mathscr{F} | B \subset G_1,\ \gamma_r(pB) \geqq k\},$$

where

$$G_1 = \{u \in E | \tfrac{1}{2}(Au, u) = 1\},$$

and set $I(u) = \frac{1}{2}\langle Lu, u\rangle$.

Proposition 3.6. *The positive eigenvalues of* (3.17) (*repeated according to multiplicity*) *are given by*

$$\mu_k = \inf_{B \in \Gamma_k(G_1)} \sup_{u \in B} I(u). \tag{3.18}$$

Proof: First note that $u = \sum_{i\in \mathbf{Z}} \xi_i \phi_i \in G_1$ if and only if $1 = \frac{1}{2}(A \sum \xi_i\phi_i, \sum \xi_i \phi_i) = \frac{1}{2}\sum_{i\in\mathbf{Z}} \xi_i^2$. Hence, for $u \in B_k = G_1 \cap \operatorname{span}\{\phi_j | j \leqq k\}$ we have

$$\begin{aligned} \tfrac{1}{2}\langle Lu, u\rangle &= \frac{1}{2}\left\langle \left(\sum_{-\infty}^{k} \xi_i \phi_i\right), \sum_{-\infty}^{k} \xi_i\phi_i\right\rangle \\ &= \tfrac{1}{2}\sum_{-\infty}^{k} \xi_i^2 \mu_i \leqq \mu_k. \end{aligned}$$

By Proposition 2.10, $\gamma_r(pB_k) = k$. Hence we see from the above inequality that

$$\inf_{\Gamma_k(G_1)} \sup I(u) \leqq \sup_{B_k} I(u) \leqq \mu_k.$$

On the other hand, any set $B \in \Gamma_k(G_1)$ has, by Corollary 2.9, a nonempty intersection with $\operatorname{span}\{\phi_i | i \geqq k\}$. Let $v \in B \cap \operatorname{span}\{\phi_i | i \geqq k\}$. Then

$$\begin{aligned} \sup_B I(u) \geqq I(v) &= \tfrac{1}{2}\sum_k^\infty |v_i|^2 \langle L\phi_i, \phi_i\rangle \\ &= \tfrac{1}{2}\sum_k^\infty |v_i|^2 \mu_i \geqq \mu_k. \end{aligned}$$

In a similar fashion we obtain solutions for the nonlinear problem

$$\mathscr{L}u = \lambda \nabla f(u), \tag{3.19}$$

where

$$f \in C^{1,\mathrm{lip}}(Z_r, \mathbb{R}), \tag{3.20}$$

Z_r being the real Hilbert space $(Z, (\cdot,\cdot)_r = \mathcal{Re}(\cdot,\cdot))$,

$$\begin{gathered} f(0)=0, \quad \nabla f(0)=0, \quad f(u)>0, \quad u \neq 0, \\ f \text{ is homogeneous of degree 2.} \end{gathered} \tag{3.21}$$

Letting $E_r = (E, \langle\cdot,\cdot\rangle_r = \mathcal{Re}\langle\cdot,\cdot\rangle)$, we then see that $G = \{u \in E_r | f(u) = 1\}$ is a C^1-manifold which is radially diffeomorphic to the unit sphere S in E_r. Furthermore, $G \in \mathscr{E}$. Also note that there exist $c, d > 0$ such that

$$c \geqq |u|^2 \geqq d, \quad \text{and} \quad (\nabla f(u), u)_r = 2 \quad \text{for all} \quad u \in G. \tag{3.22}$$

The solutions of (3.19) can be obtained by solving the constrained variational problem

$$\nabla I(u)|_G = 0 \quad \text{in} \quad E_r.$$

In fact, if we define the operator N in E by $\langle N(u), v\rangle_r = (\nabla f(u), v)_r$ for all $v \in E$, then $N \in C^{0,1}(E, E)$ and is compact. Also, N is the gradient of f in E_r. Now, if u is a critical point of $I|_G$ in E_r, then

$$\langle Lu, v\rangle_r = \lambda \langle Nu, v\rangle_r,$$

for some $\lambda \in \mathbb{R}$ and for all $v \in E$, i.e.,

$$Lu = \lambda Nu. \tag{3.23}$$

Before stating our result, let us first show

LEMMA 3.7. *$I|_G$ satisfies the Palais–Smale condition* (PS).

Proof: We first note that the assumption $\nabla I|_G(u_n) \to 0$ implies the existence of $\lambda_n \in \mathbb{R}$ such that

$$z_n = Lu_n - \lambda_n N(u_n) \to 0 \quad \text{in} \quad E, \tag{3.24}$$

while, writing $u = u^+ + u^0 + u^-$, $u^+ = \sum_{k>0} \langle u, \phi_k\rangle \phi_k$, $u^- = \sum_{\lambda_k<0} \langle u, \phi_k\rangle \phi_k$, $u^0 \in \ker L$, the boundedness of $\langle Lu_n, u_n\rangle$ implies the existence of constants c_1, $c_2 > 0$ such that

$$-d + c_1 \|u_n^+\| \leqq \|u_n^-\| \leqq c_2 \|u_n^+\| + d. \tag{3.25}$$

Now, in view of (3.24), using (3.22), we obtain

$$\begin{aligned} |\lambda_n| &\leqq \tfrac{1}{2}(|\langle Lu_n, u_n\rangle| + \|z_n\| \cdot \|u_n\|) \\ &\leqq c + d\|u_n\|. \end{aligned} \tag{3.26}$$

Then, multiplying (3.24) by u_n^+ and using (3.20), (3.22) and (3.26) we get

$$\begin{aligned}\lambda_1\|u_n^+\|^2 &\leqq \langle Lu_n^+, u_n^+\rangle\\ &\leqq \|z_n\|\,\|u_n^+\| + |\lambda_n|\,|f'(u_n)|\,|u_n^+|\\ &\leqq c + d\|u_n\|;\end{aligned}$$

that is $\|u_n\| \leqq c$ for all $n \in \mathbb{N}$, in view of (3.25) and the fact that $\|u_n^0\| = |u_n^0|$ is bounded. Since E is compactly embedded in Z, we obtain therefore convergent subsequences $u_n \to u$ in Z for some $u \in E$, and $\lambda_n \to \lambda$. Relation (3.24) now yields the convergence of $u_n^\pm$ in E, while (u_n^0) contains a convergent subsequence since $|u_n^0|$ is bounded and $\dim \ker L < \infty$.

Our aim is to prove that $I|_G$ has infinitely many distinct positive critical levels. Letting $p: E\backslash\{0\} \to S$ and E^+ be as above, and setting $\gamma_r(\cdot) = \gamma_r(\cdot|E^+)$, set

$$\Gamma_k(G) = \{B \in \mathscr{F} | B \subset G,\ \gamma_r(pB) \geqq k\}.$$

Furthermore, let $K_c = \{u \in G | \nabla I|_G(u) = 0,\ I(u) = c\}$, the set of critical points of I at level c.

THEOREM 3.8. *Let*

$$c_k = \inf_{\Gamma_k(G)} \sup I(u), \qquad k \in \mathbb{N}. \tag{3.27}$$

Then $K_{c_k} \neq \emptyset$. Furthermore, if $c_k = c_{k+1} = \cdots = c_{k+p-1}$, then $\gamma(K_{c_k}) \geqq p$.

Remark 3.9. Since $\gamma_r(pB) \geqq 1$ implies $B \cap E^+ \neq \emptyset$ and hence

$$\sup_B I(u) \geqq \inf_{G\cap E^+} I(u) \geqq \frac{\lambda_1}{2(\lambda_1+1)} \inf \|u\|^2 > 0,$$

we get $0 < c_1 \leqq c_2 \leqq \cdots$, and therefore $K_{c_j} \cap E^0 = \emptyset$ for all $j \in \mathbb{N}$, since $E_0 \subset (E^+)^\perp$.

Thus, the multiplicity statement implies that if two minimax levels coincide, then there are in fact infinitely many critical points at that level.

Proof of Theorem 3.8: The proof follows the same pattern as that of Proposition 3.1, and we use the same notations. The only difference from Proposition 3.1 is that here we require the deformation η to be of the form

$$\eta(u) = Mu + K(u), \tag{3.28}$$

where $M: E \to E$ is an equivariant isomorphism with $M(E^+)^\perp = (E^+)^\perp$, and $P^-K: G \to E^- = (E^0 + E^+)^\perp$ is compact. In fact the deformation is constructed following the trajectories of

$$\frac{d\xi}{dt} = -L\xi(t) + \frac{\langle L\xi(t), N(\xi(t))\rangle_r}{\|N(\xi(t))\|^2} N(\xi(t)),$$
$$\xi(0) = u \in E\backslash\{0\}. \tag{3.29}$$

The flow $\xi(t, u)$ (which is globally defined, since the vector field has linear growth) leaves invariant the level sets of f and the map $\eta(u) = \xi(\bar{t}, u)$ satisfies (3.5), (3.6) and (3.8) for some $\bar{t} > 0$. Furthermore, $\xi(t, u)$ is of the form

$$\xi(t, u) = e^{-tL}u + K(t, u),$$

with $K(t, \cdot)$ compact (cf. Benci [3]), and $K(t, \rho u) = \rho K(t, u)$, $\rho \in \mathbb{R}^+$, since $N(\rho u) = \rho N(u)$, $\rho \in \mathbb{R}^+$, and in view of the unique solvability of (3.29).

We claim that $A \in \Gamma_k(G)$ implies that $\eta(A) \in \Gamma_k(G)$. In fact, we have

$$p \circ \eta \circ p^{-1}: S \to S,$$

with

$$p \circ \eta \circ p^{-1}(u) = \rho(u)(e^{-\bar{t}L} + K)u,$$

where K is compact, and $\rho(u) = \|p^{-1}u\| / \|e^{-\bar{t}L} + K)p^{-1}u\|$. It is easily checked that $0 < c \leqq \rho(u) \leqq d < \infty$, and hence we have by Propositions 2.11, 2.12 and 2.13

$$\gamma_r(\rho\eta(A)) = \gamma_r(p\eta p^{-1}(pA)) \geqq \gamma_r(pA).$$

The remaining arguments are the same as in Proposition 3.1.

Finally, we give a comparison of the eigenvalues μ_k of the linear problem (3.17) with the critical levels c_k of the nonlinear problem (3.19).

LEMMA 3.10. *Assume that* $\frac{1}{2}(1/\beta^2)(Au, u) \leqq f(u) \leqq \frac{1}{2}(Au, u)$ *for all* $u \in E$ *and some* $\beta > 1$. *Then*

$$\mu_k \leqq c_k \leqq \beta^2 \mu_k \quad \textit{for all} \quad k \in \mathbb{N}. \tag{3.30}$$

Proof: As before, let $G_1 = \{u \in E \mid \frac{1}{2}(Au, u) = 1\}$, and $G = \{u \in E \mid f(u) = 1\}$, and let $p_1: G \to G_1$ be the radial projection onto G_1. We first show that

$$\sup_{p_1(A)} I(u) \leqq \sup_{A} I(u) \leqq \beta^2 \sup_{p_1(A)} I(u) \quad \text{for all} \quad A \in \mathcal{F}, A \subset G. \tag{3.31}$$

To see this, let $u \in G$ and $tu \in G_1$, $t > 0$. Then

$$\frac{1}{\beta} \leqq t \leqq 1. \tag{3.32}$$

In fact, for $u \in G$ we get

$$1 = \tfrac{1}{2}(Atu, tu) = t^2 \tfrac{1}{2}(Au, u) \geqq t^2 f(u) = t^2.$$

From $f(u) \geqq (1/2\beta^2)(Au, u)$ we derive the other inequality.

Now, let $A \in \mathscr{F}$, $A \subset G$. If $u \in A$ and $tu \in p_1(A)$, we have

$$I(u) = (1/t^2)I(tu) \leqq \beta^2 I(tu) \leqq \beta^2 \sup_{p_1(A)} I(u),$$

and hence the right inequality in (3.31) follows. From the inequality on the right in (3.32) we obtain the left inequality in (3.31).

Since $A \in \Gamma_k(G)$ if and only if $p_1(A) \in \Gamma_k(G_1)$, (3.30) follows from (3.31).

4. Estimates for the Minimal Period of Solutions of Differential Equations

In this section we derive lower bounds for the length of some closed curves and for the period of solutions of some differential equations, collecting in the same simple framework various known and new results. These bounds are obtained from an inequality of Poincaré-Wirtinger type.

LEMMA 4.1. *Let $x \in H^1(S_T, \mathbb{R}^p)$ with $S_T = \mathbb{R}/T\mathbb{Z}$ and $y \in L^2(0, T; \mathbb{R}^p)$ such that*

$$\int_0^T y(t)\, dt = 0. \tag{4.1}$$

Then

$$T|\dot{x}|_{L^2} \cdot |y|_{L^2} \geqq 2\pi|(x, y)_{L^2}|, \tag{4.2}$$

where

$$(x, y)_{L^2} = \int_0^T x(t)y(t)\, dt \quad \textit{and} \quad |x|^2_{L^2} = (x, x)_{L^2}.$$

Proof: Fourier expansion and the Plancherel equality yield the Wirtinger inequality for $z \in H^1(S_T, \mathbb{R}^p)$ satisfying $\int_0^T z(t)\, dt = 0$:

$$2\pi|z|_{L^2} \leqq T|\dot{z}|_{L^2}. \tag{4.3}$$

Now let $c \in \mathbb{R}$ be such that $\int_0^T (x - c)\, dt = 0$. Then, by the Schwarz inequality and (4.3),

$$\begin{aligned} |(x, y)_{L^2}| &= |(x - c, y)_{L^2}| \\ &\leqq |x - c|_{L^2}|y|_{L^2} \\ &\leqq \frac{T}{2\pi}|\dot{x} - \dot{c}|_{L^2}|y|_{L^2} \\ &= \frac{T}{2\pi}|\dot{x}|_{L^2}|y|_{L^2}. \end{aligned}$$

Remark 4.2. Actually, equation (4.3) is a special case of (4.2). In fact, by $\int_0^T z\,dt = 0$, there exists $x \in H^1(S_T, \mathbb{R}^p)$ such that $\dot{x} = z$. Then (4.2) with $y = \dot{z}$ can be written in the form

$$\begin{aligned} T|z|_{L^2}|\dot{z}|_{L^2} &\geqq 2\pi|(x, \dot{z})_{L^2}| \\ &= 2\pi|(\dot{x}, z)_{L^2}| = 2\pi|z|_{L^2}^2. \end{aligned}$$

Let us now derive some consequences of inequality (4.2) (or the special case (4.3)). In particular, we shall give simple proofs of results of J. Yorke (Theorem 4.3) and C. B. Croke and A. Weinstein (Theorem 4.11).

THEOREM 4.3. (J. Yorke [22]). *Let x be a nontrivial* (i.e., $x \neq$ *constant*) *T-periodic solution of the differential equation*

$$\dot{x} = f(x), \tag{4.4}$$

where $f : \mathbb{R}^p \to \mathbb{R}^p$ is a k-Lipschitz continuous map. Then one has $T \geqq 2\pi/k$.

This result was pointed out to us by M. Willem, and the following proof was derived together with him.

Proof: By (4.4) one has, for all t, s,

$$\begin{aligned} |\dot{x}(t+s) - \dot{x}(t)| &= |f(x(t+s)) - f(x(t))| \\ &\leqq k|x(t+s) - x(t)|. \end{aligned}$$

Hence, $\dot{x}$ is differentiable for almost all t with

$$|\ddot{x}(t)| \leqq k|\dot{x}(t)|. \tag{4.5}$$

From (4.5) and (4.3) with $z = \dot{x}$ one gets

$$2\pi|\dot{x}|_{L^2} \leqq T|\ddot{x}|_{L^2} \leqq Tk|\dot{x}|_{L^2}.$$

Hence, $Tk \geqq 2\pi$.

The proof shows that the differential equation was just used to prove that

$$|\ddot{x}|_{L^2} \leqq k|\dot{x}|_{L^2}. \tag{4.6}$$

This argument works also for differential equations with memory. For example,

THEOREM 4.4. *Let x be a nontrivial T-periodic solution of the differential equation with delay*

$$\dot{x}(t) = f(x(t - \tau)), \tag{4.7}$$

where $f : \mathbb{R}^p \to \mathbb{R}^p$ is a k-Lipschitz continuous map and τ a real constant (*the delay*). *Then one has $T \geqq 2\pi/k$.*

Proof: As in the proof of Theorem 4.3, one shows that, for almost all t,

$$|\ddot{x}(t)| \leqq k|\dot{x}(t-\tau)|. \tag{4.8}$$

Thus

$$\int_0^T |\ddot{x}(t)|^2 \, dt \leqq k^2 \int_0^T |\dot{x}(t-\tau)|^2 \, dt.$$

But from the T-periodicity of x one derives

$$\int_0^T |\dot{x}(t-\tau)|^2 \, dt = \int_0^T |\dot{x}(t)|^2 \, dt.$$

We are back to (4.6).

Remark 4.5. This lower bound $T \geqq 2\pi/k$ does not depend on the delay τ. If more is known about f one could get (sharper) bounds depending on the delay. But in the general case this bound is reached: For example, the linear system in $\mathbb{R}^2 \cong \mathbb{C}$, with the 1-Lipschitz continuous f defined by

$$f(z) = i\, e^{i\tau} z \quad \text{for all} \quad z \in \mathbb{C},$$

has a 2π-periodic solution $(t \to e^{it})$.

Let us now turn to differential equations with other types of conditions on the map f.

THEOREM 4.6. *Let $x \in H^1(S_T, \mathbb{R}^p)$ be a nontrivial T-periodic solution of*

$$\dot{x} = f(x), \tag{4.9}$$

where the map $f: \mathbb{R}^p \to \mathbb{R}^p$ satisfies

$$f(z) \cdot Az \geqq |f(z)|^2 \quad \textit{for all} \quad z \in \mathbb{R}^p, \tag{4.10}$$

for some $p \times p$ matrix A. Then

$$T|\tfrac{1}{2}(A - A^*)| \geqq 2\pi, \tag{4.11}$$

where A^* is the adjoint of A.

Proof: Let us first remark that integration by parts yields

$$\begin{aligned} \int_0^T A^* x(t) \dot{x}(t) \, dt &= \int_0^T x(t) A\dot{x}(t) \, dt \\ &= -\int_0^T \dot{x}(t) A x(t) \, dt. \end{aligned}$$

Hence, $(\dot{x}, (A-A^*)x)_{L^2} = 2(\dot{x}, Ax)_{L^2}$. Relations (4.9) and (4.10) together establish

$$(4.12) \qquad \begin{aligned}(\dot{x}, (A-A^*)x)_{L^2} &= 2\int_0^T f(x(t))Ax(t)\,dt \\ &\geqq 2\int_0^T |f(x(t))|^2\,dt = 2|\dot{x}|^2_{L^2}.\end{aligned}$$

Let us now write (4.2) with $y = (A-A^*)\dot{x}$:

$$(4.13) \qquad T|\dot{x}|_{L^2}|y|_{L^2} \geqq 2\pi|(x, (A-A^*)\dot{x})_{L^2}|.$$

Using $|y|_{L^2} \leqq |A-A^*|_{L^2}|\dot{x}|^2_{L^2}$ and (4.3) and (4.12) one gets

$$T|A-A^*|_{L^2}|\dot{x}|^2_{L^2} \geqq 4\pi|\dot{x}|^2_{L^2}.$$

Hence we have obtained the result.

Remark 4.7. If $f:\mathbb{R}^p \to \mathbb{R}^p$ is invertible, then condition (4.10) is equivalent to

$$(A^*y, f^{-1}(y)) \geqq |y|^2 \quad \text{for all} \quad y \in \mathbb{R}^p.$$

This will be satisfied if Af^{-1} is a 1-monotone function and $f^{-1}(0) = 0$.

Remark 4.8. If A is selfadjoint, then (4.11) cannot be satisfied: in this case there are no nontrivial periodic solutions, as is well known.

THEOREM 4.9. *Let $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ be such that*

$$(4.14) \qquad (H'(y), y) \geqq \gamma|H'(y)|^2 \quad \textit{for all} \quad y \in \mathbb{R}^{2N},$$

for some constant $\gamma > 0$. Let x be a nontrivial T-periodic solution of

$$(4.15) \qquad \dot{x} = JH'(x),$$

(J as before). Then,

$$(4.16) \qquad T \geqq 2\pi\gamma.$$

Proof: Let us apply Theorem 4.6 with $f(y) = JH'(y)$ for all $y \in \mathbb{R}^p$, and $A = J/\gamma$. From (4.14) one gets (4.10) ($J^2 = -Id$). Hence, we establish (4.16) from (4.11) and $|A - A^*| = 2/\gamma$.

The next theorem specifies geometric conditions under which (4.14) is satisfied.

THEOREM 4.10. *Let $H \in C^2(\mathbb{R}^{2N}\backslash\{0\}, \mathbb{R})$, $H(z) > 0$ for all $z \neq 0$, and $H(tz) = t^2H(z)$ for all z and for all $t > 0$. Set $\Sigma = \{u \in \mathbb{R}^{2N};\ H(u) \equiv 1\}$, and assume that, for some $\rho > 0$, (1.5) holds, i.e.,*

$$T_x\Sigma \cap \mathring{B}_\rho = \varnothing \quad \textit{for all} \quad x \in \Sigma.$$

Then (4.14) *is satisfied with* $\gamma = \frac{1}{2}\rho^2$, *and hence we have for any nontrivial T-periodic solution of* (4.15):

$$T \geqq \pi\rho^2. \tag{4.17}$$

Proof: (1.5) implies $\xi \cdot \nabla H(\xi) \geqq \rho|\nabla H(\xi)|$ for all $\xi \in \Sigma$. By homogeneity, $\xi \cdot \nabla H(\xi) = 2H(\xi) = 2$ for all $\xi \in \Sigma$, and thus $|\nabla H(\xi)| \leqq 2/\rho$ for all $\xi \in \Sigma$, by the above inequality. Hence

$$\xi \cdot \nabla H(\xi) \geqq \rho|\nabla H(\xi)| \geqq \tfrac{1}{2}\rho^2|\nabla H(\xi)|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^{2N},$$

by homogeneity. Consequently, (4.14) is satisfied with $\gamma = \frac{1}{2}\rho^2$, and we conclude that $T \geqq \pi\rho^2$ by Theorem 4.9.

We conclude this section by giving a simple proof of a result by C. B. Croke and A. Weinstein [9] on the length of certain closed curves on manifolds.

THEOREM 4.11. *Let* Σ *be a* C^2*-manifold, boundary of an open set* $\Omega \subset \mathbb{R}^p$, $0 \in \Omega$. *Assume that* (1.5),

$$T_x\Sigma \cap \mathring{B}_\rho = \varnothing \quad \textit{for all} \quad x \in \Sigma.$$

Let N_x *be the exterior normal at* $x \in \Sigma$ *and* $\gamma : [0, T] \to \Sigma$, *and assume*

$$|\dot{\gamma}| = 1, \quad \gamma(0) = \gamma(T) \quad \textit{and} \quad \int_0^T N_{\gamma(t)}\, dt = 0. \tag{4.18}$$

Then $T \geqq 2\pi\rho$.

Proof: Take $y = N_{\gamma(t)}$, and $x = y$. Since (1.5) implies $\gamma(t) \cdot N_{\gamma(t)} \geqq \rho|N_{\gamma(t)}|$ for all t, Lemma 4.1 yields

$$T \geqq \frac{2\pi(\gamma, N_\gamma)_{L^2}}{|\dot{\gamma}|_{L^2}|N_\gamma|_{L^2}} \geqq 2\pi\rho.$$

5. Proof of the Main Theorem

In this section we give a proof of Theorem 1.1 relying on the relative index introduced in subsection 2.3. In the next section we shall give another proof which is based on the methods of convex analysis.

5.1. The gauge function associated with Σ. We recall that $\Sigma = \partial\mathcal{R}$ and $\mathcal{R}$ is a compact strictly star-shaped region. Let $H_2(\cdot)$ be the square of the gauge associated with $\mathcal{R}$:

$$H_2(u) = \inf\{\lambda^2 | \lambda \in \mathbb{R}^+, u \in \lambda\mathcal{R}\}.$$

Note that $H_2(0)=0$ and $H_2 \in C^{1,\mathrm{lip}}(\mathbb{R}^{2N}, \mathbb{R})$. H_2 is positively homogeneous of degree 2. Since $\{u \in \mathbb{R}^{2N};\ H_2(u)=1\}=\Sigma$, it is classical that the systems (1.1),

$$\dot{z}=JH'(z),$$

and

$$\dot{z}=JH_2'(z) \tag{5.1}$$

have the same periodic orbits on Σ (see e.g. Rabinowitz [18] or Ekeland-Lasry [11]). Therefore, in what follows, the function appearing in (1.1) will be assumed to be the square of the gauge associated with $\mathscr{R}$.

Finally, the geometric assumption (1.4) on Σ, $\mathscr{R}$ made in subsection 1.1 implies the following inequalities on H:

$$\frac{1}{2\beta^2}(\Omega u, u) \leqq H(u) \leqq \frac{1}{2\alpha^2}(\Omega u, u) \quad \text{for all} \quad u \in \mathbb{R}^{2N}. \tag{5.2}$$

5.2. The functional framework and a variational principle. For convenience, we introduce first the following complex notations. We identify $\mathbb{R}^{2N}$ with $\mathbb{C}^N$ through the isomorphism $(p, q) \leftrightarrow p+iq, p, q \in \mathbb{R}^N$; $(\xi, \eta)_{\mathbb{C}^N}=\sum_{j=1}^N \xi_j \bar{\eta}_j$ is the usual hermitian product with corresponding norm $|\cdot|$. For a function $G \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, $\nabla G(p, q)$ denotes the gradient with respect to the real structure, but $u \to \nabla G(u) = (G_p+iG_q)(p, q)$ will be thought of as a map in $\mathbb{C}^N$. Equation (1.1) can now be written in complex form (1.1)

$$-i\dot{z}=\nabla H(z), \qquad z=p+iq \in C^1(S^1, \mathbb{C}^N).$$

Let us now set $Z=L^2(S^1, \mathbb{C}^N)$ with scalar product

$$(u, v)=\frac{1}{2\pi}\int_0^{2\pi}(u(t), \bar{v}(t))_{\mathbb{C}^N}\, dt, \qquad |u|=(u, u)^{1/2},$$

and let us consider the densely defined selfadjoint linear operator $\mathscr{L}: Z \supset \mathscr{D}(\mathscr{L}) \to Z$ given by $\mathscr{L}z=-i\dot{z}$. Note that $\sigma(\mathscr{L})=\mathbb{Z}$, $\ker \mathscr{L} \simeq \mathbb{C}^N$, and the normalized eigenvectors corresponding to $k \in \mathbb{Z}$ are $\phi_{kj}=e^{ikt}\varepsilon_j$, $\{\varepsilon_1, \cdots, \varepsilon_N\}$ canonical basis in $\mathbb{C}^N$. Any $z=(z_1, \cdots, z_N) \in Z$ has the Fourier expansion

$$z=\sum_{k \in \mathbb{Z}} z_k e^{ikt}=\sum_{\substack{k \in \mathbb{Z} \\ j=1, \cdots, N}} z_{kj}\phi_{kj}, \qquad z_k \in \mathbb{C}^N,\ z_{kj} \in \mathbb{C}, \tag{5.3}$$

and we set, following subsection 3.2,

$$E=H^{1/2}(S^1, \mathbb{C}^N)=\left\{z \in Z \,\middle|\, \sum_{k \in \mathbb{Z}}(1+|k|)|z_k|^2<+\infty\right\}.$$

The scalar product in E is given by $\sum_{k \in \mathbb{Z}}(1+|k|)(u_k, v_k)_{\mathbb{C}^N}$. As in subsection 3.2 we denote by L the extension of $\mathscr{L}$ to E and we set $I(u)=\frac{1}{2}\langle Lu, u\rangle$, $u \in E$.

Furthermore, we set $Z_r = (Z, (\cdot,\cdot)_r = \mathcal{R}e\,(\cdot,\cdot))$, i.e., Z equipped with the real structure, and denote by $f \in C^{1,\mathrm{lip}}(Z_r, \mathbb{R})$ the functional $z \to (1/2\pi)\int_0^{2\pi} H(z)$, $z \in Z$. Note that, if $z = p + iq \in Z$, then

$$\mathcal{R}e\,(\nabla f(z), z) = \frac{1}{2\pi}\int_0^{2\pi} (H_p p + H_q q) \geqq \frac{\omega_1}{\beta^2}|z|^2$$

by (5.2), so that $G = \{u \in E \mid f(u) = 1\}$ is a C^1-manifold, radially diffeomorphic to the unit sphere in E_r.

PROPOSITION 5.1. *If $u \in G$ is a critical point of $I|_G$ and $\sigma = I(u) > 0$, then $z(t) = u(t/\sigma)$ is a $2\pi\sigma$-periodic solution of* (1.1) *lying on* Σ.

Proof: A straightforward application of the Lagrange multiplier rule yields the existence of $\lambda \in \mathbb{R}$ such that

$$Lu = \lambda N(u), \tag{5.4}$$

where N is defined by the relation

$$\mathcal{R}e\,\langle Nu, v\rangle = \mathcal{R}e\,(\nabla f(u), v) = \frac{1}{2\pi}\int_0^{2\pi} H_p(p,q)\hat{p} + H_q(p,q)\hat{q},$$
$$u = p + iq,\ v = \hat{p} + i\hat{q}.$$

From this it follows by standard regularity arguments that $u \in C^1$, and then $-i\dot{u} = \lambda \nabla H(u)$.

Now, by (5.4) we get $I = \lambda f(u)$, and since $u \in G$, we obtain $\lambda = \sigma$. Hence $z(t) = u(t/\sigma)$ is a $2\pi\sigma$-periodic solution of (1.1). Finally, if $h = H(p,q)$, $u = p + iq$, it follows from $u \in G$ that $h = 1$, i.e., (p,q) lies on Σ.

5.3. Proof of Theorem 1.1. There is a natural S^1-action in the present framework: the time shifts. For $\theta \in \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$ and $u \in E$, we denote $T_\theta u = u(\cdot + \theta)$. Clearly, the operator L introduced in subsection 5.2 is equivariant, while G and I are invariant. Also, L and f satisfy the assumptions made in subsection 3.2 and, defining $(Au)(t) = \Omega(u(t))$, we obtain a bounded, equivariant, selfadjoint linear isomorphism in Z which commutes with L. Note that in this case the positive eigenvalues of (3.17),

$$\mathcal{L}u = \mu_k A u,$$

are of the form k/ω_i, $k \in \mathbb{N}$, $i = 1, \cdots, N$.

Applying Proposition 3.6 we obtain the characterization (3.18) for μ_k, and by Theorem 3.8 we have a sequence of positive critical values for $I|_G$.

We remark that if u is a critical point of $I|_G$, then $u_k(t) = u(kt)$, $k \in \mathbb{N}$, is a critical point with $I(u_k) = kI(u)$. But the u_k all give rise to the same orbit on Σ. Similarly, if $\bar{u}$ is a critical point having minimal period $2\pi/k$, then $u(t) = \bar{u}(t/k)$ is a critical point having minimal period 2π and $\bar{u} = u_k$. We shall call such a u the *primitive critical point* corresponding to $\bar{u}$.

To complete the proof it is enough to find N distinct primitive critical points. The crucial argument relies on the comparison between the μ_k and the c_k which was given in Lemma 3.10 (the assumption there easily follows from (5.2)).

To simplify notation, we assume that $\alpha = 1$ in what follows. This is in fact no loss of generality, since we can always redefine $\{\omega_1, \cdots, \omega_N\}$ to be $\{(\omega_1/\alpha), \cdots, (\omega_N/\alpha)\}$ so that for the new ellipsoid

$$\mathscr{E}_\alpha = \left\{ \sum_{i=1}^{N} \frac{\omega_i}{2\alpha} (x_i^2 + x_{i+N}^2) \leqq 1 \right\}$$

one has

$$\mathscr{E}_\alpha \subset \mathscr{R} \subset \frac{\beta}{\alpha} \mathscr{E}_\alpha = \tilde{\beta} \mathscr{E}_\alpha.$$

Consider the critical levels of $I|_{G_1}$ given by $1/\omega^1, \cdots, 1/\omega^l$ defined as in (1.6). In view of Proposition 3.6, p_i critical levels of minimax type coincide with $1/\omega^i$, $i = 1, \cdots, l$. By (3.30) there are at least p_i critical levels of $I|_G$ in $\mathscr{T}_i = [1/\omega^i, \beta^2(1/\omega^i)]$, $i = 1, \cdots, l$. Set $U_i = \{u \in G | u$ is a primitive critical point of $I|_G$, $u(jt) \in \mathscr{T}_i$ for some $j \in \mathbb{N}\}$.

LEMMA 5.2. *Assume* (1.4) *holds for some* $\beta^2 < 1 + \delta$, $(\alpha = 1)$, δ *given in* (1.9). *Then*

$$\# U_i \geqq p_i, \qquad i = 1, \cdots, l.$$

Proof: Since the subscript i remains fixed in the subsequent argument, we drop it to simplify the notation. As a consequence of the multiplicity statement in Theorem 3.8, if two (or more) critical levels coincide, then the corresponding critical set has index greater than or equal to 2, so that there are infinitely many distinct closed orbits on Σ. Hence we can assume that there are p distinct critical levels in $\mathscr{T}$. Thus, let $u_1, \cdots, u_p$ be the corresponding primitive critical points $u_j(t) = \bar{u}_j(t/h_j)$, $1 \leqq j \leqq p$. We recall that to u_j there corresponds a $2\pi \cdot I(\bar{u}_j)/h_j$ periodic solution to (1.1), so that by Theorem 4.10 we have $I(\bar{u}_j)/h_j \geqq \frac{1}{2}\rho^2$ in view of assumption (1.5). We claim that this implies

$$h_j < \frac{1}{\delta_1} + 1, \qquad \delta_1 \text{ given by (1.7).} \tag{5.5}$$

In fact, let $c = I(\bar{u}) = I(\bar{u}_j)$ (dropping subscript j). Since $(1/\omega)(2/\rho^2) \leqq 1/\delta_1$, we get from $\beta^2 < 1 + \delta_1$ that $\frac{1}{2}\rho^2(1+\delta_1)/\delta_1 > \beta^2/\omega$, while by assumption $c \leqq \beta^2/\omega$. Hence

$$\tfrac{1}{2}\rho^2 h \leqq c \leqq \frac{\beta^2}{\omega} < \tfrac{1}{2}\rho^2\left(1 + \frac{1}{\delta_1}\right).$$

We now prove that the primitive critical points u_j, $j = 1, \cdots, p$, are all distinct. In fact, otherwise we would have $w = u_j = u_m$ for some $1 \leqq j < m \leqq p$, i.e., $w(h_j t) =$

$\bar{u}_j(t)$, $w(h_m t) = \bar{u}_m(t)$, with, say, $h_j < h_m < 1/\delta_1 + 1$. Then

$$\frac{1}{\omega} \leqq I(\bar{u}_j) = h_j I(w) = \frac{h_j}{h_m} I(\bar{u}_m) \leqq \frac{h_j}{h_m} \frac{\beta^2}{\omega},$$

which implies

$$\beta^2 \geqq \frac{h_m}{h_j} \geqq \frac{h_m}{h_{m-1}} > \frac{1/\delta_1 + 1}{1/\delta_1} = 1 + \delta_1,$$

in contradiction to our assumption.

We have found l groups of primitive critical points U_i, $i = 1, \cdots, l$, corresponding to the l families $[\omega_i]$, $i = 1, \cdots, l$. Since $\sum_{i=1}^{l} p_i = N$, the following lemma will complete the proof of the theorem.

LEMMA 5.3. $U_i \cap U_j = \varnothing, \qquad i \neq j.$

Proof: We argue by contradiction, assuming there is a $w \in U_n \cap U_m$, i.e., there exist $\bar{u} \in I^{-1}(\mathcal{T}_n)$ and $\bar{v} \in I^{-1}(\mathcal{T}_m)$, such that $w(t) = \bar{u}(t/h) = \bar{v}(t/k)$, $h, k < 1/\delta_1 + 1$ (cf. (5.5)). Since $I(w) = (1/h) I(\bar{u}) \in (1/h)\mathcal{T}_n$, and also $I(w) = (1/k) I(\bar{v}) \in (1/k)\mathcal{T}_m$, we have $(1/h)\mathcal{T}_h \cap (1/k)\mathcal{T}_m \neq \varnothing$. On the other hand, assuming $(k/h)(\omega^m/\omega^n) > 1$ and using the inequality $\beta^2 < 1 + \delta_2$ in (1.3) we see by (1.8) that $(k/h)(\omega^m/\omega^n) \geqq 1 + \delta_2 > \beta^2$; this in turn implies

$$\frac{1}{k}\left[\frac{1}{\omega^m}, \beta^2 \frac{1}{\omega^m}\right] \cap \frac{1}{h}\left[\frac{1}{\omega^n}, \beta^2 \frac{1}{\omega^n}\right] = \varnothing,$$

which contradicts the above statement.

6. An Alternate Proof of the Main Theorem

In this section we shall give another proof of Theorem 1.1, which will be based on the methods of convex analysis. In order to apply these methods, the corresponding functional will have to be suitably modified. The critical points will then be found by the use of the index of subsection 2.2 and the abstract results of subsection 3.1.

6.1. A dual variational principle. The variational formulation we use here and, in particular, the duality technique without convexity is in the spirit of the work of Ekeland and Lasry [12]. For other works concerning the duality technique in related problems the reader is referred to the survey of H. Brezis [8] and the bibliography therein.

For this variational formulation and for some technical reasons which appear later on we require an auxiliary function. In the sequel, a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is termed *admissible* if it satisfies the conditions

(6.1) $$\phi \in C^2(\mathbb{R}_+, \mathbb{R}_+),$$

(6.2) $$\phi(0)=0, \qquad \phi'(+\infty)=\lim_{s\to+\infty}\frac{\phi(s)}{s}=\lim_{s\to+\infty}\phi'(s)>0,$$

(6.3) $$\phi \text{ is strictly concave,}$$

(6.4) $$\sup_{s\in\mathbb{R}^+}|s^2\phi''(s)|<+\infty.$$

Given an admissible ϕ and H as in subsection 5.1 there exists a positive real constant K such that

$$G(z)=\phi\circ H(z)+\tfrac{1}{2}K|z|^2$$

is strictly convex in $\mathbb{R}^{2N}$. The artificial quadratic term $\frac{1}{2}K|z|^2$ will allow us to use "duality arguments" similar to the Clarke-Ekeland duality principle (see more comments in [12]). We denote by G^* the convex conjugate of G:

$$G^*(\zeta)=\sup_{z\in\mathbb{R}^{2N}}\{z\cdot\zeta-G(z)\}.$$

Under our assumptions, G^* is finite, of class C^1, strictly convex, and

$$G^*(z)>G^*(0)=0 \quad \text{for all} \quad z\neq 0.$$

Since

(6.5) $$G(z)\geqq\tfrac{1}{2}\mu|z|^2 \quad \text{for all} \quad z\in\mathbb{R}^{2N} \quad \text{for some} \quad \mu>0,$$

we have

(6.6) $$G^*(z)\leqq\frac{1}{2\mu}|z|^2 \quad \text{for all} \quad z\in\mathbb{R}^{2N}.$$

Furthermore, G^* is C^1, and since $|G'(z)|\geqq K|z|$ we get

(6.7) $$|G^{*\prime}(z)|\leqq K^{-1}|z| \quad \text{for all} \quad z\in\mathbb{R}^{2N}.$$

We again use the complex notation, identifying $\mathbb{R}^{2N}$ with $\mathbb{C}^N$, and again set $Z=L^2(S^1,\mathbb{C}^N)$ (cf. subsection 5.2). Now, let

$$X=H^1(S^1,\mathbb{C}^N)=\left\{z\in Z \,\middle|\, \sum_{k\in\mathbb{Z}}(1+k^2)|z_k|^2<\infty\right\},$$

where $z_k\in\mathbb{C}^N$ are the Fourier coefficients $z=\sum_{k\in\mathbb{Z}} z_k e^{ikt}$. The scalar product in X is given by $\langle u, v\rangle=\sum_{k\in\mathbb{Z}}(1+k^2)(u_k, v_k)_{\mathbb{C}^N}$.

For $u\in X$ we define the functional

$$f(u)=\int_0^{2\pi}\{\tfrac{1}{2}\mathcal{R}e\,(i\dot{u}-Ku, u)_{\mathbb{C}^N}+G^*(-i\dot{u}+Ku)\}\,dt.$$

For simplicity, we choose $K>0$ such that $K \notin \mathbb{N}$. By (6.6) and (6.7) it is clear that

$$f \in C^1(X_r, \mathbb{R}), \tag{6.8}$$

where $X_r = (X, \langle\ ,\ \rangle_r = \mathcal{R}e\langle\ ,\ \rangle)$ is the space X with real structure.

PROPOSITION 6.1. *Let $u \in X$, $u \neq 0$, be a critical point of $f: f'(u) = 0$. Then u is of class C^1, and setting $\tau = H(u)$, $\lambda = \phi'(\tau)^{-1}$ and $\mu = \tau^{-1/2}$, the function $z(t) = \mu \cdot u(\lambda t)$ is a solution of* (1.1) *which is $2\pi/\lambda$ periodic. Furthermore, $z(t) \in \Sigma$ for all t.*

Proof: The Euler-Lagrange equation associated with the functional f reads

$$u = (G^*)'(-i\dot{u} + Ku). \tag{6.9}$$

Since $G' = [(G^*)']^{-1}$ (Fenchel's relation), (6.9) is equivalent to

$$-i\dot{u} + Ku = G'(u) = \phi'(H(u))H'(u) + Ku,$$

that is

$$-i\dot{u} = \phi'(H(u))H'(u). \tag{6.10}$$

This implies that $H(u)$ is constant and the proposition follows.

6.2. Estimates on f. The purpose of introducing the auxiliary functional f is to obtain suitable bounds. In fact we shall show that there exists a finite-dimensional space $V \subset X$ such that f is bounded from below on $V^\perp$. We remark that $I|_G$ of subsection 2.1 does not enjoy any property of this kind.

Using the expansions (5.3) we have

$$\int_0^{2\pi} \mathcal{R}e\,(-i\dot{u} + Ku, u)_{\mathbb{C}^N}\,dt = \sum_{\substack{n \in \mathbb{Z} \\ k = 1, \cdots, N}} (n+K)|u_{nk}|^2, \tag{6.11}$$

$$\int_0^{2\pi} \mathcal{R}e\,(A(-i\dot{u} + Ku), -i\dot{u} + Ku)_{\mathbb{C}^N}\,dt = \sum_{\substack{n \in \mathbb{Z} \\ k = 1, \cdots, N}} a_k(n+K)^2|u_{nk}|^2, \tag{6.12}$$

if A is the diagonal matrix with real coefficients a_k. Using (5.2), the assumptions on ϕ and standard arguments in convex analysis, one obtains, given $0 < a < \phi'(0)$, $b = \phi'(\infty)$,

$$G^*(z) \geqq \tfrac{1}{2} \sum_{k=1}^{N} (b\omega_k + K)^{-1}|z_k|^2 - c \quad \text{for all} \quad z \in \mathbb{C}^N, \tag{6.13}$$

$$G^*(z) \leqq \tfrac{1}{2} \sum_{k=1}^{N} \left(\frac{a}{\beta^2}\omega_k + K\right)^{-1} |z_k|^2 \quad \text{for all} \quad z \in \mathbb{C}^N, \tag{6.14}$$

with $|z| \leqq \delta$, for some $\delta = \delta(a) > 0$.

LEMMA 6.2. *f is bounded from below on the subspace V of X defined as the orthogonal of*

$$V^{\perp} = \operatorname{span}\{e^{int}\varepsilon_k \mid -K < n < b\omega_k\}.$$

Proof: From the definition of f and (6.13) we derive (in terms of the expansion (5.3) and using (6.12))

$$f(u) \geqq \tfrac{1}{2} \sum_{\substack{n \in \mathbf{Z} \\ k=1,\cdots,N}} (n+K)[(b\omega_k+K)^{-1}(n+K)-1]|u_{n_k}|^2 - c. \tag{6.15}$$

For $u \in V$, all the coefficients in the right-hand side expansion in (6.15) are positive and the lemma is proved.

LEMMA 6.3. *Let W be the subspace of X defined by*

$$W = \operatorname{span}\left\{ e^{int}\varepsilon_k \,\middle|\, -K < n < \frac{a}{\beta^2}\omega_k \right\}.$$

Then, for some $\delta = \delta(a) > 0$,

$$f(u) < 0 \quad \textit{for all} \quad u \in W, \quad \|u\| = \delta.$$

Proof: Since W is finite-dimensional, all norms are equivalent. Thus, by (6.14), we know that if $\delta > 0$ is sufficiently small, for all $u \in W$ with $\|u\| \leqq \delta$,

$$f(u) \leqq \tfrac{1}{2}\sum (n+K)\left[\left(\frac{a}{\beta^2}\omega_k + K\right)^{-1}(n+K)-1\right]|u_{nk}|^2. \tag{6.16}$$

For $u \in W$, all the coefficients in the right-hand side of (6.16) are negative, and the lemma thus obtains.

6.3. The case of rational dependence of the frequencies. In order to make the argument more transparent, we first prove Theorem 1.1 under the additional assumption that the frequencies $\omega_1, \cdots, \omega_N$ are rationally dependent, that is we assume (1.6) for $j = 1, \cdots, N$:

$$\omega_j = n_j \omega, \qquad j = 1, \cdots, N, \tag{6.17}$$

$\omega > 0$ being the largest real satisfying (6.17).

Let us now choose an admissible function ϕ (cf. subsection 6.1) by specifying $\phi'(0)$ and $\phi'(+\infty)$. We assume that $1 < \beta^2 < 1 + \delta$ and let $0 < r < (1/\omega)(1 + \delta - \beta^2)$. Now set

$$\begin{aligned} \phi'(0) &= \frac{\beta^2}{\omega} + \tfrac{1}{2}r, \qquad & a &= \frac{\beta^2}{\omega} + \tfrac{1}{4}r, \\ \phi'(+\infty) &= \frac{1}{\omega} - s, \qquad & s &\in (0, \tfrac{1}{2}r), \end{aligned} \tag{6.18}$$

where $s \in (0, \frac{1}{2}r)$ is chosen such that $-J\dot{v} = ((1/\omega) - s)H'(v)$ has no (nontrivial) 2π-periodic solution v. We can, in fact, assume the existence of such an $s \in (0, \frac{1}{2}r)$, since otherwise our problem would be solved: if we had a nontrivial 2π-periodic solution v_s for all $s \in (0, \frac{1}{2}r)$, we would obtain a continuum of non-constant solutions $u_s(t) \in \Sigma$ of $-J\dot{u}_s = H'(u_s)$, with corresponding periods $((1/\omega) - s)2\pi$, given by $u_s(t) = (1/H^{1/2}(v_s))v_s(t((1/\omega) - s)^{-1})$.

LEMMA 6.4. *If $\phi'(+\infty)$ is chosen as above, then f satisfies the Palais-Smale condition* (PS).

Proof: In what follows we write $L^2 = L^2(S^1, \mathbb{C}^N)$, $H^1 = H^1(S^1, \mathbb{C}^N)$, and $H^{-1} = H^{-1}(S^1, \mathbb{C}^N)$.

Let now (u_n) be a sequence in H^1 such that $f'(u_n) \to 0$ in H^{-1}, i.e.,

$$Au_n - A(G^*)'(-Au_n) = \eta_n \to 0 \quad \text{in} \quad H^{-1},$$

where $A: H^1 \to H^{-1}$ is the linear operator given by $Au = J\dot{u} - Ku$. Since A is invertible (recall that $A^{-1}: H^{-1} \to L^2$ is bounded), we get

$$u_n - (G^*)'(-Au_n) = A^{-1}\eta_n = \varepsilon_n \to 0 \quad \text{in} \quad L^2.$$

Using $[(G^*)']^{-1}(y) = G'(y) = \phi'(H(y))H'(y) + Ky$ we obtain

$$\phi'(H(u_n - \varepsilon_n))H'(u_n - \varepsilon_n) + J\dot{u}_n = K\varepsilon_n.$$

We claim that $\|u_n\|_{C^{0,\alpha}} \leqq c$ for some fixed $\alpha \in (0, \frac{1}{2})$, for all $n \in \mathbb{N}$. We assume to the contrary that $\|u_n\|_{C^{0,\alpha}} \to +\infty$. Dividing the above equation by $\|u_n\|_{C^{0,\alpha}}$ and setting

$$v_n = \frac{u_n}{\|u_n\|_{C^{0,\alpha}}}, \qquad w_n = v_n - \frac{\varepsilon_n}{\|u_n\|_{C^{0,\alpha}}},$$

we obtain

$$\phi'(H(u_n - \varepsilon_n))H'(w_n) + J\dot{v}_n = K\frac{\varepsilon_n}{\|u_n\|_{C^{0,\alpha}}}.$$

Since $\|w_n\|_{L^2} \leqq c$ for all $n \in \mathbb{N}$, and $|\phi'(s)| \leqq c$ for all $s \in \mathbb{R}_+$, we deduce from this that $\|\dot{v}_n\|_{L^2} \leqq c$ for all $n \in \mathbb{N}$, and hence $\|v_n\|_{H^1} \leqq c$ for all $n \in \mathbb{N}$. Therefore, (v_n) contains a strongly convergent subsequence $v_n \to v$ in $C^{0,\alpha}$, $\|v\|_{C^{0,\alpha}} = 1$. Since $\dot{v}_n \to \dot{v}$ weakly in L^2 and $\phi'(H(u_n - \varepsilon_n)) = a_n \to a$ weakly in $L^2(S^1, \mathbb{R})$, we see that in the (weak) limit

$$aH'(v) = -J\dot{v}, \qquad v \neq 0, \qquad a \neq 0.$$

Since $H'(0) = 0$, we obtain by uniqueness that $|v(t)| \geqq \delta > 0$ for all $t \in [0, 2\pi]$, and hence $|v_n(t)| \geqq \frac{1}{2}\delta > 0$ for all $t \in [0, 2\pi]$ and for all $n \geqq n_0$. Therefore, $|u_n(t)| = \|u_n\|_{C^{0,\alpha}}|v_n(t)| \to +\infty$ uniformly in t, and, since $|\varepsilon_n| \leqq \psi \in L^2_+$ for a.e. $t \in [0, 2\pi]$ for a subsequence, $|(u_n - \varepsilon_n)(t)| \to +\infty$ for a.e. $t \in [0, 2\pi]$. Since also $|H'(u_n - \varepsilon_n)| \leqq$

$c(1+|u_n|+|\varepsilon_n|)\leqq c'+c''\psi$ for a.e. $t\in[0,2\pi]$, Lebesgue's dominated convergence theorem implies that $\phi'(H(u_n-\varepsilon_n)H'(w_n)\to\phi'(+\infty)H'(v)$ in L^2, and hence the equation for v has the form

$$-J\dot{v}=\phi'(+\infty)H'(v), \qquad \|v\|_{C^{0,\alpha}}=1, \qquad v\ 2\pi\text{-periodic}.$$

But this contradicts our assumption.

Hence $\|u_n\|_{C^{0,\alpha}}\leqq c$ for all $n\in\mathbb{N}$. Therefore, $u_n\to u$ in L^∞ for a subsequence, and one concludes as above that $\phi'(H(u_n-\varepsilon_n))H'(u_n-\varepsilon_n)\to\phi'(H(u))H'(u)$ in L^2. But then $\dot{u}_n\to\dot{u}$ in L^2 by the equation for u_n, and hence $u_n\to u$ in H^1.

Note that (6.18) implies that

$$\#\left\{(n,j)\,\middle|\,\frac{\omega_j}{\omega}\leqq n<\frac{a}{\beta^2}\,\omega_j, j=1,\cdots,N\right\}\geqq N, \tag{6.19}$$

since for $n=n_j$, we have $n_j=\omega_j/\omega<\omega_j/\omega+\frac{1}{4}r(\omega_j/\beta^2)=(a/\beta^2)\omega_j$. Then, in view of Lemmas 6.2 and 6.3, we obtain from Theorem 3.4 the existence of at least N distinct critical orbits in X. Let us call $u_1,\cdots,u_N$ these orbits: $f'(u_j)=0$ and $f(u_j)<0, j=1,\cdots,N$. By Proposition 6.1 we know that $z_j(t)=\mu_j u_j(\lambda_j t)$ is a periodic orbit of (5.1) on Σ, where

$$\mu_j=\tau_j^{-1/2}, \qquad \lambda_j=\phi'(\tau_j)^{-1}, \qquad \tau_j=H(u_j). \tag{6.20}$$

We now prove that our construction implies that the u_j give rise to periodic orbits z_j which are pairwise disjoint.

We first remark that the minimal period T_j of u_j is of the form

$$T_j=\frac{2\pi}{n_j} \quad\text{with}\quad 1\leqq n_j\leqq\frac{2\phi'(\tau_j)}{\rho^2}\leqq\frac{2\phi'(0)}{\rho^2}=d. \tag{6.21}$$

In fact, if $2\pi/n_j$ is the minimal period of u_j, then the solution z_j to (5.1) has minimal period $2\pi\phi'(\tau_j)/n_j$, which is greater or equal to $\pi\rho^2$ by Theorem 4.10.

Suppose now by way of contradiction that z_j and z_k are the same orbit for $j\neq k$. This implies that

$$u_j(t)=\mu u_k\left(\frac{\lambda_k}{\lambda_j}t+\nu\right), \tag{6.22}$$

with some constants $\mu>0$, $\nu\in\mathbb{R}$. Thus (since $u_j\neq u_k$), $\lambda_k\neq\lambda_j$, and we may assume that $\lambda_k<\lambda_j$. From (6.22) we conclude that the minimal periods T_j and T_k of u_j and u_k, respectively, satisfy

$$\frac{\lambda_k}{\lambda_j}T_j=T_k. \tag{6.23}$$

Whence, by using (6.21)

$$\frac{\lambda_k}{\lambda_j}=\frac{T_k}{T_j}=\frac{n_j}{n_k}, \tag{6.24}$$

and thus, by (6.20),

$$\frac{\phi'(\tau_k)}{\phi'(\tau_j)}=\frac{n_k}{n_j}. \tag{6.25}$$

Since $1 \leqq n_j < n_k \leqq d \equiv 2\phi'(0)/\rho^2$, we know that $n_k/n_j \geqq d/(d-1)$, and therefore (6.25) shows that

$$\frac{\phi'(0)}{\phi'(+\infty)}>\frac{d}{d-1}, \tag{6.26}$$

which implies

$$\phi'(0)-\tfrac{1}{2}\rho^2>\phi'(\infty). \tag{6.27}$$

Now, by using (6.18), we derive from (6.27)

$$\frac{\beta^2}{\omega}+r-\tfrac{1}{2}\rho^2>\frac{1}{\omega}. \tag{6.28}$$

By definition (1.7), (6.28) yields

$$\omega r>\tfrac{1}{2}\rho^2\omega+1-\beta^2=1+\delta-\beta^2. \tag{6.29}$$

But by assumption, $1+\delta-\beta^2>r\omega$, which contradicts (6.29).

Therefore the assumption that z_j and z_k represent the same orbit is false. We have thus shown that (5.1) has at least N distinct periodic orbits on Σ. The proof of Theorem 1.1 is thereby complete in the case of rational dependence of the frequencies.

6.4. General case. In the general case one has, as defined in subsection 1.2, l families of rationally dependent frequencies $\{\omega_j^i;\ j=1,\cdots,p_i\}$, $i=1,\cdots,l$. Assuming δ as in (1.9) we construct an auxiliary function ϕ_i, $i=1,\cdots,l$, for each family, with

$$\phi_i'(0)=\frac{\beta^2}{\omega^i}+\tfrac{1}{2}r,\qquad a_i=\frac{\beta^2}{\omega_i}+\tfrac{1}{4}r,\qquad \phi_i'(+\infty)=\frac{1}{\omega_i}-s_i,\qquad s_i\in(0,\tfrac{1}{2}r), \tag{6.30}$$

where s_i satisfies a nonresonance condition as in (6.18), and $r>0$ satisfies

$$r<\frac{1}{\omega}(1+\delta-\beta^2),\qquad \omega=\max_{i=1,\cdots,l}\omega^i. \tag{6.31}$$

By subsection 6.3 we know that for each i, $i=1,\cdots,l$, there exist p_i distinct periodic solutions of (5.1) on Σ: $z_1^i,\cdots,z_{p_i}^i$, $i=1,\cdots,l$. Each of the z_j^i, $j=$

$1, \cdots, p_i$, has a minimal period p_j^i of the form

$$p_j^i = 2\pi \frac{\phi_i'(\tau_j^i)}{n_j^i}, \qquad \tau_j^i > 0, \qquad n_j^i \in \mathbb{N}, \tag{6.32}$$

and hence

$$\begin{aligned} 1 \leqq n_j^i \leqq d^i &= \frac{2\phi_i'(0)}{\rho^2} = \frac{2}{\rho^2}\left(\frac{\beta^2}{\omega^i} + \tfrac{1}{2}r\right) \\ &< \frac{2}{\rho^2}\frac{1+\delta_1}{\omega^i} \leqq \frac{1+\delta_1}{\delta_1}, \end{aligned}$$

by (6.31), that is

$$1 \leqq n_j^i < \frac{1+\delta_1}{\delta_1} \quad \text{for all} \quad j = 1, \cdots, p_i, \qquad i = 1, \cdots, l. \tag{6.33}$$

Let us now prove that the *orbit* z_j^i cannot coinicide with another *orbit* z_q^k with $i \neq k$. Indeed it is sufficient to check that $p_j^i \neq p_q^k$, for all $i \neq k$, for all j, q. We argue by contradiction and assume that $p_j^i = p_q^k$. By (6.32) this means that

$$\frac{\phi_i'(\tau)}{n} = \frac{\phi_k'(\sigma)}{m}, \tag{6.34}$$

for some τ, $\sigma > 0$, $n, m \in \mathbb{N}$ and $1 \leqq n$, $m < (1+\delta_1)/\delta_1$. Since $(1/\omega^i) - \frac{1}{2}r \leqq \phi_i' \leqq \beta^2/\omega^i + \frac{1}{2}r$, (6.34) yields

$$\frac{1}{n}\left[\frac{1}{\omega^i} - \frac{1}{2}r, \frac{2}{\omega^i} + \frac{1}{2}r\right] \cap \frac{1}{m}\left[\frac{1}{\omega^k} - \frac{1}{2}r, \frac{2}{\omega^k} + \frac{1}{2}r\right] \neq \varnothing. \tag{6.35}$$

Without loss of generality we may assume that $1/n\omega^i < 1/n\omega^k$ (since $\omega^i/\omega^k \neq m/n$). Hence (6.35) means

$$\frac{1}{n}\frac{\beta^2}{\omega^i} + \frac{r}{n} \geqq \frac{1}{m}\frac{1}{\omega^k},$$

which implies

$$\beta^2 + r\omega^i \geqq \frac{n}{m}\frac{\omega^i}{\omega^k} \geqq 1 + \delta,$$

using the assumption (1.8). But this contradicts the choice of r in (6.31). Hence the proof of Theorem 1.1 is complete.

7. Further Comments

7.1. Generalization. As mentioned in Remark 1.2, we can define δ_p, $1 \leqq p \leqq N$, with

$$\delta = \delta_N \leqq \delta_{N-1} \leqq \cdots \leqq \delta_1 = +\infty, \tag{7.1}$$

and such that if Σ satisfies (1.3)-(1.5) for $1<\beta^2/\alpha^2<1+\delta_p$, then (1.1) has at least p periodic orbits on Σ. In fact, if in the definition of δ_1 and δ_2 we only use say the first p frequencies (instead of N), then it is clear that $\delta_N \leqq \delta_{N-1} \leqq \cdots \leqq \delta_1$. It is easy to see that such a choice of δ_p will yield p distinct orbits on Σ by the proofs in Section 5 or 6. Lastly, in the above construction, δ_1 can be chosen to be $+\infty$. Indeed, this means that Σ only satisfies (1.3), i.e., Σ is strictly star shaped with respect to the origin. By Theorem 3.8 (or Theorem 3.4) we still obtain infinitely many (or N) distinct critical orbits. Hence, by Proposition 5.1 (or Proposition 6.1) we know that (1.1) has at least one periodic orbit on Σ, for any given star-shaped Σ. The above construction thus allows one to also recover this result of Rabinowitz [18].

7.2. Almost commensurable frequencies. As seen above, the choice of $\delta>0$ is made to insure that one finds N *distinct* periodic orbits on Σ. In many cases the choice in (1.7)-(1.9) is satisfactory (within the framework of our method) but in other cases it may be poor. The typical example for this is when the ω^i defined in (1.6) are "nearly" rationally dependent. If, say, ω^i/ω^j, $i\neq j$, is very near to a rational of the form n/m with $1\leqq n$, $m<1+1/\delta_1$, then the constant δ_2 in (1.8) is near zero, and δ may thus be arbitrarily small. In this situation a better choice of δ is available. For this purpose we state the following result.

THEOREM 7.1. *Assume that there exist two reals $\underline{\omega}$, $\bar{\omega}>0$ and integers $1\leqq n_1,\cdots,n_N\leqq 1+1/\delta_1$ such that*

$$\underline{\omega}<\frac{n_j}{\omega_j}<\bar{\omega} \quad \text{for all} \quad k=1,\cdots,N, \tag{7.2}$$

and such that

$$1+\delta=(1+\delta_1)\frac{\underline{\omega}}{\bar{\omega}}>0, \qquad \delta_1=\tfrac{1}{2}\rho^2\,\underline{\omega}. \tag{7.3}$$

Then, if Σ satisfies (1.3)-(1.5) *with* $1<\beta^2/\alpha^2<1+\delta$, (1.1) *has at least N distinct orbits.*

Proof: Since n_j/ω^j are critical values of $I|_{G_1}$, it follows arguing as in subsection 5.3 that $I|_G$ has at least N (distinct) critical values of minimax type in $[\underline{\omega},\beta^2\bar{\omega}]$ (assuming $\alpha=1$). Let $\bar{u}_j$, $j=1,\cdots,N$, be the critical points with $I(\bar{u}_j)\in[\underline{\omega},\beta^2\bar{\omega}]$, and let $u_j(t)=\bar{u}_j(t/m_j)$ be the corresponding primitive critical points. As in Lemma 5.2, one can prove that $m_j<1+1/\delta_1$, $j=1,\cdots,N$, and this in turn implies, again as in Lemma 5.2, that the u_j, $j=1,\cdots,N$, are distinct.

Remark 7.2. Note that if $\bar{\omega}-\underline{\omega}$ becomes small, then δ_2 (as defined in (1.8)) tends to zero. In contrast, δ as defined in (7.3) has a positive limit when $\underline{\omega}/\bar{\omega}\to 1$, namely $\delta\to\delta_1$.

From these discussions one can see that the method developed here has a certain flexibility. Sharper estimates are to be derived in more particular cases involving more structure.

However, we conjecture that more general results hold. We even conjecture that Theorem 1.1 remains valid under the sole assumption (1.3).

Acknowledgment. The work of the first author was supported in part by NSF Grant MCS-8104242, that of the third by a CNR fellowship, and that of the fourth by Schweizerischer National-Fond.

Bibliography

[1] Ambrosetti, A., and Mancini, G., *On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories*, J. Diff. Eq. 43, 1981, pp. 1-6.
[2] Benci, V., *A geometrical index for the group* S^1 *and some applications to the research of periodic solutions of* ODE's, Comm. Pure Appl. Math. 34, 1981, pp. 393-432.
[3] Benci, V., *On the critical point theory for indefinite functions in the presence of symmetries*, Trans. Am. Math. Soc. 274, 1982, pp. 533-572.
[4] Berestycki, H., and Lasry, J.-M., *Orbites périodiques de systemes conservatifs*, Séminaire Goulaouic-Meyer-Schwartz, 1981-82, Exposé XXIV, Ecole Polytechnique, Palaiseau, 91128 France.
[5] Berestycki, H., and Lasry, J.-M., *A topological method for the existence of periodic orbits to conservative systems*, to appear.
[6] Berestycki, H., and Lasry, J.-M., *Orbites periodiques de systemes conservatifs: résolution de systeme nonlineaires équivariants sous l'action de* S^1, C.R. Acad. Sc., to appear.
[7] Berestycki, H., Lasry, J.-M., Mancini, G., and Ruf, B., *Sur le nombre des orbites périodiques des équations de Hamilton sur une surface étoilée*, Note aux C.R. Acad. Sci. 296, 1983, pp. 15-18.
[8] Brezis, H., Bull. A.M.S., 8, 1983, pp. 409-426.
[9] Croke, C. B., and Weinstein, A., *Closed curves on convex hypersurfaces and periods of nonlinear oscillations*, Inv. Math. 64, 1981, pp. 199-202.
[10] Ekeland, I., *A perturbation theory near convex Hamiltonian systems*, Tech. Report 82-1, 1982, U.B.C., Vancouver, Canada.
[11] Ekeland, I., and Lasry, J.-M., *On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface*, Ann. Math. 112, 1980, pp. 283-319.
[12] Ekeland, I., and Lasry, J.-M., Ann. Math. Décision, Birkhäuser, 1983.
[13] Fadell, E., Husseini, S., and Rabinowitz, P. H., *Borsuk-Ulam theorem for arbitrary* S^1*-actions and applications*, MRC Tech. Summary Report 2301, Madison, WI 53706, USA.
[14] Hofer, H., *A simple proof for a result of I. Ekeland and J.-M. Lasry concerning the number of periodic Hamiltonian trajectories on a prescribed energy surface*, preprint.
[15] Moser, J., *Periodic orbits near an equilibrium and a theorem by A. Weinstein*, Comm. Pure Appl. Math. 29, 1976, pp. 727-747.
[16] Nirenberg, L., *Comments on nonlinear problems*, Conference Proceedings, Catania, Sept. 1981.
[17] Rabinowitz, P. H., *A variational method for nonlinear eigenvalue problems*, CIME, Lecture Notes on "Nonlinear Eigenvalue Problems", Ed. Cremonese, Roma, 1974, pp. 141-195.
[18] Rabinowitz, P., *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. 31, 1978, pp. 157-184.
[19] Weinstein, A., *Lagrangian submanifolds and Hamiltonian systems*, Ann. Math. 98, 1973, pp. 377-410.

[20] Weinstein, A., *Normal modes for nonlinear Hamiltonian systems*, Inv. Math. 20, 1973, pp. 47-57.
[21] Weinstein, A., *Periodic orbits for convex Hamiltonian systems*, Annals of Math. 108, 1978, pp. 507-518.
[22] Yorke, J. A., *Periods of periodic solutions and the Lipschitz constant*, Proc. AMS 22, 1963, pp. 509-512.

Received June, 1983.