

EXISTENCE OF SOLUTIONS AND GALERKIN APPROXIMATIONS FOR NONLINEAR FUNCTIONAL EVOLUTION EQUATIONS

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1. Introduction—Preliminaries. In this paper we are concerned with existence and approximation results for nonlinear functional evolution equations in Banach spaces. Let X be a Banach space with norm $\|\cdot\|$, and let $C = C([-r, 0], X)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$, for some $r > 0$, into X with norm $\|\psi\|_C = \sup_{\theta \in [-r, 0]} \|\psi(\theta)\|$. Let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. In [9] we examined the existence of a unique strong solution of the abstract initial value problem

$$(FDE) \quad x'(t) + A(t)x(t) = G(t, x_t), \quad t \in [0, T], \quad x_0 = \phi,$$

where $A(t): D(A(t)) = D \subset X \rightarrow X$, G satisfies a global Lipschitz condition with respect to both variables, and $\phi \in C$ is such that $\phi' \in C$ and $\phi(0) \in D$. Furthermore, we required that X^* , the dual of X , be uniformly convex and for each $t \in [0, T]$, $A(t)$ be m -accretive (see definition below) and satisfy a Kato time-dependence condition of the form

$$(*) \quad \|A(t)x - A(s)x\| \leq |t - s|L(\|x\|)(1 + \|A(s)x\|)$$

for all $t, s \in [0, T]$ and $x \in D$, where $L: R_+ = [0, \infty) \rightarrow R_+$ is a given increasing function.

By a "strong solution" of (FDE) on $[0, T]$ we mean an absolutely continuous X -valued function which, for almost all $t \in [0, T]$, is strongly differentiable and satisfies (FDE). The unique strong solution $x(t)$ of (FDE), whose existence was known from previous results, was shown in [9] to be the uniform limit of strongly continuously differentiable solutions of approximating equations for (FDE) involving the Yosida approximants of $A(t)$. In [10] a method of lines for the approximation of the solution $x(t)$ of (FDE) was developed.

Our purpose in this paper is two-fold. We first establish a local existence result for a more general nonlinear abstract functional problem of the type:

$$(DE) \quad x'(t) + A(t, x_t)x(t) = 0, \quad t \in [0, T], \quad x_0 = \phi,$$

where $A(t, \phi)v$ is m -accretive in v for every $(t, \phi) \in [0, T) \times C_0$, C_0 a certain closed subset of C , and satisfies a local Lipschitz-type condition in t and ϕ . As an important example of our result, we obtain the local existence of a unique strong solution of (FDE) under the given conditions, but with G satisfying now a *local* Lipschitz condition. This result is still new if the Lipschitz condition is global, and an application of it is given in Section 4.

Our second goal is to establish a Galerkin method for the approximation of the solutions of (FDE) for the case of a Hilbert space X , under the additional assumptions that $A(t)$ be defined on the whole of X and map bounded subsets of X into bounded sets. Our result, Theorem 2, is an improvement of the corresponding result of Kartsatos [8], and is illustrated in Section 4 by an example involving nonlinear partial elliptic operators of order $2m$.

For $x \in X$, $x^* \in X^*$, let $\langle x, x^* \rangle$ denote the number $x^*(x)$. We define the "duality mapping" $J: X \rightarrow 2^{X^*}$ as follows:

$$Jx = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The set Jx is nonempty by the Hahn Banach theorem. However, if X^* is uniformly convex, then the duality mapping J is single valued and is uniformly continuous on bounded subsets of X . An operator $B: D(B) \subset X \rightarrow X$ is called "accretive" if

$$\operatorname{Re} \langle Bu - Bv, J(u - v) \rangle \geq 0$$

for every $u, v \in D(B)$. An accretive operator B is " m -accretive" if $R(I + \lambda B) = X$ for some (equivalently, all) $\lambda > 0$. For further properties of m -accretive operators the reader is referred to Kato [11]. We denote by \bar{D} the strong closure of the set $D \subset X$.

2. Existence. In this section we give a local existence result for the initial value problem

$$(DE) \quad x'(t) + A(t, x_t)x(t) = 0, \quad t \in [0, T), \quad x_0 = \phi$$

under the following assumptions:

(A.1) X^* is uniformly convex.

(A.2) The domain of $A_1(\cdot, \cdot, \cdot)$ with $A_1(t, \psi, v) = A(t, \psi)v$ is the set $[0, T) \times C_0 \times D$, where D is a subset of X and C_0 consists of all $f \in C$ with $f(t) \in \bar{D} \cup M$, $t \in [-r, 0]$. Here $M = \{\phi(t); t \in [-r, 0]\}$.

(A.3) For every $(t, \psi) \in [0, T) \times C_0$, $A(t, \psi)v$ is m -accretive in v .

(A.4) For every $t, s \in [0, T)$, $\psi_1, \psi_2 \in C_0$, $v \in D$,

$$\begin{aligned} & \|A(t, \psi_1)v - A(s, \psi_2)v\| \\ & \leq l(\|\psi_1\|_C, \|\psi_2\|_C, \|v\|)[|t - s|(1 + \|A(s, \psi_2)v\|) + \|\psi_1 - \psi_2\|_C], \end{aligned}$$

where $l: R_+^3 \rightarrow R_+$ is increasing in all three variables.

(A.5) $\phi \in C_0$ is a given function with $\phi(0) \in D$ satisfying a Lipschitz condition on $[-r, 0]$ with Lipschitz constant K .

Our method in proving the existence of $x(t)$ follows that of Kartsatos [7], where the equation $x'(t) + A(t, x(t))x(t) = 0$ was studied. We first ensure the existence of the solution $x_u(t)$ of the problem

$$(DE)_u \quad x'(t) + A(t, u_t)x(t) = 0, \quad x_0 = \phi$$

on an interval $[0, T_1]$, where u is taken from a suitable metric space S of continuous functions. We then show that, for T_1 sufficiently small, the operator $U: u \rightarrow x_u$ maps the space S into itself and is a strict contraction. The resulting fixed point of this operator is the desired unique strong solution of (DE).

THEOREM 1. *Assume that Conditions (A.1)-(A.5) are satisfied. Then there exists $T_1 < T$ such that the initial value problem (DE) has a unique strong solution $x(t)$, $t \in [0, T_1]$, which is also Lipschitz continuous on $[0, T_1]$.*

PROOF. Let $N = 1 + \|A(0, \phi)\phi(0)\|$ and L be a positive constant with $L/N < T$. Let T_1 be such that $0 < T_1 \leq L/N$. Consider the set

$$S = \{u: [-r, T_1] \rightarrow \bar{D} \cup M; u(t) \text{ is continuous, } u(t) = \phi(t) \text{ for } t \in [-r, 0] \text{ and } \|u(t_1) - u(t_2)\| \leq N|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T_1]\}.$$

$S \neq \emptyset$ because the function $u(t)$ such that $u(t) = \phi(t)$ for $t \in [-r, 0]$ and $u(t) \equiv \phi(0)$ for $t \in [0, T_1]$ belongs to S . Now, let $u \in S$ be given and consider the problem $(DE)_u$ on the interval $[0, T_1]$. Let $N_1 = \max\{N, K\}$. The operator $B_u(t)v \equiv A(t, u_t)v$ is m -accretive in v by Condition (A.3). Also, by Condition (A.4),

$$(1) \quad \|B_u(t)v - B_u(s)v\| \leq l(\|u_t\|_C, \|u_s\|_C, \|v\|)[|t - s|(1 + \|A(s, u_s)v\|) + \|u_t - u_s\|_C]$$

for every $t \in [0, T_1]$. Now, in order to show that B_u satisfies a condition like (*) (in the introduction), we first observe that

$$\|u_t - u_s\|_C = \sup_{\theta \in [-r, 0]} \|u(t + \theta) - u(s + \theta)\|$$

for every $t, s \in [0, T_1]$. Suppose that $t, s \geq r$. Then, for each $\theta \in [-r, 0]$, $\|u(t + \theta) - u(s + \theta)\| \leq N|t - s| \leq N_1|t - s|$. Suppose that $t, s < r$. Without loss of generality, assume that $t > s$. If $\theta \in [-r, -t]$, then $t + \theta \in [t - r, 0]$ and $s + \theta \in [s - r, s - t]$. For such θ , $\|u(t + \theta) - u(s + \theta)\| = \|\phi(t + \theta) - \phi(s + \theta)\| \leq K|t - s| \leq N_1|t - s|$. If $\theta \in [-t, -s]$, then $t + \theta \in [0, t - s]$

and $s + \theta \in [s - t, 0]$. For such θ , $\|u(t + \theta) - u(s + \theta)\| \leq \|u(t + \theta) - u(0)\| + \|u(0) - u(s + \theta)\| \leq N|t + \theta| + K|s + \theta| \leq N_1|t - s|$. If $\theta \in [-s, 0]$, then $t + \theta \in [t - s, t]$ and $s + \theta \in [0, s]$, which implies again that the above inequality is true. Hence, for all $t, s < r$ and $\theta \in [-r, 0]$, we have $\sup_{\theta \in [-r, 0]} \|u(t + \theta) - u(s + \theta)\| \leq N_1|t - s|$. The same inequality holds if we assume that $t \geq r$ and $s \leq r$. The proof of this fact is similar to the above. It is therefore omitted.

In order to obtain a bound for u_t , we observe that since $u \in S$, we have $\|u(t + \theta) - u(0)\| \leq Nt \leq L$ for every $t \in [0, T_1]$ and every $\theta \in [-r, 0]$ such that $t + \theta \geq 0$. Thus, for such t and θ , $\|u(t + \theta)\| \leq \|\phi(0)\| + L \leq \|\phi\|_C + L$. For t and θ such that $t + \theta < 0$, $\|u(t + \theta) - \phi(0)\| \leq \|\phi(t + \theta)\| + \|\phi(0)\| \leq 2\|\phi\|_C$. It follows that for all $t \in [0, T_1]$, $\theta \in [-r, 0]$ we have the bound:

$$\|u_t\|_C = \sup_{\theta \in [-r, 0]} \|u(t + \theta)\| \leq 2\|\phi\|_C + L.$$

Using these estimates and (1), we obtain

$$(2) \quad \|B_u(t)v - B_u(s)v\| \leq l_1(\|v\|)|t - s|(1 + \|B_u(s)v\|),$$

where $l_1(\|v\|) = (1 + N_1)l(2\|\phi\|_C + L, 2\|\phi\|_C + L, \|v\|)$. Consequently, the conditions of Theorems 1 and 2 of Kato [11] are satisfied. Thus, the problem $(DE)_u$ has a unique strong solution $x_u(t)$ on $[0, T_1]$. The function $x_u(t)$ is also weakly continuously differentiable on $[0, T_1]$ and such that $A(t, u_t)x_u(t)$ is weakly continuous in t . Furthermore, $x_u(t)$ satisfies $(DE)_u$ everywhere on $[0, T_1]$ if $x'(t)$ denotes now the weak derivative of $x(t)$.

We are planning to show that the operator $U: u \rightarrow x_u$ is a strict contraction on S if T_1 is chosen sufficiently small. To this end, fix $u \in S$ and consider the approximating equations

$$(E)_u \quad x'_n(t) + A_n(t)x_n(t) = 0, \quad x_{n_0} = \phi,$$

where $A_n(t) = A_n(t, u_t) = A(t, u_t)[I + (1/n)A(t, u_t)]^{-1}$, $n = 1, 2, \dots$, are the Yosida approximants of $A(t, u_t)$. The operators $A_n(t)$ are defined and Lipschitz continuous on X with Lipschitz constants $\leq 2n$. Moreover, the operators $J_n(t) = [I + (1/n)A(t, u_t)]^{-1}: X \rightarrow D$ are also Lipschitz continuous on X with Lipschitz constants ≤ 1 . Since $B_u(t)$ is m -accretive for each $t \in [0, T_1]$, so are the operators $A_n(t)$ [11, Lemma 2.3]. Also, as in Lemma 4.1 of the same reference, we obtain

$$\|A_n(t)v - A_n(s)v\| = l_1(\|J_n(s)v\|)|t - s|(1 + \|A_n(s)v\|).$$

Since, by (2),

$$\begin{aligned}
 (1/n)\|A_n(s)v\| &\leq (1/n)\|A_n(s)\phi(0)\| + 2\|v - \phi(0)\| \\
 &\leq \|A_n(s, u_s)\phi(0)\| + 2(\|v\| + \|\phi(0)\|) \\
 &\leq \|A(0, \phi)\phi(0)\| + l_1(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|) \\
 &\quad + 2(\|v\| + \|\phi(0)\|) \\
 &= K_1 + 2\|v\|
 \end{aligned}$$

and $\|J_n(s)v\| \leq \|v\| + (1/n)\|A_n(s)v\|$, we finally arrive at

$$\begin{aligned}
 (3) \quad \|A_n(t, u_t)v - A_n(s, u_s)v\| &= \|A_n(t)v - A_n(s)v\| \\
 &\leq l_2(\|v\|)|t - s|(1 + \|A_n(s)v\|),
 \end{aligned}$$

where $l_2(\|v\|) = l_1(3\|v\| + K_1)$. Hence each of the equations $(E)_n$ has a unique strongly continuously differentiable solution $x_n(t)$ defined on $[0, T_1]$ and such that $\lim_{n \rightarrow \infty} x_n(t) = x_u(t)$ strongly and uniformly on $[0, T_1]$ (cf. Kato [11]).

We shall show that the sequence $\{x_n(t)\}$, $n = 1, 2, \dots$, is uniformly bounded and uniformly Lipschitz continuous on $[0, T_1]$ independently of $u \in S$. To this end, using [11, Lemma 1.3], the accretiveness of $A_n(t)$ and (3), we get

$$\begin{aligned}
 2\|x_n(t) - \phi(0)\|(d/dt)\|x_n(t) - \phi(0)\| &= (d/dt)\|x_n(t) - \phi(0)\|^2 \\
 &= 2 \operatorname{Re} \langle x'_n(t), J(x_n(t) - \phi(0)) \rangle \\
 &= -2 \operatorname{Re} \langle A_n(t)x_n(t) - A_n(t)\phi(0), J(x_n(t) - \phi(0)) \rangle \\
 &\quad - 2 \operatorname{Re} \langle A_n(t)\phi(0), J(x_n(t) - \phi(0)) \rangle \\
 &\leq 2\|A_n(t)\phi(0)\|\|x_n(t) - \phi(0)\| \\
 &\leq 2[\|A_n(t, u_t)\phi(0) - A_n(0, \phi)\phi(0)\| + \|A_n(0, \phi)\phi(0)\|]\|x_n(t) - \phi(0)\| \\
 &\leq 2[\|A_n(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)T_1(1 + \|A_n(0, \phi)\phi(0)\|)]\|x_n(t) - \phi(0)\| \\
 &\leq 2[\|A(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|)]\|x_n(t) - \phi(0)\|.
 \end{aligned}$$

This inequality holds a.e. in $[0, T_1]$. Dividing by $2\|x_n(t) - \phi(0)\|$ and integrating from 0 to $t \leq T_1$, we obtain

$$(4) \quad \|x_n(t) - \phi(0)\| \leq K_2 T_1,$$

where $K_2 = \|A(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|)$ is independent of T_1 , n and $u \in S$. In order to find a uniform upper bound for the derivatives $x'_n(t)$, we consider the function $z_n(t) \equiv x_n(t+h) - x_n(t)$, $0 \leq t, t+h < T_1$. Using again Lemma 1.3 of Kato [11], the accretiveness of $A_n(t+h)$, the uniform boundedness of $\{x_n(t)\}$ from (4), and the appraisal (3), we get

$$\begin{aligned}
 (1/2)(d/dt)\|z_n(t)\|^2 &= \operatorname{Re} \langle z'_n(t), J(z_n(t)) \rangle \\
 &= -\operatorname{Re} \langle A_n(t+h)x_n(t+h) - A_n(t)x_n(t), J(z_n(t)) \rangle
 \end{aligned}$$

$$\begin{aligned} &= -\operatorname{Re} \langle A_n(t+h)x_n(t+h) - A_n(t+h)x_n(t), J(z_n(t)) \rangle \\ &\quad - \operatorname{Re} \langle A_n(t+h)x_n(t) - A_n(t)x_n(t), J(z_n(t)) \rangle \\ &\leq \|A_n(t+h, u_{t+h})x_n(t) - A_n(t, u_t)x_n(t)\| \|z_n(t)\| \\ &\leq |h| l_2(\|x_n(t)\|)(1 + \|A_n(t)x_n(t)\|) \|z_n(t)\| \\ &\leq |h| l_2(\|\phi(0)\| + K_2 T_1)(1 + \|x'_n(t)\|) \|z_n(t)\|. \end{aligned}$$

Dividing through by $|h| \|z_n(t)\|$, integrating and then passing to the limit as $h \rightarrow 0$, we get

$$\|x'_n(t)\| = \|x'_n(0)\| + \int_0^t l_2(\|\phi(0)\| + K_2 T_1) \|x'_n(s)\| ds + l_2(\|\phi(0)\| + K_2 T_1) T_1.$$

Applying Gronwall's inequality, we find

$$(5) \quad \|x'_n(t)\| \leq (K_3 T_1 + K_4) e^{K_3 T_1},$$

where $K_3 = l_2(\|\phi(0)\| + K_2(L/N))$ is independent of T_1 , n and $u \in S$, and $K_4 = \|A(0, \phi)\phi(0)\|$. From (4) and (5) we conclude that

$$\|x_u(t)\| \leq \|\phi(0)\| + K_2 T_1, \quad \|x_u(t_1) - x_u(t_2)\| \leq K_5 |t_1 - t_2|$$

for every $t_1, t_2 \in [0, T_1]$, where K_5 is the right hand side of (5).

Now, let $u_1, u_2 \in S$ be given and let x_1, x_2 be the corresponding solutions of $(DE)_{u_i}$, $i = 1, 2$. Then we have

$$\begin{aligned} (6) \quad &(1/2)(d/dt) \|x_1(t) - x_2(t)\|^2 \\ &= -\operatorname{Re} \langle A(t, u_{1t})x_1(t) - A(t, u_{2t})x_2(t), J(x_1(t) - x_2(t)) \rangle \\ &\leq -\operatorname{Re} \langle A(t, u_{1t})x_2(t) - A(t, u_{2t})x_2(t), J(x_1(t) - x_2(t)) \rangle \\ &\leq l(\|u_{1t}\|_C, \|u_{2t}\|_C, \|x_2(t)\|) \|u_{1t} - u_{2t}\|_C \|x_1(t) - x_2(t)\| \end{aligned}$$

from which, dividing by $\|x_1(t) - x_2(t)\|$ and then integrating, we arrive at

$$\|x_1(t) - x_2(t)\| \leq K_6 \sup_{t \in [0, T_1]} \|u_{1t} - u_{2t}\|_C,$$

where $K_6 = T_1 l(2\|\phi\|_C + L, 2\|\phi\|_C + L, \|\phi(0)\| + K_2(L/N))$. Since $\sup_{t \in [0, T_1]} \|u_{1t} - u_{2t}\|_C = \sup_{t \in [0, T_1]} \|u_1(t) - u_2(t)\|$, we conclude that

$$\sup_{t \in [0, T_1]} \|x_1(t) - x_2(t)\| \leq K_6 \sup_{t \in [0, T_1]} \|u_1(t) - u_2(t)\|.$$

Now, we choose T_1 so small that $K_5 \leq N$ and $K_6 < 1$. Then the operator $U: u \rightarrow x_u$ is a strict contraction on a complete metric space. Let $x(t)$, $t \in [0, T_1]$ be the unique fixed point of U . Then $x(t)$ is the desired solution of the problem (DE). Its uniqueness follows from (6) by replacing u_1, u_2 by x_1, x_2 , respectively.

The above result can be extended to include the infinite (unbounded)

delay version of (DE). In fact, in that case we let C equal the space of all bounded and uniformly continuous functions $f: (-\infty, 0] \rightarrow X$ with the sup-norm. Moreover, we let C_0 be now the space of all $f \in C$ such that $f(t) \in \bar{D} \cup M$, $t \in (-\infty, 0]$, where $M = \{\phi(t); t \in (-\infty, 0]\}$. The proof of this result follows as above and is therefore omitted.

3. Galerkin approximations. In this section we consider a Galerkin approximation scheme for the solution of the abstract initial value problem

$$(FDE) \quad x'(t) + A(t)x(t) = G(t, x_t), \quad t \in [0, T]$$

with $X = H$, a real Hilbert space, $\phi \in C$ such that $\phi' \in C$ and the operators $A(t)$ and G satisfying the following conditions:

(C.1) For each $t \in [0, T]$, $A(t): H \rightarrow H$ is m -accretive.

(C.2) There exists a nondecreasing function $L_1: R_+ \rightarrow R_+$ such that

$$\|A(t)x - A(s)x\| \leq |t - s|L_1(\|x\|)(1 + \|A(s)x\|).$$

(C.3) $A(0)$ maps bounded sets into bounded sets.

(C.4) There exists a constant $b > 0$ such that for every $\phi, \psi \in C$, $t \in [0, T]$, $\|G(t, \phi) - G(t, \psi)\| \leq b\|\phi - \psi\|_C$.

(C.5) There exists $L_2: R_+ \rightarrow R_+$, nondecreasing and such that for every $s, t \in [0, T]$, $\phi \in C$, $\|G(t, \phi) - G(s, \phi)\| \leq L_2(\|\phi\|_C)|t - s|$.

Under the assumptions (C.1), (C.2), (C.4) and (C.5) we have the existence of a unique strong solution of (FDE) on $[0, T]$, for example, by [9, Theorem 2.1]. In what follows, the space H is separable. Let e_1, e_2, \dots be a basis of H and let H_n be the subspace of H spanned by the vectors e_1, e_2, \dots, e_n . Let $P_n: H \rightarrow H_n$ be the projection on H_n . We consider the (finite dimensional) approximating problems

$$(FDE)_n \quad \begin{aligned} x'_n(t) + P_n A(t)x_n(t) &= P_n G(t, x_{n,t}), & t \in [0, T], \\ x_n(t) &= P_n \phi(t), & t \in [-r, 0]. \end{aligned}$$

The Galerkin method has been already used by other authors to obtain the existence and/or approximation of solutions of nonlinear evolution equations. We should mention here the paper of Browder [3], where the Galerkin method was used to obtain the existence of the unique solution of the problem

$$\begin{aligned} x'(t) + A(t)x(t) &= 0, & t \in R_+, \\ x(0) &= x_0. \end{aligned}$$

Here, $A(t)$ is a continuous m -accretive (thus maximal monotone) operator defined on the whole of H and mapping bounded sets into bounded sets.

Gajewski and Zacharias established in [5] the convergence of the Galerkin approximants for the unique strong solution of the perturbed evolution equation

$$\begin{aligned}x'(t) + A(t)x(t) &= G(t, x(t)), & t \in [0, T], \\x(0) &= x_0.\end{aligned}$$

Their results were extended by Kartsatos [8] to operators $A(t)$ defined on a proper subset of H . Abstract semigroup theory has been the setting for applying the Galerkin method in Banks [1], [2], Kappel and Schappacher [6] and Webb [13]. These authors have considered equations that fall into the type:

$$x'(t) = f(t, x_t) + g(t), \quad t \in [0, T].$$

Their approach in these papers is to consider an abstract equation in the space of initial functions involving an operator which generates a nonlinear semigroup on that space. The Galerkin approximations are then given for that equation.

We note that, in our case, since $A(t)$ is defined on the whole space, it is demicontinuous, i.e., it is continuous from the strong topology of H to the weak topology of H [12, p. 107].

In what follows the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product of H . We should also remark that $P_n y(t) \rightarrow y(t)$ strongly and uniformly as $n \rightarrow \infty$ for any continuous function $y: [a, b] \subset [-r, T] \rightarrow H$.

THEOREM 2. *Assume that Conditions (C.1)–(C.5) are satisfied. Then the sequence $\{x_n(t)\}$ of the Galerkin approximants satisfying (FDE) $_n$ exists and converges strongly and uniformly to the unique solution $x(t)$ of (FDE).*

PROOF. As we mentioned above, the unique strong solution $x(t)$ of (FDE) exists by [9, Theorem 2.1]. We note that in (FDE) $_n$ the operator $P_n A(t)$ is accretive on H_n . Since it is also demicontinuous on H , it is continuous on H_n . Thus, by a well known result, $P_n A(t)$ is m -accretive on H_n . It is also easy to see that $P_n A(t)$ is Lipschitz continuous in t , satisfying a condition similar, but not identical, to (C.3). Since the projection P_n has norm 1, the function $P_n G(t, \phi)$ satisfies the Lipschitz conditions (C.4) and (C.5). With these facts established, the existence of the unique strong solution of (FDE) $_n$ is guaranteed by the following argument. Consider the equations

$$\begin{aligned}(\text{FDE})_{mn} \quad u'_{mn}(t) + P_n A_m(t)u_{mn}(t) &= P_n G(t, u_{mn_t}), & t \in [0, T], \\u_{mn_0} &= x_{n_0},\end{aligned}$$

where $x_{n_0}(\theta) = x_n(\theta) = P_n\phi(\theta)$, $\theta \in [-r, 0]$, and $A_m(t)$ are the Yoshida approximants of $A(t)$. Following the proof of Lemma 2.3 of [9], we can show that, for a fixed n , the (unique) solutions $u_{mn}(t)$ ($u_{mn}(t) \in H_n$, $m = 1, 2, \dots, t \in [0, T]$) of the problems $(FDE)_{mn}$ are uniformly bounded. On the other hand, since $P_n A(t)x$ is continuous on the set $[0, T] \times H_n$, it maps bounded subsets of it into bounded subsets of H_n . Using this fact, we can easily see that there exists a constant $K_n > 0$ such that $\|P_n A_m(t)u_{mn}(t)\| \leq K_n$ for every $m = 1, 2, \dots$ and every $t \in [0, T]$. This in turn implies that there exists a constant $L_n > 0$ such that the functions $u'_{mn}(t)$, given by $(FDE)_{mn}$, satisfy: $\|u'_{mn}(t)\| \leq L_n$ for every $m = 1, 2, \dots$ and every $t \in [0, T]$. The uniform convergence of $u_{mn}(t)$ and $u'_{mn}(t)$ to $x_n(t)$ and $x'_n(t)$, for $m \rightarrow \infty$, respectively, follows now almost exactly as in [9]. Its proof is therefore omitted.

In order to show that the sequence $\{x_n(t)\}$ is uniformly bounded, we start with the inequality

$$\begin{aligned} (1/2)(d/dt)\|x_n(t) - P_n\phi(0)\|^2 &= \langle x'_n(t), x_n(t) - P_n\phi(0) \rangle \\ &= -\langle P_n A(t)x_n(t), x_n(t) - P_n\phi(0) \rangle + \langle P_n G(t, x_{n_t}), x_n(t) - P_n\phi(0) \rangle \\ &= -\langle A(t)x_n(t), x_n(t) - P_n\phi(0) \rangle + \langle G(t, x_{n_t}), x_n(t) - P_n\phi(0) \rangle \\ &= -\langle A(t)x_n(t) - A(t)P_n\phi(0), x_n(t) - P_n\phi(0) \rangle \\ &\quad - \langle A(t)P_n\phi(0), x_n(t) - P_n\phi(0) \rangle + \langle G(t, x_{n_t}), x_n(t) - P_n\phi(0) \rangle \\ &\leq \|A(t)P_n\phi(0)\| \|x_n(t) - P_n\phi(0)\| + \|G(t, x_{n_t}) - G(t, x_{n_0})\| \|x_n(t) - P_n\phi(0)\| \\ &\quad + \|G(t, x_{n_0})\| \|x_n(t) - P_n\phi(0)\| \\ &\leq \|A(0)P_n\phi(0)\| \|x_n(t) - P_n\phi(0)\| \\ &\quad + TL_1(\|P_n\phi(0)\|)(1 + \|A(0)P_n\phi(0)\|) \|x_n(t) - P_n\phi(0)\| \\ &\quad + b\|x_{n_t} - x_{n_0}\|_C \|x_n(t) - P_n\phi(0)\| + \|G(t, x_{n_0})\| \|x_n(t) - P_n\phi(0)\| \end{aligned}$$

which implies

$$\begin{aligned} (d/dt)\|x_n(t) - P_n\phi(0)\| &\leq \|A(0)P_n\phi(0)\| + TL_1(\|P_n\phi(0)\|)(1 + \|A(0)P_n\phi(0)\|) + \|G(t, x_{n_0})\| + b\|x_{n_t} - x_{n_0}\|_C \\ &\leq K + b\|x_{n_t} - x_{n_0}\|_C, \end{aligned}$$

a.e. in $[0, T]$, where K is a positive constant. Here we have used the boundedness of $A(0)$ and $G(t, x_{n_0})$ on $[0, T]$. Integrating, we obtain

$$\|x_n(t) - P_n\phi(0)\| \leq KT + b \int_0^t \|x_{n_s} - x_{n_0}\|_C ds.$$

Thus, for any $t_1 \in [0, t]$, we have

$$\|x_n(t_1) - P_n\phi(0)\| \leq KT + b \int_0^{t_1} \|x_{n_s} - x_{n_0}\|_C ds$$

$$\leq KT + b \int_0^t \|x_{n_s} - x_{n_0}\|_C ds .$$

If $t_1 \in [-r, 0]$, then $\|x_n(t_1) - P_n\phi(0)\| \leq \|P_n\phi(t_1)\| + \|P_n\phi(0)\| \leq 2\|\phi\|_C$.
Hence

$$\sup_{\theta \in [-r, 0]} \|x_n(t + \theta) - P_n\phi(0)\| \leq KT + 2\|\phi\|_C + b \int_0^t \|x_{n_s} - x_{n_0}\|_C ds ,$$

which implies

$$\begin{aligned} \|x_{n_t} - x_{n_0}\|_C &= \sup_{\theta \in [-r, 0]} \|x_n(t + \theta) - x_{n_0}(\theta)\| \\ &\leq \sup_{\theta \in [-r, 0]} \|x_n(t + \theta) - P_n\phi(0)\| + \sup_{\theta \in [-r, 0]} \|P_n\phi(0) - x_{n_0}\| \\ &\leq KT + 4\|\phi\|_C + b \int_0^t \|x_{n_s} - x_{n_0}\|_C ds . \end{aligned}$$

Applying Gronwall's inequality above, we obtain the boundedness of $\{x_{n_t} - x_{n_0}\}$, which implies the boundedness of $\{x_n(t)\}$. We are now ready to show the convergence of $x_n(t)$ to $x(t)$ uniformly on $[0, T]$. We first observe that

$$(7) \quad \begin{aligned} \langle x'_n(t), x_n(t) - x(t) \rangle + \langle P_n A(t)x_n(t), x_n(t) - x(t) \rangle \\ = \langle P_n G(t, x_{n_t}), x_n(t) - x(t) \rangle \end{aligned}$$

$$(8) \quad \begin{aligned} \langle x'(t), x_n(t) - x(t) \rangle + \langle A(t)x(t), x_n(t) - x(t) \rangle \\ = \langle G(t, x_t), x_n(t) - x(t) \rangle . \end{aligned}$$

Subtracting (8) from (7), we find

$$\begin{aligned} \langle x'_n(t) - x'(t), x_n(t) - x(t) \rangle \\ = -\langle A(t)x_n(t), x_n(t) - P_n x(t) \rangle + \langle A(t)x(t), x_n(t) - x(t) \rangle \\ + \langle G(t, x_{n_t}), x_n(t) - P_n x(t) \rangle - \langle G(t, x_t), x_n(t) - x(t) \rangle \\ = -\langle A(t)x_n(t) - A(t)x(t), x_n(t) - x(t) \rangle - \langle A(t)x_n(t), x(t) - P_n x(t) \rangle \\ + \langle G(t, x_{n_t}) - G(t, x_t), x_n(t) - x(t) \rangle + \langle G(t, x_{n_t}), x(t) - P_n x(t) \rangle , \end{aligned}$$

which implies

$$(1/2)(d/dt) \|x_n(t) - x(t)\|^2 \leq \|A(t)x_n(t)\| \|x(t) - P_n x(t)\| + b \|x_{n_t} - x_t\|_C^2 + \|G(t, x_{n_t})\| \|x(t) - P_n x(t)\| .$$

Integrating this inequality, we arrive at

$$(9) \quad \|x_n(t) - x(t)\|^2 \leq \|x_n(0) - x(0)\|^2 + \int_0^t h(s) ds + 2b \int_0^t \|x_{n_s} - x_s\|_C^2 ds ,$$

where $h(t) = 2\|A(t)x_n(t)\| \|x(t) - P_n x(t)\| + 2\|G(t, x_{n_t})\| \|x(t) - P_n x(t)\|$.

Since the above inequality holds for any $t_1 \in [0, t]$, and since, for $t_1 \in [-r, 0]$,

$$\|x_n(t_1) - x(t_1)\| = \|P_n\phi(t_1) - \phi(t_1)\| \leq \sup_{\theta \in [-r, 0]} \|P_n\phi(\theta) - \phi(\theta)\| ,$$

we actually have

$$\begin{aligned} \|x_n(t) - x(t)\|^2 &\leq \|x_n(0) - x(0)\|^2 + \sup_{\theta \in [-r, 0]} \|P_n\phi(\theta) - \phi(\theta)\|^2 \\ &\quad + \int_0^t h(s)ds + 2b \int_0^t \|x_{n_s} - x_s\|_C^2 ds , \quad t \in [-r, T] . \end{aligned}$$

Consequently, by Gronwall's inequality, we get

$$\begin{aligned} &\sup_{\theta \in [-r, 0]} \|x_{n_i}(\theta) - x_i(\theta)\| \\ &\leq \left[\|x_n(0) - x(0)\|^2 + \sup_{\theta \in [-r, 0]} \|P_n\phi(\theta) - \phi(\theta)\|^2 + \int_0^T h(t)dt \right] e^{2bt} . \end{aligned}$$

Now, $x_n(0) - x(0) = P_n\phi(0) - \phi(0) \rightarrow 0$ as $n \rightarrow \infty$ and $P_n\phi(\theta) - \phi(\theta) \rightarrow 0$ uniformly on $[-r, 0]$. In addition, for the three normed expressions in $h(t)$ we have the following properties. From the boundedness of $A(0)$ and the inequality $\|A(t)x_n(t)\| \leq \|A(0)x_n(t)\| + TL_1(\|x_n(t)\|)(1 + \|A(0)x_n(t)\|)$ we obtain the uniform boundedness of $\{A(t)x_n(t)\}$. The uniform boundedness of $\{x_{n_i}\}$ implies the same property for $\{G(t, x_{n_i})\}$. Finally, $P_nx(t) \rightarrow x(t)$ uniformly on $[0, T]$. Thus, an application of Lebesgue's bounded convergence theorem shows that

$$\sup_{\theta \in [-r, 0]} \|x_{n_i}(\theta) - x_i(\theta)\| \rightarrow 0 \text{ as } n \rightarrow \infty ,$$

which in turn shows that $x_n(t) \rightarrow x(t)$ uniformly on $[0, T]$.

It should be noted here that we do not assume that $A(t)P_nx \rightarrow A(t)x$ for every $x \in H$. This assumption is actually included in the result of Kartsatos [8] if the domain of $A(t)$ there is the whole of H . Also, the constant b in (C.5) can be replaced by a Lebesgue integrable function $b: [0, T] \rightarrow R_+$.

4. Applications. As an example to which we can apply our result of Section 3, we cite the nonlinear initial-boundary value problem:

$$\begin{aligned} \text{(E)} \quad &(\partial/\partial t)u(x, t) + A(t, x, u(x, t)) = f(t, x, u(x, t - r)) , \quad t \in (0, T) , \quad x \in \Omega , \\ &u(x, \theta) = \phi(x, \theta) , \quad x \in \Omega , \quad \theta \in [-r, 0] , \\ &D^\alpha u(x, t) = 0 , \quad x \in \partial\Omega , \quad t \in (0, T] , \quad |\alpha| < m , \end{aligned}$$

where u is a real valued function, r is a positive constant, Ω is a bounded open subset of R^n ($R = (-\infty, \infty)$, $n \geq 2$) with sufficiently smooth

boundary, $A(t, x, u)$, $f(t, x, u)$ are nonlinear elliptic partial differential operators in divergence form:

$$A(t, x, u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} b_\alpha(t) D^\alpha A_\alpha(x, \xi(u)),$$

$$f(t, x, u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(t, x, \xi(u)),$$

and $\phi: \Omega \times [-r, 0] \rightarrow R$ is a given function. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers we adopt the notation

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad D_i = (\partial/\partial x_i), \quad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

By R^{nm} we denote the space of all real vectors of the form $\xi = \{\xi_\alpha; |\alpha| \leq m\}$. Thus, $\xi(u) = \{D^\alpha u; |\alpha| \leq m\}$.

For the results concerning such partial differential operators the reader is referred, for example, to Browder [4] and Pascali and Sburlan [12].

Now, let $W^{m,2}(\Omega)$ be the Sobolev space of all real valued functions u such that $D^\alpha u \in L^2(\Omega)$ for every α with $|\alpha| \leq m$. $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$\langle u, v \rangle_m = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)}.$$

Let $C_c^\infty(\Omega)$ be the space of all $f \in C^\infty(\Omega)$ with compact support. We denote by $W_0^{m,2}(\Omega)$ the closure of the space $C_c^\infty(\Omega)$ in $W^{m,2}(\Omega)$. The space $W_0^{m,2}(\Omega)$ is thus another separable Hilbert space. We let H denote this space and we make the following additional assumptions:

(i) for each α , $A_\alpha: \Omega \times R^{nm} \rightarrow R$ satisfies the Caratheodory conditions and there exists a function $g \in L^2(\Omega)$ and a constant $c > 0$ such that

$$|A_\alpha(x, \xi)| \leq c|\xi| + g(x), \quad (x, \xi) \in \Omega \times R^{nm},$$

where $|\xi| = (\sum_{|\alpha| \leq m} \xi_\alpha^2)^{1/2}$.

(ii) For $x \in \Omega$ and $\xi, \xi' \in R^{nm}$ we have

$$\sum_{|\alpha| \leq m} [A_\alpha(x, \xi) - A_\alpha(x, \xi')](\xi_\alpha - \xi'_\alpha) \geq 0.$$

(iii) Each $b_\alpha: [0, T] \rightarrow R_+$ is Lipschitz continuous on $[0, T]$, $\phi(\cdot, \theta) \in W_0^{m,2}(\Omega)$ for every $\theta \in [-r, 0]$, $\phi(x, \theta)$ is continuous and satisfies a Lipschitz condition with respect to θ uniformly in $x \in \Omega$.

(iv) The functions $f_\alpha: [0, T] \times \Omega \times R^{nm} \rightarrow R$ are continuous and such that: there exists a nonnegative function $h \in L^2(\Omega)$ and a constant $L > 0$ with

$$|f_\alpha(t, x, \xi) - f_\alpha(t', x, \xi')| \leq h(x)|t - t'| + L|\xi - \xi'|$$

for every $t, t' \in [0, T]$, $x \in \Omega$, and $\xi, \xi' \in R^{nm}$.

If for each $t \in [0, T]$, $u, v \in W_0^{m,2}(\Omega)$ we let

$$a^t(u, v) = \sum_{|\alpha| \leq m} b_\alpha(t) \int_\Omega A_\alpha(x, \xi(u(x))) D^\alpha v(x) dx,$$

then $a^t(u, v)$ is a bounded linear functional in v . By the Riesz representation theorem, there exists a nonlinear operator $T(t): H \rightarrow H$ such that

$$\langle T(t)u, v \rangle_m = a^t(u, v), \quad (u, v) \in H \times H.$$

The operator $T(t)$ is continuous, m -accretive, and maps bounded subsets of H into bounded sets for each $t \in [0, T]$. The proof of this fact follows as in [12, p. 275]. It is also easy to see that T satisfies the condition (C.2). Similarly, we can obtain an operator $F(t): H \rightarrow H$ such that

$$\langle F(t)u, v \rangle_m = \sum_{|\alpha| \leq m} \int_\Omega f_\alpha(t, x, \xi(u(x))) D^\alpha v(x) dx$$

for every $t \in [0, T]$, $u, v \in W_0^{m,2}(\Omega)$. In order to show that $F(t)u$ satisfies a global Lipschitz condition on $[0, T] \times W_0^{m,2}(\Omega)$, we observe that

$$\begin{aligned} & \left| \int_\Omega [f_\alpha(t, x, \xi(u(x))) - f_\alpha(t', x, \xi(v(x)))] D^\alpha v(x) dx \right| \\ & \leq \left(\int_\Omega [f_\alpha(t, x, \xi(u(x))) - f_\alpha(t', x, \xi(v(x)))]^2 dx \right)^{1/2} \cdot \|v\|_{m,2} \\ & \leq \left(\int_\Omega [h(x)|t - t'| + L|\xi(u(x)) - \xi(v(x))|^2] dx \right)^{1/2} \cdot \|v\|_{m,2} \\ & \leq \left(\left(\int_\Omega h^2(x) dx \right)^{1/2} |t - t'| + L \|u - v\|_{m,2} \right) \|v\|_{m,2} \\ & = (K|t - t'| + L \|u - v\|_{m,2}) \|v\|_{m,2}, \end{aligned}$$

where $\|\cdot\|_{m,2}$ is the norm of $W_0^{m,2}(\Omega)$ and K is an obvious constant. Adding these inequalities, we obtain our assertion.

Now, we consider the abstract problem

$$\begin{aligned} \text{(AE)} \quad & u'(t) + T(t)u(t) = G(t, u_t), \quad t \in [0, T], \\ & u_0 = \phi, \end{aligned}$$

where $G(t, \psi) = F(t)\psi(-r)$, for any $\psi \in C$, and $u'(t)$ denotes the weak derivative of $u(t)$. Since the conditions (C.1)–(C.5) are satisfied, the unique strong solution of (AE) can be approximated by the Galerkin method.

As an application of Theorem 1, we consider the initial-boundary value problem consisting of the equation

$$(\partial/\partial t)u(x, t) + A(t, x, u(x, t - r), u(x, t)) = 0, \quad t \in (0, T), \quad x \in \Omega$$

and the initial and boundary conditions in (E). We assume that the initial and boundary conditions satisfy the hypotheses made above, and we let the elliptic differential operator A have the form

$$A(t, x, u, v) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha b_\alpha(t, x, \xi(u)) A_\alpha(x, \xi(v)).$$

We assume, further, that the following conditions hold:

- (1) Each A_α satisfies (i) and (ii) above with $g(x)$ constant and $c = 0$.
- (2) Each b_α is defined and continuous on $[0, T) \times \Omega \times R^{n_m}$, it has values in R_+ and, for some constants $K > 0$, $L > 0$,

$$|b_\alpha(t, x, \xi) - b_\alpha(t', x, \xi')| \leq K|t - t'| + L|\xi - \xi'|$$

for every $t, t' \in [0, T)$, $x \in \Omega$, $\xi, \xi' \in R^{n_m}$.

Now, let $T(t, u)v$ be defined on $W_0^{m,2}(\Omega)$ from the equation

$$\langle T(t, u)v, w \rangle_m = \sum_{|\alpha| \leq m} \int_\Omega b_\alpha(t, x, \xi(u(x))) A_\alpha(x, \xi(v(x))) D^\alpha w(x) dx.$$

It is easy to see that $T(t, u)v$ is continuous, monotone and bounded in v and satisfies the following Lipschitz condition:

$$\|T(t, u)v - T(t', u')v\|_{m,2} \leq K_1|t - t'| + L_1\|u - u'\|_{m,2}$$

for all $t, t' \in [0, T)$, $u, u', v \in W_0^{m,2}$, where K_1, L_1 are positive constants. Setting $T(t, \phi)v = T(t, \phi(-r))v$ for $(t, \phi, v) \in [0, T) \times C \times W_0^{m,2}$, we see that all the conditions of Theorem 1 are satisfied for the abstract problem

$$u'(t) + T(t, u_t)u(t) = 0, \quad t \in [0, T),$$

$$u_0 = \phi.$$

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