M. NAKAI AND S. SEGAWA
KODAI MATH. J.
33 (2010), 99–115

EXISTENCE OF SINGULAR HARMONIC FUNCTIONS[1]

Mitsuru Nakai and Shigeo Segawa

Abstract

An afforested surface $W := \langle P, (T_n)_{n\in\mathbf{N}}, (\sigma_n)_{n\in\mathbf{N}}\rangle$, $\mathbf{N}$ being the set of positive integers, is an open Riemann surface consisting of three ingredients: a hyperbolic Riemann surface P called a plantation, a sequence $(T_n)_{n\in\mathbf{N}}$ of hyperbolic Riemann surfaces T_n each of which is called a tree, and a sequence $(\sigma_n)_{n\in\mathbf{N}}$ of slits σ_n called the roots of T_n contained commonly in P and T_n which are mutually disjoint and not accumulating in P. Then the surface W is formed by foresting trees T_n on the plantation P at the roots for all $n \in \mathbf{N}$, or more precisely, by pasting surfaces T_n to P crosswise along slits σ_n for all $n \in \mathbf{N}$. Let $\mathcal{O}_s$ be the family of hyperbolic Riemann surfaces on which there are no nonzero singular harmonic functions. One might feel that any afforested surface $W := \langle P, (T_n)_{n\in\mathbf{N}}, (\sigma_n)_{n\in\mathbf{N}}\rangle$ belongs to the family $\mathcal{O}_s$ as far as its plantation P and all its trees T_n belong to $\mathcal{O}_s$. The aim of this paper is, contrary to this feeling, to maintain that this is not the case.

1. Introduction

We denote by $HP(R)$ the vector subspace of the vector space $H(R)$ of harmonic functions on a Riemann surface R consisting of essentially positive harmonic functions on R, where u is essentially positive if u is expressed as $u = u_1 - u_2$ with $u_j \in H(R)^+ := \{v \in H(R) : v \geqq 0\}$ $(j = 1, 2)$, or equivalently, u is essentially positive if $|u|$ admits a harmonic majorant on R. We denote by $u \vee v$ ($u \wedge v$, resp.) the least (greatest, resp.) harmonic majorant (minorant, resp.) of u and v on R for u and v in $HP(R)$ so that $u \vee v$ and $u \wedge v$ also belong to $HP(R)$ and $u \wedge v = -((-u) \vee (-v))$. With respect to these lattice operations of the join $\vee$ and the meet $\wedge$, the vector space $HP(R)$ forms a vector lattice. Then the Jordan decomposition

$$u = u^+ - u^- \quad (u^+ := u \vee 0,\ u^- := -(u \wedge 0)) \tag{1.1}$$

of $u \in HP(R)$ gives the canonical way of expressing u as a function in

[1] 2000 *Mathematics Subject Classification*. Primary 30F20; Secondary 30F15, 30F25, 30C20.

Key words and phrases. afforested surface, essentially positive, hyperbolic, Joukowski mapping, parabolic, Parreau decomposition, Riemann surface, quasibounded, singular, Wiener compactification, Wiener (harmonic) boundary.

Received April 28, 2009.

$$(1.2)\qquad HP(R) = HP(R)^+ - HP(R)^+.$$

In view of the fact that the vector space $HB(R)$ of bounded harmonic functions on R is an important vector sublattice of $HP(R)$, we say that a $u \in HP(R)$ is *quasibounded* if

$$(1.3)\qquad u = \lim_{s,t \in \mathbf{R}^+, s,t \uparrow +\infty} (u \wedge s) \vee (-t)$$

locally uniformly on R, where $\mathbf{R}$ is the real number field and $\mathbf{R}^+ := \{t \in \mathbf{R} : t \geqq 0\}$, so that every $u \in HB(R)$ is trivially quasibounded on R. On the other hand, a $u \in HP(R)$ is said to be *singular* if

$$(1.4)\qquad (u \wedge s) \vee (-t) = 0$$

identically on R for every s and t in $\mathbf{R}^+$. We denote by $HP_q(R)$ ($HP_s(R)$, resp.) the vector subspace of $HP(R)$ consisting of quasibounded (singular, resp.) harmonic functions on R and we have the Parreau decomposition of $HP(R)$:

$$(1.5)\qquad HP(R) = HP_q(R) \oplus HP_s(R) \quad \text{(the direct sum decomposition).}$$

It can happen that $HP_q(R) = HP(R)$, or equivalently, $HP_s(R) = \{0\}$. We denote by $\mathcal{O}_s$ the class of hyperbolic Riemann surfaces R with $HP_s(R) = \{0\}$. The examples of R in the null class $\mathcal{O}_s$ is furnished by the following inclusion relation:

$$(1.6)\qquad \mathcal{O}_{HP} \backslash \mathcal{O}_G < \mathcal{O}_s \quad \text{(the strict inclusion),}$$

where $\mathcal{O}_{HP}$ is the family of open Riemann surfaces R with $HP(R) = \mathbf{R}$ and $\mathcal{O}_G$ is the family of parabolic Riemann surfaces R so that $R \notin \mathcal{O}_G$ means that R is hyperbolic in the sense that R carries the Green function $g(\cdot, \zeta; R)$ on R with its pole at any point ζ in R characterized as the minimal positive harmonic function on $R \backslash \{\zeta\}$ with

$$(1.7)\qquad -\Delta g(\cdot, \zeta; R) = 2\pi\delta_\zeta \quad \text{(the Dirac measure supported at } \zeta\text{).}$$

We denote by $\dim R$ the *harmonic dimension* of R that is given by the cardinal number of the Martin minimal boundary of R if $R \notin \mathcal{O}_G$ and the cardinal number of the Martin minimal boundary of R less arbitrary fixed parametric disc lying over the ideal boundary of R if $R \in \mathcal{O}_G$. At this point we must recall the strict inclusion relation $\mathcal{O}_G < \mathcal{O}_{HP}$ (cf. e.g. [8]). In connection with the result [4] of Masaoka and the second named author of the present paper that

$$(1.8)\qquad \sup_{R \in \mathcal{O}_s} \dim R \leqq \aleph_0 := \operatorname{card} \mathbf{N} \quad \text{(the cardinal number of } \mathbf{N}\text{),}$$

there arose the question whether the relation $\leqq$ is in fact the genuine inequality $<$ or the equality $=$ in the above (1.8). We have settled the question in [5] that the equality holds in (1.8) and in fact we have shown that

$$(1.9)\qquad \{\dim R : R \in \mathcal{O}_s\} = [1, \aleph_0] := \mathbf{N} \cup \{\aleph_0\}.$$

In the course of the proof of (1.9) we introduced a notion of, what we call, afforested surfaces, by the aid of which we succeeded in showing the existence of an $R \in \mathcal{O}_s$ with $\dim R = \aleph_0$.

By a *slit* γ in a Riemann surface X we mean a simple arc γ in X such that there exists a parametric disc $U := \{|z| < 1\}$ on X in which γ is represented as $\gamma = [-r, r] := \{z \in U : \operatorname{Im} z = 0, |\operatorname{Re} z| \leqq r\}$ $(0 < r < 1)$. We now state what we mean by an afforested surface. Let X and Y be two Riemann surfaces. We say that γ is a *common slit* in X and Y if there exists a simply connected Jordan region V_X (V_Y, resp.) contained in X and $\mathbf{C}$ (Y and $\mathbf{C}$, resp.) such that $\gamma = [-r, r] = \{t \in \mathbf{R} : -r \leqq t \leqq r\} \subset V_X \cap V_Y$. We denote by

$$(X \backslash \gamma) \bowtie_\gamma (Y \backslash \gamma)$$

the Riemann surface obtained by pasting $X \backslash \gamma$ to $Y \backslash \gamma$ crosswise along γ. As above $\mathbf{N}$ stands for the class of positive integers. For each $n \in \mathbf{N}$ we set $\mathbf{N}_n := \{i \in \mathbf{N} : i < n + 1\}$ and $\mathbf{N}_{\aleph_0} := \mathbf{N}$ so that $\mathbf{N}_\xi = \{i \in \mathbf{N} : i < \xi + 1\}$ for $\xi \in \mathbf{N} \cup \{\aleph_0\}$. An *afforested surface* $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle$ consists of three ingredients: an open Riemann surface $P \notin \mathcal{O}_G$ called a *plantation*, a finite or infinite sequence (according to $\xi \in \mathbf{N}$ or $\xi = \aleph_0$) $(T_i)_{i \in \mathbf{N}_\xi}$ of mutually disjoint open Riemann surfaces $T_i \notin \mathcal{O}_G$ for $i \in \mathbf{N}_\xi$ called *trees*, and a finite or infinite sequence $(\sigma_i)_{i \in \mathbf{N}_\xi}$ of common slits σ_i in P and T_i for $i \in \mathbf{N}_\xi$ called *roots* of trees T_i. Here σ_i are assumed to be mutually disjoint, isolated, and not accumulating in P. To determine W we define a sequence $(W_i)_{i \in \mathbf{N}_\xi}$ inductively as follows. First let

$$W_1 := \left(P \Big\backslash \bigcup_{i \in \mathbf{N}_\xi} \sigma_i\right) \bowtie_{\sigma_1} (T_1 \backslash \sigma_1)$$

and if $W_1, \ldots, W_{i-1}$ $(i \in \mathbf{N}_\xi,\ i \geqq 2)$ have been defined, then let

$$W_i := W_{i-1} \bowtie_{\sigma_i} (T_i \backslash \sigma_i)$$

for every $i \in \mathbf{N}_\xi$, and we define an afforested surface $W := W_\xi$ for $\xi \in \mathbf{N}$ and $W := \lim_{i \uparrow \infty} W_i$ for $\xi = \aleph_0$. In fact,

$$W := \cdots \left(\left(\left(P \Big\backslash \bigcup_{i \in \mathbf{N}_\xi} \sigma_i \right) \bowtie_{\sigma_1} (T_1 \backslash \sigma_1) \right) \bowtie_{\sigma_2} (T_2 \backslash \sigma_2) \right) \cdots, \tag{1.10}$$

and the Riemann surface $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle$ is called the *afforested surface* formed by foresting each tree T_i to P at its root σ_i for every $i \in \mathbf{N}_\xi$. We can see that $W \notin \mathcal{O}_G$ along with P and T_i.

For an afforested surface $W := \langle P, (T_i)_{i \in \mathbf{N}}, (\sigma_i)_{i \in \mathbf{N}} \rangle$ we consider the following condition

$$\sum_{i \in \mathbf{N}} (4M_i + 1) \frac{\sup_{P \backslash V_i} g(\cdot, \zeta_i; P)}{\inf_{\sigma_i} g(\cdot, \zeta_i; P)} < 1, \tag{1.11}$$

where $\zeta_i \in P$ corresponds to the center 0 of $\sigma_i = [-s_i, s_i]$ $(s_i > 0)$ with respect to a parametric disc V_i at ζ_i such that $\overline{V}_i = \{|z| \leqq 1\} \subset P$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ $(i \neq j)$ for every i and j in $\mathbf{N}$, $g(\cdot, \zeta; P)$ is the Green function on P, and M_i is the Harnack constant of $\{o\} \cup \partial V_i$ with a reference point $o \in P \setminus \bigcup_{i \in \mathbf{N}} (1/2)\overline{V}_i$ with respect to the family $H(P \setminus \bigcup_{i \in \mathbf{N}} (1/2)\overline{V}_i)^+$. We have obtained the following result from which the conclusion (1.9) was derived ([5]):

Theorem A. *Suppose that P and T_i belong to $\mathcal{O}_s$ for every $i \in \mathbf{N}_\xi$. If the sequence $(\sigma_i)_{i \in \mathbf{N}_\xi}$ is finite or else shrinks so rapidly as to satisfy* (1.11), *then the afforested surface $W := \langle P, (T_i)_{i \in \mathbf{N}_\xi}, (\sigma_i)_{i \in \mathbf{N}_\xi} \rangle$ also belongs to $\mathcal{O}_s$ and*

$$\dim W = \xi + 1 \tag{1.12}$$

when in particular P and all the trees T_i $(i \in \mathbf{N}_\xi)$ belong to $\mathcal{O}_{HP} \setminus \mathcal{O}_G$.

Concerning the above result we observe the following two points. First, if $\xi \in \mathbf{N}$, then $W \in \mathcal{O}_s$ without any additional condition such as (1.11) no matter how $\xi \in \mathbf{N}$ is large. Second, the condition (1.11) seems to be too technical. Even in the case of $\xi \in \mathbf{N}$ the corresponding condition to (1.11) may not be valid, i.e. $\sum_{i \in \mathbf{N}_\xi} (4M_i + 1) \sup_{P \setminus V_i} g(\cdot, \zeta_i; P) / \inf_{\sigma_i} g(\cdot, \zeta_i; P) \geqq 1$ can happen for $\xi \in \mathbf{N}$. In view of these observations one might be tempted to say that W is always a member of $\mathcal{O}_s$ for all $\xi \leqq \aleph_0$ without any further restriction such as (1.11). As a matter of fact we got several inquiries including one from the (of course unknown) referee of our former paper [5] in his/her referee report whether $W \in \mathcal{O}_s$ is always true without any additional condition even if $\xi = \aleph_0$. We took it for granted that some additional requirement on the size of $(\sigma_i)_{i \in \mathbf{N}_\xi}$ for $\xi = \aleph_0$ is in order to conclude that $W \in \mathcal{O}_s$ without giving any deeper consideration when we completed the paper [5]. After starting the trial to give such an example of an afforested surface $W \notin \mathcal{O}_s$, we recognized that the work is even harder than the original work [5] but fortunately we have been successful in constructing the required one, to exhibit which is the purpose of the present paper. Namely, we will prove the following result.

The Main Theorem. *There exists an afforested surface $W := \langle P, (T_i)_{i \in \mathbf{N}}, (\sigma_i)_{i \in \mathbf{N}} \rangle$ such that P and T_i $(i \in \mathbf{N})$ are all in the class $\mathcal{O}_s$ and yet W does not belong to the class $\mathcal{O}_s$.*

The proof of this main theorem will be divided into four parts and given as consecutive 4 sections in the sequel. The basic material of our construction is the special surface in $\mathcal{O}_{HP} \setminus \mathcal{O}_G$, called the Sario-Tôki disc, and therefore it is essential to understand the structure of these kind of surfaces. This will be described in the next §2 to an extent we really need in our construction. The plantation and holes in it to forest trees are prepared in §3 together with the prototype of the singular function on it to be constructed. Trees and the extension of the above preparatory function to trees are given in §4. In the final §5, the fact that the afforested surface and the singular function on it constructed based upon the preparations in §§2–4 really satisfy the required properties in the main theorem will be proven.

2. Sario-Tôki discs

We will make an essential use of special type of Riemann surfaces in the class $\mathcal{O}_{HP}\backslash\mathcal{O}_G$, which we call Sario-Tôki discs. We state the structure of such surfaces to an extent we need in our construction of an afforested surface carrying singular harmonic functions.

Let γ_1 and γ_2 be two radial slits of the unit disc $\mathbf{D} : |z| < 1$ formed by the points $re^{i\theta_1}$ and $re^{i\theta_2}$ respectively with $0 < a \leqq r \leqq b < 1$. Each slit γ_j $(j = 1, 2)$ has a left edge γ_j^+ corresponding to $\theta = \theta_j + 0$ and a right edge γ_j^- corresponding to $\theta = \theta_j - 0$. We then identify γ_1^+ with γ_2^- and γ_2^+ with γ_1^-, i.e. we paste a small slitted neighborhood of γ_1 to that of γ_2 crosswise along identified $\gamma_1 = \gamma_2$, which defines a Riemann surface as usual.

More generally we can consider a cyclic identification of any finite number of radial slits $\gamma_1, \dots, \gamma_k$, all extending between $|z| = a$ and $|z| = b$. In this case γ_1^+ is identified with γ_2^-, γ_2^+ with γ_3^-, etc. and finally γ_k^+ with γ_1^-. The identified end points will have neighborhoods consisting of k full discs. Such identifications may be performed simultaneously for several pairs or cycles, even for infinitely many, under the assumption that they do not intersect or accumulate inside $\mathbf{D}$. To be complete in formality, we even identify a slit with itself, i.e. a cyclic identification with $k = 1$. Needles to say, this trivial identification produces no change at all.

We denote by Γ the union of all radial slits in $\mathbf{D}$ which are isolated in $\mathbf{D}$. The identified slits from slits in Γ form a set $\hat{\Gamma}$ which is a union of isolated simple arcs with only end points in common. Let $\hat{\mathbf{D}}$ be the resulting Riemann surface obtained from the above identifying process. It is seen that $\hat{\mathbf{D}}\backslash\hat{\Gamma} = \mathbf{D}\backslash\Gamma$ not only as sets but also as Riemann surfaces. The coordinate function z for $\mathbf{D}$ is thus a well defined holomorphic function on $\hat{\mathbf{D}}\backslash\hat{\Gamma}$ but not continuous on $\hat{\Gamma}$ or not even defined on $\hat{\Gamma}$. However $\log|z|$ is well defined on all of $\hat{\mathbf{D}}$ by understanding $\log|z| = -\infty$ for $z = 0$ and harmonic on $\hat{\mathbf{D}}\backslash\{0\}$. In other words there is a harmonic function $\hat{g}$ on $\hat{\mathbf{D}}\backslash\{0\}$ such that $\log|z| = -\hat{g}(z)$ for $z \in \hat{\mathbf{D}}\backslash\hat{\Gamma} = \mathbf{D}\backslash\Gamma$ and $\hat{g} = g(\cdot, 0; \hat{\mathbf{D}})$, which is the Green function on $\hat{\mathbf{D}}$ with its pole at $z = 0$. Thus regardless of the choice of Γ and hence of $\hat{\Gamma}$, $\hat{\mathbf{D}}$ is of hyperbolic, i.e.

$$\hat{\mathbf{D}} \notin \mathcal{O}_G. \tag{2.1}$$

We now give a specific rule for constructing the required $\hat{\mathbf{D}}$. It will be determined by two sequences $(r_\nu)_{\nu\in\mathbf{N}}$ of strictly increasing sequence in the open interval $(0, 1)$ converging to 1 and $(n_\nu)_{\nu\in\mathbf{N}}$ from $\mathbf{N}$. By a suitable choice of these sequences it is seen that $HP(\hat{\mathbf{D}})$ consists of only constants so that with (2.1) we have

$$\hat{\mathbf{D}} \in \mathcal{O}_{HP}\backslash\mathcal{O}_G. \tag{2.2}$$

As for the concrete indication of $(r_\nu)_{\nu\in\mathbf{N}}$ and $(n_\nu)_{\nu\in\mathbf{N}}$ for (2.2) and the detailed proof for it we refer the reader to any one of e.g. the following monographs [1], [8], and [9].

Observe that every natural number ν has a unique representation $\nu = \nu(h,k) = (2h+1)2^k$ with h and k in $\mathbf{Z}^+ = \{m \in \mathbf{Z} : m \geqq 0\}$ with $\mathbf{Z}$ the set of integers. With each $\nu = \nu(h,k)$ we associate 2^{k+n_ν} radial slits with end points on $|z| = r_{2\nu}$ and $|z| = r_{2\nu+1}$. These slits are equally spaced one of which is on the positive real axis. Each of the above slits is said to be *rank* ν and *type* k. To complete the description of the constructing rule of $\hat{\mathbf{D}}$, we write $\theta_k = 2\pi/2^k$. The sectors $j\theta_k \leqq \theta \leqq (j+1)\theta_k$ $(0 \leqq j \leqq 2^k)$ are denoted by Σ_{jk}. The slits of type k which lie on the rays $\theta = j\theta_k$ are identified cyclically. The remaining slits of the same type will be identified pairwise within each sector Σ_{jk} symmetrically about its bisecting ray.

A Riemann surface $\hat{\mathbf{D}}$ constructed as described above is referred to as a *Sario-Tôki disc* since it is originally constructed by Sario [7] and also by Tôki [10] independently. Since (2.2) is a property of ideal boundary (cf. [8]) in the sense that if a Riemann surface $R_1 \in \mathcal{O}_{HP}\backslash\mathcal{O}_G$ and if another Riemann surface R_2 gives the complement in R_2 of a compact subset of R_2 coincident with the complement in R_1 of a compact subset of R_1, then $R_2 \in \mathcal{O}_{HP}\backslash\mathcal{O}_G$, we can always replace $(r_\nu)_{\nu\in\mathbf{N}}$ by any its end part subsequence $(r_\nu)_{\nu\geqq\nu_0}$ for any $\nu_0 \in \mathbf{N}$. Hence we can say that there exists a Sario-Tôki disc $\hat{\mathbf{D}}$ such that

$$\hat{\mathbf{D}} \supset \overline{\mathbf{D}(a)} \tag{2.3}$$

for any given $a \in (0,1)$, where $\mathbf{D}(a) := \{|z| < a\}$. From the construction of $\hat{\mathbf{D}}$ it follows the existence of an exhaustion $(\hat{\mathbf{D}}_\nu)_{\nu\geqq 0}$ of $\hat{\mathbf{D}}$ such that $\hat{\mathbf{D}}_0 = \mathbf{D}(a) \subset \overline{\mathbf{D}(a)} \subset \hat{\mathbf{D}}$ and $\partial\hat{\mathbf{D}}_\nu$ is a concentric circle in $\mathbf{D}$ with

$$\partial\hat{\mathbf{D}}_\nu = \{|z| = t_\nu\} \subset \{r_{2\nu-1} < |z| < r_{2\nu}\} \subset \hat{\mathbf{D}}\backslash\hat{\Gamma} \quad (t_\nu \in (r_{2\nu-1}, r_{2\nu}), \nu \in \mathbf{N}). \tag{2.4}$$

Once more we restate (2.1) as

$$g(z,0;\hat{\mathbf{D}}) = -\log|z| \quad (z \in \hat{\mathbf{D}}\backslash\hat{\Gamma} = \mathbf{D}\backslash\Gamma), \tag{2.5}$$

where $g(\cdot,0;\hat{\mathbf{D}})$ is the Green function on $\hat{\mathbf{D}}$ with its pole at $z = 0 \in \hat{\mathbf{D}} \cap \mathbf{D}$.

3. A plantation P with root holes σ_n and a basic function h

Choose an arbitrary but then fixed Sario-Tôki disc $\hat{\mathbf{D}}$ given by $(r_\nu)_{\nu\in\mathbf{N}}$ and $(n_\nu)_{\nu\in\mathbf{N}}$ (cf. §2) which we afresh denote by P. The Riemann surface P will play the role of the plantation for the afforested surface W with required properties in the main theorem that will be constructed in the sequel. Let $(P_n)_{n\geqq 0}$ be an exhaustion of P such that $P_0 = \mathbf{D}(a) \subset \overline{\mathbf{D}(a)} \subset P$ and $\partial P_\nu = \{|z| = t_\nu\} \subset \{r_{2\nu-1} < |z| < r_{2\nu}\}$ $(\nu \in \mathbf{N})$. We choose a decreasing sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ of positive numbers $\varepsilon_n \in (0, \pi/4)$, which will be a bit more specified below. We denote by

$$\alpha_n := \partial P_n = \{t_n e^{i\theta} : 0 \leqq \theta \leqq 2\pi\} \quad (n \in \mathbf{N}) \tag{3.1}$$

and we take a subarc β_n of α_n given by

$$\beta_n = \{t_n e^{i\theta} : |\theta| \leqq \varepsilon_n\} \quad (n \in \mathbf{N}). \tag{3.2}$$

For a compact subset K of P such that $P \backslash K$ is connected, the function

$$w(z, K; P) = \inf_s s(z),$$

where s runs over continuous positive superharmonic functions on P with $s|K \geqq 1$, is referred to as the *harmonic measure* of K on P. If K is a non-degenerate continuum with connected $P \backslash K$, then $w(\cdot, K; P) \in C(P) \cap H(P \backslash K)^+$, $0 < w(\cdot, K; P) < 1$ on $P \backslash K$, and $w(\cdot, K; P) \,|\, K = 1$. For any fixed $n \in \mathbf{N}$, $w(\cdot, \beta_n; P) \downarrow 0$ as $\varepsilon_n \downarrow 0$ and therefore we can choose the sequence $(\varepsilon_n)_{n \in \mathbf{N}}$ so rapidly decreasingly convergent as to satisfy

$$\sum_{n \in \mathbf{N}} w(0, \beta_n; P) < +\infty. \tag{3.3}$$

Since each $w(\cdot, \beta_n; P)$ is a potential, (3.3) assures that

$$w := \sum_{n \in \mathbf{N}} w(\cdot, \beta_n; P)$$

is locally uniformly convergent on P and hence w is a potential on P (cf. e.g. [3]). Finally we set

$$\sigma_n := \overline{\alpha_n \backslash \beta_n} \quad (n \in \mathbf{N}), \tag{3.4}$$

each of which is a simple arc in P. Of course, $\sigma_n \cap \sigma_m = \emptyset$ $(n \neq m)$, and $\{\sigma_n : n \in \mathbf{N}\}$ does not accumulate in P. Pick a suitable parametric disc $U_n := \{|z| < 1\}$ such that $\sigma_n \subset U_n$ and

$$\sigma_n := [-s_n, s_n] = \{z \in U_n : |\mathrm{Re}\, z| \leqq s_n, \mathrm{Im}\, z = 0\} \quad (s_n \in (0,1)) \tag{3.5}$$

in terms of local parameter z in U_n for every $n \in \mathbf{N}$. Here we moreover choose $\{U_n : n \in \mathbf{N}\}$ in such a fashion that $\overline{U}_n \cap \overline{U}_m = \emptyset$ $(n \neq m)$. Each σ_n in P plays the role of the hole into which the root s_n of the tree T_n will be put to forest T_n to P in the afforested surface W to be constructed. We set

$$\Sigma := \bigcup_{n \in \mathbf{N}} \sigma_n.$$

We denote by $\delta = \delta P$ the *Wiener harmonic boundary* of P (cf. e.g. [2], [8]). In view of $P \in \mathcal{O}_{HP} \backslash \mathcal{O}_G$, $\delta = \delta P$ is a one point set. The closure of a subset X of the Wiener compactification P^* of P will also be denoted by $\overline{X}$. We maintain the following result.

Claim 3.6. *The set Σ accumulates to δ:*

$$\delta \subset \overline{\Sigma}. \tag{3.7}$$

Proof. Since δ is a one point set, (3.7) is equivalent to that $\delta \cap \bar{\Sigma} \neq \emptyset$. Hence, by assuming $\delta \cap \bar{\Sigma} = \emptyset$, we only have to derive a contradiction. Take a $\varphi \in C(P^*)$ with $\varphi|\delta = 0$ and $\varphi|\bar{\Sigma} = 1$, the existence of which is assured by the fact $\delta \cap \bar{\Sigma} = \emptyset$. By applying the Wiener decomposition (cf. e.g. [8]) to φ, we obtain, as the harmonic part of φ, the function $c \in HB(P \backslash \Sigma)$ such that $c|\bar{\Sigma} = 1$ and $c|\delta = 0$. By the maximum principle (cf. e.g. [8]), $c \geqq 0$ on P^*. Based upon the fact (cf. e.g. [8]) that a nonnegative superharmonic function vanishes on δ if and only if it is a potential, we see that c is a potential on P. Recall that w is also a potential on P. Hence the function

$$s := c + w$$

is a potential on P and

$$s|\alpha_n \geqq 1 \quad (n \in \mathbf{N}).$$

Let (α_n, α_{n+1}) be the subregion of P bounded by α_n and α_{n+1} and also (α_1) the subregion of P bounded by α_1. By the usual minimum principle for superharmonic functions

$$s \,|\, (\alpha_n, \alpha_{n+1}) \geqq 1 \quad \text{and} \quad s|(\alpha_1) \geqq 1.$$

In view of

$$P = (\alpha_1) \cup \left(\bigcup_{n \in \mathbf{N}} (\alpha_n, \alpha_{n+1}) \right),$$

we conclude that $s \geqq 1$ on P. Hence, by the fact that s is a potential on one hand and $s \geqq 1$ on P on the other hand, we deduce

$$0 = \lim_{z \in P, z \to \delta} s(z) \geqq \liminf_{z \in P, z \to \delta} s(z) \geqq 1,$$

which is clearly a contradiction and we have shown $\delta \cap \bar{\Sigma} \neq \emptyset$ so that (3.7). □

Recall that SO_{HB} is the family of bordered Riemann surfaces (R, Γ), R is a Riemann surface and Γ a specific part of the border ∂R of R including the case $\Gamma = \partial R$ but not $\Gamma = \emptyset$, such that the class

$$HB(R, \Gamma) := \{u \in HB(R) \cap C(R \cup \Gamma) : u|\Gamma = 0\}$$

reduces to $\{0\}$ (cf. e.g. [8]). If R is a subsurface of a Riemann surface S, every point of whose nonempty relative boundary ∂R relative to S is regular with respect to the Dirichlet problem, then $(R, \partial R) \in SO_{HB}$ if and only if $(\bar{R} \backslash \overline{\partial R}) \cap \delta S = \emptyset$ (cf. e.g. [8]). Thus (3.7) implies (and in fact is equivalent to) that

$$(P \backslash \Sigma, \Sigma) \in SO_{HB}. \tag{3.8}$$

Based upon these properties we can obtain the following result on the existence of a basic function h which plays an essential role in the proof of our main theorem.

CLAIM 3.9. *There exists a continuous function h on P such that $h \in H(P\backslash\Sigma)^+\backslash\{0\}$ and*

$$h|\Sigma = 0, \quad \liminf_{z \in P, z \to \delta P} h(z) = 0 \tag{3.10}$$

so that $h \in HP_s(P\backslash\Sigma)$.

Proof. We denote by ζ_n the center of the arc β_n, i.e. the point corresponding to t_n in (3.2). Using the Green function $g(\cdot, \zeta_n; P\backslash\Sigma)$ on $P\backslash\Sigma$ with its pole at ζ_n for every $n \in \mathbf{N}$, we consider the function

$$g_n := \frac{g(\cdot, \zeta_n; P\backslash\Sigma)}{g(0, \zeta_n; P\backslash\Sigma)}$$

on P by understanding $g_n(\zeta_n) = +\infty$ and $g_n|\Sigma = 0$ for every $n \in \mathbf{N}$. By $g_n(0) = 1$ and $g_n > 0$ on $P\backslash\Sigma$, the Harnack inequality assures that the family $\{g_n : n \in \mathbf{N}\}$ forms a normal family on $P\backslash\Sigma$ and thus we can find a subsequence $(n(\nu))_{\nu\in\mathbf{N}}$ of $\mathbf{N}$ such that $(g_{n(\nu)})_{\nu\in\mathbf{N}}$ is convergent to an $h \in H(P\backslash\Sigma)^+$ locally uniformly on $P\backslash\Sigma$. Hence, on setting $h_\nu := g_{n(\nu)} \in H((P\backslash\Sigma)\backslash\{\zeta_{n(\nu)}\})^+$, we have

$$h_\nu(0) = 1 \quad (\nu \in \mathbf{N}), \tag{3.11}$$

$h_\nu \in C(P\backslash\{\zeta_{n(\nu)}\})$ $(\nu \in \mathbf{N})$ and

$$h_\nu|\Sigma = 0 \quad (\nu \in \mathbf{N}), \tag{3.12}$$

and we see that

$$h = \lim_{\nu\to\infty} h_\nu \in H(P\backslash\Sigma)^+ \tag{3.13}$$

locally uniformly on $P\backslash\Sigma$. By the above (3.13) and (3.11) we trivially deduce

$$h(0) = 1. \tag{3.14}$$

Again by (3.13) and (3.12) we can conclude that $h \in C(P)$ and

$$h|\Sigma = 0. \tag{3.15}$$

This can be seen as follows. For each $i \in \mathbf{N}$, by the maximum principle, since $|h_\mu - h_\nu| = 0$ on σ_i, we have

$$\sup_{\bar{U}_i} |h_\mu - h_\nu| = \sup_{\partial U_i} |h_\mu - h_\nu| \to 0 \quad (\mu, \nu \to \infty)$$

because ∂U_i is compact in $P\backslash\Sigma$ and $h_\mu - h_\nu \to h - h = 0$ $(\mu, \nu \to \infty)$ uniformly on ∂U_i. Thus $\sup_{\bar{U}_i} |h_\mu - h| \to 0$ $(\mu \to \infty)$ assures that $h \in C(\overline{U}_i)$ along with $h_\mu \in C(\overline{U}_i)$ and (3.15) is deduced as a consequence of (3.12). By (3.7) and (3.15) it is clear that the second equality in (3.10) holds.

For any $t \in \mathbf{R}^+$, as functions on $P\backslash\Sigma$, $h \wedge t \in HB(P\backslash\Sigma) \cap C(P)$ and therefore $h \wedge t$, as a bounded subharmonic function on P, is continuous on P^*. Hence (3.10) assures that

$$(h \wedge t) \,|\, \Sigma \cup \delta P = 0.$$

By the maximum principle (cf. e.g. [8]), $h \wedge t = 0$. This shows that h is singular on $P \backslash \Sigma$, i.e. $h \in HP_s(P \backslash \Sigma)$. $\square$

4. Superharmonic extension of the basic function

Starting from the plantation P and the basic function h on it given in §3, we will forest with suitable trees T_n in the class $\mathcal{O}_{HP} \backslash \mathcal{O}_G$ at their roots σ_n to the root holes σ_n in P and construct the harmonic function k_n on $T_n \backslash \sigma_n$ with vanishing boundary values on σ_n and -1 on the harmonic boundary δT_n of T_n such that the new function given by h on $P \backslash \bigcup_{n \in \mathbf{N}} \sigma_n$ and k_n on each T_n $(n \in \mathbf{N})$ is superharmonic on the afforested surface $W := \langle P, (T_n)_{n \in \mathbf{N}}, (\sigma_n)_{n \in \mathbf{N}} \rangle$. For the purpose we prepare the following extension result both for the domain of definition and the function on it.

Let $U := \mathbf{D}$ the unit disc in the complex plane $\mathbf{C}$ and $\sigma = [-s, s]$ the slit in U on the real line so that $0 < s < 1$. Let $h \in C(\overline{U}) \cap H(U \backslash \sigma)^+$ vanishing on σ. Let T be an open Riemann surface with $T \notin \mathcal{O}_G$. We say that the slit σ in U is contained in T if there is a simply connected region D in T such that there is a parametric disc (V, z) in T satisfying $\sigma \subset D \subset V$ with $\sigma = \{z \in V : |\mathrm{Re}\, z| < s, \mathrm{Im}\, z = 0\}$. Then we can form a new Riemann surface $(U \backslash \sigma) \bowtie_\sigma (T \backslash \sigma)$, which we call the surface formed from U by foresting the tree T with root σ at the root hole σ in U. Let $k \in C(T^*) \cap H(T \backslash \sigma)$ be such that $k|\sigma = 0$ and $k \,|\, \delta T = -1$ so that $-k$ is the harmonic measure of the Wiener harmonic boundary δT on $T \backslash \sigma$, where T^* is, as before, the Wiener compactification of T. For convenience the function k will be referred to as the *associated function* with T. To consider h and k on $(U \backslash \sigma) \bowtie_\sigma (T \backslash \sigma)$ we understand that $h \,|\, (T \backslash \sigma) = 0$ and $k \,|\, (U \backslash \sigma) = 0$ so that $h + k$ can be considered on $(U \backslash \sigma) \bowtie_\sigma (T \backslash \sigma)$ with $(h+k) \,|\, U = h$ and $(h+k) \,|\, T = k$. We wish to have the situation where the hybridized function $h + k$ is superharmonic.

LEMMA 4.1 (Hybridizing Lemma). *For any triple (U, σ, h) of the unit disc U, a slit σ of length $2s$ on the real line symmetric about the origin of U, and a positive harmonic function h on $U \backslash \sigma$ with vanishing (continuous, resp.) boundary values on σ (∂U, resp.), there is a Riemann surface T belonging to the class $\mathcal{O}_{HP} \backslash \mathcal{O}_G$ with the slit σ in T identified with the above σ in U and the associated function k with T such that $h + k$ is superharmonic on the afforested surface $(U \backslash \sigma) \bowtie_\sigma (T \backslash \sigma)$.*

Proof. Since $h \geqq 0$ is continuous on $\overline{U}$ with $h|\sigma = 0$ and harmonic on $U \backslash \sigma$, we can deduce that

$$M := \max_{\overline{U}} h = \max_{\partial U} h \in (0, +\infty).$$

We choose arbitrary but then fixed numbers ρ and ρ_1 in $(0, 1)$ satisfying

$$\left(\frac{s}{1 + \sqrt{1 - s^2}} \right)^{1/M} < \rho < \rho_1 < 1. \tag{4.2}$$

The number ρ plays the lead and ρ_1 the support. Let $w = j(z)$ be the Joukowski mapping of the extended z-plane $\hat{\mathbf{C}}_z := \hat{\mathbf{C}}$ onto the extended w-plane $\hat{\mathbf{C}}_w := \hat{\mathbf{C}}$ given by

$$w = j(z) := \frac{s}{2}\left(\frac{z}{\rho} + \frac{\rho}{z}\right).$$

Then the circle $C_\rho : |z| = \rho$ in the z-plane $\hat{\mathbf{C}}_z$ is mapped onto the slit $\sigma = [-s, s]$. Let σ^+ (σ^-, resp.) be the upper (lower, resp.) edge of σ. If we view $\sigma^+ \cup \sigma^-$ a Jordan curve in the Carathéodory compactification of $\hat{\mathbf{C}}\backslash\sigma$, then $w = j(z)$ maps C_ρ homeomorphically onto $\sigma^+ \cup \sigma^-$. We denote by D_ρ the disc bounded by C_ρ. We set $j_0 := j|\overline{D}_\rho$ and $j_\infty := j\,|\,(\hat{\mathbf{C}}\backslash D_\rho)$ with $j_0|C_\rho = j_\infty|C_\rho = j|C_\rho$. Then $w = j_0(z)$ ($w = j_\infty(z)$, resp.) maps D_ρ ($\hat{\mathbf{C}}\backslash\overline{D_\rho}$, resp.) onto $\hat{\mathbf{C}}\backslash\sigma$ conformally and $\overline{D_\rho}$ ($\hat{\mathbf{C}}\backslash D_\rho$, resp.) onto $(\hat{\mathbf{C}}\backslash\sigma) \cup (\sigma^+ \cup \sigma^-)$ homeomorphically. Actually $w = j(z)$ is a conformal mapping of $\hat{\mathbf{C}}_z$ onto the Riemann surface $(\hat{\mathbf{C}}_w\backslash\sigma) \bowtie_\sigma (\hat{\mathbf{C}}_w\backslash\sigma)$ so that $w = j_0(z)$ ($w = j_\infty(z)$, resp.) is the conformal mapping of $\overline{D}_\rho$ ($\hat{\mathbf{C}}\backslash D_\rho$, resp.) onto $(\hat{\mathbf{C}}\backslash\sigma) \cup (\sigma^+ \cup \sigma^-)$. Observe that the circle $C_r : |z| = r$ $(0 < r < \rho)$ is mapped by $w = j_0(z)$ onto the ellipse E_r with the major axis $[-s(\rho^2 + r^2)/2\rho r, s(\rho^2 + r^2)/2\rho r]$ on the real axis and minor axis $[-s(\rho^2 - r^2)/2\rho r, s(\rho^2 - r^2)/2\rho r]i$ on the imaginary axis. Since the circle family $\{C_r : 0 < r < \rho\}$ covers $D_\rho\backslash\{0\}$, i.e.

$$\bigcup_{0<r<\rho} C_r = D_\rho\backslash\{0\}, \tag{4.3}$$

we have the corresponding situation for $\mathbf{C}\backslash\sigma$ via $w = j_0(z)$ that the ellipse family $\{E_r : 0 < r < \rho\}$ covers $\mathbf{C}\backslash\sigma$, i.e.

$$\bigcup_{0<r<\rho} E_r = \mathbf{C}\backslash\sigma. \tag{4.4}$$

We next consider the annulus $j_0^{-1}(U\backslash\sigma)$ bounded by two Jordan curves. One is the circle C_ρ corresponding to σ and the other c_ρ corresponds to the unit circle ∂U. Observe that

$$c_\rho := j_0^{-1}(\partial U) = j_0^{-1}(|w| = 1)$$

is an analytic Jordan curve in D_ρ. By (4.3) and (4.4) there is a unique ellipse $E_{\tau\rho}$ $(0 < \tau < 1)$ touching ∂U at 1 (and also at -1) so that $C_{\tau\rho}$ is enclosing c_ρ touching at $\tau\rho$ (and also at $-\tau\rho$). Then $j_0(\tau\rho) = 1$, from which we deduce

$$\tau = \frac{s}{1 + \sqrt{1 - s^2}}. \tag{4.5}$$

We denote by $\hat{U}$ the annulus bounded by the outer boundary circle C_ρ and the inner boundary analytic Jordan curve c_ρ:

$$\hat{U} := j_0^{-1}(U\backslash\sigma) \quad \text{and} \quad \partial\hat{U} = C_\rho - c_\rho.$$

The function h on U can be harmonically transplanted to $\hat{U}$ as a function $\hat{h}$ in the class $C(\hat{U} \cup C_\rho \cup c_\rho) \cap H(\hat{U})^+$ with vanishing boundary values on C_ρ and the continuous boundary values on c_ρ:

$$\hat{h} = h \circ j_0.$$

By the definition of τ in (4.5) we see that

$$\hat{U} = D_\rho \backslash \overline{(c_\rho)} \supset \{\tau\rho < |z| < \rho\},$$

where (c_ρ) is the region bounded by c_ρ. In view of the above inclusion relation we see, by the maximum principle, that

$$\hat{h}(re^{i\theta}) \leqq \frac{M}{\log(\rho/\tau\rho)} \log(\rho/r)$$

for $\tau\rho \leqq r \leqq \rho$ and therefore we deduce, keeping the fact that two functions on the both sides of the above inequality vanishing on $C_\rho : |z| = \rho$ can be harmonically continued across C_ρ in mind,

$$\left[\frac{\partial}{\partial r}\hat{h}(re^{i\theta})\right]_{r=\rho} \geqq \frac{M}{\rho \log \tau}. \tag{4.6}$$

By (2.3) we can find a Sario-Tôki disc $\hat{\mathbf{D}}$ with

$$\overline{\mathbf{D}(\rho)} \subset \mathbf{D}(\rho_1) \subset \overline{\mathbf{D}(\rho_1)} \subset \hat{\mathbf{D}}$$

and using this $\hat{\mathbf{D}}$ we consider

$$\hat{V} := \hat{\mathbf{D}} \backslash \overline{\mathbf{D}(\rho)}.$$

Weld $\hat{U}$ to $\hat{V}$ by identifying $C_\rho = \{|z| = \rho\}$ with $\partial\hat{V} = \{|z| = \rho\}$, which amounts to the same that we are identifying D_ρ with $\mathbf{D}(\rho)$. The resulting surface is just

$$\hat{U} \cup C_\rho \cup \hat{V} = \hat{\mathbf{D}} \backslash (c_\rho).$$

Consider the function

$$\hat{k} := \frac{1}{\log(1/\rho)} g(\cdot, 0; \hat{\mathbf{D}}) - 1$$

on $\hat{V} \cup C_\rho$, where $g(\cdot, 0; \hat{\mathbf{D}})$ is the Green function on $\hat{\mathbf{D}}$ with its pole at 0. Clearly $\hat{k}\,|\,\delta\hat{\mathbf{D}} = -1$, where $\delta\hat{\mathbf{D}}$ is the Wiener harmonic boundary of $\hat{\mathbf{D}}$, and $\hat{k}|C_\rho = 0$. Since, by (2.5), we have

$$\hat{k}(re^{i\theta}) = \frac{1}{\log \rho} \log r - 1$$

for $\rho \leqq r \leqq \rho_1$, we see that

$$\left[\frac{\partial}{\partial r}\hat{k}(re^{i\theta})\right]_{r=\rho} = \frac{1}{\rho \log \rho}. \tag{4.7}$$

By (4.2) and (4.5), we can deduce from (4.6) and (4.7) that

$$\left[\frac{\partial}{\partial r-}\hat{h}(re^{i\theta})\right]_{r=\rho} > \left[\frac{\partial}{\partial r+}\hat{k}(re^{i\theta})\right]_{r=\rho}. \tag{4.8}$$

Here it is essentially important that we are computing the left (right, resp.) derivative $\partial/\partial r-$ ($\partial/\partial r+$, resp.) with respect to the common local parameter $re^{i\theta}$ on $\hat{U} \cup C_\rho \cup (\mathbf{D}(\rho_1)\backslash\overline{\mathbf{D}(\rho)})$, on which $\hat{h}$, $\hat{k}$, and $\hat{h}+\hat{k}$ are defined as follows. The function $\hat{h}$ is as it is on $\hat{U} \cup C_\rho$ but we set $\hat{h} \equiv 0$ on $\hat{V}$. The function $\hat{k}$ is as it is on $C_\rho \cup \hat{V}$ but we put $\hat{k} \equiv 0$ on $\hat{U}$. Then $\hat{h}+\hat{k}$ is $\hat{h}$ on $\hat{U} \cup C_\rho$ and $\hat{k}$ on $C_\rho \cup \hat{V}$ and anyhow $\hat{h}+\hat{k}$ is well defined on $\hat{U} \cup C_\rho \cup \hat{V} = \hat{\mathbf{D}}\backslash(c_\rho)$ and superharmonic there by virtue of (4.8).

Observe that $\mathbf{D}(\rho_1)\backslash\overline{\mathbf{D}(\rho)}$ is mapped by $w = j_\infty(z)$ onto the annulus $V\backslash\sigma$, where V is a Jordan region in the w-plane. Since $j_\infty(\partial(\hat{\mathbf{C}}\backslash\overline{\mathbf{D}(\rho_1)})) = \partial V$ and j_∞ is a conformal mapping of a vicinity of $\partial(\hat{\mathbf{D}}\backslash\overline{\mathbf{D}(\rho_1)})$ onto a vicinity of ∂V, we can weld V to $\hat{\mathbf{D}}\backslash\overline{\mathbf{D}(\rho_1)}$ at ∂V and $\partial(\hat{\mathbf{D}}\backslash\overline{\mathbf{D}(\rho_1)})$ identified by j_∞ (cf. [6]) and we denote by T the resulting Riemann surface. Since being a member of $\mathcal{O}_{HP}$ and that of $\mathcal{O}_G$ for a Riemann surface are ideal boundary properties (cf. [8]), we see that

$$T \in \mathcal{O}_{HP}\backslash\mathcal{O}_G \subset \mathcal{O}_s \tag{4.9}$$

along with $\hat{\mathbf{D}}$ because T and $\hat{\mathbf{D}}$ have the common identical ideal boundary neighborhood $\hat{\mathbf{D}}\backslash\overline{\mathbf{D}(\rho_1)}$. Since $T\backslash\sigma$ is conformally equivalent to $\hat{\mathbf{D}}\backslash\overline{\mathbf{D}(\rho)}$ and $\sigma^+ \cup \sigma^-$ correspond to $\partial\mathbf{D}(\rho) = C_\rho$, $\hat{k}$ can be conformally transplanted to a function k on $T\backslash\sigma$ such that $k \in C(T^*) \cap H(T\backslash\sigma)^+$, T^* being the Wiener compactification of T, with $k|\sigma = 0$ and $k\,|\,\delta T = -1$, δT being the Wiener harmonic boundary of T. Similarly h is viewed as being conformally transplanted to U from $\hat{h}$ on $\hat{U}$ such that $h \in C(\overline{U}) \cap H(U\backslash\sigma)^+$ with $h|\sigma = 0$ (in reality, starting from h, $\hat{h}$ was given by $\hat{h} = h \circ j_0$). Since the part $\hat{U} \cup C_\rho \cup (\mathbf{D}(\rho_1)\backslash\overline{\mathbf{D}(\rho)}) \subset \hat{U} \cup C_\rho \cup \hat{V} = \hat{\mathbf{D}}\backslash\overline{(c_\rho)}$ is mapped conformally onto $(U\backslash\sigma) \bowtie_\sigma (V\backslash\sigma) \subset (U\backslash\sigma) \bowtie_\sigma (T\backslash\sigma)$ and $(h+k) \circ j = \hat{h}+\hat{k}$ there under the definition $h|T = 0$ and $k|U = 0$, the superharmonicity of $\hat{h}+\hat{k}$ on $\hat{\mathbf{D}}\backslash\overline{(c_\rho)}$ implies that of $h+k$ on $(U\backslash\sigma) \bowtie_\sigma (T\backslash\sigma)$. $\square$

5. Construction of a nonzero singular function

We take the plantation P adopted in §3 so that, first of all, we have

$$P \in \mathcal{O}_{HP}\backslash\mathcal{O}_G \subset \mathcal{O}_s$$

and hence the Wiener harmonic boundary δP of P in the Wiener compactification P^* of P consists of a single point d, i.e. $\delta P = \{d\}$; there is a sequence $(U_n)_{n\in\mathbf{N}}$ of parametric discs $U_n = \{|z| < 1\}$ such that $\overline{U}_n \cap \overline{U}_m = \emptyset$ $(n \neq m)$ and $(\overline{U}_n)_{n\in\mathbf{N}}$ does not accumulate in P, i.e. for any compact subset L of P, the class $\{i \in \mathbf{N} : \overline{U}_i \cap L \neq \emptyset\}$ is either empty or at most finite subset of $\mathbf{N}$; there is a sequence $(\sigma_n)_{n\in\mathbf{N}}$ of slits $\sigma_n = [-s_n, s_n] \subset U_n$ $(0 < s_n < 1)$ such that

$$\delta P \subset \bar{\Sigma} \quad \left(\Sigma := \bigcup_{n\in\mathbf{N}} \sigma_n\right).$$

Moreover we have, what we call, a fundamental function h on P characterized by $h \in C(P) \cap H(P \backslash \Sigma)^+$ with $h|\Sigma = 0$ and by the most important property

$$h \in HP_s(P \backslash \Sigma)^+ \backslash \{0\}. \tag{5.1}$$

As a consequence, if a $v \in HB(P \backslash \Sigma)$, the class of bounded harmonic functions on $P \backslash \Sigma$, satisfies $h \geqq v$ on $P \backslash \Sigma$, then $v \leqq 0$ on $P \backslash \Sigma$. We now choose an exhaustion $(Q_i)_{i \in \mathbf{N}}$ of P consisting of relatively compact subregions Q_i of P where relative boundaries ∂Q_i are analytic Jordan curves (cf. §2) such that

$$Q_n \supset \bigcup_{1 \leqq i \leqq n} \bar{U}_i \quad \text{and} \quad P \backslash \bar{Q}_n \supset \bigcup_{n < i < \infty} \bar{U}_i.$$

Next we use the result in §4. For each $n \in \mathbf{N}$, by the hybridizing lemma 4.1, we can choose a tree $T_n \in \mathcal{O}_{HP} \backslash \mathcal{O}_G \subset \mathcal{O}_s$ containing the slit σ_n identified with that in $U_n \subset P$ and a $k_n \in C(T_n^*) \cap HB(T_n \backslash \sigma_n)$ with $k_n|\sigma_n = 0$ and $k_n\,|\,\delta T_n = -1$, T_n^* being the Wiener compactification of T_n and δT_n the Wiener harmonic boundary of T_n consisting of a single point d_n so that $\delta T_n = \{d_n\}$, such that $h + k_n$ is superharmonic on $(U_n \backslash \sigma_n) \uplus_{\sigma_n} (T_n \backslash \sigma_n)$ by extending h to T_n by $h|T_n = 0$ and k_n to P by $k_n|P = 0$. Let W be the afforested surface $\langle P, (T_n)_{n \in \mathbf{N}}, (\sigma_n)_{n \in \mathbf{N}} \rangle$. Let k be the function on W such that $k|T_n = k_n$ $(n \in \mathbf{N})$ so that $k|P = 0$. Similarly h is extended to W by setting $h = 0$ on $\bigcup_{n \in \mathbf{N}} T_n$. Then $h + k$ is a superharmonic function on W such that $h + k \geqq -1$ on W.

At this point we pause to recall the notion of harmonic measure functions. A function ω on W is referred to as a *harmonic measure function* if $\omega \in H(W)$ and

$$\omega \wedge (1 - \omega) = 0 \tag{5.2}$$

on W. The condition (5.2) implies $0 \leqq \omega \leqq 1$ on W so that $\omega \in HB(W)^+$ and therefore $\omega \in C(W^*)$, where W^* is the Wiener compactification of W. Since $f \mapsto f\,|\,\delta W$ is a bijective linear mapping of $HB(W)$ onto $C(\delta W)$, where δW is the Wiener harmonic boundary of W, the compact subset δW of W^* is known to be a Stonean space characterized by the property that the closure of any open subset of δW is again open so that *clopen* (i.e. closed and open) subsets of δW constitute a base of topology of δW. Then the condition (5.2) can be seen to be equivalent to that $\omega\,|\,\delta W$ is the characteristic function of some clopen subset of δW.

We now return to our present work of constructing a function u in the class $HP_s(W)^+ \backslash \{0\}$. Since $\delta T_n = \{d_n\}$ is an isolated one point set in δW and hence is an open subset of δW for every $n \in \mathbf{N}$, the set $\bigcup_{n \in \mathbf{N}} \delta T_n$ is an open subset of δW and thus the set

$$K := \overline{\bigcup_{n \in \mathbf{N}} \delta T_n}$$

is a clopen subset of δW. Then there exists a unique $w \in C(W^*) \cap HB(W)^+$ such that

$$w \,|\, \delta W = \chi_K : \text{the characteristic function of } K \text{ on } \delta W.$$

Hence w thus constructed is a harmonic measure function on W and thus the property corresponding to (5.2) for w is valid, i.e. we have

$$w \wedge (1 - w) = 0 \tag{5.3}$$

on W. For each $n \in \mathbf{N}$ we form an auxiliary afforested surface W_n:

$$W_n := \langle Q_n, (T_i)_{1 \leqq i \leqq n}, (\sigma_i)_{1 \leqq i \leqq n} \rangle,$$

which may be viewed as a subsurface of W with $\partial W_n = \partial Q_n$. Then $(W_n)_{n \in \mathbf{N}}$ forms an "exhaustion" of W in a generalized sense. Let $w_n \in C(\overline{W}_n) \cap HB(W_n))^+$ with $w_n \,|\, \partial W_n = 0$ and $w_n \,|\, (\bigcup_{1 \leqq i \leqq n} \delta T_i) = 1$, where $\overline{W}_n$ is the closure of W_n in W. We set $w_n \,|\, (W \backslash W_n) = 0$. We maintain the following important relation:

$$w = \lim_{n \to \infty} w_n \tag{5.4}$$

locally uniformly on W. By the maximum principle, we see on comparing the boundary values of w_n and w_{n+1} on $\partial W_n \cup (\bigcup_{1 \leqq i \leqq n} \delta T_i)$ that $(w_n)_{n \in \mathbf{N}}$ is an increasing sequence on W with $0 \leqq w_n \leqq 1$ on $\overline{W}_n$ for every $n \in \mathbf{N}$, and hence we see that $(w_n)_{n \in \mathbf{N}}$ converges to a $p \in HB(W)^+$ with $0 \leqq p \leqq 1$ on W locally uniformly. In view of $w_n \leqq p \leqq 1$ on W_n for every $n \in \mathbf{N}$, we see that $p \,|\, (\bigcup_{i \in \mathbf{N}} \delta T_i) = 1$. By the continuity we clearly have $p | K = 1$, and trivially $p \,|\, (\delta W \backslash K) \geqq 0$. Since $w = p = 1$ on K and $w = 0 \leqq p$ on $\delta W \backslash K$, the maximum principle assures that $w \leqq p$ on W. On the other hand, again by the maximum principle, we see that $w_n \leqq w$ on W by comparing the boundary values of w_n and w on $\partial W_n \cup (\bigcup_{1 \leqq i \leqq n} \delta T_i)$, and a fortiori we deduce $\lim_{n \to \infty} w_n \leqq w$ on W, or equivalently $p \leqq w$. We have thus shown that $w \leqq p$ and $p \leqq w$ on W, from which (5.4) follows.

We are now in the final stage of our proof of the main theorem stated in the introduction. Observe that $k + w \geqq 0$ on W and thus

$$h + k + w \geqq h$$

on W. Since the term on the left hand side $h + k + w$ is superharmonic on W along with $h + k$ on W (cf. §4) and the term h on the right hand side of the above is subharmonic on W, we can find a harmonic majorant u of h satisfying

$$h + k + w \geqq u \geqq h \geqq 0$$

on W. Hence $u \in HP(W)^+ \backslash \{0\}$ and the proof will be over if we can show that $u \in HP_s(W)^+$. For the purpose we choose any $v \in HB(W)^+$ with $u \geqq v \geqq 0$ on W and we are to show that $v \equiv 0$ on W. Replacing v by $(1/m)v$ with suitably large $m \in \mathbf{N}$, if necessary, we can assume without loss of generality not essentially but technically convenient condition that

$$0 \leqq v < 1 \tag{5.5}$$

on W in addition to the essential restraint

$$h + k + w \geqq v \geqq 0 \tag{5.6}$$

on v considered on W. Since (5.6) takes the form $h + w \geqq v$ on $P \backslash \Sigma$ or $h \geqq v - w$ on $P \backslash \Sigma$ with $v - w \in HB(P \backslash \Sigma)$, the fact that $h \in HP_s(P \backslash \Sigma)^+$ in (5.1) established at the end of §3 assures that $v - w \leqq 0$ on $P \backslash \Sigma$. Since $k \leqq 0$ and $h = 0$ on $\bigcup_{i \in \mathbf{N}} T_i$, (5.6) shows that $w \geqq k + w \geqq v$ on $\bigcup_{i \in \mathbf{N}} T_i$. Hence, anyway, we deduce

$$w \geqq v \tag{5.7}$$

on W. On δT_i, $k + w = -1 + 1 = 0$ and $h|T_i = 0$ yield with (5.6) that $v = 0$, i.e. $v \,|\, (\bigcup_{1 \leqq i \leqq n} \delta T_i) = 0 = (1 - w_n) \,|\, (\bigcup_{1 \leqq i \leqq n} \delta T_i)$. As an effect of the technical requirement (5.5) we see that $v < 1 = 1 - w_n$ on $\partial W_n = \partial Q_n$. Thus the maximum principle assures that $v < 1 - w_n$ on W_n. Hence $v \leqq \lim_{n \to \infty} (1 - w_n)$ on W and by (5.4) we deduce

$$1 - w \geqq v \tag{5.8}$$

on W. Thus, by (5.3), we conclude that (5.7) and (5.8) yield

$$0 \leqq v \leqq w \wedge (1 - w) = 0$$

on W so that $v \equiv 0$ on W, as required.

References

[1] L. V. Ahlfors and L. Sario, Riemann surfaces, Princeton mathematical series **26**, Princeton Univ. Press, 1960.

[2] C. Constantinescu und A. Cornea, Ideale Ränder Riemannscher Flächen, Ergebnisse der Mathematik und ihre Grenzgebiete **32**, Springer-Verlag, 1963.

[3] F.-Y. Maeda, Dirichlet Integrals on Harmonic Spaces, Lecture notes in mathematics **803**, Springer-Verlag, 1980.

[4] H. Masaoka and S. Segawa, On several classes of harmonic functions on a hyperbolic Riemann surfaces, Proc. of the 15th ICFIDCAA, Osaka, 2007, OCAMI Studies **2** (2008), 289–294.

[5] M. Nakai and S. Segawa, Types of afforested surfaces, Kodai Math. J. **32** (2008), 109–116.

[6] K. Oikawa, Welding of polygons and the type of Riemann surfaces, Kōdai Math. Sem. Rep. **13** (1961), 37–52.

[7] L. Sario, Positive harmonic functions, Lectures on functions of a complex varible, Univ. Michigan Press, Ann Arbor, 1955, 257–263.

[8] L. Sario and M. Nakai, Classification theory of Riemann surfaces, Grund-lehren der mathematischen Wissenschaften in Einzeldarstellungen **164**, Springer-Verlag, 1970.

[9] L. Sario, M. Nakai, C. Wang and L. O. Chung, Classification theory of Riemannian manifolds; harmonic, quasiharmonic, and biharmonic functions, Lecture notes in mathematics **605**, Springer-Verlag, 1970.

[10] Y. Tôki, On examples in the classification of Riemann surfaces, I, Osaka Math. J. **5** (1953), 267–280.

Mitsuru Nakai
Professor Emeritus at:
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya 466-8555
Japan

Mailing Address:
52 Eguchi, Hinaga
Chita 478-0041
Japan
E-mail: nakai@daido-it.ac.jp

Shigeo Segawa
Department of Mathematics
Daido University
Takiharu, Minami, Nagoya 457-8530
Japan
E-mail: segawa@daido-it.ac.jp