Applications of Mathematics

Sahbi Boussandel

Existence of solutions for evolution equations in Hilbert spaces with anti-periodic boundary conditions and its applications

Applications of Mathematics, Vol. 63 (2018), No. 4, 423–437

Persistent URL: `http://dml.cz/dmlcz/147319`

EXISTENCE OF SOLUTIONS FOR EVOLUTION EQUATIONS IN HILBERT SPACES WITH ANTI-PERIODIC BOUNDARY CONDITIONS AND ITS APPLICATIONS

Sahbi Boussandel, Bizerte

Received August 26, 2017. Published online May 20, 2018.

Abstract. We establish the existence of solutions for evolution equations in Hilbert spaces with anti-periodic boundary conditions. The energies associated to these evolution equations are quadratic forms. Our approach is based on application of the Schaefer fixed-point theorem combined with the continuity method.

Keywords: existence; anti-periodic boundary condition; Schaefer fixed-point theorem; continuity method; diffusion equation

MSC 2010: 35K10, 35K55, 35K57, 35K59, 35K90, 47J35

1. Introduction

In this paper we are concerned with the quasilinear nonmonotone problem

$$\begin{cases} u'(t) + Q(t,u(t))^{-1}\nabla E(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T), \end{cases} \tag{1.1}$$

where ∇E denotes the gradient in a Hilbert space H with respect to a fixed inner product $\langle\cdot,\cdot\rangle_H$ of a quadratic form E defined on a Hilbert space V, and $Q\colon [0,T]\times H \to \mathcal{L}(H)$ is a mapping such that $Q(t,u)$ is invertible for every $(t,u)\in[0,T]\times H$. We are interested in solutions of problem (1.1) in the sense of L^2, that is, in functions u satisfying u', $\nabla E(u) \in L^2(0,T;H)$. Our approach is based on application of the Schaefer fixed-point theorem which is useful for proving the existence of solutions for evolution problems.

DOI: 10.21136/AM.2018.0233-17

Lemma 1. *Let X be a Banach space and let $S\colon X \to X$ be a compact mapping. Assume that the Schaefer set $\mathcal{C} := \{u \in X\colon u = \lambda Su$ for some $\lambda \in [0,1]\}$ is bounded. Then S has a fixed-point.*

In order to apply this theorem, we need to introduce the linear nonautonomous problem

$$\begin{cases} u'(t) + A(t)(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T) \end{cases} \tag{1.2}$$

with $A(t) = Q(t, v(t))^{-1}\nabla E$, $v \in L^2(0,T;H)$ being fixed. We show that problem (1.2) admits a unique solution by means of the continuity method. This method consists in approximating problem (1.2) by the family of linear problems

$$\begin{cases} u'(t) + (1-\alpha)\nabla E(u(t)) + \alpha A(t)(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T), \end{cases} \tag{1.3}$$

where α takes on values from the interval $[0,1]$. Let us denote

$$Z = \{u \in W^{1,2}(0,T\colon\ H) \cap L^2(0,T;D(\nabla E))\colon\ u(0) = -u(T)\}$$

and

$$Y = L^2(0,T;H).$$

We prove that the operator

$$\begin{aligned} K\colon\ & Z \to Y, \\ & u \mapsto u' + A(\cdot)u, \end{aligned}$$

is continuously connected by the family of operators $(K_\alpha)_{\alpha\in[0,1]}$,

$$\begin{aligned} K_\alpha\colon\ & Z \to Y, \\ & u \to u' + (1-\alpha)\nabla E + \alpha A(\cdot)u, \end{aligned}$$

to the operator K_0 for which it is known that the solution of (1.3) exists and is unique. This permits us to define a mapping $S\colon Y \to Y$ for which we apply Lemma 1, and we obtain an eventual fixed-point, which is a solution to problem (1.1).

Anti-periodic problems have been widely studied by many authors. Okochi [12] initiated the study for anti-periodic solutions of multivalued evolution equations in Hilbert spaces. More precisely, he considered an abstract evolution problem of the form

$$u'(t) + \partial\varphi(u(t)) \ni f(t), \quad t \in \mathbb{R}, \tag{1.4}$$

where $\partial\varphi$ denotes the subdifferential of a functional φ defined on a Hilbert space H. Under the assumptions that φ is proper l.s.c. convex and even, it was proved that for every $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ such that $f(t+T) = -f(t)$ for a.e. $t \in \mathbb{R}$ there exists an anti-periodic solution of (1.1). We note that this result includes the result obtained in this paper for $Q(t,u)$ being constant. In [11], under some additional compactness condition, the author used the Schauder fixed-point theorem in order to prove the existence of anti-periodic solutions for the nonmonotone nonlinear problem

$$u'(t) + \partial\varphi(u(t)) - \lambda u(t) \ni f(t), \quad t \in \mathbb{R},$$

where $\lambda \geqslant 0$ is given.

In [3], the author studied the existence of solutions for the following evolution equation

$$\begin{cases} u'(t) + Au(t) + \nabla G(u(t)) + F(t, u(t)) = 0, & t \in \mathbb{R}, \\ u(t+T) = -u(t), & t \in \mathbb{R}, \end{cases}$$

where $A\colon D(A) \to H$ is a self-adjoint operator, ∇G is the gradient of a mapping $G\colon H \to \mathbb{R}$ and $F\colon \mathbb{R} \times H \to H$ is a nonlinear mapping. Applying the Schauder fixed-point theorem, an existence result was obtained under the assumptions that $D(A)$ is compactly embedded into H, ∇G is a bounded continuous mapping and F is a continuous mapping which is bounded by an L^2 function. Recently, Zhenhai [17] established an existence result for the following nonlinear evolution equation with nonmonotone perturbations

$$\begin{cases} u'(t) + Au(t) + G(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T) \end{cases}$$

in a reflexive Banach space V, where A is monotone and G is not. His approach consists in applying the theory of maximal monotone and pseudomonotone operators. He applied his main result in order to solve the diffusion equation involving the p-Laplace operator with Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = f(x,u) & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = -u(T,\cdot) & \text{in } \Omega, \end{cases}$$

where $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies some suitable conditions. Still in the setting of Banach spaces, in [8] the authors considered the problem

$$\begin{cases} u'(t) = -Au(t) + g(t, u(t)), & t \in \mathbb{R}, \\ u(t+T) = -u(t), & t \in \mathbb{R} \end{cases} \tag{1.5}$$

in a Banach space X, where $A\colon\ X \to X$ is a linear continuous accretive mapping and $g\colon\ \mathbb{R}\times X \to X$ is a continuous mapping which is even with respect to the second variable. Using a combination of the contraction mapping principle, the theory of accretive operators and the homotopy property of the Leray-Schauder degree, the authors obtained an existence result for problem (1.5).

For further results on anti-periodic solutions for evolution equations we invite the reader to consult [1], [9], [4], [5], [7], [6], [14], [13], [15], [16] and the references therein.

This paper consists of four sections. In Section 2 we summarize the relevant material on gradients, we give all assumptions needed in this paper and we state our main result. The proof of the main result will be given in Section 3. Finally, in Section 4 we discuss an application of the preceding abstract theory to a quasilinear diffusion equation.

2. Functional setting, assumptions and main result

Consider a real Hilbert space V with inner product $\langle\cdot,\cdot\rangle_V$ and associated norm $\|\cdot\|_V$ and a second real Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and associated norm $\|\cdot\|_H$. We suppose that V is densely and continuously embedded into H. Then there exists a constant $c>0$ such that

$$\|u\|_H \leqslant c\|u\|_V \tag{2.1}$$

for all $u \in V$. We shall consider also a continuous and symmetric bilinear form $a\colon\ V\times V \to \mathbb{R}$. Recall that a is continuous if and only if there exists $C \geqslant 0$ such that

$$|a(u,v)| \leqslant C\|u\|_V\|v\|_V$$

for all $u,v\in V$. Define the functional $E\colon\ V \to \mathbb{R}$ by

$$E(u) = \tfrac{1}{2}a(u,u)$$

for all $u\in V$. The functional E is called a *quadratic form.* Recall that since the bilinear form is continuous on $V\times V$, the functional E is continuously differentiable on V, and using the fact that a is symmetric we have

$$E'(u)v = a(u,v)$$

for all $u,v\in V$. We shall define the gradient of E in H with respect to the inner product $\langle\cdot,\cdot\rangle_H$ by

$$\begin{aligned} &D(\nabla E) = \{u\in V\colon\ \ \exists\, w\in H \quad \forall\, v\in V,\ a(u,v)=\langle w,v\rangle_H\},\\ &\nabla E(u) = w. \end{aligned}$$

Notice that since E is a quadratic form, the gradient ∇E is a linear operator on H with domain $D(\nabla E)$, which is a linear subspace of H. The operator ∇E is closed, and $D(\nabla E)$ equipped with the graph norm is a Banach space. Notice that since V is densely embedded into H, the element $\nabla E(u)$ is uniquely determined. We let further a function

$$\begin{aligned} Q\colon\ [0,T]\times H &\to \mathcal{L}(H),\\ (t,u) &\mapsto Q(t,u), \end{aligned}$$

be such that $Q(t,u)$ is symmetric positive definite for every $(t,u)\in[0,T]\times H$. Here $\mathcal{L}(H)$ denotes the space of linear bounded operators from H into H. Denoting by $\mathrm{Inner}(H)$ the set of all inner products on H, this allows us to define the function

$$\begin{aligned} g\colon\ [0,T]\times H &\to \mathrm{Inner}(H),\\ (t,u) &\mapsto \langle\cdot,\cdot\rangle_{g(t,u)}, \end{aligned}$$

with $\langle v,w\rangle_{g(t,u)} = \langle Q(t,u)v,w\rangle_H$. We denote by $\|\cdot\|_{g(t,u)}$ the norm associated with the inner product $\langle\cdot,\cdot\rangle_{g(t,u)}$. Our basic assumptions on V, H, a, Q and g are the following.

(A1) There exists a constant $c_1>0$ such that for every $u\in V$

$$c_1\|u\|_V^2 \leqslant a(u,u).$$

(A2) For every $(t,u)\in[0,T]\times H$, $Q(t,u)$ is invertible.

(A3) For every u, v, $w\in H$, the function $t\to\langle v,w\rangle_{g(t,u)}$ is measurable on $[0,T]$.

(A4) There exist two constants $c_2,c_3>0$ such that for every every $t\in[0,T]$ and every $u,v\in H$

$$c_2\|v\|_H \leqslant \|v\|_{g(t,u)} \leqslant c_3\|v\|_H.$$

(A5) There exists a constant $c_4>0$ such that for every $t\in[0,T]$ and every $u,v\in H$

$$c_4\|v\|_H^2 \leqslant \langle Q(t,u)^{-1}v,v\rangle_H.$$

(A6) If $(u_n)\subset H$ is such that

$$u_n\to u \quad \text{in } H,$$

then we have for every $t\in[0,T]$ and every $v,w\in H$

$$\langle v,w\rangle_{g(t,u_n)} \to \langle v,w\rangle_{g(t,u)}.$$

(A7) The mapping Q satisfies

$$\left.\begin{aligned} u_n &\rightharpoonup u \text{ in } L^2(0,T;H)\\ v_n &\to v \text{ in } L^2(0,T;H)\end{aligned}\right\} \Rightarrow Q(\cdot,v_n)u_n \rightharpoonup Q(\cdot,v)u \text{ in } L^2(0,T;H).$$

(A8) The embedding

$$W^{1,2}(0,T;H)\cap L^2(0,T;V)\hookrightarrow L^2(0,T;H)$$

is compact.

We note that under assumptions (A3), (A4), and (A6), we have the following fact: if $u,v,w\colon [0,T]\mapsto H$ are three measurable functions, then the function $t\to\langle v(t),w(t)\rangle_{g(t,u(t))}$ is measurable on $[0,T]$. For the proof of this fact we invite the reader to see [2], Remarks 6 and 7. We note also, from assumption (A4), that we have for every $(t,u)\in[0,T]\times H$

$$\|Q(t,u)^{-1}\|_{\mathcal{L}(H)}\leqslant\frac{1}{c_2}.$$

Let us consider the evolution problem

$$\begin{cases} u'(t)+Q(t,u(t))^{-1}\nabla E(u(t))=f(t) & \text{for a.e. } t\in(0,T),\\ u(0)=-u(T), \end{cases}\tag{2.2}$$

where $f\colon [0,T]\to H$ is a given function. The identity $u(0)=-u(T)$ is called an *anti-periodic condition*. We are concerned in solutions of problem (2.2) given in the following sense.

A function $u\colon [0,T]\to V$ is called a solution of problem (2.2) if

$$\begin{aligned}&u\in W^{1,2}(0,T;H)\cap L^2(0,T;D(\nabla E)),\\ &u \text{ satisfies problem (2.2)}.\end{aligned}$$

Recall from the Sobolev embedding theorem that $W^{1,2}(0,T;H)$ is a subspace of $C([0,T];H)$, so if u is a solution of problem (2.2), the anti-periodic condition $u(0)=-u(T)$ makes sense. Recall also that the space

$$W^{1,2}(0,T;H)\cap L^2(0,T;D(\nabla E))$$

equipped with the norm

$$\|u\|^2=\int_0^T\|u(t)\|_H^2\,\mathrm{d}t+\int_0^T\|u'(t)\|_H^2\,\mathrm{d}t+\int_0^T\|\nabla E(u(t))\|_H^2\,\mathrm{d}t$$

is continuously embedded into $C([0,T];V)$.

The purpose of this paper is to prove the following theorem.

Theorem 2. *Assume that assumptions* (A1)–(A8) *are satisfied. Then for every* $f \in L^2(0,T;H)$, *problem* (2.2) *admits a solution.*

3. Proof of Theorem 2

We set

$$Z := \{u \in W^{1,2}(0,T;H) \cap L^2(0,T;D(\nabla E))\colon\ u(0) = -u(T)\}$$

equipped with the norm satisfying

$$\|u\|_Z^2 = \int_0^T \|u(t)\|_H^2 \,\mathrm{d}t + \int_0^T \|u'(t)\|_H^2 \,\mathrm{d}t + \int_0^T \|\nabla E(u(t))\|_H^2 \,\mathrm{d}t,$$

and

$$Y := L^2(0,T;H)$$

equipped with the norm

$$\|u\|_Y = \Big(\int_0^T \|u(t)\|_H^2 \,\mathrm{d}t\Big)^{1/2}.$$

For every $\alpha \in [0,1]$ and every $v \in Y$ we consider the linear bounded operator

$$\begin{aligned} A_\alpha\colon\ & Z \to Y,\\ & u \to (1-\alpha)\nabla E(u) + \alpha Q(\cdot, v(\cdot))^{-1}\nabla E(u), \end{aligned}$$

and the linear nonautonomous problem

$$\begin{cases} u'(t) + A_\alpha(u(t)) = f(t) & \text{for a.e. } t \in (0,T),\\ u(0) = -u(T). \end{cases} \tag{3.1}$$

For every $\alpha \in [0,1]$ we define the linear bounded operator

$$\begin{aligned} S_\alpha\colon\ & Z \to Y,\\ & u \mapsto u' + A_\alpha u. \end{aligned}$$

We introduce the set

$$M = \{\alpha \in [0,1]\colon\ \forall\, f \in Y,\ \exists!\, u \in Z \text{ which is a solution of problem (3.1)}\}.$$

We note that $\alpha \in M$ if and only if the operator S_α is invertible.

Proposition 3. *The set* M *is nonempty.*

P r o o f. Denote $A = \nabla E$. It suffices to prove that $0 \in M$, which is equivalent to the fact that for every $f \in L^2(0,T;H)$ there exists a unique $u \in W^{1,2}(0,T;H) \cap L^2(0,T;D(A))$, which is a solution of the linear autonomous problem

$$\begin{cases} u'(t) + A(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = -u(T). \end{cases} \tag{3.2}$$

It is known that for every $f \in L^2(0,T;H)$ and every $x \in V$ the initial value problem

$$\begin{cases} u'(t) + A(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = x \end{cases}$$

admits a unique solution $u \in W^{1,2}(0,T;H) \cap L^2(0,T;D(A))$ and this solution is given by

$$u(t) = \mathrm{e}^{-tA}x + \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s, \quad t \in [0,T],$$

where $(\mathrm{e}^{-tA})_{t \in [0,T]}$ denotes the positive definite semigroup on H generated by $-A$. Hence, the solution of the boundary value problem (3.2) is obtained by writing the condition

$$(\mathrm{e}^{-TA} + I_H)u(0) = -\int_0^T \mathrm{e}^{-(T-s)A} f(s)\,\mathrm{d}s,$$

where the operator on the left-hand side is invertible, since -1 does not belong to the spectrum of positive definite operator e^{-TA}. Here I_H denotes the identity mapping of H. This yields the existence and uniqueness of solutions for problem (3.2). $\square$

Proposition 4. *The set M is open.*

P r o o f. Denote by $\mathcal{L}(Z,Y)$ and $\mathcal{I}(Z,Y)$ the space of linear bounded operators and the set of linear bounded invertible operators from Z into Y, respectively. The desired result follows from the fact that $\mathcal{I}(Z,Y)$ is open in $\mathcal{L}(Z,Y)$ and from the estimate

$$\exists\, C > 0 \quad \forall\, \alpha,\ \beta \in [0,1],\ \|S_\alpha - S_\beta\|_{\mathcal{L}(Z,Y)} \leqslant C|\alpha - \beta|. \tag{3.3}$$

$\square$

Proposition 5. *If $u \in W^{1,2}(0,T;H) \cap L^2(0,T;D(\nabla E))$ then $E(u(\cdot)) \in W^{1,1}(0,T)$ and we have for a.e. $t \in (0,T)$*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = \langle \nabla E(u(t)), u'(t) \rangle_H.$$

Proof. The proof of this result is similar to that of [10], Lemma 10.4. □

Proposition 6. *The set M is closed.*

Proof. Let (α_n) be any sequence in M which converges to $\alpha \in [0,1]$. We prove that $\alpha \in M$, that is, S_α is invertible. It suffices to show that the sequence of invertible operators (S_{α_n}) converges strongly to S_α in $\mathcal{L}(Z,Y)$ and there exists $c > 0$ such that for every $n \in \mathbb{N}$, $\|S_{\alpha_n}^{-1}\|_{\mathcal{L}(Y,Z)} \leqslant c$. From property (3.3), we deduce that (S_{α_n}) converges strongly to S_α in $\mathcal{L}(Z,Y)$. Hence, it suffices to prove the uniform estimates for $(S_{\alpha_n}^{-1})$. From the definition of the set M, for every $n \in \mathbb{N}$ there exists a unique $u_n \in Z$, which is a solution of the linear nonautonomous problem

$$\begin{cases} u_n'(t) + (1-\alpha_n)\nabla E(u_n(t)) + \alpha_n Q(t,v(t))^{-1}\nabla E(u_n(t)) = f(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u_n(0) = -u_n(T). \end{cases} \tag{3.4}$$

Multiply (3.4) by $\nabla E(u_n(t))$ with respect to the inner product $\langle\cdot,\cdot\rangle_H$, and then integrate over $(0,T)$ to get

$$\int_0^T \langle \nabla E(u_n), u_n'\rangle_H \,\mathrm{d}t + (1-\alpha_n)\int_0^T \|\nabla E(u_n)\|_H^2 \,\mathrm{d}t + \alpha_n \int_0^T \langle Q(t,v(t))^{-1}\nabla E(u_n), \nabla E(u_n)\rangle_H \,\mathrm{d}t = \int_0^T \langle f(t), \nabla E(u_n)\rangle_H \,\mathrm{d}t.$$

Using Proposition 5 and the fact that $u_n(0) = -u_n(T)$, we have

$$\int_0^T \langle \nabla E(u_n), u_n'\rangle_H \,\mathrm{d}t = \frac{1}{2}a(u_n(T),u_n(T)) - \frac{1}{2}a(u_n(0),u_n(0)) = 0.$$

It follows from assumption (A5) that

$$(1-\alpha_n + c_4\alpha_n)\int_0^T \|\nabla E(u_n)\|_H^2 \,\mathrm{d}t \leqslant \int_0^T \|f(t)\|_H \|\nabla E(u_n)\|_H \,\mathrm{d}t,$$

which implies

$$C\int_0^T \|\nabla E(u_n)\|_H^2 \,\mathrm{d}t \leqslant \int_0^T \|f(t)\|_H \|\nabla E(u_n)\|_H \,\mathrm{d}t$$

with $C = \min\{1, c_4\}$. Therefore,

$$\int_0^T \|\nabla E(u_n)\|_H^2 \,\mathrm{d}t \leqslant C' \int_0^T \|f(t)\|_H^2 \,\mathrm{d}t \tag{3.5}$$

with $C' > 0$ being independent of n. Multiply (3.4) by $u_n'(t)$ with respect to the inner product $\langle\cdot,\cdot\rangle_H$, and then integrate over $(0,T)$ to get

$$\int_0^T \|u_n'\|_H^2 \,\mathrm{d}t + (1-\alpha_n)\int_0^T \langle\nabla E(u_n), u_n'\rangle_H \,\mathrm{d}t + \alpha_n \int_0^T \langle Q(t,v(t))^{-1}\nabla E(u_n), u_n'\rangle_H \,\mathrm{d}t = \int_0^T \langle f(t), u_n'\rangle_H \,\mathrm{d}t.$$

From Proposition 5 and assumption (A4) it follows that

$$\int_0^T \|u_n'\|_H^2 \,\mathrm{d}t \leqslant \int_0^T \Big(\frac{1}{c_2}\|\nabla E(u_n)\|_H + \|f(t)\|_H\Big)\|u_n'\|_H \,\mathrm{d}t.$$

Using estimate (3.5), we obtain

$$\int_0^T \|u_n'\|_H^2 \,\mathrm{d}t \leqslant C'' \int_0^T \|f(t)\|_H^2 \,\mathrm{d}t \tag{3.6}$$

with $C'' > 0$ being independent of n. From assumption (A1) and estimate (2.1), we have

$$c_1 \int_0^T \|u_n\|_V^2 \,\mathrm{d}t \leqslant \int_0^T a(u_n,u_n)\,\mathrm{d}t = \int_0^T \langle\nabla E(u_n), u_n\rangle_H \leqslant c\int_0^T \|\nabla E(u_n)\|_H \|u_n\|_V \,\mathrm{d}t.$$

Employing again estimate (3.5), we get

$$\int_0^T \|u_n\|_V^2 \,\mathrm{d}t \leqslant C''' \int_0^T \|f(t)\|_H^2 \,\mathrm{d}t \tag{3.7}$$

with $C''' > 0$ being independent of n. Combining estimates (3.7), (3.6), and (3.5) proves that $(S_{\alpha_n}^{-1})$ is uniformly bounded in $\mathcal{L}(Y,Z)$. This completes the proof. □

Proposition 7. *For every $f \in Y$ there exists a unique $u \in Z$, which is a solution of the linear nonautonomous problem*

$$\begin{cases} u'(t) + Q(t,v(t))^{-1}\nabla E(u(t)) = f(t) & \text{for a.e. } t \in (0,T),\\ u(0) = -u(T). \end{cases} \tag{3.8}$$

P r o o f. Since $[0,1]$ is connected, we can deduce from Propositions 6, 4, and 3 that $M = [0,1]$, and the conclusion follows by taking $\alpha = 1$ in problem (3.1). □

Let $f \in Y$ and $v \in Y$. From Proposition 7, there exists a unique $u \in Z$, which is a solution of problem (3.8). This permits us to define the solution mapping

$$\begin{aligned} S\colon\ Y &\to Y, \\ v &\mapsto Sv = u, \end{aligned}$$

where $u \in Z$ is the unique solution of problem (3.8).

Proposition 8. *The mapping S is continuous from Y into Y.*

Proof. Let $(v_n) \subset Y$ be such that

$$v_n \to v \quad \text{in } Y, \tag{3.9}$$

and let $u_n = Sv_n$ and $u = Sv$. We prove that $u_n \to u$ in Y. It suffices to prove that $u_n \to u$ in Y for a subsequence. By definition of the mapping S, we have

$$\begin{cases} u_n' + Q(t, v_n(t))^{-1} \nabla E(u_n) = f(t) & \text{for a.e. } t \in (0,T), \\ u_n(0) = -u_n(T). \end{cases} \tag{3.10}$$

Multiplying (3.10) by $u_n'(t)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t, v_n(t))}$ and integrating over $(0,T)$, we get

$$\int_0^T \|u_n'\|^2_{g(t,v_n)} \,\mathrm{d}t + \int_0^T \langle \nabla E(u_n), u_n' \rangle_H \,\mathrm{d}t = \int_0^T \langle f(t), u_n' \rangle_{g(t,v_n)} \,\mathrm{d}t.$$

Proposition 5, the fact that $u_n(0) = -u_n(T)$, and assumption (A4), implies

$$c_2^2 \int_0^T \|u_n'\|_H^2 \,\mathrm{d}t \leqslant c_3^2 \int_0^T \|f(t)\|_H \|u_n'\|_H \,\mathrm{d}t,$$

which yields

$$\int_0^T \|u_n'\|_H^2 \,\mathrm{d}t \leqslant C \int_0^T \|f(t)\|_H^2 \,\mathrm{d}t \tag{3.11}$$

with $C > 0$ independent of n. Multiply (3.10) by $u_n(t)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t, v_n(t))}$ and then integrate over $(0,T)$ to get

$$\int_0^T \langle u_n', u_n \rangle_{g(t,v_n)} \,\mathrm{d}t + \int_0^T a(u_n, u_n) \,\mathrm{d}t = \int_0^T \langle f(t), u_n \rangle_{g(t,v_n)} \,\mathrm{d}t.$$

From assumptions (A1), (A4), and estimate (2.1), we deduce

$$c_1 \int_0^T \|u_n\|_V^2 \, \mathrm{d}t \leqslant \int_0^T c_3^2(\|f(t)\|_H + c\|u_n'\|_H)\|u_n\|_V \, \mathrm{d}t.$$

It follows from (3.11) that there exists $C' > 0$, which is independent of n, such that

$$\int_0^T \|u_n\|_V^2 \, \mathrm{d}t \leqslant C' \int_0^T \|f(t)\|_H^2 \, \mathrm{d}t. \tag{3.12}$$

Employing (3.12), (3.11), and assumption (A4), we can see from (3.10) that

$$(\nabla E(u_n)) \text{ is bounded in } L^2(0,T;H).$$

Consequently,

$$(u_n) \text{ is bounded in } Z.$$

Hence, passing to a subsequence if necessary, we deduce that

$$u_n \rightharpoonup w \quad \text{in } W^{1,2}(0,T;H), \tag{3.13}$$

$$u_n \rightharpoonup w \quad \text{in } L^2(0,T;V), \tag{3.14}$$

$$\nabla E(u_n) \rightharpoonup \nabla E(w) \quad \text{in } L^2(0,T;H), \tag{3.15}$$

$$u_n \to w \quad \text{in } L^2(0,T;H). \tag{3.16}$$

Convergence (3.16) is guaranteed by assumption (A8).

Multiplying (3.10) by $Q(t, v_n(t))$, we find that

$$Q(t, v_n(t))u_n' + \nabla E(u_n) = Q(t, v_n(t))f(t) \text{ a.e. } t \in (0,T).$$

This implies for every $x \in L^2(0,T;H)$

$$\int_0^T \langle Q(t,v_n)u_n', x\rangle_H \, \mathrm{d}t + \int_0^T \langle \nabla E(u_n), x\rangle_H \, \mathrm{d}t = \int_0^T \langle Q(t,v_n)f(t), x\rangle_H \, \mathrm{d}t.$$

Letting $n \to \infty$ and employing (3.15), (3.13), (3.9), and assumption (A7), we obtain for every $x \in L^2(0,T;H)$

$$\int_0^T \langle Q(t,v)w', x\rangle_H \, \mathrm{d}t + \int_0^T \langle \nabla E(w), x\rangle_H \, \mathrm{d}t = \int_0^T \langle Q(t,v)f(t), x\rangle_H \, \mathrm{d}t.$$

Since $x \in L^2(0,T;H)$ is arbitrary, this gives

$$Q(t, v(t))w' + \nabla E(w) = Q(t, v(t))f(t) \quad \text{a.e. } t \in (0,T),$$

that is,

$$w' + Q(t, v(t))^{-1}\nabla E(w) = f(t) \quad \text{a.e. } t \in (0, T).$$

From convergence (3.13) and the fact that $u_n(0) = -u_n(T)$ we get that $w(0) = -w(T)$. Therefore, $w \in Z$ is a solution of the problem

$$\begin{cases} w' + Q(t, v(t))^{-1}\nabla E(w) = f & \text{for a.e. } t \in (0, T), \\ w(0) = -w(T). \end{cases}$$

Since u is the unique solution of this last problem, we conclude that $w = u$. From convergence (3.16) we get

$$u_n \to u \ \text{ in } \ L^2(0, T; H),$$

which proves that S is a continuous mapping. □

Proposition 9. *The mapping S is relatively compact.*

P r o o f. We prove that SK is a relatively compact set in Y for any bounded set K in Y, which is equivalent to the fact that for any bounded sequence (v_n) in Y we can extract a subsequence, denoted again by (v_n), such that (Sv_n) converges strongly in Y. Let (v_n) be any bounded sequence in Y and put $u_n = Sv_n$. Then u_n is a solution of problem (3.10). As in the proof of Proposition 8, we can show that (u_n) is bounded in $W^{1,2}(0, T; H) \cap L^2(0, T; V)$. Using assumption (A8), we can extract from (u_n) a subsequence converging strongly in $L^2(0, T; H)$. This shows that S is a relatively compact mapping. □

Proposition 10. *The Schaefer set*

$$\mathcal{C} := \{u \in Y \colon \ u = \lambda S u \quad \textit{for some } \lambda \in [0, 1]\}$$

is bounded in Y.

P r o o f. Let $u \in \mathcal{C}$. Then there exists $\lambda \in [0, 1]$ such that

$$\begin{cases} u'(t) + Q(t, u(t))^{-1}\nabla E(u(t)) = \lambda f(t) & \text{for a.e. } t \in (0, T), \\ u(0) = -u(T). \end{cases} \tag{3.17}$$

As in the proof of Proposition 8, by multiplying problem (3.17) by $u'(t)$ and $u(t)$, respectively, with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t, u(t))}$, there exists $C > 0$, which is independent of u, such that

$$\int_0^T \|u\|_V^2 \, \mathrm{d}t \leqslant C \int_0^T \|f(t)\|_H^2 \, \mathrm{d}t.$$

This last estimate implies that $\mathcal{C}$ is a bounded set in Y. □

P r o o f of Theorem 2. Combining Propositions 8, 9, and 10, we claim from Lemma 1 that there exists $u \in Y$ which is a fixed-point of S, that is, $u \in Z$ is a solution of problem (2.2). The proof of Theorem 2 is complete. $\square$

4. Application

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $\varepsilon \in (0,1)$, and let

$$m\colon [0,T] \times \Omega \times \mathbb{R} \to \Big[\varepsilon, \frac{1}{\varepsilon}\Big]$$

be a measurable function such that $m(t,x,\cdot)$ is continuous for every $(t,x) \in [0,T]\times\Omega$.

Let us consider the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\Delta u = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = -u(T,\cdot) & \text{in } \Omega, \end{cases} \tag{4.1}$$

where Δ is the Laplace operator. We put $V = W_0^{1,2}(\Omega)$ equipped with the norm $\|u\|_V = \|\nabla u\|_{L^2(\Omega)^N}$ and $H = L^2(\Omega)$ equipped with the usual inner product and norm. Let $E\colon V \to \mathbb{R}$ be the function defined for every $u \in V$ by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

We define the Laplace operator with Dirichlet boundary conditions on $L^2(\Omega)$ by

$$D(\Delta) = \Big\{u \in W_0^{1,2}(\Omega)\colon \exists\, w \in L^2(\Omega)\ \forall\, v \in W_0^{1,2}(\Omega),\ \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x = -\int_\Omega wv\,\mathrm{d}x\Big\},$$

$\Delta u = w$.

Let $Q\colon [0,T]\times H \to \mathcal{L}(H)$ be defined for every $(t,u) \in [0,T]\times H$ by

$$Q(t,u) = \frac{1}{m(t,\cdot,u)} I_H,$$

where $I_H\colon H \to H$ denotes the identity mapping of H. Then the function $g\colon [0,T]\times H \to \operatorname{Inner}(H)$ is given by

$$\langle v,w\rangle_{g(t,u)} = \int_\Omega v(x)w(x)\frac{\mathrm{d}x}{m(t,x,u(x))}.$$

We check that all assumptions of Theorem 2 are satisfied. The verification of assumption (A7) can be made as in [2], Proof of Corollary 10. As a consequence of Theorem 2, we obtain:

Corollary 11. *For every* $f \in L^2(0,T;L^2(\Omega))$, *problem* (4.1) *admits a solution* $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;D(\Delta))$.

Acknowledgement. I would like to thank the reviewers for valuable remarks, comments, and suggestions.

References

[1] *S. Aizicovici, M. McKibben, S. Reich*: Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *43* (2001), 233–251. zbl MR doi

[2] *S. Boussandel*: Global existence and maximal regularity of solutions of gradient systems. J. Differ. Equations *250* (2011), 929–948. zbl MR doi

[3] *Y. Chen*: Anti-periodic solutions for semilinear evolution equations. J. Math. Anal. Appl. *315* (2006), 337–348. zbl MR doi

[4] *Y. Chen, Y. J. Cho, J. S. Jung*: Antiperiodic solutions for semilinear evolution equations. Math. Comput. Modelling *40* (2004), 1123–1130. zbl MR doi

[5] *Y. Chen, J. J. Nieto, D. O'Regan*: Anti-periodic solutions for fully nonlinear first-order differential equations. Math. Comput. Modelling *46* (2007), 1183–1190. zbl MR doi

[6] *Y. Chen, J. J. Nieto, D. O'Regan*: Anti-periodic solutions for evolution equations associated with maximal monotone mappings. Appl. Math. Lett. *24* (2011), 302–307. zbl MR doi

[7] *Y. Chen, D. O'Regan, R. P. Agarwal*: Anti-periodic solutions for evolution equations associated with monotone type mappings. Appl. Math. Lett. *23* (2010), 1320–1325. zbl MR doi

[8] *Y. Chen, D. O'Regan, R. P. Agarwal*: Anti-periodic solutions for semilinear evolution equations in Banach spaces. J. Appl. Math. Comput. *38* (2012), 63–70. zbl MR doi

[9] *Y. Chen, X. Wang, H. Xu*: Anti-periodic solutions for semilinear evolution equations. J. Math. Anal. Appl. *273* (2002), 627–636. zbl MR doi

[10] *R. Chill, E. Fašangová*: Gradient Systems. Lecture Notes of the 13th International Internet Seminar. Matfyzpress, Praha, 2010.

[11] *A. Haraux*: Anti-periodic solutions of some nonlinear evolution equations. Manuscr. Math. *63* (1989), 479–505. zbl MR doi

[12] *H. Okochi*: On the existence of periodic solutions to nonlinear abstract parabolic equations. J. Math. Soc. Japan *40* (1988), 541–553. zbl MR doi

[13] *H. Okochi*: On the existence of anti-periodic solutions to a nonlinear evolution equation associated with odd subdifferential operators. J. Funct. Anal. *91* (1990), 246–258. zbl MR doi

[14] *H. Okochi*: On the existence of anti-periodic solutions to nonlinear parabolic equations in noncylindrical domains. Nonlinear Anal., Theory Methods Appl. *14* (1990), 771–783. zbl MR doi

[15] *Ph. Souplet*: An optimal uniqueness condition for the antiperiodic solutions of parabolic evolution equations. C. R. Acad. Sci., Paris, Sér. I *319* (1994), 1037–1041. (In French.) zbl MR

[16] *Ph. Souplet*: Optimal uniqueness condition for the antiperiodic solutions of some nonlinear parabolic equations. Nonlinear Anal., Theory Methods Appl. *32* (1998), 279–286. zbl MR doi

[17] *L. Zhenhai*: Anti-periodic solutions to nonlinear evolution equations. J. Funct. Anal. *258* (2010), 2026–2033. zbl MR doi

Author's address: *Sahbi Boussandel*, Faculty of Sciences of Bizerte, Department of Mathematics, 7021 Jarzouna Bizerte, University of Carthage, Bizerte, Tunisia, Laboratore EDP et Applications LR03ES04, e-mail: `sboussandels@gmail.com`.