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EXISTENCE OF SOLUTIONS IN A CONE FOR NONLINEAR ALTERNATIVE PROBLEMS

JUAN J. NIETO

ABSTRACT. Using the alternative method we present sufficient conditions for the existence of positive solutions to nonlinear equations at resonance and extend a well-known result of Cesari and Kannan.

Introduction. Cesari and Kannan [**2**] proved an abstract result in terms of the alternative method. Their result and some of its ramifications (see [**1**]) have been applied to a large class of problems at resonance to prove the existence of solutions.

Let E be a Banach space. We say that C is a cone in E if C is a nonempty, convex subset of E such that $\lambda C \subset C$ for every $\lambda \geqslant 0$.

Here we prove the existence of solutions in a cone for equations at resonance of the form $Lu = Nu$, where L is a linear operator and N is a (nonlinear) operator. In the case when the cone is E, we obtain the well-known result of Cesari and Kannan [**2**].

In applications, for instance, if L is an elliptic operator on a bounded domain Ω of $\mathbf{R}^n$, one usually takes E as a subspace of $L^2(\Omega)$ and the cone $C = \{u \in E: u \geqslant 0 \text{ a.e. in } \Omega\}$.

Also, our result is related to that of Gaines and Santanilla [**3**] concerning the existence of solutions in a convex set.

Main result. Let E and F be Banach spaces with norms $\| \ \|_E$ and $\| \ \|_F$, respectively. Let $L: D(L) \subset E \to F$ be a linear operator and $N: E \to F$ a continuous (nonlinear) operator such that N maps bounded sets into bounded sets. Assume that C is a cone in E and

(1) there exists a continuous map $\gamma: E \to C$ such that $\gamma(c) = c$ for every $c \in C$, and γ maps bounded sets in E into bounded sets in E.

In addition, suppose that L is a Fredholm map of index 0 and there exist projections $P: E \to E$, $Q: F \to F$, and a linear map $H: (I - Q)F \to (I - P)E$

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satisfying

$$(2)\qquad \begin{aligned} &H(I-Q)Lu = (I-P)u && \text{for every } u \in D(L),\\ &QLu = LPu && \text{for every } u \in D(L),\\ &LH(I-Q)Nu = (I-Q)Nu && \text{for every } u \in E. \end{aligned}$$

Thus, it is well known that $Lu = Nu$ is equivalent to the coupled system of equations

$$\begin{aligned} &QNu = 0 && \text{(bifurcation equation)},\\ &u = Pu + H(I-Q)Nu && \text{(auxiliary equation)}. \end{aligned}$$

We can write the spaces E and F as the direct sums $E = E_0 \oplus E_1$, $F = F_0 \oplus F_1$, where $E_0 = PE$, $E_1 = (I-P)E$, $F_0 = QF$, and $F_1 = (I-Q)F$. Also, we assume

(3) $E_0 = \operatorname{Ker} L$, $F_1 = \operatorname{Im} L$, $D(H) = \operatorname{Im} L$ and $\operatorname{Im} H = E_1 \cap D(L)$.

(4) $\dim E_0 = \dim F_0 < +\infty$.

(5) H is completely continuous.

(6) There exist continuous maps $B\colon E \times F \to \mathbf{R}$ and $J\colon F_0 \to E_0$ such that B is bilinear, J is one-to-one and onto, and

(i) for $v_0 \in F_0$, $v_0 = 0$ iff $B(u_0, v_0) = 0$ for all $u_0 \in E_0$,

(ii) $B(Jv_0, v_0) \geqslant 0$ for every $v_0 \in F_0$ and $B(Jv_0, v_0) = 0$ iff $v_0 = 0$,

(iii) $Jv_0 = 0$ iff $v_0 = 0$,

(iv) $B(u_0, J^{-1}u_0) = 0$ iff $u_0 = 0$,

(v) $B(u_0, v_0) = B(Jv_0, J^{-1}u_0)$ for every $u_0 \in E_0$, $v_0 \in F_0$.

REMARK. If $E \subset F$ and F is a Hilbert space with inner product $\langle u, v\rangle$, then one can define $B(u_0, v_0) = \langle u_0, v_0\rangle$. Thus, if $F = L^2(\Omega)$,

$$B(u_0, v_0) = \int_\Omega u_0(x) \cdot v_0(x)\, dx.$$

For $u \in E$ we write $u = u_0 + u_1$, with $u_0 \in E_0$, $u_1 \in E_1$. With this, the auxiliary and bifurcation equations become $QN(u_0 + u_1) = 0$ and $u_1 = H(I-Q)N(u_0 + u_1)$, respectively. We are now in a position to prove our result.

THEOREM. *Let conditions* (1)–(6) *hold. In addition, assume there exists*

(7) $J_0 > 0$ *such that* $\|Nu\| \leqslant J_0$ *for every* $u \in C$,

(8) $R_0 > 0$ *such that* $B(u_0, QN(u)) \leqslant 0$ *for every* $u = u_0 + u_1 \in C$, *with* $\|u_0\| = R_0$ *and* $u_1 = H(I-Q)N(u_0 + u_1)$, *and*

(9) $r_0 \geqslant \|H(I-Q)\| \cdot J_0$ *such that* $(P + JQN)\gamma u \in C$ *and* $H(I-Q)N\gamma(u) \in C$ *for every* $u \in S$, *where*

$$S = \{u = u_0 + u_1 \in E\colon \|u_0\| \leqslant R_0, \|u_1\| \leqslant r_0\}.$$

Then $Lu = Nu$ *has at least one solution* $u \in S \cap C$.

PROOF. The set S is closed, bounded, and convex. Define the homotopy $T\colon [0,1] \times S \to E$ by $T(\lambda, u) = \lambda P\gamma(u) + H(I-Q)N\gamma(u) + \lambda JQN\gamma(u)$. Note that $T(\lambda, \cdot)$ is compact for every $\lambda \in [0,1]$ since P and Q are projections with finite-dimensional range and H is compact. For $\lambda = 0$, $T(0, u) = H(I-Q)N\gamma(u) \in E_1$.

Thus, by (9),

$$\|T(0, u)\| \leqslant \|H(I - Q)\| \cdot \|N\gamma(u)\| < r_0,$$

which shows that $T(0, \partial S) \subset S$.

We shall now prove that $T(\lambda, u) \neq u$ for every $(\lambda, u) \in [0,1) \times \partial S$. Indeed, let $T(\lambda, u) = u$ and, consequently,

$$u_0 = \lambda P\gamma(u) + \lambda JQN\gamma(u), \tag{10}$$

$$u_1 = H(I - Q)N\gamma(u). \tag{11}$$

If $u \in \partial S$, then either $\|u_1\| = r_0$ or $\|u_0\| = R_0$. In the first case, using (11), we get

$$r_0 = \|u_1\| = \|H(I - Q)N\gamma(u)\| < r_0,$$

which is a contradiction.

In the second case, $\|u_0\| = R_0$. Hence, by (9), $(P + JQN)\gamma(u) \in C$ and $u_0 \in C$ since C is a cone. Also by (9), $u_1 = H(I - Q)N\gamma(u) \in C$ and, consequently, $u = u_0 + u_1 \in C$. This implies, by the property of γ, that $\gamma(u) = u$ and $u_0 = \lambda Pu + \lambda JQNu$. This last inequality is equivalent to $(1 - \lambda)u_0 = \lambda JQNu$. We assume that $\lambda > 0$ since $\lambda = 0$ implies $(1 - \lambda)u_0 = 0$ and $u_0 = 0$. Thus, by (8) and (11), $B(u_0, QN\gamma(u)) \leqslant 0$. On the other hand,

$$\lambda B(u_0, QNu) = \lambda B\big(JQNu, J^{-1}u_0\big) = B\big((1 - \lambda)u_0, J^{-1}u_0\big) > 0,$$

which is again a contradiction. Therefore, $T(\lambda, u) \neq u$ for every $(\lambda, u) \in [0,1) \times \partial S$, and we can conclude [**4**, Theorem 4.4.11] that $T(1, \cdot)$ has a fixed point. Hence, there exists $u \in S$ satisfying

$$u = P\gamma(u) + H(I - Q)N\gamma(u) + JQN\gamma(u).$$

Reasoning as before, $u \in C$ and satisfies the auxiliary and bifurcation equations, that is, u is a solution of $Lu = Nu$ such that $u \in S \cap C$. This completes the proof of the Theorem.

If $C = E$ we obtain the result of [**2**].

COROLLARY. *Let conditions* (1)–(6) *hold. In addition, assume there exists*

(7)′ $J_0 > 0$ *such that* $\|Nu\| \leqslant J_0$ *for every* $u \in E$,

(8)′ $R_0 > 0$ *such that* $B(u_0, QNu) \leqslant 0$ *for every* $u = u_0 + u_1 \in E$, *with* $\|u_0\| = R_0$ *and* $u_1 = H(I - Q)N(u_0 + u_1)$.

Then $Lu = Nu$ *has at least one solution.*

For some particular cases of our result and applications to nonlinear boundary value problems, see [**5**].

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DEPARTAMENTO DE TEORIA DE FUNCIONES, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SANTIAGO, SANTIAGO, SPAIN