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Existence of positive solutions for the fractional q-difference boundary value problem

Yue Liang[1*], He Yang[2] and Hong Li[1]

*Correspondence: liangyuegsau@163.com
[1] Center for Quantitative Biology, College of Science, Gansu Agricultural University, Lanzhou 730070, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the existence of positive solutions for a class of fractional boundary value problems involving q-difference. By using the fixed point theorem of cone mappings, two existence results are obtained. Examples are given to illustrate the abstract results.

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1 Introduction

The theory of q-calculus or quantum calculus was initially developed by [6, 7] and it has many applications in the fields of hypergeometric series, particle physics, quantum mechanics and complex analysis. For a general introduction of q-calculus or quantum calculus, we refer to [1, 2, 8]. Recently, fractional boundary value problems with q-difference have been investigated by many authors; see [3, 4, 9–11] and the references therein. In [3], Ferreira considered the existence of positive solutions for the nonlinear q-fractional boundary value problem (BVP)

$$\begin{cases} D_q^\alpha u(t) = -f(t,u(t)), & t \in I := (0,1), \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \ge 0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $2 < \alpha \le 3$, $f : I^* \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $I^* = [0,1]$, $\mathbb{R}^+ = [0,+\infty)$. By utilizing a fixed point theorem in cones, he obtained the following existence theorem.

Theorem A *Let $\tau = q^n$ with $n \in \mathbb{N}$. Suppose that $f(t,u)$ is a nonnegative continuous function on $[0,1] \times \mathbb{R}^+$. If there exist two positive constants $r_2 > r_1 > 0$ such that the function f satisfies*

$(P1)$ $\frac{\beta}{[\alpha-1]_q} + M \max_{(t,u)\in[0,1]\times[0,r_1]} f(t,u) \le r_1$;

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(*P*2) $\frac{\beta}{[\alpha-1]_q} + N \max_{(t,u)\in[\tau,1]\times[\tau^{\alpha-1}r_2,r_2]} f(t,u) \geq r_2$,
where

$$[\alpha-1]_q = \frac{1-q^{\alpha-1}}{1-q},$$
$$M = \int_0^1 G(1,qs)\,d_qs,$$
$$N = \max_{t\in[0,1]} \int_\tau^1 G(t,qs)\,d_qs,$$

$G(t,qs)$ is the Green's function which will be specified later, then the BVP (1.1) has a solution satisfying $u(t) > 0$ for $t \in (0,1]$.

Clearly, the conditions (*P*1) and (*P*2) are strong in application. In 2015, Li et al. [9] studied a class of fractional Schrödinger equations with q-difference of the form

$$D_q^\alpha u(t) + \frac{n}{\hbar}\big(\aleph - \rho(t)\big)u(t) = 0, \quad t \in I, \tag{1.2}$$

where $\rho(t)$ is the trapping potential, n is the mass of a particle, $\hbar$ is the Planck constant, $\aleph$ is the energy of a particle. Let $\lambda = \frac{n}{\hbar}$ and $h(t) = \aleph - \rho(t)$. They transformed Eq. (1.2) to

$$D_q^\alpha u(t) + \lambda h(t) f\big(u(t)\big) = 0, \quad t \in I, \tag{1.3}$$

subject to the boundary conditions

$$u(0) = D_q u(0) = D_q u(1) = 0, \tag{1.4}$$

where $0 < q < 1$, $2 < \alpha \leq 3$, $f : I^* \times \mathbb{R} \to (0,\infty)$ is continuous, $h : I \to (0,\infty)$ is continuous. By applying a fixed point theorem in cones, they proved several theorems for the existence of positive solutions of the problem (1.3)–(1.4). Here, we just list two important results of [9].

Theorem B *Suppose that* (*H*1) *and one of* (*H*2) *and* (*H*3) *hold, where*
(*H*1) *$h(t)$ is continuous for $t \in (0,1)$ such that $\int_0^1 h(t)\,d_qt < +\infty$;*
(*H*2) $\lim_{u\to 0} \frac{f(u)}{u} = \infty$;
(*H*3) $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$.
Then the problem (1.3)–(1.4) has at least one positive solution provided that

$$0 < \lambda < \frac{\sup_{r>0} \frac{r}{\max_{0\leq u\leq r} f(u)}}{\max_{t\in[0,1]} \int_0^1 G(t,qs)h(s)\,d_qs}, \tag{1.5}$$

where $r > 0$ is constant.

Theorem C *Suppose that* (*H*1) *and one of* (*H*4) *and* (*H*5) *hold, where*
(*H*4) $\lim_{u\to 0} \frac{f(u)}{u} = 0$;
(*H*5) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

Then the problem (1.3)–(1.4) *has at least one positive solution provided that*

$$\frac{\inf_{r>0}\frac{r}{\min_{\tau r\le u\le r}f(u)}}{\min_{t\in[\tau,1]}\int_{\tau}^{1}G(t,qs)h(s)\,d_qs}<\lambda<\infty, \tag{1.6}$$

where $r>0$ *is constant.*

It is obvious that the conditions $(H2)$–$(H5)$ are weaker than $(P1)$–$(P2)$, but (1.5) and (1.6) are not easy to verify in application.

In the present work, we consider the fractional boundary value problem (Fr-BVP) with q-difference of the form

$$\begin{cases} D_q^{\alpha}u(t)+\omega(t)f(t,\delta(t)u(t))=0, & t\in I,\\ u(0)=D_qu(0)=D_qu(1)=0, \end{cases} \tag{1.7}$$

where $0<q<1$, $2<\alpha\le 3$, $\omega\in C[0,1]$, $\delta\in C(I^*,(0,+\infty))$, $f\in C(I\times\mathbb{R}^+,\mathbb{R}^+)$, f may be singular at $t=0$ and/or 1. Here, $\delta(t)$ is a scaling function of u in the nonlinearity f.

For the sake of simplicity, denote

$$\delta_m=\min_{t\in I^*}\delta(t),\qquad \delta_M=\max_{t\in I^*}\delta(t).$$

Throughout this paper, we always assume that the functions f and ω satisfy the following conditions.

$(A1)$ $\omega\in C[0,1]$ and there exists $\xi>0$ such that $\omega(t)\ge\xi$ for $t\in I$;

$(A2)$ $\int_0^1 G(1,qs)f(s,\delta_M)\,d_qs<+\infty$;

$(A3)$ $f(t,\delta_m)>0$ for any $t\in I$ and there exist constants $\sigma_1\ge\sigma_2>1$ such that, for every $\tau\in(0,1]$,

$$\tau^{\sigma_1}f(t,x)\le f(t,\tau x)\le\tau^{\sigma_2}f(t,x) \tag{1.8}$$

for any $t\in I$ and $x\in\mathbb{R}^+$;

$(A4)$ $f(t,\delta_m)>0$ for any $t\in I$ and there exist constants $0<\sigma_3\le\sigma_4<1$ such that, for every $\tau\in(0,1]$,

$$\tau^{\sigma_4}f(t,x)\le f(t,\tau x)\le\tau^{\sigma_3}f(t,x) \tag{1.9}$$

for any $t\in I$ and $x\in\mathbb{R}^+$.

Remark 1.1

(1) If f satisfies the assumption $(A3)$ or $(A4)$, then $f(t,x)$ is non-decreasing with respect to $x\in\mathbb{R}^+$ for every $t\in I$.

(2) The condition (1.8) is equivalent to

$$\tau^{\sigma_2}f(t,x)\le f(t,\tau x)\le\tau^{\sigma_1}f(t,x),\quad\forall\tau\ge 1. \tag{1.10}$$

(3) The condition (1.9) is equivalent to

$$\tau^{\sigma_3}f(t,x)\le f(t,\tau x)\le\tau^{\sigma_4}f(t,x),\quad\forall\tau\ge 1.$$

Remark 1.2 The assumptions $(A3)$ and $(A4)$ are order conditions. They are much easier to verify in application than the conditions $(H2)$–$(H5)$ and (1.5), (1.6).

Remark 1.3 If $\delta(t) \equiv 1$ for $t \in [0,1]$ and $f(t,u) = f(u)$, then the Fr-BVP (1.7) becomes to the problem (1.3)–(1.4) with $\omega(t) = \lambda h(t)$. Therefore, the Fr-BVP (1.7) is more general than the problem (1.3)–(1.4).

By using the fixed point theorem of cone mappings, we obtain the following theorems.

Theorem 1.1 *Let the assumptions* $(A1)$–$(A3)$ *hold. Then the Fr-BVP* (1.7) *has at least one positive solution* $u \in C[0,1]$.

Theorem 1.2 *Let the assumptions* $(A1)$, $(A2)$ *and* $(A4)$ *hold. Then the Fr-BVP* (1.7) *has at least one positive solution* $u \in C[0,1]$.

The rest of this paper is organized as follows. In Sect. 2 we introduce some preliminaries and notations which are useful in our proof. In Sect. 3, we will prove Theorems 1.1 and 1.2. Examples are given in Sect. 4 to illustrate the abstract results.

2 Preliminaries

In this section, we introduce some definitions and notations on fractional q-difference equations. Some related lemmas are also given in this section. For $q \in (0,1)$ and $a, b, \alpha \in \mathbb{R}$, we denote

$$[\alpha]_q := \frac{1-q^\alpha}{1-q}$$

and

$$(a-b)^{(\alpha)} := a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{n+\alpha}}.$$

The q-analogue of the power function $(a-b)^n$ is defined by

$$(a-b)^0 = 1$$

and

$$(a-b)^n = \prod_{k=1}^{\infty}\bigl(a-bq^k\bigr), \quad n \in \mathbb{N}.$$

The q-gamma function is given by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$

and it satisfies $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

Let ℓ be a function defined on $[0,1]$. The q-derivative of ℓ is

$$(D_q\ell)(t) = \frac{\ell(t) - \ell(qt)}{(1-q)t}, \quad t > 0,$$

and

$$(D_q\ell)(0) = \lim_{t\to 0}(D_q\ell)(t).$$

The q-derivative of ℓ of high order is given by

$$\bigl(D_q^0\ell\bigr)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\bigl(D_q^n\ell\bigr)(t) = D_q\bigl(D_q^{n-1}\ell\bigr)(t), \quad t \in [0,1], n \in \mathbb{N}.$$

The following definitions of fractional q-calculus are cited from [3].

Definition 2.1 The fractional q-integral of the Riemann–Liouville type of order $\alpha \geq 0$ for the function ℓ is defined by

$$\bigl(I_q^0\ell\bigr)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\bigl(I_q^\alpha\ell\bigr)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\ell(s)\,d_qs, \quad \alpha > 0, t \in [0,1].$$

Definition 2.2 The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ for the function ℓ is defined by

$$\bigl(D_q^0\ell\bigr)(t) = \ell(t), \quad t \in [0,1],$$

and

$$\bigl(D_q^\alpha\ell\bigr)(t) = \bigl(D_q^m I_q^{m-\alpha}\ell\bigr)(t)\alpha > 0, \quad t \in [0,1],$$

where $m := \lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

We refer the reader to the papers [3, 10] and the monographs [1, 2] for more details on the definitions of fractional q-calculus.

In order to prove the existence of positive solutions of the Fr-BVP (1.7), for any $h \in C[0,1]$, we first consider the linear Fr-BVP

$$\begin{cases} D_q^\alpha u(t) + h(t) = 0, & t \in I, \\ u(0) = D_q u(0) = D_q u(1) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 ([3]) *Let $0 < q < 1$ and $2 < \alpha \le 3$. For any $h \in C[0,1]$, the linear Fr-BVP* (2.1) *has a unique solution expressed by*

$$u(t) = \int_0^1 G(t,qs)h(s)\,d_q s,$$

where

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \le qs \le t < 1, \\ (1-qs)^{(\alpha-2)}t^{\alpha-1}, & 0 \le t \le qs < 1, \end{cases} \tag{2.2}$$

is the Green's function of the linear Fr-BVP (2.1).

Lemma 2.2 ([3]) *The Green's function $G(t,qs)$ has the following properties*:

(i) $G(t,qs) \ge 0$ *for all* $t,s \in I^*$;

(ii) $t^{\alpha-1}G(1,qs) \le G(t,qs) \le G(1,qs)$ *for all* $t,s \in I^*$.

By Lemma 2.1, we can define the solution of the Fr-BVP (1.7) as follows.

Definition 2.3 A function $u \in C[0,1]$ is called a solution of the Fr-BVP (1.7) if it satisfies the integral equation

$$u(t) = \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_q s, \quad t \in I^*.$$

If $u(t) > 0$ for $t \in I$, then it is called a positive solution of the Fr-BVP (1.7).

Let $E := C[0,1]$. Then E is a Banach space endowed with the norm

$$\|u\| = \max_{t\in I^*}\bigl|u(t)\bigr|, \quad \forall u \in E.$$

Let $\eta \in (0,1)$. Define a cone K in E by

$$K = \Bigl\{u \in E : u(t) \ge 0, t \in I^*, \min_{t\in[\eta,1]} u(t) \ge \eta^{\alpha-1}\|u\|\Bigr\}.$$

Then K is a nonempty closed convex cone of E.

Define an operator $Q : K \to E$ by

$$(Qu)(t) = \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_q s, \quad t \in I^*. \tag{2.3}$$

Lemma 2.3 *Let the assumptions* (*A*1)–(*A*3) *hold. Then $Q : K \to E$ is well defined, and $u \in E$ is a positive solution of the Fr-BVP* (1.7) *if and only if u is a positive fixed point of Q.*

Proof For fixed $u \in E$ with $u(t) \geq 0$ for all $t \in I^*$, choosing a constant $a \in (0,1)$ such that $a\|u\| < 1$. Then, for any $t \in I^*$, by (1.8) and (1.10), we have

$$\begin{aligned} f\bigl(t,\delta(t)u(t)\bigr) &\leq \left(\frac{1}{a}\right)^{\sigma_1} f\bigl(t,a\delta(t)u(t)\bigr) \\ &\leq \left(\frac{1}{a}\right)^{\sigma_1} \bigl[au(t)\bigr]^{\sigma_2} f\bigl(t,\delta(t)\bigr) \\ &\leq a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2} f(t,\delta_M). \end{aligned}$$

So, for any $t \in I^*$, by (2.2), we have

$$\begin{aligned} 0 &< \int_0^1 G(t,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\leq \int_0^1 G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\leq a^{\sigma_2-\sigma_1}\|u\|^{\sigma_2}\|\omega\| \int_0^1 G(1,qs)f(s,\delta_M)\,d_qs \\ &< +\infty. \end{aligned}$$

This implies that the operator $Q : K \to E$ is well defined. By Definition 2.3, $u \in E$ is a positive solution of the Fr-BVP (1.7) if and only if u is a positive fixed point of Q. □

Lemma 2.4 *If the assumptions* (*A*1), (*A*2) *and* (*A*4) *hold, then* $Q : K \to E$ *is well defined, and* $u \in E$ *is a positive solution of the Fr-BVP* (1.7) *if and only if* u *is a positive fixed point of* Q.

Lemma 2.5 $Q : K \to K$ *is a completely continuous operator.*

Proof For any $u \in K$ and $t \in I^*$, by Lemma 2.2 and (1.10), we have $(Qu)(t) \geq 0$ on I^* and

$$\begin{aligned} \min_{t\in[\eta,1]}(Qu)(t) &\geq \min_{t\in[\eta,1]} \int_0^1 t^{\alpha-1}G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &= \eta^{\alpha-1}\int_0^1 G(1,qs)\omega(s)f\bigl(s,\delta(s)u(s)\bigr)\,d_qs \\ &\geq \eta^{\alpha-1}\|Qu\|. \end{aligned}$$

Hence, $Q : K \to K$. By the Ascoli–Arzela theorem, one can prove that $Q : K \to K$ is completely continuous. □

At last, we state a fixed point theorem of cone mapping to end this section, which is useful in the proof of our main results.

Lemma 2.6 ([5]) *Let* E *be a Banach space,* $P \subset E$ *a cone in* E. *Assume that* Ω_1 *and* Ω_2 *are two bounded and open subset of* E *with* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. *If*

$$Q : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$$

is a completely continuous operator such that either

(i) $\|Qu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2$, *or*

(ii) $\|Qu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2$,

Then Q *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Proof of the main results

In this section, we will apply Lemma 2.6 to prove the existence of positive solutions of the Fr-BVP (1.7). For any $0 < r < R$, let

$$\Omega_r = \{u \in E : \|u\| < r\}, \qquad \Omega_R = \{u \in E : \|u\| < R\}.$$

Then $\partial\Omega_r = \{u \in E : \|u\| = r\}$, $\partial\Omega_R = \{u \in E : \|u\| = R\}$.

Proof of Theorem 1.1 On the one hand, defining an operator $Q : K \to E$ as in (2.3), we prove that there exists a constant $r \in (0, 1]$ such that

$$\|Qu\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_r.$$

In fact, for $u \in K$ with $\|u\| \leq 1$, we have

$$f\bigl(t, \delta(t)u(t)\bigr) \leq u^{\sigma_2}(t) f\bigl(t, \delta(t)\bigr) \leq \|u\|^{\sigma_2} f(t, \delta_M), \quad \forall t \in I^*.$$

So, by Lemma 2.2, we have

$$\begin{aligned} \|Qu\| &= \max_{t \in I^*} \left| \int_0^1 G(t, qs)\omega(s) f\bigl(s, \delta(s)u(s)\bigr)\, d_q s \right| \\ &\leq \|u\|^{\sigma_2} \|\omega\| \int_0^1 G(1, qs) f(s, \delta_M)\, d_q s \\ &= \beta_1 \|u\|^{\sigma_2}, \end{aligned}$$

where $\beta_1 = \|\omega\| \int_0^1 G(1, qs) f(s, \delta_M)\, d_q s$.

If $\beta_1 > 1$, choosing $r = (\frac{1}{\beta_1})^{\frac{1}{\sigma_2 - 1}}$, then $r \in (0, 1)$. For any $u \in K \cap \partial\Omega_r$, we have

$$\|Qu\| \leq \beta_1 \|u\|^{\sigma_2} = \beta_1^{1 - \frac{\sigma_2}{\sigma_2 - 1}} = r = \|u\|.$$

If $\beta_1 \leq 1$, choosing $r = 1$, then, for any $u \in K \cap \partial\Omega_r$, we have

$$\|Qu\| \leq \beta_1 \|u\|^{\sigma_2} = \beta_1 \leq 1 = r = \|u\|.$$

On the other hand, we prove that there exists a constant $R > 1$ such that

$$\|Qu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_R.$$

In fact, for $u \in K$ with $u(t) \geq 1$ for $t \in I^*$, we have

$$f\bigl(t, \delta(t)u(t)\bigr) \geq u^{\sigma_2}(t) f\bigl(t, \delta(t)\bigr) \geq u^{\sigma_2}(t) f(t, \delta_m), \quad \forall t \in I^*.$$

Thus, for every $u \in K \cap \partial\Omega_R$, by Lemma 2.5, we have $Qu \in K$ and, for any $\eta \in (0,1)$,

$$\begin{aligned}
\|Qu\| &\geq \min_{t\in[\eta,1]} (Qu)(t) \\
&= \min_{t\in[\eta,1]} \int_0^1 G(t,qs)\omega(s)f\big(s,\delta(s)u(s)\big)\,d_qs \\
&\geq \min_{t\in[\eta,1]} \int_0^1 G(t,qs)\omega(s)u^{\sigma_2}(s)f(s,\delta_m)\,d_qs \\
&\geq \min_{t\in[\eta,1]} t^{\alpha-1}\xi \int_\eta^1 G(1,qs)u^{\sigma_2}(s)f(s,\delta_m)\,d_qs \\
&\geq \eta^{(\sigma_2+1)(\alpha-1)}\xi\|u\|^{\sigma_2} \int_\eta^1 G(1,qs)f(s,\delta_m)\,d_qs \\
&= \beta_2\|u\|^{\sigma_2},
\end{aligned}$$

where $\beta_2 = \eta^{(\sigma_2+1)(\alpha-1)}\xi \int_\eta^1 G(1,qs)f(s,\delta_m)\,d_qs$.

If $\beta_2 < 1$, choosing $R = (\frac{1}{\beta_2})^{\frac{1}{\sigma_2-1}}$, then $R > 1 \geq r$. For any $u \in K \cap \partial\Omega_R$, we have

$$\|Qu\| \geq \beta_2\|u\|^{\sigma_2} = \beta_2^{1-\frac{\sigma_2}{\sigma_2-1}} = R = \|u\|.$$

If $\beta_2 \geq 1$, choosing $R = \beta_2 + 1$, then $R > 1 \geq r$. For $u \in K \cap \partial\Omega_R$, we have

$$\|Qu\| \geq \beta_2\|u\|^{\sigma_2} \geq \beta_2\|u\| \geq \|u\|.$$

Hence, by Lemma 2.6, Q has at least one fixed point $u^* \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$ satisfying $0 < r \leq \|u^*\| \leq R$. Hence for $\eta \in (0,1)$, $\min_{t\in[\eta,1]} u^*(t) \geq \eta^{\alpha-1}\|u^*\| > 0$ and it is a positive solution of the Fr-BVP (1.7). □

Proof of Theorem 1.2 Similar to the proof of Theorem 1.1, we can prove this theorem. So we omit the details here. □

Remark 3.1 If $\omega \in L^\infty[0,T]$, the results in Theorem 1.1 and 1.2 are still true.

4 Examples

Example 4.1 Consider the following BVP:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{5}{2}}u(t) + \frac{5-\sin\pi t^2}{t(1-t)}(e^{3t}u^3(t) + e^{2t}u^2(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad D_{\frac{1}{2}}u(0) = D_{\frac{1}{2}}u(1) = 0. \end{cases} \tag{4.1}$$

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $f(t,\delta(t)u(t)) = \frac{1}{t(1-t)}(e^{3t}u^3(t) + e^{2t}u^2(t))$ and $\omega(t) = 5 - \sin\pi t^2$, where $\delta(t) = e^t > 0$. Then $\xi = 4$, $\delta_m = 1$ and $\delta_M = e$. Clearly, $f(t,\delta_m) = \frac{2}{t(1-t)} > 0$ and

$$\int_0^1 G\left(1,\frac{1}{2}s\right)f(s,\delta_M)\,d_{\frac{1}{2}}s \leq \int_0^1 \frac{(1-\frac{1}{2}s)^{(\frac{1}{2})}(e^2+e^3)}{\Gamma_{\frac{1}{2}}(\frac{5}{2})s(1-s)}\,d_{\frac{1}{2}}s < +\infty,$$

where $\Gamma_{\frac{1}{2}}(\frac{5}{2}) = \frac{(\frac{1}{2})^{(\frac{3}{2})}}{(\frac{1}{2})^{\frac{3}{2}}}$. Hence the conditions $(A1)$ and $(A2)$ hold.

For $\tau \in (0,1]$, since

$$\frac{\tau^3}{t(1-t)}(x^3+x^2) \le f(t,\tau x) = \frac{1}{t(1-t)}(\tau^3 x^3 + \tau^2 x^2) \le \frac{\tau^2}{t(1-t)}(x^3+x^2),$$

then the condition $(A3)$ is satisfied with $\sigma_1 = 3$, $\sigma_2 = 2$. Hence, by Theorem 1.1, the BVP (4.1) has at least one positive solution $u \in C[0,1]$.

Example 4.2 Consider the following BVP:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{5}{2}} u(t) + \frac{\cos 3t^2+2}{t(1-t)}(e^{\frac{t}{3}} u^{\frac{1}{3}}(t) + e^{\frac{t}{4}} u^{\frac{1}{4}}(t)) = 0, & t \in (0,1), \\ u(0) = 0, \qquad D_{\frac{1}{2}} u(0) = D_{\frac{1}{2}} u(1) = 0. \end{cases} \tag{4.2}$$

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $f(t,\delta(t)u(t)) = \frac{1}{t(1-t)}(e^{\frac{t}{3}} u^{\frac{1}{3}}(t) + e^{\frac{t}{4}} u^{\frac{1}{4}}(t))$ and $\omega(t) = \cos 3t^2 + 2$, where $\delta(t) = e^t > 0$. Then $\xi = 1$, $\delta_m = 1$ and $\delta_M = e$. Only we verify $(A4)$. For $\tau \in (0,1]$, since

$$\frac{\tau^{\frac{1}{3}}}{t(1-t)}(x^{\frac{1}{3}} + x^{\frac{1}{4}}) \le f(t,\tau x) = \frac{1}{t(1-t)}(\tau^{\frac{1}{3}} x^{\frac{1}{3}} + \tau^{\frac{1}{4}} x^{\frac{1}{3}}) \le \frac{\tau^{\frac{1}{4}}}{t(1-t)}(x^{\frac{1}{3}} + x^{\frac{1}{4}}),$$

then the condition $(A4)$ is satisfied with $\sigma_3 = \frac{1}{4}$, $\sigma_4 = \frac{1}{3}$. Hence, by Theorem 1.2, the BVP (4.2) has at least one positive solution $u \in C[0,1]$.

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None of the authors have any competing interests in the manuscript.

Authors' contributions
All authors contributed equally in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Center for Quantitative Biology, College of Science, Gansu Agricultural University, Lanzhou 730070, P.R. China. [2]College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P.R. China.

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References

1. Ahmad, B., Ntouyas, S., Tariboon, J.: Quantum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities. World Scientific, Singapore (2016)
2. Ernst, T.: A Comprehensive Treatment of q-Calculus. Springer, Basel (2012)
3. Ferreira, R.: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**, 367–373 (2011)
4. Gao, C., Lv, L., Wang, Y.: Spectra of a discrete Sturm–Liouville problem with eigenparameter-dependent boundary conditions in Pontryagin space. Quaest. Math. (2019). https://doi.org/10.2989/16073606.2019.1680456
5. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, New York (1988)
6. Jackson, D., Fukuda, T., Dunn, O., Majors, E.: On q-definite integrals. J. Pure Appl. Math. **41**, 193–203 (1910)

7. Jackson, F.: On q-functions and a certain difference operator. Trans. R. Soc. Edinb. **46**, 253–281 (1908)
8. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2001)
9. Li, X., Han, Z., Li, X.: Boundary value problems of fractional q-difference Schrödinger equations. Appl. Math. Lett. **46**, 100–105 (2015)
10. Mao, J., Zhao, Z., Wang, C.: The unique iterative positive solution of fractional boundary value problem with q-difference. Appl. Math. Lett. **10**, 106002 (2020)
11. Wang, G.: Twin iterative positive solutions of fractional q-difference Schrödinger equations. Appl. Math. Lett. **76**, 103–109 (2018)

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