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Existence of solutions and monotone iterative method for infinite systems of parabolic differential-functional equations

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Abstract. We consider the Fourier first boundary value problem for an infinite system of weakly coupled nonlinear differential-functional equations. To prove the existence and uniqueness of solution, we apply a monotone iterative method using J. Szarski's results on differential-functional inequalities and a comparison theorem for infinite systems.

1. Introduction. We consider an infinite system of weakly coupled differential-functional equations of the form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z(t,\cdot)), \quad i \in S, \tag{1}$$

where

$$\mathcal{F}^i := \frac{\partial}{\partial t} - \mathcal{A}^i, \quad \mathcal{A}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\frac{\partial^2}{\partial x_j \partial x_k},$$

$x = (x_1, \ldots, x_m)$, $(t,x) \in (0,T) \times G := D$, $T < \infty$, $G \subset \mathbb{R}^m$ and G is an open bounded domain with $C^{2+\alpha}$ $(0 < \alpha \le 1)$ boundary.

Let $B(S)$ be the Banach space of mappings

$$v : S \ni i \to v^i \in \mathbb{R},$$

with the finite norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\},$$

where S is a denumerable set of indices (finite or infinite). The case of finite systems ($B(S) = \mathbb{R}^r$) was treated in [3, 4]. For infinite countable S we have $B(S) = l^\infty$ and we now focus on such infinite systems. Thus,

$$\|v\|_{B(S)} = \|v\|_{l^\infty}.$$

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Denote by $C_S(\overline{G})$ the real Banach space of mappings

$$w : \overline{G} \ni x \to (w(x) : S \ni i \to w^i(x) \in \mathbb{R}) \in l^\infty,$$

where w^i are continuous in $\overline{G}$, with the finite norm

$$\|w\| := \sup\{|w^i(x)| : x \in \overline{G},\ i \in S\}.$$

For any fixed $t \in [0,T)$, we denote by $z(t,\cdot) = (z^1(t,\cdot), z^2(t,\cdot), \ldots)$ the function

$$z(t,\cdot) : \overline{G} \ni x \to z(t,x) \in l^\infty$$

which is an element of the space $C_S(\overline{G})$. We denote the space of these functions by $C_S(\overline{D})$.

For system (1) we consider the following *Fourier first boundary value problem*:

Find a regular solution $z = z(t,x)$ of (1) in $\overline{D}$ satisfying the boundary condition

$$z(t,x) = g(t,x) \quad \text{for } (t,x) \in \Sigma, \tag{2}$$

where $\sigma := (0,T) \times \partial G$, $D_0 := \{(t,x) : t = 0,\ x \in \overline{G}\}$, $\Sigma := D_0 \cup \sigma$, $\overline{D} := D \cup \Sigma$ and $g = (g^1, g^2, \ldots)$.

To prove the existence and uniqueness of the solution, we apply an iterative successive approximations method (see [3, 4]). We use J. Szarski's results [9, 10] on differential-functional inequalities and a comparison theorem for infinite systems and parabolic differential inequalities [8].

Infinite systems of parabolic differential and differential-integral equations are used to describe polymerization-type chemical reaction phenomena (coagulation and fragmentation of clusters) [2], [6]. An infinite system of ordinary differential equations was introduced by M. Smoluchowski ([7], 1917) as a model for coagulation of colloids moving according to a Brownian motion.

2. Notations, definitions and assumptions. A mapping $z \in C_S(\overline{D})$ will be called *regular* if the functions z^i $(i \in S)$ are continuous in $\overline{D}$ and have continuous derivatives $\partial z^i/\partial t$, $\partial^2 z^i/\partial x_i \partial x_k$ in D for $j,k = 1,\ldots,m$. We write briefly $z \in C_S^{\mathrm{reg}}(\overline{D})$.

A regular mapping will be called a *regular solution* of problem (1), (2) in $\overline{D}$ if the above equations are satisfied in D and the boundary condition (2) is satisfied.

The Hölder space $C^{l+\alpha}(\overline{D}) := C^{(l+\alpha)/2,l+\alpha}(\overline{D})$ $(l = 0,1,2,\ldots;\ 0 < \alpha \le 1)$ is the space of continuous functions f whose derivatives $\partial^{r+s} f/\partial t^r \partial x^s := D_t^r D_x^s f(t,x)$ $(0 \le 2r + s \le l)$ all exist and are Hölder continuous with exponent α $(0 < \alpha \leq 1)$ in D, with the finite norm

$$|f|^{l+\alpha} := \sup_{\substack{(t,x)\in D\\ 0\leq 2r+s\leq l}} |D_t^r D_x^s f(t,x)| + \sup_{\substack{P,P'\in D\\ 2r+s=l\\ P\neq P'}} \frac{|D_t^r D_x^s f(t,x) - D_t^r D_x^s f(t',x')|}{[d(P,P')]^\alpha},$$

where $P = (t,x)$, $P' = (t',x')$ and $d(P,P')$ is the parabolic distance defined by

$$d(P,P') := (|t-t'| + \|x-x'\|_{\mathbb{R}^m}^2)^{1/2},$$

and $\|x\|_{\mathbb{R}^m} := (\sum_{j=1}^m x_j^2)^{1/2}$.

We denote by $C_S^{l+\alpha}(\overline{D})$ the Banach space of mappings z such that $z^i \in C^{l+\alpha}(\overline{D})$ for all $i \in S$.

In the space $C_S(\overline{D})$ the following order is introduced: for $z, \widetilde{z} \in C_S(\overline{D})$ the inequality $z \leq \widetilde{z}$ means that $z^i(t,x) \leq \widetilde{z}^i(t,x)$ for all $(t,x) \in \overline{D}$, $i \in S$.

We assume that the operators $\mathcal{A}^i$ $(i \in S)$ are *uniformly elliptic* in $\overline{D}$, i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^m a_{jk}^i(t,x)\xi_j\xi_k \geq \mu \sum_{j=1}^m \xi_j^2$$

for all $\xi = (\xi_1,\ldots,\xi_m) \in \mathbb{R}^m$, $(t,x) \in \overline{D}$, $i \in S$.

We say that the operators $\mathcal{F}^i = \partial/\partial t - \mathcal{A}^i$ $(i \in S)$ are *uniformly parabolic* in $\overline{D}$ when the operators $\mathcal{A}^i$ are uniformly elliptic in $\overline{D}$.

Functions $u = u(t,x)$ and $v = v(t,x) \in C_S^{reg}(\overline{D})$ satisfying the systems of inequalities

$$\begin{cases} \mathcal{F}^i[u^i](t,x) \leq f^i(t,x,u(t,\cdot)) & \text{for } (t,x) \in D,\ i \in S, \\ u(t,x) \leq g(t,x) & \text{for } (t,x) \in \Sigma, \end{cases} \tag{3}$$

$$\begin{cases} \mathcal{F}^i[v^i](t,x) \geq f^i(t,x,v(t,\cdot)) & \text{for } (t,x) \in D,\ i \in S, \\ v(t,x) \geq g(t,x) & \text{for } (t,x) \in \Sigma \end{cases} \tag{4}$$

are called, respectively, a *lower* and an *upper function* for problem (1), (2) in $\overline{D}$.

We assume that the functions

$$f^i : D \times C_S(\overline{G}) \ni (t,x,s) \to f^i(t,x,s) \in \mathbb{R}, \quad i \in S,$$

satisfy the following assumptions:

(H_f) $f^i(\cdot,\cdot,s) \in C^{0+\alpha}(\overline{D})$ $(i \in S)$;

(L) f^i $(i \in S)$ satisfy the Lipschitz condition with respect to s, i.e., for all $s, \widetilde{s}$ we have

$$|f^i(t,x,s) - f^i(t,x,\widetilde{s})| \leq L\|s - \widetilde{s}\| \quad \text{for } (t,x) \in D,$$

where $L > 0$ is constant;

(W) f^i $(i \in S)$ are increasing with respect to s.

(H_a) The coefficients $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ $(j,k = 1,\ldots,m,\ i \in S)$ in (1) are Hölder continuous with respect to t and x in $\overline{D}$, i.e., $a^i_{jk} \in C^{0+\alpha}(\overline{D})$.

(H_g) $g^i \in C^{2+\alpha}(\Sigma)$ for $i \in S$.

We remark that if $g^i \in C^{2+\alpha}(\Sigma)$ and $\partial G \in C^{2+\alpha}$ then, without loss of generality, we can consider the homogeneous boundary condition

$$z(t,x) = 0 \quad \text{for } (t,x) \in \Sigma. \tag{5}$$

Accordingly, in what follows we confine ourselves to considering the homogeneous problem (1), (5) in $\overline{D}$ only.

Assumption (A). There exists at least one pair $u_0 = u_0(t,x)$, $v_0 = v_0(t,x)$ of a lower and an upper function for problem (1), (5) in $\overline{D}$.

3. Existence and uniqueness theorem

Theorem. *Let all the above assumptions hold. Consider the following infinite systems of linear equations*:

$$\mathcal{F}^i[u^i_n](t,x) = f^i(t,x,u_{n-1}(t,\cdot)), \tag{6}$$

$$\mathcal{F}^i[v^i_n](t,x) = f^i(t,x,v_{n-1}(t,\cdot)), \qquad i \in S, \tag{7}$$

for $n = 1,2,\ldots$ *with boundary condition* (5). *Then*

(i) *there exist unique regular solutions* u_n *and* v_n $(n = 1,2,\ldots)$ *of systems* (6) *and* (7) *with boundary condition* (5) *in* $\overline{D}$;

(ii) *the inequalities*

$$u_{n-1}(t,x) \le u_n(t,x), \qquad v_n(t,x) \le v_{n-1}(t,x) \tag{8}$$

hold for $(t,x) \in \overline{D}$ $(n = 1,2,\ldots)$;

(iii) *the functions* u_n *and* v_n $(n = 1,2,\ldots)$ *are lower and upper functions for problem* (1), (5) *in* $\overline{D}$, *respectively*;

(iv) $\lim_{n\to\infty}[v_n(t,x) - u_n(t,x)] = 0$ *uniformly in* $\overline{D}$;

(v) *the function*

$$z(t,x) = \lim_{n\to\infty} u_n(t,x)$$

is a unique regular solution of problem (1), (5) *in* $\overline{D}$ *and* $z \in C^{2+\alpha}_S(\overline{D})$.

Before going into the proof we introduce the Nemytskiĭ operator and prove some lemmas. Nemytskiĭ operators play an important role in the theory of nonlinear equations. For more information see [1].

Remark 1. If u and v are lower and upper functions for problem (1), (5) in $\overline{D}$, respectively, and z is a regular solution of this problem, then by the

Szarski theorem on differential-functional inequalities for infinite systems of parabolic type [10] we have

$$u(t,x) \le z(t,x) \le v(t,x) \quad \text{for } (t,x) \in \overline{D}. \tag{9}$$

In particular we have

$$u_0(t,x) \le z(t,x) \le v_0(t,x) \quad \text{for } (t,x) \in \overline{D}. \tag{10}$$

Let $\beta \in C_S(\overline{D})$ be a sufficiently regular function. Denote by $\mathcal{P}$ the operator

$$\mathcal{P} : \beta \to \gamma = \mathcal{P}\beta,$$

where γ is the (supposedly unique) solution of the boundary value problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,\cdot)) & \text{for } (t,x) \in D,\ i \in S, \\ \gamma(t,x) = 0 & \text{for } (t,x) \in \Sigma. \end{cases} \tag{11}$$

The operator $\mathcal{P}$ is the composition of the nonlinear Nemytskiĭ operator $\mathbf{F} = (\mathbf{F}^1, \mathbf{F}^2, \ldots)$,

$$\mathbf{F} : \beta \to \delta = \mathbf{F}\beta,$$

where

$$\mathbf{F}^i\beta(t,x) := f^i(t,x,\beta^1(t,\cdot),\beta^2(t,\cdot),\ldots) = \delta^i(t,x), \qquad i \in S,$$

and $\delta = (\delta^1, \delta^2, \ldots)$, and the linear operator

$$\mathcal{G} : \delta \to \gamma,$$

where γ is the (supposedly unique) solution of the linear problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & \text{in } D,\ i \in S, \\ \gamma(t,x) = 0 & \text{on } \Sigma. \end{cases} \tag{12}$$

Hence $\mathcal{P} = \mathcal{G} \circ \mathbf{F}$.

LEMMA 1. *If $\beta \in C_S^{0+\alpha}(\overline{D})$ and the function $f = (f^1, f^2, \ldots)$ generating the Nemytskiĭ operator $\mathbf{F}$ satisfies conditions (H_f) and (L), then*

$$\delta = \mathbf{F}\beta \in C_S^{0+\alpha}(\overline{D}).$$

Proof. Since $\beta \in C_S^{0+\alpha}(\overline{D})$, we have

$$|\beta^i(t,x) - \beta^i(t',x')| \le K(|t-t'| + \|x-x'\|^2_{\mathbb{R}^m})^{\alpha/2}$$

for all $(t,x), (t',x') \in \overline{D}$ and $i \in S$, where K is some nonnegative constant. Hence

$$\|\beta(t,\cdot) - \beta(t',\cdot)\| = \sup_{\substack{i \in S \\ x \in \overline{G}}} \{|\beta^i(t,x) - \beta^i(t',x)|\} \le K|t-t'|^{\alpha/2}.$$

From (H_f) and (L) it follows that

$$
\begin{aligned}
&|\delta^i(t,x)-\delta^i(t',x')| \\
&\quad = |\mathbf{F}^i\beta(t,x)-\mathbf{F}^i\beta(t',x')| \\
&\quad = |f^i(t,x,\beta(t,\cdot))-f^i(t',x',\beta(t',\cdot))| \\
&\quad \le |f^i(t,x,\beta(t,\cdot))-f^i(t',x',\beta(t,\cdot))|+|f^i(t',x',\beta(t,\cdot))-f^i(t',x',\beta(t',\cdot))| \\
&\quad \le K_1(|t-t'|+\|x-x'\|^2_{\mathbb{R}^m})^{\alpha/2}+L\|\beta(t,\cdot)-\beta(t',\cdot)\| \\
&\quad \le K_1(|t-t'|+\|x-x'\|^2_{\mathbb{R}^m})^{\alpha/2}+LK|t-t'|^{\alpha/2} \\
&\quad \le K^*(|t-t'|+\|x-x'\|^2_{\mathbb{R}^m})^{\alpha/2},
\end{aligned}
$$

where $K^* = K_1 + LK$ for all $(t,x),(t',x') \in \overline{D}$, $i \in S$. Therefore $\delta \in C_S^{0+\alpha}(\overline{D})$. ∎

Lemma 2. *If $\delta \in C_S^{0+\alpha}(\overline{D})$ and the coefficients satisfy assumption (H_α), then problem (1), (5) has a unique regular solution $\gamma \in C_S^{2+\alpha}(\overline{D})$.*

Proof. Observe that system (12) has the following property: the ith equation depends on the ith unknown function only. Therefore, applying the theorem on the existence and uniqueness of solution of Fourier's first problem for a linear parabolic equation (see A. Friedman [5], Theorems 6 and 7, p. 65), the statement of the lemma follows immediately. ∎

Lemmas 1 and 2 yield

Corollary. $\mathcal{P} = \mathcal{G}\circ\mathbf{F} : C_S^{0+\alpha}(\overline{D}) \ni \beta \to \gamma = \mathcal{P}\beta \in C_S^{2+\alpha}(\overline{D})$. ∎

Lemma 3. *If β is an upper function (resp. a lower function) for problem (1), (5) in $\overline{D}$, then $\mathcal{P}\beta(t,x) \le \beta(t,x)$ (resp. $\mathcal{P}\beta(t,x) \ge \beta(t,x)$) in $\overline{D}$.*

Proof. If β is an upper function, then by (4) we have

$$\mathcal{F}^i[\beta^i](t,x) \ge f^i(t,x,\beta(t,\cdot)) \quad \text{in } D,\ i\in S.$$

From the definition of the operator $\mathcal{P}$ (see (12)) it follows that

$$\mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,\cdot)) \quad \text{in } D,\ i\in S.$$

Therefore

$$\mathcal{F}^i[\gamma^i-\beta^i](t,x) \le 0 \quad \text{in } D,\ i\in S.$$

and

$$\gamma(t,x)-\beta(t,x) \le 0 \quad \text{on } \Sigma.$$

Hence, by the Szarski theorem [10], we have $\gamma(t,x)-\beta(t,x) \le 0$ in $\overline{D}$ so

$$\mathcal{P}\beta(t,x) = \gamma(t,x) \le \beta(t,x) \quad \text{in } \overline{D}. \ ∎ \tag{13}$$

Lemma 4. *If β is an upper (resp. a lower) function for problem (1), (5) in $\overline{D}$, then $\gamma = \mathcal{P}\beta$ is also an upper (resp. a lower) function for problem (1), (5) in $\overline{D}$.*

P r o o f. From (11), (13) and condition (W),

$$\mathcal{F}^i[\gamma^i](t,x) - f^i(t,x,\gamma(t,\cdot)) = f^i(t,x,\beta(t,\cdot)) - f^i(t,x,\gamma(t,\cdot)) \ge 0$$

in D, $i \in S$, and $\gamma(t,x) = 0$ on Σ. From the Corollary it follows that γ is a regular function, so it is an upper function for problem (1), (5) in $\overline{D}$.

Proof of Theorem. Starting from a lower function u_0 and an upper function v_0, we define by induction

$$u_1 = \mathcal{P}u_0, \quad u_n = \mathcal{P}u_{n-1},$$
$$v_1 = \mathcal{P}v_0, \quad v_n = \mathcal{P}v_{n-1}, \quad n = 1, 2, \ldots$$

From Lemmas 1, 2 and 4 it follows that u_n and v_n $(n = 1, 2, \ldots)$ are respectively a lower and an upper function for problem (1), (5) in $\overline{D}$.

By induction, from Lemma 3 we have

$$u_{n-1}(t,x) \le \mathcal{P}u_{n-1}(t,x) = u_n(t,x),$$
$$v_n(t,x) = \mathcal{P}v_{n-1}(t,x) \le v_{n-1}(t,x) \quad (n = 1, 2, \ldots) \text{ for } (t,x) \in \overline{D}.$$

Therefore

$$u_0(t,x) \le u_1(t,x) \le \ldots \le u_n(t,x)$$
$$\le \ldots \le v_n(t,x) \le \ldots \le v_1(t,x) \le v_0(t,x)$$

for $(t,x) \in \overline{D}$.

We now show by induction that

$$w_n^i(t,x) \le N_0 \frac{(Lt)^n}{n!}, \quad n = 0, 1, 2, \ldots, \text{ for } (t,x) \in \overline{D},\ i \in S, \tag{14}$$

where by (9) and (10),

$$w_n^i(t,x) = v_n^i(t,x) - u_n^i(t,x) \ge 0 \quad \text{in } \overline{D} \tag{15}$$

and

$$N_0 = \max_{i \in S} \max_{(t,x) \in \overline{D}} [v_0^i(t,x) - u_0^i(t,x)] \ge 0;$$

owing to the regularity of u_0 and v_0 we have $N_0 < \infty$.

It is obvious that (14) holds for w_0. Suppose it holds for w_n. Since the functions f^i $(i \in S)$ satisfy the Lipschitz condition (L), by (6), (7), (8), (14) and (15), we get

$$\mathcal{F}^i[w_{n+1}^i](t,x) = f^i(t,x,v_n(t,x)) - f^i(t,x,u_n(t,\cdot)) \le L\|w_n(t,\cdot)\|.$$

By the definition of the norm in $C_S(\overline{D})$ and by (14) we get

$$\|w_n(t,\cdot)\| \le \frac{(Lt)^n}{n!},$$

so we finally obtain

$$\mathcal{F}^i[w_{n+1}^i](t,x) \le N_0 \frac{L^{n+1}t^n}{n!} \quad \text{for } (t,x) \in D,\ i \in S, \tag{16}$$

and

$$w_{n+1}(t,x)=0 \quad \text{for } (t,x)\in\Sigma.$$

Consider the comparison system

$$\mathcal{F}^i[M^i_{n+1}](t,x)=N_0\frac{L^{n+1}t^n}{n!} \quad \text{for } (t,x)\in D,\ i\in S, \tag{17}$$

with the boundary condition

$$M_{n+1}(t,x)\ge 0 \quad \text{on } \Sigma. \tag{18}$$

It is obvious that the functions

$$M^i_{n+1}(t,x)=N_0\frac{(Lt)^{n+1}}{(n+1)!}, \quad i\in S.$$

are regular solutions of (17), (18) in $\overline{D}$.

Applying a theorem on differential inequalities of parabolic type ([8], Theorem 64.1, p. 195) to systems (16) and (17) we get

$$w^i_{n+1}(t,x)\le M^i_{n+1}(t,x)=N_0\frac{(Lt)^{n+1}}{(n+1)!} \quad \text{for } (t,x)\in\overline{D},\ i\in S,$$

so the induction step is proved and so is inequality (14).

As a direct consequence of (14) we obtain

$$\lim_{n\to\infty}[v_n(t,x)-u_n(t,x)]=0 \quad \text{uniformly in } \overline{D}. \tag{19}$$

The functional sequences $\{u_n\}$ and $\{v_n\}$ are monotone and bounded, and (19) holds, so there exists a continuous function $U=U(t,x)$ in $\overline{D}$ such that

$$\lim_{n\to\infty}u_n(t,x)=U(t,x), \quad \lim_{n\to\infty}v_n(t,x)=U(t,x) \quad \text{uniformly in } \overline{D}. \tag{20}$$

Since the functions f^i $(i\in S)$ are monotone (condition (W)), from (8) it follows that the functions $f^i(t,x,\ u_{n-1}(t,\cdot))$ $(i\in S)$ are uniformly bounded in D with respect to n. Hence we conclude by Lemma 2 that all the functions $u_n\in C^{2+\alpha}_S(\overline{D})$ for $n=1,2,\ldots$ satisfy the Hölder condition with a constant independent of n. Hence $U\in C^{0+\alpha}_S(\overline{D})$.

If we now consider the system of equations

$$\mathcal{F}^i[z^i](t,x)=f^i(t,x,U(t,\cdot))=\mathbf{F}^iU(t,x) \quad \text{for } (t,x)\in D,\ i\in S \tag{21}$$

with boundary condition (5), then by Lemma 1 we have $\mathbf{F}^iU\in C^{0+\alpha}_S(\overline{D})$. Therefore by Lemma 2 this problem has a unique regular solution $z\in C^{2+\alpha}_S(\overline{D})$.

Let us now consider systems (6) and (21) together, and apply Szarski's theorem ([8], Theorem 51.1, p. 147) on the continuous dependence of solution of the first problem on initial and boundary values and on the right-hand sides of the systems.

Since the f^i $(i \in S)$ satisfy the Lipschitz condition (L), by (20) we have

$$\lim_{n\to\infty} f^i(t,x,u_n(t,\cdot)) = f^i(t,x,U(t,\cdot)) \quad \text{uniformly in } \overline{D}.$$

Hence

$$\lim_{n\to\infty} u_n(t,x) = z(t,x). \tag{22}$$

By (20) and (22),

$$z = z(t,x) = U(t,x) \quad \text{for } (t,x) \in \overline{D}$$

is a regular solution of problem (1), (5) in $\overline{D}$ and $z \in C_S^{2+\alpha}(\overline{D})$.

The uniqueness of the solution follows from Szarski's uniqueness criterion [9]. It also follows directly from inequality (14). ∎

Remark 2. Instead of Assumption (A), one may use the following stronger assumption (see [4]):

Assumption (A^*). There exists at least one pair $u_0 = u_0(t,x)$, $v_0 = v_0(t,x)$ of a lower and an upper function for problem (1), (5) in $\overline{D}$ such that

$$u_0(t,x) \le v_0(t,x) \quad \text{for } (t,x) \in \overline{D}.$$

Under Assumption (A^*), the other assumptions on the functions f^i may be weakened in such a way that conditions (H_f), (L) and (W) hold only locally in the set $\mathcal{K}$, where $\mathcal{K} := \{(t,x,s) : (t,x) \in \overline{D},\ s \in \langle u_0, v_0\rangle\}$ and $\langle u_0, v_0\rangle$ is the segment defined by

$$\langle u_0, v_0\rangle := \{s \in C_S(\overline{D}) : u_0(t,x) \le s(t,x) \le v_0(t,x) \text{ for } (t,x) \in \overline{D}\}.$$

The existence and uniqueness of the solution will then be guaranteed in the set $\langle u_0, v_0\rangle$, though. ∎

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