Existence of solutions for the nonlinear partial differential equation arising in the optimal investment problem

By Ryo ABE*) and Naoyuki ISHIMURA**)

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Abstract: We are concerned with the solvablity of certain nonlinear partial differential equation (PDE), which is derived from the optimal investment problem under the random risk process. The equation describes the evolution of the Arrow-Pratt coefficient of absolute risk aversion with respect to the optimal value function. Employing the fixed point approach combined with the convergence argument we show the existence of solutions.

Key words: Absolute risk aversion; nonlinear partial differential equations; solvability.

1. Introduction. We deal with the existence of solutions for the singular parabolic partial differential equation (PDE) of the form

$$\text{(1.1)}\qquad \frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\left\{\left(1+\frac{1}{r^2}\right)\frac{\partial r}{\partial x} - r^2\right\},$$
$$r = r(x,t) \quad \text{in } (x,t) \in \Omega_T := \boldsymbol{R}^+ \times (0,T)$$

where $T > 0$ and $\boldsymbol{R}^+ = \{x > 0\}$. The unknown function r is related to the Arrow-Pratt coefficient of absolute risk aversion [11] for the optimal value function; it is natural to assume that r is positive and non-increasing. We thus impose the next condition for r.

$$\text{(1.2)}\qquad r \ge 0, \qquad \frac{\partial r}{\partial x}(0,t) = 0,$$
$$r(x,t) \to \alpha \quad \text{as } x \to \infty,$$

where α denotes non-negative constant. The derivation of (1.1) and other properties are recalled in §2.

The problem (1.1)(1.2) is supplemented by the initial condition.

$$\text{(1.3)}\qquad r(x,0) = r_0(x) \quad \text{on } x \in \boldsymbol{R}^+,$$

where the non-increasing initial datum r_0 belongs to $H^1(\boldsymbol{R}^+)$ and satisfies the compatibility condition (1.2).

The aim of the current article is to solve (1.1)(1.2)(1.3) in a weak sense. To begin with we clarify the notion of weak solutions (see for instance [4]).

Definition 1. We say r a weak solution of (1.1)(1.2)(1.3) if the following conditions are fulfilled.

(1) $r - \alpha \in L^\infty(0,T;L^2(\boldsymbol{R}^+)) \cap L^2(0,T;H^1(\boldsymbol{R}^+))$, $\partial r/\partial t \in L^2(0,T;H^{-1}(\boldsymbol{R}^+))$.

(2) There hold for each $\varphi \in H^1(\boldsymbol{R}^+)$ and almost every $0 \le t \le T$

$$\text{(1.4)}\qquad \int_{\boldsymbol{R}^+} \frac{\partial r}{\partial t}(x,t)\varphi(x)dx = -\int_{\boldsymbol{R}^+}\left\{\left(1+\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + 2r\frac{\partial r}{\partial x}\varphi\right\}dx.$$

(3) $r(x,0) = r_0(x)$ in $L^2(\boldsymbol{R}^+)$.

The main result of this paper is then stated as follows:

Theorem 2. *For any positive and non-increasing $r_0 \in H^1(\boldsymbol{R}^+)$ satisfying $(\partial r_0/\partial x)(0) = 0$ and $r_0(x) \to \alpha$ as $x \to \infty$ with $\alpha > 0$, there corresponds $T = T(r_0) > 0$ such that there exists a positive solution r for (1.1)(1.2)(1.3), which is non-increasing in x, in the sense of Definition* 1.

The proof of Theorem, which is performed in §3, is based on an approximation argument with the combination of fixed point approach. §4 is devoted to the classification of steady state solutions of (1.1). We conclude by Discussions in §5 with a comment on the interpretation in economics.

2. Model equation. Here we briefly sketch the derivation and survey the background issues of (1.1).

The basic model we follow is due to Browne [2].

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*) Simplex Technology Inc., Nihonbashi, Tokyo 103-0027, Japan.
**) Department of Mathematics, Graduate School of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan.

It is assumed that there is only one risky stock available for investment, whose price P_t at time t is governed by the stochastic differential equation of Black-Scholes-Merton type [3,10]

$$dP_t = P_t(\mu dt + \sigma dW_t^{(1)}),$$

where μ and σ are constants and $\{W_t^{(1)}\}_{t\geq 0}$ is a standard Brownian motion. There is also a risk process, which is denoted by Y_t and assumed to be modeled as

$$dY_t = \alpha dt + \beta dW_t^{(2)},$$

where α and β ($\beta > 0$) are constants and $\{W_t^{(2)}\}_{t\geq 0}$ is another standard Brownian motion. It is allowed these two Brownian motions to be correlated with the correlation coefficient ρ. We prescribe $0 \leq \rho^2 < 1$ in the sequel.

The company invests in the risky stock under an investment policy f, where $f = \{f_t\}_{0\leq t\leq T}$ is a suitable, admissible adapted control process. T stands for the maturity date. Let X_t^f denote the wealth of the company at time t with $X_0 = x$, whose evolution process is given by

$$dX_t^f = f_t \frac{dP_t}{P_t} + dY_t, \quad X_0 = x.$$

The generator $\mathcal{A}^f$ of this wealth process is then expressed as

$$(\mathcal{A}^f g)(x,t) = \frac{\partial g}{\partial t} + (f\mu + \alpha)\frac{\partial g}{\partial x} + \frac{1}{2}(f^2\sigma^2 + \beta^2 + 2\rho\sigma\beta f)\frac{\partial^2 g}{\partial x^2}.$$

Suppose that the investor wants to maximize the utility $u(x)$ from his terminal wealth. The utility function $u(x)$ is customarily assumed to satisfy $u' > 0$ and $u'' < 0$. Let $V(x,t) = \sup_f E[u(X_T^f)\,|\,X_t^f = x]$. Then the Hamilton-Jacobi-Bellman equation becomes

$$\text{(2.1)} \qquad \sup_f\{\mathcal{A}^f V(x,t)\} = 0, \quad V(x,T) = u(x).$$

Suppose that (2.1) has a classical solution V with $\partial V/\partial x > 0$, $\partial^2 V/\partial x^2 < 0$. We then infer that

$$\text{(2.2)} \qquad f_t^* = -\frac{\mu}{\sigma^2}\frac{\partial V/\partial x}{\partial^2 V/\partial x^2} - \frac{\rho\beta}{\sigma},$$

where $\{f_t^*\}_{0\leq t\leq T}$ denotes the optimal policy. Placing (2.2) back into (2.1) we obtain

$$\text{(2.3)} \qquad \frac{\partial V}{\partial t} + \left(\alpha - \frac{\rho\beta\mu}{\sigma}\right)\frac{\partial V}{\partial x} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{(\partial V/\partial x)^2}{\partial^2 V/\partial x^2} + \frac{1}{2}\beta^2(1-\rho^2)\frac{\partial^2 V}{\partial x^2} = 0 \quad \text{for } 0 < t < T$$
$$V(T,x) = u(x).$$

Browne [2] shows that (2.3) possesses a solution in the case $u(x) = \lambda - (\gamma/\theta)e^{-\theta x}$ with positive constants λ, γ, θ. This utility has constant absolute risk aversion parameter θ; precisely stated, $-u''(x)/u'(x) = \theta$. Abe [1] made a preliminary research whether (2.3) has other solutions. Here we proceed further in the analysis of (2.3).

Let $v(x,t)$ be defined by $V(x, T-t) = v(E(x+Ft), Gt)$, where

$$E = \sqrt{\frac{\mu^2}{(1-\rho^2)\beta^2\sigma^2}}, \quad F = \alpha - \frac{\rho\beta\mu}{\sigma}, \quad G = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2.$$

It follows that after a calculation

$$\text{(2.4)} \qquad \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - \frac{(\partial v/\partial x)^2}{\partial^2 v/\partial x^2}, \quad v(x,0) = u(E^{-1}x).$$

The study of this singular parabolic PDE (2.4) seems interesting. Here we additionally introduce

$$\text{(2.5)} \qquad r(x,t) := -\frac{\partial^2 v/\partial x^2}{\partial v/\partial x} = -\frac{\partial}{\partial x}\log\left|\frac{\partial v}{\partial x}(x,t)\right|.$$

A little tedious computation then finally leads us to the equation (1.1). It should be noted that (2.5) is related to the coefficient of absolute risk aversion.

3. Proof of Theorem. Now we prove Theorem 2. Since we seek for a solution r which tends to α (> 0) as $x \to \infty$, we make a translation $r \mapsto r + \alpha$ so that we consider the next problem.

$$\text{(3.1)} \qquad \frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\left\{\left(1 + \frac{1}{(r+\alpha)^2}\right)\frac{\partial r}{\partial x} - (r+\alpha)^2\right\},$$
$$r = r(x,t) \quad \text{in } (x,t) \in \Omega_T$$
$$\frac{\partial r}{\partial x}(0,t) = 0, \quad \frac{\partial r}{\partial x}(x,t) \leq 0 \quad \text{for } (x,t) \in \Omega_T$$
$$r \to 0 \qquad \text{as } x \to \infty \quad \text{for } 0 < t < T$$
$$r(x,0) = r_0(x) - \alpha \qquad \text{for } x \in \boldsymbol{R}^+.$$

Fix $L > 1$ and let $\Omega_T^L := (0,L)\times(0,T)$. We approximate $r_0 - \alpha \in H^1(\boldsymbol{R}^+)$ by $r_0^L - \alpha \in C^1[0,L]$ with $\partial r_0^L/\partial x(0) = 0$ and $r_0^L(L) - \alpha = 0$. We introduce the convex set E^L defined as

$$E^L := \{r \in C^1(\overline{\Omega_T^L})\,|\,0 \leq r \leq K,$$
$$-K \leq \partial r/\partial x \leq 0 \text{ in } \overline{\Omega_T^L},$$

$$\partial r(0,t)/\partial x = 0,\ r(L,t) = 0 \text{ for } 0 < t < T,$$
$$r(x,0) = r_0^L(x) - \alpha \text{ for } 0 \le x \le L\},$$

where $K := 2(\|r_0 - \alpha\|_{C^1[0,L]} + 1)$.

Take $s \in E^L$ and try to find a solution $r \in E^L$ of the problem

$$\text{(3.2)} \quad \frac{\partial r}{\partial t} = \left(1 + \frac{1}{(s+\alpha)^2}\right)\frac{\partial^2 r}{\partial x^2} - \frac{2}{(s+\alpha)^3}\left(\frac{\partial r}{\partial x}\right)^2 - 2(r+\alpha)\frac{\partial r}{\partial x} \quad \text{in } (x,t) \in \Omega_T^L.$$

A standard theory of PDE tells us that an a-priori bound of r and $\partial r/\partial x$ implies the existence of solutions [5]. We first see that a simple application of the maximum principle yields $0 \le r \le \|r_0^L - \alpha\|_{L^\infty(0,L)}$.

Moreover we derive that the equation for $q := \partial r/\partial x$ becomes

$$\frac{\partial q}{\partial t} = \left(1 + \frac{1}{(s+\alpha)^2}\right)\frac{\partial^2 q}{\partial x^2} - \frac{2}{(s+\alpha)^3}\left(\frac{\partial s}{\partial x} + 2q\right)\frac{\partial q}{\partial x} - 2(r+\alpha)\frac{\partial q}{\partial x} - 2\left(\frac{-3}{(s+\alpha)^3}\frac{\partial s}{\partial x} + 1\right)q^2$$
$$\text{in } (x,t) \in \Omega_T^L.$$

Since $N := 3\alpha^{-3}K + 1 \ge -3(s+\alpha)^{-3}(\partial s/\partial x) + 1 \ge 0$, it follows that $-K \le \partial r/\partial x \le 0$ in Ω_T^L with $T := (4NK)^{-1}$ by virture that $-2NK^2 t - 2^{-1}K$ gives a lower barrier. Gradient bound at $x = L$ is provided by a supersolution $-2^{-1}K(x - L + 1)^2 + 2^{-1}K$ for (3.2) on $L - 1 < x < L$. Thus we deduce that $\|r(\cdot,t)\|_{C^1[0,L]} \le K$ for every $0 \le t \le T$, which ensures the existence of a solution to (3.2).

We define an operator $B : E^L \to E^L$ by $s \mapsto r = Bs$ is a solution of (3.2). Parabolic regularity shows that B is compact. Since E^L is a closed convex set in a Banach space $C^1(\overline{\Omega_T^L})$ we infer that there exists a fixed point r^L of B in E^L thanks to the Leray-Schauder fixed point theorem. See Corollary 11.2 of [6].

Now that we have found a non-increasing solution r^L for the equation (1.1) on Ω_T^L with $r^L(L) = \alpha$, we want to take a limit of $L \to \infty$. To do so we first extend r^L to be defined on $\boldsymbol{R}^+$ by setting $r^L(x,t) = \alpha$ for $x \ge L$. Take any monotone increasing sequence $0 < L_1 < L_2 < \cdots < L_n < \cdots \to \infty$ and we write $r_n(x,t) = r^{L_n}(x,t)$ for simplicity. Plugging $\varphi = r_n - r_m \in H^1(\boldsymbol{R}^+)$ $(n > m)$ into (1.4) for the equations of r_n and r_m respectively, and subtracting term by term, we compute

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(r_n - r_m)(t)\|^2_{L^2(\boldsymbol{R}^+)} + \left\|\frac{\partial}{\partial x}(r_n - r_m)(t)\right\|^2_{L^2(\boldsymbol{R}^+)} \\
&= -\int_{\boldsymbol{R}^+}\left(\frac{\partial r_n/\partial x}{r_n^2} - \frac{\partial r_m/\partial x}{r_m^2} - r_n^2 + r_m^2\right) \times \frac{\partial(r_n - r_m)}{\partial x}\,dx \\
&\quad + \left\{-\left(1 + \frac{1}{r_m^2}\right)\frac{\partial r_m}{\partial x} + r_m^2\right\}(r_n - r_m)\Big|_{x=L_m} \\
&\le 2(K+\alpha)\left(\frac{K}{\alpha^4} + 1\right)\|(r_n - r_m)(t)\|_{L^2(\boldsymbol{R}^+)} \cdot \left\|\frac{\partial}{\partial x}(r_n - r_m)(t)\right\|_{L^2(\boldsymbol{R}^+)} \\
&\quad + \left(\left(1 + \frac{1}{\alpha^2}\right)K + \alpha^2\right)|(r_n - \alpha)(L_m,t)| \\
&\le \frac{1}{2}\left\|\frac{\partial}{\partial x}(r_n - r_m)(t)\right\|^2_{L^2(\boldsymbol{R}^+)} + 2(K+\alpha)^2\left(\frac{K}{\alpha^4} + 1\right)^2\|(r_n - r_m)(t)\|^2_{L^2(\boldsymbol{R}^+)} \\
&\quad + \left(\left(1 + \frac{1}{\alpha^2}\right)K + \alpha^2\right)|(r_n - \alpha)(L_m,t)|.
\end{aligned}$$

Gronwall lemma implies immediately that

$$\begin{aligned}
&\|(r_n - r_m)(t)\|^2_{L^2(\boldsymbol{R}^+)} + \int_0^t e^{C(t-\tau)}\left\|\frac{\partial}{\partial x}(r_n - r_m)(\tau)\right\|^2_{L^2(\boldsymbol{R}^+)} d\tau \\
&\le e^{Ct}\|(r_0)_n - (r_0)_m\|^2_{L^2(\boldsymbol{R}^+)} + \left(\left(1 + \frac{1}{\alpha^2}\right)K + \alpha^2\right)\int_0^t e^{C(t-\tau)}|(r_n - \alpha)(L_m,\tau)|d\tau,
\end{aligned}$$

where $C := 4(K+\alpha)^2(\alpha^{-4}K + 1)^2$.

Since $\|(r_0)_n - (r_0)_m\|^2_{L^2(\boldsymbol{R}^+)} \to 0$ as well as $|(r_n - \alpha)(L_m,\cdot)| \to 0$ as $n, m \to \infty$, we see that $\{r_n\}$ is a Cauchy sequence in $L^\infty(0,T;L^2(\boldsymbol{R}^+)) \cap L^2(0,T;H^1(\boldsymbol{R}^+))$. The limiting function r is seen to verify (1.4) for each $\varphi \in H^1(\boldsymbol{R}^+)$ and almost every $0 \le t \le T$. The existence of solution r claimed in Theorem 2 is thereby established.

4. Steady state solutions. In this section we analyze the structure of steady state solutions of (1.1); that is, we determine the set of solutions for

$$\left(1 + \frac{1}{r^2}\right)\frac{\partial r}{\partial x} - r^2 = C, \qquad r = r(x),$$

where C denotes a constant independent of x and t. There are three possibilities according to the sign of C. We note that, however, the first two cases are meaningless for the economics because r takes a negative value.

If $C > 0$ then we write $C = M^2$ to obtain

$$(4.1)\qquad -\frac{1/M^2}{r(x)} + \left(\frac{1}{M} - \frac{1}{M^3}\right)\tan^{-1}\frac{r(x)}{M} = x - \frac{1/M^2}{r(0)} + \left(\frac{1}{M} - \frac{1}{M^3}\right)\tan^{-1}\frac{r(0)}{M}.$$

It follows that $r(x) \sim -M^{-2}x^{-1}$ as $x \to \infty$.

If $C = 0$ then we have

$$(4.2)\qquad -\frac{1}{r(x)} - \frac{1}{3r(x)^3} = x - \frac{1}{r(0)} - \frac{1}{3r(0)^3}.$$

It follows that $r(x) \sim -(3x)^{-1/3}$ as $x \to \infty$.

Consequently if $C \geq 0$ then there is no steady state solution suitable to the finance. Only the next last case fits into our requirement.

If $C < 0$ then we write $C = -M^2$ to obtain

$$(4.3)\qquad \frac{1/M^2}{r(x)} + \frac{1+M^2}{2M^3}\log\left|\frac{r(x)-M}{r(x)+M}\right| = x + \frac{1/M^2}{r(0)} + \frac{1+M^2}{2M^3}\log\left|\frac{r(0)-M}{r(0)+M}\right|,$$

provided $r(x) \neq M$. In this case $r(x) \sim M^{-2}x^{-1}$ as $x \to \infty$. It is also clear that $r(x) \equiv M$ gives one of steady state solutions, which has a character of constant absolute risk aversion. It should be noted that the last steady state solutions correspond to those presented in [2].

5. Discussions. The solvability is discussed for certain singular parabolic partial differential equation (PDE), which is related to the Arrow-Pratt coefficient of absolute risk aversion for the optimal value function. We prove the existence of solutions, which tend to a positive constant α. However, the analysis of steady state solutions exhibited in §4 indicates that the case $\alpha = 0$ may be possible. The problem whether the equation admit a solution which tends to zero or not is worth further investigation. We will return to this topic from the computational standpoint in the next paper [8].

The singular parabolic PDE of the form (2.3) or (2.4) has been often observed in the stochastic control theory. Indeed as mentioned in Hipp [7] the achievement of Browne [2] is one of first papers on this basis appeared in insurance mathematics, which is now an important area of applications of the stochastic control. We therefore believe that the advanced qualitative study of such singular PDEs is indispensable from the viewpoint of applications. For more details on stochastic control applied in insurance mathematics, see for instance a nice review of Hipp [7] and the references cited therein. We should point out, however, that the analysis of these singular equations is much more challenging than that of usual possible nonlinear Black-Scholes equations (see for example [9,12]). We hope that our paper has made a first step toward the better understanding of these PDEs.

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