PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

EXISTENCE OF SETS OF UNIQUENESS OF l^p FOR GENERAL ORTHONORMAL SYSTEMS[1]

LEONARDO COLZANI

ABSTRACT. It is proved that for every orthonormal complete system in $L^2(0, 1)$ there exists a set A, of measure arbitrarily close to 1, which carries no nonzero function with Fourier transform in l^p, for every $p < 2$.

1. Suppose $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal complete system (ONC) in $L^2(0, 1)$. We call a Lebesgue measurable set $E \subset (0, 1)$ a set of uniqueness of l^p if no nonzero function $f \in L^2(0, 1)$, vanishing almost everywhere in the complement of E, satisfies the condition

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p < +\infty,$$

where $\{\hat{f}(n)\}_{n=1}^{\infty}$ denotes the Fourier transform of f with respect to the system $\{\phi_n\}$, i.e.

$$\hat{f}(n) = \int_0^1 f(x)\,\overline{\phi_n(x)}\,dx.$$

Y. Katznelson [6] first proved that the trigonometric system admits sets of uniqueness of l^p, for every $p < 2$, of Lebesgue measure arbitrarily close to 1 (see also [3]). Katznelson's theorem has been subsequently generalized to the system of characters of a nondiscrete locally compact abelian group by A. Figà-Talamanca and G. I. Gaudry [4], and to every uniformly bounded ONC by the author [2].

The aim of this paper is to prove a further extension of this result to every ONC. As a consequence we give a new proof of the generalization (due to W. Orlicz and A. M. Olevskiĭ) of a well-known theorem of Carleman stating that there exists a continuous function f such that

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p = +\infty \quad \text{for every } p < 2.$$

2. For $1 \leqslant p \leqslant +\infty$ we use $\|f\|_p$ and $\|\hat{f}\|_p$ in their usual meanings. The following lemmas hold.

Received by the editors October 30, 1980.
1980 *Mathematics Subject Classification*. Primary 42C15, 42C25.
[1] This work was partially supported by the C.N.R.

LEMMA 1. *Suppose* $\phi_1, \ldots, \phi_N$ *are functions in* $L^2(0, 1)$, *and* E *is an interval contained in* $(0, 1)$. *If* $\varepsilon > 0$ *and* $\delta > 0$, *there exists a function* $\Psi \in L^2(0, 1)$ *such that*:

(i) $\Psi(x) = 0$ if $x \notin E$;
(ii) $|\{x \in E/\Psi(x) \neq 1\}| < \delta|E|$;
(iii) $\|\Psi\|_2 < (2|E|/\delta)^{1/2}$;
(iv) $|\int_0^1 \Psi(x)\overline{\phi_j(x)}\, dx| < \varepsilon, j = 1, \ldots, N$.

PROOF. Let $k = [1/\delta] + 1$ and let n be a positive integer. We split E in k^n intervals $E_1, \ldots, E_{k^n}$ of the same measure. Set

$$\begin{aligned} \psi_n(x) &= 0 && \text{if } x \notin E, \\ &= 1 - k && \text{if } x \in E_1 \cup E_{k+1} \cup E_{2k+1} \cup \cdots \cup E_{k^n - k + 1}, \\ &= 1 && \text{if } x \in E \setminus E_1 \cup E_{k+1} \cup \cdots \cup E_{k^n-k+1}. \end{aligned}$$

A direct computation shows that ψ_n's satisfy (i)–(iii) for every n; moreover, $\{\psi_n\}_{n=1}^{\infty}$ tends to 0 weakly, and putting $\Psi = \psi_n$, with n large enough, (iv) is satisfied too. □

LEMMA 2. *Suppose* $\{\phi_n\}$ *is an ONC. Then, for every* $\delta > 0$, $q > 2$, $0 < a < 1$, *there exists a function* $\Psi \in L^2(0, 1)$ *such that*:

(i) $\Psi(x) = 0$ *if* $x \notin (0, a)$;
(ii) $|\{x \in (0, a)/\Psi(x) \neq 1\}| < \delta$;
(iii) $\|\hat{\Psi}\|_q < \delta$.

PROOF. Let ε and η be positive numbers to be specified later. Divide $(0, a)$ into m intervals $E_1, \ldots, E_m$ of measure less than η. We shall define the required function Ψ piecewise on every E_i.

Let

$$\begin{aligned} \psi_1(x) &= 1 && \text{if } x \in E_1, \\ &= 0 && \text{if } x \notin E_1, \end{aligned}$$

and put $n_1 = 1$.

Suppose now $\psi_1, \ldots, \psi_{i-1}$ have already been defined. Then there exists an integer n_i such that

$$\left| \sum_{j=1}^{i-1} \hat{\psi}_j(n) \right| < \varepsilon \quad \text{for every } n > n_i, \tag{1}$$

and, via Lemma 1, it is possible to construct a function $\psi_i \in L^2(0, 1)$ such that:

$\psi_i(x) = 0$ if $x \notin E_i$;
$|\{x \in E_i/\psi_i(x) \neq 1\}| < \delta|E_i|$;
$\|\psi_i\|_2 < (2|E_i|/\delta)^{1/2}$, and

$$|\hat{\psi}_i(n)| < \varepsilon/2^i \quad \text{for every } n < n_i. \tag{2}$$

Put $\Psi = \sum_{i=1}^m \psi_i$. It is easy to see that Ψ satisfies (i) and (ii). Moreover,

$$\|\Psi\|_2 < (2/\delta)^{1/2}. \tag{3}$$

In order to prove (iii) we observe that for every i and every n,

$$|\hat{\psi}_i(n)| \leq \|\psi_i\|_2 < (2|E_i|/\delta)^{1/2} < (2\eta/\delta)^{1/2} < \varepsilon \tag{4}$$

if $\eta = \eta(\varepsilon)$ is chosen small enough.

Then, if $n \geq n_m$, from (1) and (4) it follows that

$$|\hat{\Psi}(n)| \leq \left|\sum_{j=1}^{m-1} \hat{\psi}_j(n)\right| + |\hat{\psi}_m(n)| < 2\varepsilon,$$

and, if $n_{i-1} \leq n < n_i$,

$$|\hat{\Psi}(n)| \leq \left|\sum_{j=1}^{i-2} \hat{\psi}_j(n)\right| + |\hat{\psi}_{i-1}(n)| + \sum_{j=i}^{m} |\hat{\psi}_j(n)| = I_1 + I_2 + I_3.$$

But, it follows from (1) that $I_1 < \varepsilon$, from (4) that $I_2 < \varepsilon$, and from (2) that $I_3 < \sum_{j=i}^{m} (\varepsilon/2^j) < \varepsilon$. Collecting these results we obtain

$$\|\hat{\Psi}\|_\infty < 3\varepsilon, \tag{5}$$

and so, from (3) and (5),

$$\|\hat{\Psi}\|_q \leq \|\hat{\Psi}\|_2^{2/q} \cdot \|\hat{\Psi}\|_\infty^{(q-2)/q} < (2/\delta)^{1/q} \cdot (3\varepsilon)^{(q-2)/q} < \delta$$

if ε is small enough. □

REMARK 1. This lemma was originally proved by Y. Katznelson for the trigonometric system, and subsequently extended to any uniformly bounded ONC by A. Figà-Talamanca and G. I. Gaudry. Our proof, which holds for any, possibly unbounded, ONC is based on an idea of G. Alexits (see [**1**, Chapter II, §11]).

THEOREM. *For every ONC* $\{\phi_n\}$, *and every* $\varepsilon > 0$, *there exists a measurable set* $A \subset (0, 1)$, *with* $|A| < \varepsilon$, *such that if* $f \in L^2(0, 1)$ *vanishes a.e. in* A, *and* $\|\hat{f}\|_p < +\infty$ *for some* $p < 2$, *then* $f(x) = 0$ *a.e. in* $(0, 1)$.

The theorem follows from Lemma 2 as in [**2**].

REMARK 2. I. I. Hirschman and Y. Katznelson [**5**] proved that the trigonometric system admits closed sets which are sets of uniqueness of l^p, but not of $l^{p'}$, with $p < p' < 2$. For an arbitrary ONC this feature fails to hold, as is shown in [**2**].

3. It is interesting to notice that, using our theorem, it is possible to prove easily the extension of a well-known theorem of T. Carleman to every ONC (see [**7**, Chapter III, §4] for the original proof of this extension).

THEOREM. *For every ONC* $\{\phi_n\}$ *there exists a continuous bounded function* f *such that* $\|\hat{f}\|_p = +\infty$ *for every* $p < 2$.

PROOF. See [**2**].

REMARK 3. The theorems stated for orthonormal systems in $L^2(0, 1)$ can be easily extended to orthonormal systems in $L^2(-\infty, +\infty)$ or to orthonormal systems of square integrable functions over more general measure spaces.

References

1. G. Alexits, *Convergence problems of orthogonal series*, Pergamon Press, Oxford and New York, 1961.

2. L. Colzani, *Sets of uniqueness of l^p for general orthonormal complete systems*, Boll. Un. Mat. Ital. B **16** (1979), 1134–1143.

3. L. De Michele and P. M. Soardi, *A remark on sets of uniqueness of l^p*, Boll. Un. Mat. Ital. **11** (1975), 64–65.

4. A. Figà-Talamanca and G. I. Gaudry, *Multipliers and sets of uniqueness of L^p*, Michigan Math. J. **17** (1970), 179–191.

5. I. I. Hirschman and Y. Katznelson, *Sets of uniqueness and multiplicity for $l^{p,\alpha}$*, Israel J. Math. **3** (1965), 221–231.

6. Y. Katznelson, *Sets of uniqueness for some classes of trigonometrical series*, Bull. Amer. Math. Soc. **70** (1964), 722–723.

7. A. M. Olevskiĭ, *Fourier series with respect to general orthogonal systems*, Springer-Verlag, Berlin and New York, 1975.

Istituto Matematico "F. Enriques", via C. Saldini 50, 20133 Milano, Italy