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EXISTENCE OF SOLUTION OF A COUPLED PROBLEM ARISING IN THE THERMOELECTRICAL SIMULATION OF ELECTRODES

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Abstract. In this paper we prove the existence of a solution to a system of partial differential equations arising from the thermoelectrical modeling of electrodes for electric furnaces. It consists of Maxwell equations coupled with the heat transfer equation through the Joule effect and the fact that thermal conductivity and electrical resistivity depend on temperature. The problem is formulated in cylindrical coordinates to take advantage of its axisymmetry.

The result is shown by introducing a regularized problem and using Schauder's fixed point theorem. Passing to the limit requires a priori estimates in weighted Sobolev spaces for an elliptic problem involving a right-hand side that is only integrable.

1. Introduction. Electrodes are among the main components of reduction furnaces for ferroalloys, calcium carbide, iron, and some others. Their purpose is to generate high temperatures that are needed for the reduction chemical reactions to take place. For this a great amount of energy is generated in an electric arc that arises on the tip of each electrode at the furnace center (Si, FeSi processes), or in a resistive layer (slag processes).

Typical diameter of electrodes is 1–2 m while their length is of the order of 10 m. High intensity alternating electric current up to 150 kA is used.

Classical electrodes extensively used in industry include pure graphite, prebaked, and Soderberg electrodes. The latter are built of paste consisting of a carbon aggregate and a tar binding that are fed into a steel casing. A number of steel fins are attached to the inside of the casing. The great amount of heat generated by the Joule effect is partially employed for baking the paste. This is a crucial process during which the initially soft/liquid non-conductive paste at the top of the electrode becomes a solid carbon conductor. Accordingly, its electrical conductivity undergoes important changes along the electrode. In fact, it depends on temperature because the latter determines the baking

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process. As a consequence it is of great importance to get a correct temperature distribution. In particular, the position of the baking zone strongly influences the electrode operation.

The advantages of Soderberg electrodes with respect to pure graphite or prebaked electrodes are that they are built in larger sizes and cost less. However, as the electrode is consumed, it has to be slipped typically 0.5 m per day. Then the casing melts and pollutes the final product and this is why they cannot be used for silicon metal. In fact, for silicon furnaces, prebaked electrodes were the only alternative used until this decade.

Recently (see [4], [7]) a new compound electrode for the production of silicon has been developed by the Ferroatlántica Company in Spain.

It consists of a central column of baked carbonaceous material, graphite or similar, that acts as a central mechanical support and an outside steel casing with the same diameter as the prebaked electrodes currently in use. A special Soderberg paste is introduced between the central column and the casing, which flows down the column until it finishes baking in the area of the contact clamps through which the electric current is introduced.

Two different slipping systems exist, one for the casing and another one for the central column; the combination of both systems is necessary so as to slip the casing as little as possible and also to carry out the correct extrusion of the carbon electrode with the central column slipping rings.

The result is that the furnace works in a completely similar way to those including prebaked electrodes and there is no appreciable pollution in the silicon due to the casing. Unlike standard Soderberg electrodes, the casing does not have fins, which are needed to bake the paste more easily. Actually, the inside of the casing is absolutely smooth so as to allow the slippage of the electrode throughout the casing. The casing is really an extrusion sleeve.

However, its overall thermoelectrical behavior changes with respect to pure Soderberg or prebaked electrodes. The reason is that graphite is a better conductor than Soderberg paste so, in particular, the skin effect is less important than in pure graphite, prebaked or classical Soderberg electrodes.

The main advantage of these compound electrodes is decreasing cost. In silicon metal production, the savings is around 12–16% (see [5]).

One inconvenience is that the slipping velocity is not free as it is for prebaked electrodes because paste has to be baked and this requires a minimum period of time between slipping. In fact, the baking of paste is a crucial point in the working of this type of electrode and mathematical models can help us to know the position of the baked/non-baked interphase.

In general, whatever the type of electrodes you consider, the great complexity in designing and operating them makes it very convenient to use numerical simulation to predict their behaviour. Accordingly, during recent years, several papers have been devoted to computing the distribution of electric current and temperatures in Soderberg electrodes (see for instance [12] and references therein). Most of them are based on cylindrical geometry and finite-difference procedures.

Thermoelectrical modeling leads to a nonlinear system of partial differential equations for electromagnetic field and temperature. Coupling between Maxwell and heat transfer equations is due to the Joule effect, which is the source term in the heat equation, and also to the fact that thermoelectrical parameters depend on temperature. As we already mentioned, Soderberg paste is non-conductive at room temperature but it is a good conductor, similar to pure graphite, at the highest temperatures (about 2500°C).

Numerical solution of industrial thermoelectrical problems is studied in several recent papers (see [10], [14], [9]) but they do not include any results concerning existence of solution. Moreover, in our case, differences in the geometry and in the physical problem under consideration lead to different partial differential equations.

These thermoelectrical problems are quite similar to the so-called *thermistor problem* which has been extensively studied in recent years from the theoretical point of view (see for instance [11] and references therein). However, three main features make it impossible for us to directly apply the techniques employed in most of the papers on the thermistor problem. First, our problem is written in cylindrical coordinates, which leads to the use of weighted Sobolev spaces. Second, electrodes we want to simulate involve two different materials and so physical parameters depend not only on temperature but on the position as well and this prevents us from using Kirchoff transformations to linearize the heat equation. Third, electric current is not direct and therefore we cannot use electric potential to characterize the electromagnetic field. Instead, a scalar equation for the tangential component of the magnetic field is considered, but the maximum principle does not hold for it. This fact does not allow us to transform the Joule term in the heat equation as in [11]. Thus we are led to using techniques for elliptic problems involving right-hand sides in the non-reflexive Banach space L^1.

The outline of the paper is as follows: in Sec. 2 we recall Maxwell and heat transfer equations and compute boundary conditions to get a "well-posed" electromagnetic problem. The main results are given in Sec. 3 where we establish an appropriate functional framework and prove existence of a solution. A numerical solution and an application to real industrial situations can be found in [2], [3], [7], and [5].

2. The physical problem. A mathematical model.

Figure 1 shows a simple sketch of an electrode. Alternating current, the intensity and frequency of which are known, enters through the contact clamp and is further distributed. Then it leaves the electrode through its bottom where an electric arc is produced. In order to compute the electromagnetic field we have to solve the Maxwell equations:

$$\operatorname{curl} \mathbf{H} = \mathbf{J}, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} = 0, \tag{2.2}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2.3}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{2.4}$$

$$\mathbf{J} = \sigma \mathbf{E}, \tag{2.5}$$

where we have neglected the term including the electric displacement. The notation used is as follows:

- $\mathbf{J}$ is the current density,
- $\mathbf{E}$ is the electric field,

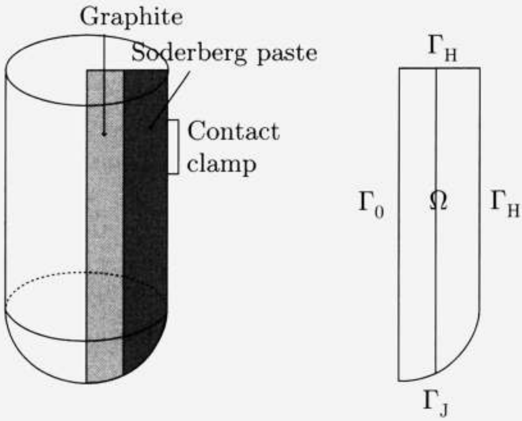

FIG. 1. A compound electrode

- **B** is the magnetic induction,
- **H** is the magnetic field,
- μ is the magnetic permeability,
- σ is the electric conductivity.

Assuming alternating current, all fields are given by the general expression

$$F(x,t) = \mathrm{Re}(e^{i\omega t}\mathbf{F}(x)), \tag{2.6}$$

ω being the angular frequency. The corresponding fields satisfy the equations

$$i\omega\mathbf{B} + \operatorname{curl}\mathbf{E} = 0, \tag{2.7}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{J}, \tag{2.8}$$

$$\mathbf{J} = \sigma\mathbf{E}. \tag{2.9}$$

By using (2.4) and (2.5) it is possible to get the following equation for the magnetic field, which is valid in the electrode:

$$i\omega\mu(x,T)\mathbf{H} + \operatorname{curl}\left(\frac{1}{\sigma(x,T)}\operatorname{curl}\mathbf{H}\right) = 0. \tag{2.10}$$

The electric conductivity depends on the temperature T. Therefore, in order to integrate this equation we need to solve the heat equation:

$$d(x,T)c(x,T)\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\operatorname{grad}T\right) - \operatorname{div}(k(x,T)\operatorname{grad}T) = \mathbf{J}\cdot\mathbf{E}, \tag{2.11}$$

where d, c, and k denote mass density, specific heat, and thermal conductivity, which also depend on temperature, and $\mathbf{v}$ is the slipping (vertical) velocity due to electrode consumption. The source term in the right-hand side represents the released heat per unit volume arising from the Joule effect. Notice that it makes a coupling between this equation and (2.10). As a first step, in the present paper we consider the steady-state model and neglect electrode consumption and slipping. Hence the first two terms in (2.11) disappear.

To get appropriate boundary conditions for the partial differential equation (2.10) is not an easy task. However, the objective of this paper is to simulate one single electrode. While this assumption requires neglecting the so-called "proximity effect" due to the presence of the other electrodes in the furnace, it allows us to consider cylindrical symmetry and, as will be seen below, to write boundary conditions yielding a "well-posed" problem.

Indeed, under this assumption, all fields are independent of the angular variable θ. Furthermore, the current density field has no component in the tangential direction $\mathbf{e}_\theta$. Then Eqs. (2.2), (2.4), and (2.5) imply that the magnetic field is proportional to $\mathbf{e}_\theta$. Let us write the curl operator in cylindrical coordinates in italics, i.e.,

$$\mathit{curl}\ H(r,z) = -\frac{\partial H_\theta}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\mathbf{e}_z \tag{2.12}$$

where (r,z) belongs to the two-dimensional domain Ω shown in Fig. 1.

Then (2.10) becomes

$$i\omega\mu H_\theta - \frac{\partial}{\partial z}\left(\frac{1}{\sigma((r,z),T)}\frac{\partial H_\theta}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{1}{\sigma((r,z),T)r}\frac{\partial(rH_\theta)}{\partial r}\right) = 0. \tag{2.13}$$

On the other hand, from (2.1) we get

$$\mathit{curl}\ H(r,z)\cdot\nu = J\cdot\nu \tag{2.14}$$

where $\nu = (\nu_r, \nu_z)$ denotes the outward unit normal vector to $\Gamma = \partial\Omega$.

The right-hand side in the previous equation is known on the part of the boundary of Ω called Γ_H (see Fig. 1), because the electric current flux on the surface of the electrode is either given (at the contact clamps) or null. Moreover, an easy computation shows that the term in the left-hand side is equal to

$$\frac{1}{r}\frac{\partial(rH_\theta)}{\partial\tau} \tag{2.15}$$

where $\tau = (\nu_x, -\nu_r)$ is a unit vector tangent to the boundary.

Equation (2.14) can be integrated to get the electromagnetic field on the boundary. Let us denote by $\mathbf{s}(u)$ a parametrization of Γ_H starting at point P. Let us suppose $\tau\cdot\dot{\mathbf{s}}(u) > 0$. Then H_θ is given by

$$s_1(u)H_\theta(\mathbf{s}(u)) = \int_0^u s_1(v)\Big(-J_r(\mathbf{s}(v))s_2'(v) + J_z(\mathbf{s}(v))s_1'(v)\Big)\,dv. \tag{2.16}$$

On the rest of the boundary $\Gamma_J = \Gamma\setminus(\Gamma_0\cup\Gamma_H)$, which is in contact with the electric arc, we assume the tangential component of the density current to be zero. This is a "natural" boundary condition for the problem as we will see below.

We notice that the input current distribution on the casing/clamp interface will significantly influence the position of the baking zone. This is why in practical simulations clamps are included and then the model computes this distribution. Similarly, the steel casing can also be modelled. However, so long as it does not have fins, its influence can be neglected, as numerical results show.

In order to write the heat equation in cylindrical coordinates we first notice that, assuming axisymmetry, the gradient operator is given by

$$grad\, T(r,z) = \frac{\partial T}{\partial r}\mathbf{e}_r + \frac{\partial T}{\partial z}\mathbf{e}_z \tag{2.17}$$

and then the heat equation (2.11) becomes

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rk((r,z),T)\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(rk((r,z),T)\frac{\partial T}{\partial z}\right)\right) = \frac{1}{\sigma((r,z),T)}|curl\, H|^2. \tag{2.18}$$

For the sake of simplicity, in the present paper we consider Dirichlet boundary conditions on $\Gamma_T = \Gamma \setminus \Gamma_0$, i.e., we assume that the temperature is given on the whole surface of the electrode. However, for practical applications, convective/radiative heat transfer conditions should be considered.

3. Weak formulation. Existence results. In this section we prove the existence of a solution to the thermoelectrical problem (2.13), (2.18). We start by introducing some functional spaces and establishing the assumptions on the physical parameters appearing in the equations.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $1 \le p < \infty$ and $s \in \mathbb{R}$ let us denote by $L^p_s(\Omega;\mathbb{K})$ the space of $\mathbb{K}$-valued measurable functions V such that

$$\|V\|_{L^p_s(\Omega)} = \left(\int_\Omega |V(r,z)|^p r^s\, dr\, dz\right)^{\frac{1}{p}} < \infty \tag{3.19}$$

and, for any positive integer m, let $W^{m,p}_s(\Omega;\mathbb{K})$ be the weighted Sobolev space defined as the set of functions in $L^p_s(\Omega;\mathbb{K})$ such that their partial derivatives of order less than or equal to m belong to $L^p_s(\Omega;\mathbb{K})$ and are provided with the norm

$$\|V\|_{W^{m,p}_s(\Omega;\mathbb{K})} = \left(\sum_{l=0}^{m} |V|^p_{W^{l,p}_s(\Omega;\mathbb{K})}\right)^{\frac{1}{p}} \tag{3.20}$$

where

$$|V|_{W^{l,p}_s(\Omega;\mathbb{K})} = \left(\sum_{j=1}^{l} \left\|\frac{\partial^l V}{\partial r^j \partial z^{l-j}}\right\|^p_{L^p_s(\Omega;\mathbb{K})}\right)^{\frac{1}{p}}. \tag{3.21}$$

As usual, for $p = 2$ the space $W^{m,p}_s(\Omega;\mathbb{K})$ will be denoted by $H^m_s(\Omega;\mathbb{K})$.

Let us denote by $H_s(curl,\Omega;\mathbb{K})$ the functional space

$$H_s(curl,\Omega;\mathbb{K}) = \{G \in L^2_s(\Omega;\mathbb{K}) : curl\, G \in (L^2_s(\Omega;\mathbb{K}))^2\} \tag{3.22}$$

where *curl* is given by (2.12). It is easy to see that

$$H_1(curl,\Omega;\mathbb{K}) = \{G : rG \in H^1_{-1}(\Omega;\mathbb{K})\}. \tag{3.23}$$

In what follows, let us suppose that $\sigma(x,T), \mu(x,T)$, and $k(x,T)$ are Carathéodory functions from $\Omega \times \mathbb{R}$ into $\mathbb{R}$, i.e., measurable with respect to x and continuous with respect to T, and furthermore

$$\alpha_0 \le \sigma(x,T) \le \alpha_1, \tag{3.24}$$

$$\beta_0 \le \mu(x,T) \le \beta_1, \tag{3.25}$$

$$\gamma_0 \le k(x,T) \le \gamma_1, \quad \text{a.e. in } \Omega, \ \forall T \in \mathbb{R}, \tag{3.26}$$

where α_i, β_i, and γ_i, $i = 1, 2$ are strictly positive constants.

For mathematical analysis and numerical purposes it is convenient to write weak formulations of the equations to be solved.

3.1. *The electromagnetic problem.* Let us start with (2.10). In what follows we drop index θ and, for the sake of simplicity, let g be a given function in $H^1_{-1}(\Omega;\mathbb{C})$.

By multiplying by the conjugate of a complex test function G null on Γ_H, integrating in Ω with respect to the measure $r\,dr\,dz$, and using a Green's formula and the boundary condition on Γ_J we get the following weak formulation of the *electromagnetic problem*:

EP: *To find a function $H \in H_1(curl,\Omega;\mathbb{C})$ such that $rH - g = 0$ on Γ_H and, furthermore,*

$$\int_\Omega i\omega\mu H\overline{G} r\,dr\,dz + \int_\Omega \frac{1}{\sigma}\left(\frac{\partial H}{\partial z}\frac{\partial \overline{G}}{\partial z} + \frac{1}{r}\frac{\partial (rH)}{\partial r}\frac{1}{r}\frac{\partial (r\overline{G})}{\partial r}\right) r\,dr\,dz = 0 \qquad \forall G \in H_{1,\Gamma_H}(curl,\Omega,\mathbb{C}), \tag{3.27}$$

where

$$H_{1,\Gamma_H}(curl,\Omega;\mathbb{C}) = \{G \in H_1(curl,\Omega;\mathbb{C}) : rG_{|\Gamma_H} = 0\}. \tag{3.28}$$

In order to prove an existence theorem for problem (3.27) it is useful to make a change of variable. Indeed, it is easy to see that H is a solution of **EP** if and only if $U = rH$ is a solution of the following *modified electromagnetic problem*:

MEP: *Find $U \in H^1_{-1}(\Omega;\mathbb{C})$ such that $U - g = 0$ on Γ_H and, furthermore,*

$$\int_\Omega i\omega\mu U\overline{V}\frac{1}{r}\,dr\,dz + \int_\Omega \frac{1}{\sigma}\left(\frac{\partial U}{\partial z}\frac{\partial \overline{V}}{\partial z} + \frac{\partial U}{\partial r}\frac{\partial \overline{V}}{\partial r}\right)\frac{1}{r}\,dr\,dz = 0 \qquad \forall V \in H^1_{-1,\Gamma_H}(\Omega;\mathbb{C}) \tag{3.29}$$

where

$$H^1_{-1,\Gamma_H}(\Omega;\mathbb{C}) = \{G \in H^1_{-1}(\Omega;\mathbb{C}) : V_{|\Gamma_H} = 0\}. \tag{3.30}$$

Notice that from a trace theorem (see for instance [16]) $V_{|\Gamma_H}$ makes sense in $L^2(\Gamma_H;\mathbb{C})$ for $V \in H^1_{-1}(\Omega;\mathbb{C})$. We have the

PROPOSITION 3.1. Assume (3.24) and (3.25) hold. Then the **MEP** has a unique solution.

Proof. It is a straightforward consequence of the Lax-Milgram theorem. Indeed, let us denote by $a(U,V)$ the sesquilinear form in the left-hand side of (3.29). Then a is continuous from $H^1_{-1}(\Omega;\mathbb{C}) \times H^1_{-1}(\Omega;\mathbb{C})$ into $\mathbb{C}$ and, furthermore,

$$\begin{aligned} |a(V,V)| &= \left[\left(\omega\int_\Omega \mu|V|^2\frac{1}{r}\,dr\,dz\right)^2 + \left(\int_\Omega \frac{1}{\sigma}|\nabla V|^2\frac{1}{r}\,dr\,dz\right)^2\right]^{\frac{1}{2}} \\ &\geq C_1\left[\left|\omega\int_\Omega \mu|V|^2\frac{1}{r}\,dr\,dz\right| + \left|\int_\Omega \frac{1}{\sigma}|\nabla V|^2\frac{1}{r}\,dr\,dz\right|\right] \\ &\geq C_2\|V\|^2_{H^1_{-1}(\Omega;\mathbb{C})}. \quad \square \end{aligned} \tag{3.31}$$

The following corollary follows from a trace theorem in ([15]).

COROLLARY 3.1. Let U be the solution of **MEP**. Then $U_{|\Gamma_0} = 0$.

COROLLARY 3.2. Let H be defined as $H = \frac{1}{r}U$, U being the solution of **MEP**. Then H is the solution of **EP** and we have

$$H \in L^2_1(\Omega;\mathbb{C}), \tag{3.32}$$

$$J = curl\ H \in (L^2_1(\Omega;\mathbb{C}))^2. \tag{3.33}$$

3.2. *The thermal problem.* In order to prove the existence of a solution to the thermal problem we notice that Kirchoff transformations cannot be used to eliminate the nonlinearity in the term of the left-hand side as is classical in the literature on the thermistor problem (see for instance [8], [11]). This is because the domain is not homogeneous and then the thermal conductivity depends not only on temperature but on the geometric variables (r,z) as well.

On the other hand, according to Corollary 3.2 and assumptions (3.24) and (3.25), the right-hand side of the heat equation (2.18) belongs to the space $L^1_1(\Omega;\mathbb{R})$. Hence classical variational techniques cannot be applied in order to prove the existence of a solution.

In the classical thermistor problem the "electrical" unknown is the scalar potential, which is the solution of a Poisson equation, and then the maximum principle holds. However, in our case this principle is not true and then we do not know whether the solution H of the problem **EP** belongs to $L^\infty(\Omega;\mathbb{C})$.

For any q, $1 \leq q < \infty$, we denote by $W^{1,q}_{1,\Gamma_T}(\Omega;\mathbb{R}) = \{W \in W^{1,q}_1(\Omega;\mathbb{R}) : W_{|\Gamma_T} = 0\}$ and, as usual, $H^1_{1,\Gamma_T}(\Omega;\mathbb{R}) = W^{1,2}_{1,\Gamma_T}(\Omega;\mathbb{R})$.

We shall look for a weak solution of the thermal problem in the following sense:
TP: *Find $T \in W^{1,q}_1(\Omega;\mathbb{R})$ such that $T - f \in W^{1,q}_{1,\Gamma_T}(\Omega;\mathbb{R})$ and, furthermore,*

$$\int_\Omega k(\cdot,T)\left(\frac{\partial T}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial W}{\partial r}\right) r\,dr\,dz = \int_\Omega \rho(\cdot,T)|curl\ H|^2 W r\,dr\,dz,$$

$$\forall W \in W^{1,q'}_{1,\Gamma_T}(\Omega;\mathbb{R}) \cap L^\infty(\Omega;\mathbb{R}) \tag{3.34}$$

for some q, $1 < q < 2$.

Here q' stands, as usual, for the conjugate exponent of q, and we have denoted by f a lifting of the boundary temperature on Γ_T. In (3.34) ρ denotes the electrical resistivity, i.e., $\rho = 1/\sigma$.

3.3. *The coupled problem.* For technical reasons, we impose the following hypothesis concerning the domain Ω, previously considered in [15]:

Ω is a Lipschitz bounded domain that can be decomposed as $\Omega = \Omega_a \cup \mathcal{O}$, where Ω_a is a trapezium having one of its bases contained in the axis $r = 0$, the distance from $\mathcal{O}$ to the axis $r = 0$ is strictly positive, and $\Omega_a \cap \mathcal{O} = \varnothing$. (H)

Our existence result for the thermoelectrical problem is the following.

THEOREM 3.1. Let $1 < q < 3/2$. Under the assumptions (H) and (3.24)–(3.26) there exists a solution $(H,T) \in H_1(curl, \Omega; \mathbb{C}) \times W_1^{1,q}(\Omega; \mathbb{R})$ to the coupled problem **EP-TP**.

In order to prove this theorem, we first regularize the problem by a technique used in [13]. It consists of replacing the right-hand side in the heat equation by

$$\frac{\rho |curl\ H|^2}{1 + \varepsilon\rho |curl\ H|^2}, \tag{3.35}$$

which is bounded by $1/\varepsilon$ for all H; so it belongs to $L^\infty(\Omega; \mathbb{R})$. We shall use Schauder's fixed point theorem to prove the existence of a solution for the following regularized problem:

Find $(H^\varepsilon, T^\varepsilon) \in H_1(curl, \Omega; \mathbb{C}) \times H_1^1(\Omega; \mathbb{R})$ such that $rH^\varepsilon - g \in H^1_{-1,\Gamma_H}(\Omega; \mathbb{C})$, $T^\varepsilon - f \in H^1_{1,\Gamma_T}(\Omega; \mathbb{R})$ and, furthermore,

$$\int_\Omega i\omega\mu(\cdot, T^\varepsilon) H^\varepsilon \overline{G} r\, dr\, dz + \int_\Omega \rho(\cdot, T^\varepsilon) \left(\frac{\partial H^\varepsilon}{\partial z}\frac{\partial \overline{G}}{\partial z} + \frac{1}{r}\frac{\partial (rH^\varepsilon)}{\partial r}\frac{1}{r}\frac{\partial (r\overline{G})}{\partial r} \right) r\, dr\, dz = 0,$$
$$\forall G \in H_{1,\Gamma_H}(curl, \Omega; \mathbb{C}), \tag{3.36}$$

$$\int_\Omega k(\cdot, T^\varepsilon) \left(\frac{\partial T^\varepsilon}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T^\varepsilon}{\partial r}\frac{\partial W}{\partial r} \right) r\, dr\, dz = \int_\Omega \frac{\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2}{1 + \varepsilon\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2} W r\, dr\, dz,$$
$$\forall W \in H^1_{1,\Gamma_T}(\Omega; \mathbb{R}), \tag{3.37}$$

and then we shall pass to the limit when $\varepsilon \longrightarrow 0$. It must be noted that this regularization only affects the thermal problem. Hence, some auxiliary results about the electric problem will be used in the proof of the existence of a solution to the regularized problem as well as for passing to the limit. For the sake of clarity we have found it convenient to split the proof of Theorem 3.1 into several lemmas. The proofs of some of them are standard; hence we have omitted them.

First some nonlinear mappings related to the electric problem are defined. For given $S \in L_1^1(\Omega; \mathbb{R})$ let $H_S \in H_1(curl, \Omega; \mathbb{C})$ be the unique solution to the electromagnetic problem:

$$rH_S - g \in H^1_{-1,\Gamma_H}(\Omega; \mathbb{C}), \tag{3.38}$$

$$\int_\Omega i\omega\mu(\cdot, S) H_S \overline{G} r\, dr\, dz + \int_\Omega \rho(\cdot, S) \left(\frac{\partial H_S}{\partial z}\frac{\partial \overline{G}}{\partial z} + \frac{1}{r}\frac{\partial (rH_S)}{\partial r}\frac{1}{r}\frac{\partial (r\overline{G})}{\partial r} \right) r\, dr\, dz = 0,$$
$$\forall G \in H_{1,\Gamma_H}(curl, \Omega; \mathbb{C}). \tag{3.39}$$

Thus we have defined a nonlinear mapping $\mathcal{H}$ by

$$S \in L_1^1(\Omega;\mathbb{R}) \to \mathcal{H}(S) = H_S \in H_1(curl,\Omega;\mathbb{C}). \tag{3.40}$$

In order to study the regularized problem we need two other mappings. For $\varepsilon > 0$ given, we define the mapping $\mathcal{L}^\varepsilon$ by

$$(S,H) \in L_1^1(\Omega;\mathbb{R}) \times H_1(curl,\Omega;\mathbb{C}) \to \mathcal{L}^\varepsilon(S,H) = L^\varepsilon_{S,H} \in (H^1_{1,\Gamma_T}(\Omega;\mathbb{R}))' \tag{3.41}$$

where $L^\varepsilon_{S,H}$ is the linear form on $H^1_{1,\Gamma_T}(\Omega;\mathbb{R})$ defined by

$$\langle L^\varepsilon_{S,H}, W\rangle = \int_\Omega \frac{\rho(\cdot,S)|curl\, H|^2}{1+\varepsilon\rho(\cdot,S)|curl\, H|^2} W r\, dr\, dz. \tag{3.42}$$

Finally, let $\mathcal{T}$ be the mapping given by

$$(S,L) \in L_1^1(\Omega;\mathbb{R}) \times (H^1_{1,\Gamma_T}(\Omega;\mathbb{R}))' \to \mathcal{T}(S,L) = T_{S,L} \in H_1^1(\Omega;\mathbb{R}) \tag{3.43}$$

where $T_{S,L}$ is the unique solution of the problem

$$T_{S,L} - f \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R}), \tag{3.44}$$

$$\int_\Omega k(\cdot,S)\left(\frac{\partial T_{S,L}}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T_{S,L}}{\partial r}\frac{\partial W}{\partial r}\right) r\, dr\, dz = \langle L, W\rangle, \quad \forall W \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R}). \tag{3.45}$$

Then we have the following results:

LEMMA 3.2. There is a constant $C > 0$ depending only on g such that

$$\|\mathcal{H}(S)\|_{H_1(curl,\Omega;\mathbb{C})} \le C, \quad \forall S \in L_1^1(\Omega;\mathbb{R}). \tag{3.46}$$

LEMMA 3.3. The mapping $\mathcal{H} : L_1^1(\Omega;\mathbb{R}) \to H_1(curl,\Omega;\mathbb{C})$ is continuous.

Proof. Let $S \in L_1^1(\Omega;\mathbb{R})$ and let $\{S_n\}$ be a sequence such that $\{S_n\} \to S$ in $L_1^1(\Omega;\mathbb{R})$. For $H_n = \mathcal{H}(S_n)$ we first prove that $\{H_n\} \to \mathcal{H}(S)$ weakly in $H_1(curl,\Omega;\mathbb{C})$. Indeed, from Lemma 3.2, $\{H_n\}$ is bounded in $H_1(curl,\Omega;\mathbb{C})$; hence it has a weakly convergent subsequence. We still denote it by $\{H_n\}$ and do the same for the corresponding subsequence of $\{S_n\}$. Let H be the weak limit of H_n. Then H_n satisfies

$$rH_n - g \in H^1_{-1,\Gamma_H}(\Omega;\mathbb{C}), \tag{3.47}$$

$$\int_\Omega i\omega\mu(\cdot,S_n)H_n\overline{G} r\, dr\, dz + \int_\Omega \rho(\cdot,S_n)\left(\frac{\partial H_n}{\partial z}\frac{\partial \overline{G}}{\partial z} + \frac{1}{r}\frac{\partial(rH_n)}{\partial r}\frac{1}{r}\frac{\partial(r\overline{G})}{\partial r}\right) r\, dr\, dz = 0,$$
$$\forall G \in H_{1,\Gamma_H}(curl,\Omega;\mathbb{C}). \tag{3.48}$$

Clearly $rH - g \in H^1_{-1,\Gamma_H}(\Omega;\mathbb{C})$. On the other hand, since $\{S_n\} \to S$ in $L_1^1(\Omega;\mathbb{R})$ we can extract a subsequence such that $\{S_n\} \to S$ a.e. in Ω which implies $\rho(\cdot,S_n) \to \rho(\cdot,S)$ a.e. in Ω. From this fact and assumption (3.24) we obtain, by using Lebesgue's dominated convergence theorem, that for any $G \in H_{1,\Gamma_H}(curl,\Omega;\mathbb{C})$:

$$\rho(\cdot,S_n)\frac{1}{r}\frac{\partial(r\overline{G})}{\partial r} \to \rho(\cdot,S)\frac{1}{r}\frac{\partial(r\overline{G})}{\partial r} \text{ strongly in } L_1^2(\Omega;\mathbb{C}). \tag{3.49}$$

From (3.49) and the weak convergence of $\frac{1}{r}\frac{\partial(rH_n)}{\partial r}$ towards $\frac{1}{r}\frac{\partial(rH)}{\partial r}$ in $L^2_1(\Omega;\mathbb{C})$, we deduce

$$\lim_{n\to\infty}\int_\Omega \rho(\cdot,S_n)\frac{1}{r}\frac{\partial(rH_n)}{\partial r}\frac{1}{r}\frac{\partial(r\overline{G})}{\partial r}r\,dr\,dz = \int_\Omega \rho(\cdot,S)\frac{1}{r}\frac{\partial(rH)}{\partial r}\frac{1}{r}\frac{\partial(r\overline{G})}{\partial r}r\,dr\,dz \tag{3.50}$$

for any $G \in H_{1,\Gamma_H}(curl,\Omega;\mathbb{C})$.

Now we may pass to the limit in the two other terms in (3.48) by using the same arguments and so we obtain $H = \mathcal{H}(\mathcal{S})$. Since this limit is unique, we have that the whole sequence $\{H_n\}$ converges weakly to H in $H_1(curl,\Omega;\mathbb{C})$.

Now we prove the strong convergence. For this purpose we subtract the variational equations verified by H_n and H and take $G = H_n - H$ as test function. Then we find

$$\begin{aligned}\int_\Omega & i\omega\mu(\cdot,S_n)|H_n - H|^2 r\,dr\,dz \\ &+ \int_\Omega \rho(\cdot,S_n)\left(\left|\frac{\partial(H_n-H)}{\partial z}\right|^2 + \left|\frac{1}{r}\frac{\partial(r(H_n-H))}{\partial r}\right|^2\right) r\,dr\,dz \\ &\qquad = -\int_\Omega i\omega(\mu(\cdot,S_n)-\mu(\cdot,S))H(\overline{H_n-H})r\,dr\,dz \\ &- \int_\Omega (\rho(\cdot,S_n)-\rho(\cdot,S))\left(\frac{\partial H}{\partial z}\frac{\partial(\overline{H_n-H})}{\partial z} + \frac{1}{r^2}\frac{\partial(rH)}{\partial r}\frac{\partial(r(\overline{H_n-H}))}{\partial r}\right) r\,dr\,dz.\end{aligned} \tag{3.51}$$

By using again Lebesgue's theorem and the fact that $\{H_n\} \to H$ weakly in $H_1(curl,\Omega;\mathbb{C})$, we obtain that the right-hand side of (3.51) converges to zero. Then the result follows by taking the modulus and recalling (3.24) and (3.25). □

LEMMA 3.4. For any $\varepsilon > 0$, the mapping $\mathcal{L}^\varepsilon$ is continuous.

LEMMA 3.5. The mapping $\mathcal{T} : L^1_1(\Omega;\mathbb{R}) \times (H^1_{1,\Gamma_T}(\Omega;\mathbb{R}))' \to H^1_1(\Omega;\mathbb{R})$ is continuous and satisfies the inequality

$$\|\mathcal{T}(S,L)\|_{H^1_1(\Omega;\mathbb{R})} \le C(\|L\|_{(H^1_{1,\Gamma_T}(\Omega;\mathbb{R}))'} + \|f\|_{H^1_1(\Omega;\mathbb{R})}). \tag{3.52}$$

Proof. The proof of the continuity of $\mathcal{T}$ follows exactly the same lines as that in Lemma 3.3.

Now we can prove the existence of a solution of the regularized problem.

THEOREM 3.6. Problem (3.36)–(3.37) has a solution $(H^\varepsilon, T^\varepsilon) \in H_1(curl,\Omega;\mathbb{C}) \times H^1_1(\Omega;\mathbb{R})$.

Proof. Let j be the injection $H^1_1(\Omega;\mathbb{R}) \hookrightarrow L^1_1(\Omega;\mathbb{R})$ and let $\mathcal{G}^\varepsilon : L^1_1(\Omega;\mathbb{R}) \to H^1_1(\Omega;\mathbb{R})$ be the mapping defined by $\mathcal{G}^\varepsilon(S) = \mathcal{T}(S,\mathcal{L}^\varepsilon(S,\mathcal{H}(S)))$. Then $\mathcal{G}^\varepsilon$ is a continuous mapping and it is clear that $(H^\varepsilon,T^\varepsilon)$ is a solution of (3.36)–(3.37) if and only if T^ε is a fixed point of the mapping $\mathcal{G}^\varepsilon \circ j$ and $H^\varepsilon = (\mathcal{H}\circ j)(T^\varepsilon)$. The injection $H^1_1(\Omega;\mathbb{R}) \hookrightarrow L^2_1(\Omega;\mathbb{R})$ is compact (see [15], Lemma 4.2) and so it is j. Therefore $\mathcal{G}^\varepsilon \circ j$ is compact. Moreover, from Lemma 3.2 and inequality (3.52) there is a constant $C = C(g,f,\varepsilon)$ such that

$$\|\mathcal{G}^\varepsilon(S)\|_{H^1_1(\Omega;\mathbb{R})} \le C \quad \forall S \in H^1_1(\Omega;\mathbb{R}) \tag{3.53}$$

and then Schauder's theorem yields the existence of a fixed point T^ε of $\mathcal{G}^\varepsilon$. □

Now we pass to the limit as $\varepsilon \to 0$. Notice that from Lemma 3.2, there is a constant $C > 0$ depending only on g such that

$$\|H^\varepsilon\|_{H_1(curl,\Omega;\mathbb{C})} \leq C \quad \forall \varepsilon > 0. \tag{3.54}$$

Let

$$\varphi^\varepsilon = \frac{\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2}{1 + \varepsilon\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2}. \tag{3.55}$$

Clearly

$$\|\varphi^\varepsilon\|_{L^\infty(\Omega;\mathbb{R})} \leq \frac{1}{\varepsilon} \tag{3.56}$$

and, using (3.24) and (3.54),

$$\int_\Omega |\varphi^\varepsilon| r\, dr\, dz \leq \int_\Omega \frac{\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2}{1 + \varepsilon\rho(\cdot, T^\varepsilon)|curl\ H^\varepsilon|^2} r\, dr\, dz \leq C, \tag{3.57}$$

that is, φ^ε is bounded in the non-reflexive space $L^1_1(\Omega; \mathbb{R})$. This is the main difficulty in passing to the limit as $\varepsilon \to 0$. We handle it by essentially using some techniques from [6]. There are some differences between the problem treated by these authors and **TP**. First, they consider homogeneous boundary conditions. Second, their operator is of the form $-\operatorname{div}(a(\cdot, \operatorname{grad} T))$. Hence in their case the differential operator may depend nonlinearly on $\operatorname{grad} T$ but must not depend explicitly on T, contrary to our case. Third, they consider standard (unweighted) Sobolev spaces. Since we use weighted ones, we must apply a Sobolev injection for this family of spaces, stated in Lemma 3.7, which leads to the restriction $1 < q < 3/2$ in Lemma 3.8 below and therefore in Theorem 3.1. Although the proof of this lemma can be easily done following the lines of subsection II.4 of [6] with some adaptations, we have included it for the sake of completeness.

LEMMA 3.7. Let Ω satisfy hypothesis (H). Then, for any $1 < q < 3$ we have the embedding

$$W_1^{1,q}(\Omega) \subset L_1^{q^*}(\Omega)$$

with $q^* = 3q/(3-q)$.

Proof. Owing to hypothesis (H), it suffices to state this injection for the domain Ω_a. From Theorem 4.3 of [15], the space $\mathcal{D}(\overline{\Omega_a})$ is dense in $W_1^{1,q}(\Omega_a)$. Using the techniques in the proof of Theorem 4.4 of that paper and a standard cut-off argument, the proof can be further reduced to that of the inequality

$$\|u\|_{L^{q^*}(Q_b)} \leq C\|u\|_{W_1^{1,q}(Q_b)} \tag{3.58}$$

for $u \in C^1(\overline{Q_b})$, where Q_b is the half-disk

$$Q_b = \{(r, z) \in \mathbb{R}^2, r > 0, r^2 + z^2 < b^2\}$$

and $b > 0$ is large enough so that $\overline{\Omega_a} \subset Q_b$. Now inequality (3.58) is just a particular case of inequality (64) in Chapter 5 of [1]. $\square$

Since $(T^\varepsilon, H^\varepsilon)$ is a solution of the regularized problem, T^ε satisfies (3.37), namely

$$\int_\Omega k(\cdot, T^\varepsilon)\left(\frac{\partial T^\varepsilon}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T^\varepsilon}{\partial r}\frac{\partial W}{\partial r}\right) r\,dr\,dz = \int_\Omega \varphi^\varepsilon W r\,dr\,dz, \qquad \forall W \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R}). \tag{3.59}$$

We have the following

LEMMA 3.8. For any $1 < q < 3/2$ and for any $B > 0$ there is a constant $C = C(q, \gamma_0, \gamma_1, \Omega, B)$ such that if $\varphi \in (H^1_{1,\Gamma_T}(\Omega;\mathbb{R}))' \cap L^1_1(\Omega;\mathbb{R})$ with $\|\varphi\|_{L^1_1(\Omega,\mathbb{R})} \leq B$ and $T \in H^1_1(\Omega, \mathbb{R})$ is a solution of

$$T - f \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R}), \tag{3.60}$$

$$\int_\Omega k(\cdot, T)\left(\frac{\partial T}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial W}{\partial r}\right) r\,dr\,dz = \int_\Omega \varphi W r\,dr\,dz, \qquad \forall W \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R}), \tag{3.61}$$

then

$$\|T\|_{W^{1,q}_1(\Omega,\mathbb{R})} \leq C. \tag{3.62}$$

Proof. For every $n \in \mathbb{N}$ let ψ_n be the function defined by

$$\psi_n(s) = \begin{cases} n & \text{if } s > n, \\ s & \text{if } -n \leq s \leq n, \\ -n & \text{if } s < -n. \end{cases} \tag{3.63}$$

Let $T_0 = T - f$. Then $T_0, \psi_n(T_0) \in H^1_{1,\Gamma_T}(\Omega;\mathbb{R})$. Taking $\psi_n(T_0)$ as a test function in (3.61) we easily get

$$\|\nabla T_0\|^2_{L^2_1(D_n)} \leq \frac{\gamma_1}{\gamma_0}\|\nabla T_0\|_{L^2_1(D_n)}\|\nabla f\|_{L^2_1(\Omega)} + \frac{Bn}{\gamma_0} \tag{3.64}$$

with

$$D_n = \{(r,z) \in \Omega : |T_0(r,z)| \leq n\}$$

and then

$$\|\nabla T_0\|^2_{L^2_1(D_n)} \leq c_0 + c_1 n \tag{3.65}$$

with $c_0 = (\gamma_1/\gamma_0)^2\|\nabla f\|^2_{L^2_1(\Omega)}$ and $c_1 = 2B/\gamma_0$. Now, we define ψ_n by

$$\psi_n(s) = \begin{cases} 1 & \text{if } s > n+1, \\ s-n & \text{if } n \leq s \leq n+1, \\ 0 & \text{if } -n \leq s \leq n, \\ s+n & \text{if } -n-1 \leq s \leq -n, \\ -1 & \text{if } s < -n-1. \end{cases} \tag{3.66}$$

In the same manner, we get now

$$\int_{B_n} |\nabla T_0|^2 r\,dr\,dz \le c_0 + c_1 \tag{3.67}$$

where

$$B_n = \{(r,z) \in \Omega : n \le |T_0(r,z)| \le n+1\}.$$

Let $1 < q < 2$. Hölder's inequality gives

$$\int_{B_n} |\nabla T_0|^q r\,dr\,dz \le \left(\int_{B_n} |\nabla T_0|^2 r\,dr\,dz\right)^{\frac{q}{2}} \left(\int_{B_n} r\,dr\,dz\right)^{\frac{2-q}{2}}. \tag{3.68}$$

Let $q^* = 3q/(3-q)$. Since

$$\int_{B_n} |T_0|^{q^*} r\,dr\,dz \ge n^{q^*} \int_{B_n} r\,dr\,dz \tag{3.69}$$

we have

$$\int_{B_n} |\nabla T_0|^q r\,dr\,dz \le \left(\int_{B_n} |\nabla T_0|^2 r\,dr\,dz\right)^{\frac{q}{2}} \left(\int_{B_n} |T_0|^{q^*} r\,dr\,dz\right)^{\frac{2-q}{2}} \frac{1}{n^{q^*(2-q)/2}}. \tag{3.70}$$

Recalling (3.67) and setting $c_2 = (c_0 + c_1)^{q/2}$ we have

$$\int_{B_n} |\nabla T_0|^q r\,dr\,dz \le c_2 \left(\int_{B_n} |T_0|^{q^*} r\,dr\,dz\right)^{\frac{2-q}{2}} \frac{1}{n^{q^*(2-q)/2}}. \tag{3.71}$$

Applying Hölder's inequality with exponents $2/(2-q)$ and $2/q$ we deduce that for any integer $n_0 \ge 1$

$$\sum_{n=n_0}^{\infty} \int_{B_n} |\nabla T_0|^q r\,dr\,dz \le c_2 \left(\sum_{n=n_0}^{\infty} \int_{B_n} |T_0|^{q^*} r\,dr\,dz\right)^{\frac{2-q}{q}} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(2-q)/q}}\right)^{\frac{q}{2}}. \tag{3.72}$$

Since $\Omega = D_{n_0} \cup \bigcup_{n=n_0}^{\infty} B_n$ and

$$\int_{D_{n_0}} |\nabla T_0|^q r\,dr\,dz \le c_3 n_0^{\frac{q}{2}} \tag{3.73}$$

where

$$c_3 = c_2 \left(\int_{\Omega} r\,dr\,dz\right)^{\frac{2-q}{2}}, \tag{3.74}$$

we have

$$\int_{\Omega} |\nabla T_0|^q r\,dr\,dz \le c_3 n_0^{\frac{q}{2}} + c_2 \|T_0\|_{L_1^{q^*}(\Omega,\mathbb{R})}^{\frac{q^*(2-q)}{2}} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(2-q)/q}}\right)^{\frac{q}{2}}. \tag{3.75}$$

Using Poincaré's inequality in $W^{1,q}_{1,\Gamma_T}(\Omega;\mathbb{R})$ and the Sobolev injection $W^{1,q}_1(\Omega) \subset L^{q^*}_1(\Omega)$ we have

$$\begin{aligned} \|T_0\|^q_{L^{q^*}_1(\Omega,\mathbb{R})} &\le c_4 \int_\Omega |\nabla T_0|^q r\, dr\, dz \\ &\le c_4 \left[c_3 n_0^{\frac{q}{2}} + c_2 \|T_0\|^{\frac{q^*(2-q)}{2}}_{L^{q^*}_1(\Omega,\mathbb{R})} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{\frac{q^*(2-q)}{q}}} \right)^{\frac{q}{2}} \right]. \end{aligned} \tag{3.76}$$

Now, $q^* = 3q/(3-q)$ and $1 < q < 3/2$ imply $\frac{q^*(2-q)}{q} > 1$ and $\frac{q^*(2-q)}{2} < q$; so taking $n_0 = 1$ and setting

$$c_5 = \sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(2-q)/q}}$$

we have

$$\|T_0\|^q_{L^{q^*}_1(\Omega,\mathbb{R})} \le c_4 c_3 + c_4 c_2 c_5 \|T_0\|^{\frac{q^*(2-q)}{2}}_{L^{q^*}_1(\Omega,\mathbb{R})}. \tag{3.77}$$

Since $q^*(2-q)/2 < q$ there is a constant c_6 such that

$$\|T_0\|_{L^{q^*}_1(\Omega,\mathbb{R})} \le c_6 \tag{3.78}$$

and recalling (3.75) we obtain for some constant c_7,

$$\int_\Omega |\nabla T_0|^q r\, dr\, dz \le c_7 \tag{3.79}$$

from which we get the result. □

Proof of Theorem 3.1. As we have seen, when $\varepsilon \to 0$, φ^ε is bounded in $L^1_1(\Omega;\mathbb{R})$. Then, by the previous lemma, T^ε is bounded in $W^{1,q}_1(\Omega;\mathbb{R})$ for any $1 < q < 3/2$ and there is a sequence $\varepsilon_n \to 0$ and $T \in W^{1,q}_1(\Omega;\mathbb{R})$ such that $T^{\varepsilon_n} \to T$ weakly in $W^{1,q}_1(\Omega;\mathbb{R})$. Since the injection $W^{1,q}_1(\Omega;\mathbb{R}) \hookrightarrow L^q_1(\Omega;\mathbb{R})$ is compact (see [17], Theorem 2.2), we have $T^{\varepsilon_n} \to T$ strongly in $L^q_1(\Omega;\mathbb{R})$. Setting $H^{\varepsilon_n} = \mathcal{H}(T^{\varepsilon_n})$ and $H = \mathscr{H}(T)$, Lemma 3.3 gives $H^{\varepsilon_n} \to H$ strongly in $H_1(curl, \Omega;\mathbb{C})$; so $|curl\, H^{\varepsilon_n}|^2 \to |curl\, H|^2$ in $L^1_1(\Omega,\mathbb{R})$. Hence we can extract a subsequence, still denoted ε_n, such that

$$\begin{aligned} T^{\varepsilon_n} &\to T \quad \text{a.e. in } \Omega, \\ |curl\ H^{\varepsilon_n}|^2 &\to |curl\ H|^2 \quad \text{a.e. in } \Omega, \end{aligned}$$

and furthermore

$$\exists h \in L^1_1(\Omega;\mathbb{R}) \text{ such that } |curl\ H^{\varepsilon_n}|^2 \le h \text{ a.e. in } \Omega, \quad \forall n \in \mathbb{N}.$$

Recalling that

$$\varphi^{\varepsilon_n} = \frac{\rho(\cdot, T^{\varepsilon_n})|curl\ H^{\varepsilon_n}|^2}{1 + \varepsilon_n \rho(\cdot, T^{\varepsilon_n})|curl\ H^{\varepsilon_n}|^2} \tag{3.80}$$

and extracting an a.e. convergent subsequence of T^{ε_n} and $|curl\ H^{\varepsilon_n}|^2$ we deduce that

$$\varphi^{\varepsilon_n} \to \rho(\cdot, T)|curl\ H|^2 \quad \text{a.e. in } \Omega, \tag{3.81}$$

and using Lebesgue's theorem we conclude that

$$\varphi^{\varepsilon_n} \to \rho(\cdot, T)|curl\ H|^2$$

in $L_1^1(\Omega)$. Finally, we may pass to the limit in the problem satisfied by T^{ε_n} to obtain that $T \in W_1^{1,q}(\Omega;\mathbb{R})$ satisfies

$$T - f \in W^{1,q}_{1,\Gamma_T}(\Omega;\mathbb{R}), \tag{3.82}$$

$$\int_\Omega k(\cdot,T)\left(\frac{\partial T}{\partial z}\frac{\partial W}{\partial z} + \frac{\partial T}{\partial r}\frac{\partial W}{\partial r}\right) r\,dr\,dz = \int_\Omega \rho(\cdot,T)|curl\ H|^2 W r\,dr\,dz,$$
$$\forall W \in W^{1,q'}_{1,\Gamma_T}(\Omega;\mathbb{R}) \cap L^\infty(\Omega;\mathbb{R}). \tag{3.83}$$

The proof is now complete.

REMARK 3.9. We do not know whether the solution is unique. In fact, the problem being nonlinear, to prove uniqueness is a difficult task that is beyond the scope of this paper.

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