EXISTENCE OF SOLUTIONS IN A CLOSED SET FOR DELAY DIFFERENTIAL EQUATIONS IN BANACH SPACES

S. Leela[1] and V. Moauro[2]

Technical Report No. 50

January, 1977

1 Department of Mathematics, SUNY College at Geneseo, Geneseo, N.Y. 14454, and University of Texas at Arlington, Arlington, Texas 76019.

2 Istituto di Matematica, "R. Caccioppoli" dell' Universita' di Napoli, Napoli, Italy, and University of Texas at Arlington, Arlington, Texas 76019.

EXISTENCE OF SOLUTIONS IN A CLOSED SET FOR DELAY

DIFFERENTIAL EQUATIONS IN A BANACH SPACE

by

S. Leela* and V. Moauro†

1. *Introduction*

The study of the Cauchy problem for ordinary differential equations in a Banach space has been extensive [1,3-7,9-12]. The two main directions that are followed in such a study are (i) finding monotonicity type conditions which guarantee the existence as well as uniqueness of solutions and (ii) finding compactness type conditions which assure only the existence of solutions [3,4]. It is also known [10,12] that in order to prove the existence of solutions in a closed subset F of the Banach space, a boundary condition of the type

$$\lim_{h\to 0^+} \frac{1}{h} \, d(x + hf(t,x),F) = 0, \quad x \in F,$$

is required. One would wonder why a corresponding theory is lacking for delay differential equations in Banach spaces which also occur in many physical problems [15-17]. One reason for such a state of affairs seems to be the difficulty in imposing the assumptions, since in this case, the domain and the range of the function involved in the differential equation are not in the same Banach space. Recently an attempt is made in [8] to overcome this difficulty and to prove the existence and uniqueness of solutions in an open set for the delay differential equations in a Banach space E of the form

* This work is partially supported by the U.S. Army Research Office, Durham, N.C.

† This work is supported by Italian Council of Research (C.N.R.).

$$(1.1) \qquad x'(t) = f(t,x_t), \quad x_{t_0} = \phi_0,$$

by requiring the function f to satisfy a monotonicity type condition over a suitable subset of the domain of f. Here f is a continuous mapping from $R_+ \times \mathcal{C}$ into E and $\phi_0 \in \mathcal{C}$ where $\mathcal{C}$ is the class of continuous functions from $[-\tau,0]$ into E, $\tau > 0$.

In this paper, our objective is to extend the results concerning the existence of solutions in a closed set to delay differential equations of the form (1.1). We employ the compactness type condition in terms of Kuratowski measure of noncompactness α and a boundary condition, namely,

$$(1.2) \qquad \lim_{h \to 0^+} \frac{1}{h} d(\phi(0) + hf(t,\phi),F) = 0,$$

for $\phi \in \mathcal{C}$ which satisfies $\phi(0) \in F$, where F is a closed subset of the Banach space E. One might think that a less restrictive boundary condition, namely, (1.2) holds for only those $\phi \in \mathcal{C}$ which satisfy $\phi(\theta) \in F$ for every $\theta \in [-\tau,0]$, could be used. However, it is shown in a counterexample provided by Professor Martin (discussed in detail at the end of Section 4) that such a condition is not enough to yield the existence of solutions in a closed set.

As in [8], we require f to satisfy the compactness type condition on a suitable subset of the domain of f. In Section 2, we give the notational preliminaries, some fundamental properties of α and the basic comparison result for delay differential equations. Section 3 deals with the existence of ε-approximate solutions for the Cauchy problem (1.1) in a closed set and showing that the limit of a uniformly convergent sequence of ε-approximate solutions is a solution of (1.1). In Section 4, we prove the main existence result by first establishing that the set of ε-approximate solutions is relatively compact. Also, the counterexample due to Martin is discussed and it is shown that the function in the example does not satisfy our boundary condition.

The assumption of uniform continuity of f has been crucial in our proof in Section 4. The question whether the existence of solutions of (1.1) in a closed set can be proved by means of compactness type condition and boundary condition, without the uniform continuity of f, is still open. Our existence results extend and generalize the analogous results obtained for ordinary differential equations in Banach spaces [1,9,11] to delay differential equations.

2. *Preliminaries*

Let E be a Banach space and let $\mathcal{C} = C[[-\tau,0],E]$, where $\tau > 0$ is a real number, be the space of continuous functions defined on the interval $[-\tau,0]$ and with values in E. If $\phi \in \mathcal{C}$, let us define

$$||\phi||_0 = \sup_{\theta \in [-\tau, 0]} ||\phi(\theta)||,$$

where $||\cdot||$ denotes the norm in E. Let F be a closed subset of E and consider the set

$$\mathcal{C}_F = \{\phi \in \mathcal{C} : \phi(0) \in F\} .$$

We note that $\mathcal{C}_F$ is a closed subset of $\mathcal{C}$.

Let $a > 0$, $t_0 \in R^+$ and $\phi_0 \in \mathcal{C}_F$. Let us consider the function $y \in C[[t_0 - \tau, t_0 + a], E]$ defined as follows:

$$y(t) = \begin{cases} \phi_0(t - t_0), & \text{if } t_0 - \tau \leq t \leq t_0 , \\ \phi_0(0) , & \text{if } t_0 \leq t \leq t_0 + a . \end{cases}$$

For $b > 0$ and $t \in [t_0, t_0 + a]$, define the set $\mathcal{C}_F^t(b)$ by

$$\mathcal{C}_F^t(b) = \mathcal{C}_F \cap \{\phi \in \mathcal{C} : ||\phi - y_t||_0 \leq b\} ,$$

where y_t denotes an element of $\mathcal{C}$ defined by $y_t(\theta) = y(t + \theta)$, $-\tau \leq \theta \leq 0$, for each t.

If $f \in C[R^+ \times \mathcal{C}_F, E]$, it is possible to prove that there exists a $b > 0$ such that the function f is bounded [6] on the set

$$\tilde{\mathcal{C}}_0(b) = \bigcup_{t \in [t_0, t_0 + a]} (\{t\} \times \mathcal{C}_F^t(b)).$$

The set $\tilde{\mathcal{C}}_0(b)$ is closed. In fact, let $\{(t_n, \phi_n)\}$ be a sequence of elements of $\tilde{\mathcal{C}}_0(b)$ such that $t_n \to \hat{t}$ and $\phi_n \to \hat{\phi}$. Obviously $\hat{t} \in [t_0, t_0 + a]$ and $\hat{\phi} \in \mathcal{C}_F$. Also, we have

$$||\hat{\phi} - y_{\hat{t}}||_0 \leq ||\hat{\phi} - \phi_n||_0 + ||\phi_n - y_{t_n}||_0 + ||y_{t_n} - y_{\hat{t}}||_0 ,$$

and for arbitrary $\varepsilon > 0$ and n sufficiently large,

$$||\hat{\phi} - y_{\hat{t}}||_0 \leq 2\varepsilon + b.$$

Since ε is arbitrary, this shows that $(\hat{t},\hat{\phi}) \in \tilde{\mathcal{C}}_0(b)$.

We wish to employ the Kuratowski measure of noncompactness of bounded subsets of E [2] to establish existence criteria for solutions of the Cauchy problem (1.1). For each bounded subset S of E, the measure of noncompactness $\alpha(S)$ is defined by

$\alpha(S) = inf\,\{d > 0 : S$ can be covered by a finite number of sets of diameter $d\}$.

The following lemma gives some of the fundamental properties of α which are needed later in our study.

Lemma 1. Let P and Q be bounded subsets of E. Then,

(i) $\alpha(P) \leq \alpha(Q)$ if $P \subset Q$;

(ii) $\alpha(P) = 0$ iff P is relatively compact;

(iii) $\alpha(\lambda P) = |\lambda|\alpha(P)$, where λ is a real number and $\lambda P = \{\lambda u : u \in P\}$;

(iv) $\alpha(CO\ P) = \alpha(P)$ where $CO\ P$ is the convex hull of P;

(v) $\alpha(P) = \alpha(\overline{P})$ where $\overline{P}$ is the closure of P;

(vi) $\alpha(P + Q) \leq \alpha(P) + \alpha(Q)$ where $P + Q = \{u + v : u \in P,\ v \in Q\}$;

(vii) $\alpha(\{p_n\}) - \alpha(\{q_n\}) \leq \alpha(\{p_n - q_n\})$, where $\{p_n\}$, $\{q_n\}$ are bounded sequences in E;

(viii) if $||p|| \leq \varepsilon$ for all $p \in P$, then $\alpha(P) \leq 2\varepsilon$.

For the convenience of future reference, let us list the following hypotheses:

(A_1) $f \in C[[t_0, t_0 + a] \times \mathcal{C}_F, E]$;

(A_2) $\liminf_{h \to 0^+} \frac{1}{h} d(\phi(0) + hf(t,\phi), F) = 0$, for each

$(t,\phi) \in [t_0, t_0 + a] \times \mathcal{C}_F$, where $d(x,F) = \inf\{||x - y||; y \in F\}$;

(A_3) for $t \in [t_0, t_0 + a]$ and $\Phi^t \subset \mathcal{C}^t_F(b)$,

$\liminf_{h \to 0^+} \frac{1}{h} [\alpha(\Phi^t(0)) - \alpha(\{\phi(0) - hf(t,\phi) : \phi \in \Phi^t\})] \leq g(t,\alpha(\Phi^t(0)))$

whenever $\alpha(\Phi^t(\theta)) \leq \alpha(\Phi^t(0))$ for every $\theta \in [-\tau, 0]$;

(A_4) $g \in C[R^+] \times [0,2b], R^+]$, $g(t,0) \equiv 0$ and $u(t) \equiv 0$ is the unique solution of $u' = g(t,u)$, $u(t_0) = 0$ on $[t_0, t_0 + a)$.

The following known comparison result [6] is useful in the sequel.

Lemma 2. Assume that

(i) $g \in C[R^+]] \times R^+, R^+]$ and $r(t,t_0,u_0)$ is the maximal solution of

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0$$

existing on $[t_0,\infty)$;

(ii) $m \in C[[t_0 - \tau,\infty), R^+]$ and for every $t_1 \geq t_0$, $t_1 \notin S$, where S is a countable subset of $[t_0,\infty)$ the differential inequality

$$D_- m(t_1) \leq g(t_1, m(t_1))$$

is satisfied provided $m_{t_1}(\theta) \leq m(t_1)$, $-\tau \leq \theta \leq 0$.

Then, $\quad m(t) \leq r(t,t_0,u_0)$, $t \geq t_0$,

whenever $m_{t_0}(\theta) \leq u_0$, $-\tau \leq \theta \leq 0$.

3. *Approximate Solutions*

In this section, we shall prove two auxiliary results which are needed to prove our main existence result. One is concerning the construction of an ε-approximate solution of the Cauchy problem (1.1) for any given $\varepsilon > 0$. The second result establishes that the limit of a uniformly convergent sequence of ε-apprxoimate solutions is indeed a solution of the Cauchy problem (1.1).

Lemma 3. Suppose that the hypotheses (A_1) and (A_2) hold. Let $\phi_0 \in \mathcal{C}_F$, $||f(t,\phi)|| \leq M - 1$ $(M > 1)$ on $\tilde{\mathcal{C}}_0(b)$ and $\gamma = min\ (a, \frac{b}{M})$. Then, for each $\varepsilon \in (0,1]$, the Cauchy problem (1.1) has an ε-approximate solution, that is, there exists a function x from $[t_0 - \tau, t_0 + \gamma]$ into E which satisfies the following properties:

(i) there exists a sequence $\{\sigma_i\}_{i=0}^{\infty}$ in $[t_0, t_0 + \gamma]$ such that $\sigma_0 = t_0$, $\sigma_i < \sigma_{i+1}$ if $\sigma_i < t_0 + \gamma$, $\sigma_{i+1} - \sigma_i \leq \varepsilon$ and $\lim_{i\to\infty} \sigma_i = t_0 + \gamma$;

(ii) $x(t) = \phi_0(t - t_0)$ for $t \in [t_0 - \tau, t_0]$ and $||x(t) - x(s)|| \leq M|t - s|$ for $t, s \in [t_0, t_0 + \gamma)$;

(iii) for each $i \geq 0$, $(\sigma_i, x_{\sigma_i}) \in \tilde{\mathcal{C}}_0(b)$ and $x(t)$ is linear on each of the intervals $[\sigma_i, \sigma_{i+1}]$;

(iv) if $t \in (\sigma_i, \sigma_{i+1})$ and $\sigma_i < t_0 + \gamma$, then $||x'(t) - f(\sigma_i, x_{\sigma_i})|| \leq \varepsilon$.

Proof. We proceed to prove the lemma by induction on i. Let us assume that we have defined $\sigma_0 = t_0, \ldots, \sigma_i$, $\sigma_i < t_0 + \gamma$, $i \geq 1$ and the function $x(t)$ on $[t_0 - \tau, \sigma_i]$ such that the properties (i)-(iv) hold on $[t_0 - \tau, \sigma_i]$. We shall show that it is possible to define σ_{i+1} and the function $x(t)$ on $[\sigma_i, \sigma_{i+1}]$ such that (i)-(iv) hold on $[t_0 - \tau, \sigma_{i+1}]$.

Choose $\delta_i \in (0,\varepsilon]$ satisfying

(1) $\sigma_i + \delta_i \leq t_0 + \gamma$;

(2) $d(x(\sigma_i) + \delta_i f(\sigma_i, x_{\sigma_i}), F) \leq \frac{\varepsilon}{2}\delta_i$; and

(3) δ_i is the largest number such that (1) and (2) hold.

Set $\sigma_{i+1} = \sigma_i + \delta_i$ and choose $x(\sigma_{i+1}) \in F$ such that

$$||x(\sigma_i) + \delta_i f(\sigma_i, x_{\sigma_i}) - x(\sigma_{i+1})|| \leq \varepsilon\delta_i$$

which is possible in view of (2). We then define

$$x(t) = \frac{x(\sigma_{i+1}) - x(\sigma_i)}{\sigma_{i+1} - \sigma_i}(t - \sigma_i) + x(\sigma_i), \qquad t \in [\sigma_i, \sigma_{i+1}].$$

It is easy to check that $x(t)$ satisfies properties (ii) and (iv) on $[t_0 - \tau, \sigma_{i+1}]$. To verify that $x(t)$ satisfies property (iii), we have to prove that

$$x_{\sigma_{i+1}} \in \mathcal{C}_F \quad \text{and} \quad ||x_{\sigma_{i+1}} - y_{\sigma_{i+1}}||_0 \leq b.$$

Since $x(\sigma_{i+1}) \in F$, it is clear that $x_{\sigma_{i+1}} \in \mathcal{C}_F$. Also, since

$$||x_{\sigma_{i+1}} - y_{\sigma_{i+1}}|| = \sup_{\theta \in [-\tau, 0]} ||x(\sigma_{i+1} + \theta) - y(\sigma_{i+1} + \theta)||,$$

let us consider the value $||x(\sigma_{i+1} + \theta) - y(\sigma_{i+1} + \theta)||$ for (i) $\sigma_{i+1} \geq t_0 + \tau$ and (ii) $\sigma_{i+1} < t_0 + \tau$. In the first case, $\sigma_{i+1} + \theta \geq t_0$ for every $\theta \in [-\tau, 0]$ and

$$\|x(\sigma_{i+1}+\theta)-y(\sigma_{i+1}+\theta)\| = \|x(\sigma_{i+1}+\theta)-\phi_0(0)\|$$

$$= \|x(\sigma_{i+1}+\theta)-x(t_0)\| \le M|\sigma_{i+1}-t_0| \le b,$$

using the definition of the function $y(\cdot)$ and the property (ii) which is satisfied by $x(t)$ on $[t_0-\tau,\sigma_{i+1}]$.

In case (ii),

$$\|x(\sigma_{i+1}+\theta)-y(\sigma_{i+1}+\theta)\| = \begin{cases} \|x(\sigma_{i+1}+\theta)-x(t_0)\| \le M|\sigma_{i+1}-t_0| \le \\ \quad \text{if } \sigma_{i+1}+\theta \ge t_0 \\ \|\phi_0(\sigma_{i+1}+\theta-t_0)-\phi_0(\sigma_{i+1}+\theta-t_0)\| \\ \quad \text{if } \sigma_{i+1}+\theta < t_0 . \end{cases}$$

Thus, in either case $\|x_{\sigma_{i+1}}-y_{\sigma_{i+1}}\|_0 \le b$ and $(\sigma_{i+1},x_{\sigma_{i+1}}) \in \tilde{\mathbf{C}}_0(b)$, proving that property (iii) is verified.

To complete the proof, it only remains to show that $\lim_{i\to\infty}\sigma_i = t_0+\gamma$. Let us assume, for contradiction, that for $i \ge 1$, $\sigma_i < t_0+\gamma$ and $\lim_{i\to\infty}\sigma_i = \bar{\sigma} < t_0+\gamma$. We shall first prove that the sequence $\{x_{\sigma_i}\}$ converges and for this, we need to consider the two cases (i) $\bar{\sigma} > t_0+\tau$ (ii) $\bar{\sigma} \le t_0+\tau$. In case (i), by choosing l,k large enough, we can have σ_l, $\sigma_k \ge t_0+\tau$. Then,

$$\|x_{\sigma_l}-x_{\sigma_k}\|_0 = \sup_{\theta\in[-\tau,0]} \|x(\sigma_l+\theta)-x(\sigma_k+\theta)\|$$

$$\le M|\sigma_k-\sigma_l|,$$

which shows that $\{x_{\sigma_i}\}$ is a Cauchy sequence.

In case (ii), let us choose $\sigma_l < \sigma_k < \bar{\sigma}$. Then, we have

$$||x_{\sigma_k} - x_{\sigma_l}||_0 = \sup_{\theta\in[-\tau,0]} ||x(\sigma_k + \theta) - x(\sigma_l + \theta)||.$$

But

$$||x(\sigma_k + \theta) - x(\sigma_i + \theta)|| = \begin{cases} ||\phi_0(\sigma_k+\theta-t_0) - \phi_0(\sigma_l+\theta-t_0)||, & \text{if } \sigma_k + \theta \leq t_0, \\ ||x(\sigma_k+\theta) - \phi_0(\sigma_l+\theta-t_0)||, & \text{if } \sigma_l + \theta \leq t_0 \leq \sigma_k + \\ ||x(\sigma_k+\theta) - x(\sigma_l+\theta)||, & \text{if } t_0 \leq \sigma_l + \theta. \end{cases}$$

Since ϕ_0 is uniformly continuous on $[-\tau,0]$, we obtain

$$||x_{\sigma_k} - x_{\sigma_l}||_0 \to 0 \quad \text{as} \quad k, l \to \infty.$$

Thus, $\overline{\phi} = \lim_{i\to\infty} x_{\sigma_i}$ exists and $\overline{\phi} \in \mathcal{C}_F$ since $\mathcal{C}_F$ is closed.

We shall now show that there exist a $\overline{\delta} \in (0,\varepsilon]$ and an index i_0 such that for $i \geq i_0$,

1) $\sigma_i + \overline{\delta} \leq t_0 + \gamma$;
2) $d(x(\sigma_i) + \overline{\delta}f(\sigma_i,x_{\sigma_i}),F) \leq \frac{\varepsilon}{2}\overline{\delta}$.

In fact, for any $h > 0$,

$$d(x(\sigma_i) + hf(\sigma_i,x_{\sigma_i}),F) \leq ||x(\sigma_i) - \overline{\phi}(0)||$$

$$+ h||f(\sigma_i,x_{\sigma_i}) - f(\overline{\sigma},\overline{\phi})|| + d(\overline{\phi}(0) + hf(\overline{\sigma},\overline{\phi}),F).$$

Let $\overline{\delta} \leq t_0 + \gamma - \overline{\sigma}$, $\overline{\delta} > 0$ be such that

$$d(\overline{\phi}(0) + \overline{\delta}f(\overline{\sigma},\overline{\phi}),F) \leq \frac{\varepsilon}{4}\overline{\delta}$$

which is possible in view of (A_2). Because of the convergence of $\{x_{\sigma_i}\}$ and $\{\sigma_i\}$ to $\overline{\phi}$ and $\overline{\sigma}$ respectively and the continuity of f at $(\overline{\sigma},\overline{\phi})$, we can deduce that there exists an index i_0 such that for every $i \geq i_0$,

$$d(x(\sigma_i) + \bar{\delta} f(\sigma_i, x_{\sigma_i}), F) \leq \frac{\varepsilon}{2} \bar{\delta}.$$

As δ_i is chosen to be the largest number such that (1) and (2) are satisfied, we have $\delta_i \geq \bar{\delta}$. But this is absurd since $\delta_i \to 0$ as $i \to \infty$. The Lemma is proved.

We now prove that if a sequence of ε-approximate solutions like the one constructed in Lemma 3 converges, then it converges to a solution of the Cauchy problem (1.1).

Lemma 4. Let the hypotheses (A_1) and (A_2) hold. Let $\{\varepsilon_n\} \subset (0,1)$ be a non-increasing sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\{x_n(t)\}$ be the sequence of ε_n-approximate solutions of the Cauchy problem (1.1) which exist by Lemma 3. If $\{x_n(t)\}$ converges uniformly on $[t_0 - \tau, t_0 + \gamma]$, then $x(t) = \lim_{n\to\infty} x_n(t)$ is a solution of (1.1).

Proof. We have $x_{t_0} = \phi_0$. Further $x(t)$ is continuous on $[t_0 - \tau, t_0 + \gamma]$ and thus, the function: $t \to x_t$, $t \in [t_0, t_0 + \gamma]$ is continuous. Let us set $\tau_n(t) = \sigma^n_{i_n}$, if $t \in [\sigma^n_{i_n}, \sigma_{i_n+1}]$. We then have

$$\lim_{n\to\infty} ||x_t - x_{n,\tau_n(t)}||_0 = 0, \text{ uniformly,}$$

since $x_{n,t} \to x_t$ uniformly, the function $t \to x_t$, $t \in [t_0, t_0 + \gamma]$, is continu and $|\tau_n(t) - t| \to 0$ as $n \to \infty$ uniformly. We also get that $x_t \in \mathfrak{C}_F$ since $x_{n,\tau_n(t)} \in \mathfrak{C}_F$ and $\mathfrak{C}_F$ is closed.

In order to complete the proof, we need to show that for $t \in [t_0, t_0 + \gamma]$,

$$||x(t) - \phi_0(0) - \int_{t_0}^{t} f(s, x_s) ds|| = 0.$$

Let $t \in [t_0, t_0 + \gamma]$. Then, for each positive integer n, we have

$$||x(t) - \phi_0(0) - \int_{t_0}^{t} f(s,x_s)ds||$$

$$\leq ||x(t) - x_n(t)|| + ||x_n(t) - \phi_0(0) - \int_{t_0}^{t} f(\tau_n(s),x_{n,\tau_n(s)})ds||$$

$$+ \int_{t_0}^{t} ||f(\tau_n(s),x_{n,\tau_n(s)}) - f(s,x_s)||ds.$$

Now, for any $\eta > 0$, we can find a positive integer $N(\eta)$ such that for $n \geq N(\eta)$

(1) $||x(t) - x_n(t)|| < \eta$;

(2) $||x_n(t) - \phi_0(0) - \int_{t_0}^{t} f(\tau_n(s), x_{n,\tau_n(s)})ds|| \leq$

$$||x_n(t) - \phi_0(0) - \int_{t_0}^{t} x_n'(s)ds|| + \int_{t_0}^{t} ||x_n'(s) - f(\tau_n(s),x_{n,\tau_n(s)})||ds$$

$$\leq \epsilon_N(t - t_0) \leq \epsilon_N\gamma < \eta;$$

(3) $$\int_{t_0}^{t} ||f(\tau_n(s),x_{n,\tau_n(s)}) - f(s,x_s)||ds < \eta,$$

as (s,x_s) varies over a compact subset of $[t_0, t_0 + \gamma] \times \mathcal{C}_F$.

Hence, for every $t \in [t_0, t_0 + \gamma]$, we have

$$||x(t) - \phi_0(0) - \int_{t_0}^{t} f(s,x_s)ds|| < 3\eta,$$

for every $\eta > 0$. The proof is therefore complete.

4. *Main Existence Results*

We are now in a position to prove our main result concerning the local existence of a solution of the Cauchy problem (1.1).

<u>*THEOREM 1.*</u> Let $t_0 \in R^+$ and $\phi_0 \in \mathcal{C}_F$. Let the hypotheses (A_1),(A_2),(A_3) and (A_4) hold. Further assume that f is uniformly continuous on $[t_0, t_0 + a] \times ($
Then, the Cauchy problem (1.1) has a solution existing on $[t_0 - \tau, t_0 + \gamma]$, where $\gamma = min\,(a, \frac{b}{M})$, $M - 1 = sup\,||f(t,\phi)||$ on $\tilde{\mathcal{C}}_0(b)$.

<u>*Proof.*</u> Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$ be a decreasing sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\{x_n(t)\}$ be the sequence of ε_n-approximate solutions of (1.1), which exists on $[t_0 - \tau, t_0 + \gamma]$ in view of Lemma 3. In virtue of Lemma 4, the theorem is proved if we show that there exists a subsequence of $\{x_n(t)\}$ which converges uniformly on $[t_0 - \tau, t_0 + \gamma]$. However, in order to apply Ascoli-Arzela's theorem to get a uniformly convergent subsequence of $\{x_n(t)\}$, it is enough to prove that, for each $t \in [t_0, t_0 + \gamma]$, the set $\{x_n(t)\}$ is a relatively compact subset of E, that is, $\alpha(\{x_n(t)\}) = 0$, since we already know by Lemma 3 that $\{x_n(t)\}$ is an equicontinuous and uniformly bounded sequence.

We define

$$p(t) = \alpha(\{x_n(t)\}), \quad t \in [t_0 - \tau, t_0 + \gamma]$$

and observe that

(1) $p(t) = \alpha(\{x_n(t) : n \geq k\}) = \alpha(\{x_n(\tau_n(t)) : n \geq k\})$ for every positive integer k;

(2) $\alpha(\{x_n(t) - hf(\tau_n(t), x_{n,\tau_n(t)})\}) = \alpha(\{x_n(\tau_n(t)) - hf(\tau_n(t), x_{n,\tau_n(t)})\})$

Let $t \in (t_0, t_0 + \gamma)$. If $t \in (\sigma^n_{i_n}, \sigma^n_{i_n+1})$ for each n, we have, by using (1), (2) and the property (vii) of α,

$$\frac{p(t) - p(t-h)}{h} = \frac{1}{h}\left[\alpha(\{x_n(\tau_n(t))\}) - \alpha(\{x_n(t-h)\})\right]$$

$$\leq \frac{1}{h}\left[\alpha(\{x_n(\tau_n(t))\}) - \alpha(\{x_n(t) - hf(\tau_n(t), x_{n,\tau_n(t)})\})\right] +$$

$$+ \frac{1}{h}\left[\alpha(\{x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\})\right]$$

$$= \frac{1}{h}\left[\alpha(\{x_n(\tau_n(t))\}) - \alpha(\{x_n(\tau_n(t)) - hf(\tau_n(t), x_{n,\tau_n(t)})\})\right]$$

$$+ \frac{1}{h}\left[\alpha(\{x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\})\right].$$

But, in view of property (iv) of the ε_n-approximate solution,

$$||x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})||$$

$$\leq \int_{t-h}^{t} ||x_n'(s) - f(\tau_n(s), x_{n,\tau_n(s)})||ds$$

$$+ \int_{t-h}^{t} ||f(\tau_n(s), x_{n,\tau_n(s)}) - f(\tau_n(t), x_{n,\tau_n(t)})||ds$$

$$\leq \varepsilon_n h + \int_{t-h}^{t} ||f(\tau_n(s), x_{n,\tau_n(s)}) - f(\tau_n(t), x_{n,\tau_n(t)})||ds.$$

The uniform continuity of f yields that for every $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that $||f(\tau_n(s), x_{n,\tau_n(s)}) - f(\tau_n(t), x_{n,\tau_n(t)})|| < \eta$ whenever $|\tau_n(s) - \tau_n(t)| < \delta$ and $||x_{n,\tau_n(s)} - x_{n,\tau_n(t)}||_0 < \delta$. However, $|\tau_n(s) - \tau_n(t)| \leq |t - s| + \varepsilon_n \leq h + \varepsilon_n$ and

$$\|x_n(\tau_n(s)+\theta) - x_n(\tau_n(t)+\theta)\| \le \begin{cases} M|\tau_n(s) - \tau_n(t)| \le M(h+\varepsilon_n), \\ \quad \text{if } t_0 \le \tau_n(s)+\theta, \\ \|\phi_0(\tau_n(s)+\theta-t_0) - \phi_0(\tau_n(t)+\theta-t_0)\|, \\ \quad \text{if } \tau_n(t)+\theta \le t_0, \\ \|\phi_0(\tau_n(s)+\theta-t_0) - \phi_0(0)\| + \|x_n(t_0) - \\ \quad x_n(\tau_n(t)+\theta)\|, \\ \quad \text{if } t_0-\tau_n(t) < \theta < t_0-\tau_n(s). \end{cases}$$

As ϕ_0 is uniformly continuous on $[-\tau, 0]$, there exists a $\bar{\delta}$ such that when $|\tau_n(s) - \tau_n(t)| \le h + \varepsilon_n < \bar{\delta}$ we have $\|\phi_0(\tau_n(s)) - \phi_0(\tau_n(t))\|_0 < \frac{\delta}{2}$, for all $\tau_n(s)$ and $\tau_n(t)$ in $[t_0, t_0+\gamma]$. Then, we get

$$\|x_n(\tau_n(s)+\theta) - x_n(\tau_n(t)+\theta)\| \le \begin{cases} M(h+\varepsilon_n), & \theta \ge t_0 - \tau_n(s) \\ \frac{\delta}{2}, & \theta \le t_0 - \tau_n(t) \\ \frac{\delta}{2} + M(h+\varepsilon_n), & t_0 - \tau_n(t) \le \theta \\ & \le t_0 - \tau_n(s). \end{cases}$$

Let us now choose $m = m(\eta)$ large enough and $\hat{h} = \hat{h}(\eta)$ small enough such that for $n \ge m$ and $h \le \hat{h}$, we have $\varepsilon_n < \eta$, $h + \varepsilon_n < \bar{\delta}$ and $M(h+\varepsilon_n) < \frac{\delta}{2}$. Then, for $\eta > 0$, $n \ge m(\eta)$ and $h \le \hat{h}(\eta)$, we obtain

$$\frac{1}{h}\|x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\| \le \varepsilon_n + \eta < 2\eta$$

and consequently, by the property (viii) of the measure α, we deduce that

$$\frac{1}{h}\alpha(\{x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\})$$

$$= \alpha\left(\left\{\frac{x_n(t) - x_n(t-h)}{h} - f(\tau_n(t), x_{n,\tau_n(t)}) : n \ge m(\eta)\right\}\right) \le 2(2\eta)$$

and

$$\liminf_{h\to 0^+} \frac{1}{h}\, \alpha(\{x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\}) \leq 4\eta.$$

Since this is true for every $\eta > 0$, we get

$$\liminf_{h\to 0^+} \frac{1}{h}\, \alpha(\{x_n(t) - x_n(t-h) - hf(\tau_n(t), x_{n,\tau_n(t)})\}) = 0.$$

Thus, we have

$$D\_p(t) \leq \liminf_{h\to 0^+} \frac{1}{h} \left[\alpha(\{x_n(\tau_n(t))\} - \alpha(\{x_n(\tau_n(t)) - hf(\tau_n(t), x_{n,\tau_n(t)})\})\right].$$

Let us now consider the subset of $\mathfrak{C}_F$ given by

$$\Phi_N^t = \{x_{n,\tau_n(t)} : n \geq N\},\quad N \text{ a positive integer},$$

and show that for N large enough, $\Phi_N^t \subset \mathfrak{C}_F^t(b)$. In fact, since

$$\|x_n(\tau_n(t)+\theta) - y(t+\theta)\| = \begin{cases} \|x_n(\tau_n(t)+\theta) - x_n(t_0)\| \leq M\gamma \leq b, \\ \quad \text{if } \tau_n(t)+\theta \geq t_0, \\ \|\phi_0(\tau_n(t)+\theta-t_0) - \phi_0(t+\theta-t_0)\|, \\ \quad \text{if } t+\theta \leq t_0, \\ \|\phi_0(\tau_n(t)+\theta-t_0) - \phi_0(0)\|, \\ \quad \text{if } \tau_n(t)+\theta \leq t_0 \leq t_0+\theta, \end{cases}$$

if we choose N sufficiently large, we can have, for $n \geq N$, $|t - \tau_n(t)| \leq \varepsilon_n \leq \varepsilon_N < \delta_{\phi_0}(b)$ and therefore, $\|x_{n,\tau_n(t)} - y_t\|_0 \leq b$, where $\delta_{\phi_0}(b)$ is the positive number associated with b in the definition of the uniform continuity of ϕ_0 on $[t_0 - \tau, t_0 + \gamma]$.

Further, if $\alpha(\Phi_N^t(\theta)) \leq \alpha(\Phi_N^t(0))$ for every $\theta \in [-\tau, 0]$, that is, $p(t + \theta) \leq p(t)$ for every $\theta \in [-\tau, 0]$, then by hypothesis (A_3) we get the differential inequality

$$D_- p(t) \leq g(t, p(t)).$$

Therefore, for every $t \in [t_0, t_0 + \gamma] \setminus S$, where S is a countable set, for which

$$||p_t||_0 \leq p(t)$$

we have

$$D_- p(t) \leq g(t, p(t)).$$

Also, $||p_{t}||_0 = \sup_\theta |p(t_0 + \theta)| = \sup_\theta \alpha(\{x_n(t_0 + \theta)\}) = \sup_\theta \alpha(\{\phi_0(\theta)\}) = 0$. Therefore, by Lemma 2, we get

$$p(t) \leq r(t, t_0, 0)$$

where $r(t, t_0, 0)$ is the maximal solution of $u' = g(t, u)$, $u(t_0) = 0$, which by hypothesis (A_4), yields that

$$p(t) = \alpha(\{x_n(t)\}) \equiv 0,$$

the proof is complete.

REMARK. The Theorem 1 is still valid if we require that (A_3) be satisfied for only those $\Phi^t \subset \mathbf{C}_F^t$ which are sequences of elements of $\mathbf{C}_F^t$. In such a situation, the condition

$$\alpha(\{f(t,\phi) : \phi \in \Phi^t\}) \leq g(t, \alpha(\Phi^t(0)))$$

is stronger than the compactness type condition in (A_3).

It is possible to give an existence theorem under a rather general compactness-type condition given in terms of a Lyapunov-like function.

Let us consider the mapping $V : \bigcup_{t\in[t_0-\tau,t_0+a]} \{t\} \times \Omega^t \to R^+$, where, for each $t \in [t_0 - \tau,\ t_0 + a]$, $\Omega^t = \{A : A \subset B[y(t),b]\}$, $B[y(t),b]$ being the ball of radius b with center at $y(t)$. Let us suppose that V has the following properties:

(i) $V(t,A)$ is continuous in t and α-continuous in A;

(ii) $V(t,A) \equiv 0$ *iff* $\bar{A}$ is compact;

(iii) $|V(t,A) - V(t,B)| \leq L|\alpha(A) - \alpha(B)|$, $t \in [t_0,\ t_0 + a]$, $A,\ B \in \Omega^t$.

An existence result under general compactness-type condition in terms of the Lyapunov-like function V is the following.

<u>*THEOREM 2.*</u> Let the hypotheses of Theorem 1 hold with the assumption (A_3) replaced by

(A_3^*) there exists a function V satisfying (i), (ii), (iii) and the condition

$$(4.1)\quad \begin{cases} D_-V(t,\Phi^t) \equiv \liminf_{h\to 0^+} \frac{1}{h}\left[V(t,\Phi^t(0)) - V(t-h,\{\phi(0) - hf(t,\phi) : \phi \in \Phi^t\})\right] \\ \leq g(t,V(t,\Phi^t(0))), \quad t \geq t_0, \end{cases}$$

whenever $\Phi^t \subset \mathcal{C}_F^t$ is such that

$$V(t + \theta,\Phi^t(\theta)) \leq V(t,\Phi^t(0)), \quad \theta \in [-\tau,0].$$

Then, the conclusion of Theorem 1 remains valid.

<u>*Proof.*</u> We shall only sketch the proof as many details follow along similar lines to those in the proof of Theorem 1.

Set $p(t) = V(t, \{x_n(t)\})$, $t \in [t_0 - \tau, t_0 + \gamma]$. It can be shown, as in Theorem 1, that

$$D_-p(t) \leq g(t, p(t))$$

for all $t \in [t_0, t_0 + \gamma] \backslash S$ (S is a countable subset of $[t_0, t_0 + a]$ such that

$$||p_t||_0 \leq p(t).$$

Also, $$||p_{t_0}|| = \sup_{\theta \in [-\tau, 0]} V(t_0 + \theta, \{x_n(t_0 + \theta)\})$$
$$= \sup_{\theta \in [-\tau, 0]} V(t_0 + \theta, \{\phi_0(t_0 + \theta)\}) = 0$$

and consequently, applying Lemma 2, we have

$$p(t) = V(t, \{x_n(t)\}) \equiv 0,$$

which yields that the set $\{x_n(t)\}$ is relatively compact.

Remark 1. If we take $V(t, A) = \alpha(A)$, $A \in \Omega^t$, the inequality (4.1) reduces to

$$\liminf_{h \to 0^+} \frac{1}{h} \left[\alpha(\phi^t(0)) - \alpha(\{\phi(0) - hf(t, \phi) : \phi \in \phi^t\})\right]$$
$$\leq g(t, \alpha(\phi^t(0)))$$

whenever $\alpha(\Phi^t(\theta)) \leq \alpha(\Phi^t(0))$, $\theta \in [-\tau, 0]$, which is the same as (A_3).

Remark 2. Observe that if we assume the following less stringent boundary condition

$$\liminf_{h \to 0^+} \frac{1}{h} d(\phi(0) + hf(t, \phi), F) = 0 \tag{4.2}$$

for every $\phi \in \mathcal{C}$ such that $\phi(s) \in F$ for every $s \in [-\tau, 0]$, Theorems 1 and 2 are not true. In fact, the counterexample due to R. H. Martin is as follows:

Let the closed set F be $(-\infty,0] \cup \{\frac{1}{n} : n \geq 1\}$. Consider the function $T : R \to R$ defined by

$$T(x) = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases}$$

and the function $f : C[[-1,0],R] \to R$ defined by

$$f(\phi) = T(\phi(-1)).$$

It is easy to show that f satisfies the condition (4.2). Let $\phi \in C[[-1,0],R]$ be such that $\phi(s) \in F$ for every $s \in [-1,0]$. The possible cases are (i) $\phi(0) > 0$ and (ii) $\phi(0) \leq 0$. In the first case, $\phi(s)$ is a continuous function on $[-1,0]$ and $\phi(s) = \frac{1}{n}$, $s \in [-1,0]$, for some integer n. Then, we have

$$\frac{1}{h}\, d(\phi(0) + hf(\phi),F) = \frac{1}{h}\, d\left(\frac{1}{n},F\right) = 0 \quad \text{for every} \quad h > 0.$$

In the second case, we could have either $\phi(0) \leq 0$, $\phi(-1) = 0$ or $\phi(0) \leq 0$, $\phi(-1) < 0$. Then,

$$\frac{1}{h}\, d(\phi(0) + hf(\phi),F) = \begin{cases} \frac{1}{h}\, d(\phi(0),F) = 0, \\ \frac{1}{h}\, d(\phi(0) - h\phi(-1),F) \end{cases}$$

and $\liminf_{h\to 0^+} \frac{1}{h}\, d(\phi(0) - h\phi(-1),F) = 0$. Thus, condition (4.2) is verified. But, if we consider $\phi_0 \in C[[-1,0],R]$ defined by $\phi_0(s) = s$, $s \in [-1,0]$, the Cauchy problem

$$(4.3) \qquad x'(t) = f(x_t),\ x_{t_0} = \phi_0,$$

does not have a solution in the closed set F, since we have, for a solution $x(t)$ of (4.3),

$$x'(t_0) = f(\phi_0) = T(-1) = 1 > 0 \quad \text{and} \quad x(t_0) = 0.$$

To see that the function f does not satisfy our boundary condition (A_2), consider a continuous function $\phi \in \mathcal{C}_F$ such that $\phi(0) = 1$, $\phi(-1) = -1$. We then have

$$\liminf_{h\to 0^+} \frac{1}{h}\, d(\phi(0) + hf(\phi),F)$$
$$= \liminf_{h\to 0^+} \frac{1}{h}\, d(1 + h,F) = \liminf_{h\to 0^+} \frac{1 + h - 1}{h} = 1.$$

Remark 3. In [14] a set F is defined to be positively invariant with respect to the equation $x'(t) = f(t,x_t)$ if for every $t_0 \in R$ and for every $\psi_0 \in \mathcal{C}$, such that $\psi_0(\theta) \in F\ \forall\ \theta \in [-\tau,0]$, every solution of (1.1) remains in F for every $t > t_0$ for which it exists. Assuming that F is convex and $f(t,\psi)$ satisfies a local Lipschitz condition in ψ, Seifert proves in [14] that the boundary condition (4.2) implies for $E = R^n$ the positive invariance of F. However, for ordinary differential equations in R^n, Nagumo's (weak invariance) result [ ], obtained with no convexity hypothesis on F, also yields (strong) positive invariance, if some uniqueness assumption is made. The existence results in this paper extend Nagumo's result to differential equations with delay. If uniqueness of solutions of (1.1) is also assumed, then the set F is (strongly) positively invariant in the following sense: for every $t_0 \in R$ and for every $\psi_0 \in \mathcal{C}_F$, any solution $x(t)$ of (1.1) with $x_{t_0} = \psi_0$ is such that $x(t) \in F$ for $t > t_0$ for which it exists.

However, as suggested by Professor Martin, one can also observe that the kind of invariance required by Seifert is assured (without convexity hypothesis on F) when one imposes the boundary condition for every $\psi \in \hat{\mathcal{C}}_F = \{\phi \in \mathcal{C}_F : \psi(s) \in \overline{Co}\, F\}$ which reduces to (4.2) when F is convex. This is true because, when $\psi_0 \in \hat{\mathcal{C}}_F$, the a-proximate solutions constructed in Lemma 3 are such that $x_n, \sigma_i^n \in \hat{\mathcal{C}}_F$.

ACKNOWLEDGMENT

The authors wish to express their thanks to Professor Martin for providing the counterexample in Remark 2 and to Professor Lakshmikantham for helpful discussions.

REFERENCES

[1] A. Cellina, On the local existence of solutions of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phy.* 20(1972), 293-296.

[2] K. Kuratowski, *Topology, Volume II*, Academic Press, New York, 1966.

[3] V. Lakshmikantham, Existence and comparison results for differential equations in a Banach space, *Proc. Int. Conf. on Diff. Eqs.*, Academic Press, New York (1975), 459-473.

[4] V. Lakshmikantham, Stability and asymptotic behavior of solutions of differential equations in a Banach space, *Lecture, Notes, CIME*, Italy, 1974.

[5] V. Lakshmikantham and J. Eisenfeld, On the existence of solutions of differential equations in a Banach space, *Revue Math. Pures. Appl.* (to appear).

[6] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities, Vol. II*, Academic Press, New York, 1969.

[7] V. Lakshmikantham, A. R. Mitchell and R. W. Mitchell, Differential Equations on closed subsets of a Banach space, *Trans. Amer. Math. Soc.* 220(1976), 103-113.

[8] V. Lakshmikantham, A. R. Mitchell and R. W. Mitchell, On the existence of solutions of differential equations of retarded type in a Banach space, *Ann. Polon. Math.*, (to appear).

[9] Tien-Yien Li., Existence of solutions for ordinary differential equations in Banach spaces, *Jour. Diff. Eq.* 18(1975), 29-40.

[10] R. H. Martin, Jr., Differential equations on closed subsets of a Banach space, *Trans Amer. Math. Soc.* 179(1973), 399-414.

[11] R. H. Martin, Jr., Approximation and existence of solutions to ordinary differential equations in Banach spaces, *Funk. Ekvac.* 16(1973), 195-211.

[12] R. H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach spaces*, John Wiley & Sons, New York, 1976.

[13] M. Nagumo, Uber die Lage der Integralkuruen gewohnlicher Differentialgleichungen, *Proc. physico-Math. Soc. Japan, Ser. 3*, 24(1942), 550-559.

[14] G. Seifert, Positively invariant closed sets for systems of delay differential equations, *Jour. Diff. Eqs.* 22(1976), 292-304.

[15] R. J. Thompson, On some functional differential equations: existence of solutions and difference approximations, *SIAM J. Numer. Anal.* 5(1968), 475-487.

[16] R. J. Thompson, *Difference Approximations for Some Functional Differential Equations*, Springer-Verlag Lecture Notes #333.

[17] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.*, to appear.