EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER SYSTEMS IN A BANACH SPACE

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Introduction

This paper is concerned with the existence of solutions of boundary value problems (BVP, for short) for nonlinear second order ordinary differential equations of the type

(1.1) $$x'' = H(t,x,x'), \quad 0 < t < 1,$$

(1.2) $$ax(0) - bx'(0) = x_0 \quad \text{and} \quad cx(1) + dx'(1) = x_1$$

where $x \in X$ is a real Banach space. In case $X = R^M$, existence was proved by first obtaining a priori bounds for $||x(t)||$, $||x'(t)||$ of a solution of (1.1) and (1.2) and then employing a theorem of Scorza-Dragoni [3,7,16]. The methods involve assuming inequalities in terms of the second derivative of Lyapunov-like functions relative to H, using comparison theorems for scalar second order equations and utilizing Leray-Schauder's alternative or equivalently the modified function approach [2,3,6,7,8,11].

In this paper, we wish to extend this fruitful method to the case when X is an arbitrary Banach space. First of all, this necessitates

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extending the basic result of Scorza-Dragoni. If we assume that H is compact operator as in [14], this extension is relatively easy. Since interest in abstract BVP's is partly due to the possibility of applications to partial differential equations, assuming compactness of H excludes many interesting examples. For example, using the method of lines (see the survey paper [12]), nonlinear elliptic BVP's may be approximated by an infinite system of BVP of the type (1.1) and (1.2). Consequently, we impose compactness-like conditions on H in terms of the Kuratowski's measure of noncompactness in extending Scorza-Dragoni's theorem. Thus, in Section 1, we develop further properties of the measure of noncompactness that are needed in our work. Utilizing these properties and the fixed point theorem of Darbo [5], we prove in Section 2 the generalization of Scorza-Dragoni's theorem. (Section 2 also contains a result concerning existence in the small.) Section 3 deals with extending the modified function approach to our problem (1.1) and (1.2). Here we use a new comparison result [3,1] and Lyapunov-like functions, and follow an argument similar to the one in [2,8]. To avoid monotony, we consider only one result, omitting variations as given in [2,8].

1. *Properties of Measure of Noncompactness*

Throughout this paper X will denote a real Banach space with norm, $||\cdot||$, $E = C^1[I,X]$ where $I = [a,b]$ and

$$|||\phi||| = max[\sup_I ||\phi(t)||, \ \sup_I ||\phi'(t)||].$$

For $A \subseteq E$, we will use the notation;

$$A(t) = \{\phi(t) | \phi \in A\},$$

$$A'(t) = \{\phi'(t) | \phi \in A\},$$

$$A' = \{\phi' | \phi \in A\},$$

$$A(I) = \{\phi(t) | \phi \in A, \quad t \in I\},$$

and
$$A'(I) = \{\phi'(t) | \phi \in A, \quad t \in I\}.$$

If A is a bounded subset of X, the Kuratowski's measure of non-compactness is defined by

$$\alpha(A) = \inf[d > 0 : A \text{ is covered by a finite number of sets with diameter } \leq d].$$

Let us list some known properties of α, [13],which we shall use in our subsequent discussion.

Theorem 1.1. Let A,B be bounded subsets of X. Then

(i) $\alpha(A) = 0$ iff $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of A;

(ii) $\alpha(A) = \alpha(\overline{A})$ and $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in R$ where $\lambda A = [\lambda x : x \in A]$;

(iii) $\alpha(A \cup B) = max(\alpha(A),\alpha(B))$,

(iv) $\alpha(A) \leq \alpha(B)$ if $A \subset B$;

(v) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ where $A + B = [x + y : x \in A$ and $y \in B]$;

(vi) α is continuous with respect to Hansdorff metric;

(vii) $\alpha(A) = \alpha(Co\ A)$ where $Co\ A$ is the convex hull of A;

(viii) if $\{A_n\}$ is a family of nonempty bounded subsets of X such that $A_{n+1} \subset A_n$, $n = 1,2,\ldots$ and $\lim_{n\to\infty} \alpha(A_n) = 0$, then $\bigcap_{n=1}^{\infty} \overline{A}_n$ is nonempty and compact;

(ix) if $H = \{x_k\}$, $x_k \in C[I,X]$, is any equicontinuous family of functions, then

$$\sup_{t\in I} \alpha(\{x_k(t) : x_k \in H\}) = \alpha(H),$$

(x) if $\sup_{a\in A} ||a|| \leq r$, then $\alpha(A) \leq 2r$.

For convenience, we shall be using below the same symbol, α to denote the measure of noncompactness in X as well as in other Banach spaces like E etc. This does not create confusion if the reader uses a little care.

We begin with the following two lemmas that are of a general nature.

Lemma 1.1. Let S be a bounded subset of the real and B a bounded subset of X. Then for $S\cdot B = \{s\cdot b \; s \in S, \; b \in B\}$ we have $\alpha(S\cdot B) = (\sup_{t\in S} |t|)\alpha(B)$.

Proof. For each $t \in S$, we see $tb \subseteq S\cdot B$ and hence $\alpha(S\cdot B) \geq \alpha(t\cdot B) = |t|\alpha(B)$. Thus $\alpha(S\cdot B) \geq (\sup_{t\in S} |t|)\alpha(B)$. For $\epsilon > 0$ there exist $t_1,\ldots,t_n \in \overline{S}$ such that for each $a \in S$ there exists t_i with $|a - t_i| < \frac{\epsilon}{\sup_{x\in B} ||x||}$. Moreover there exists $S_1,\ldots,S_m \subseteq X$ with $\operatorname{diam} S_i < \alpha(B) + \epsilon$ and $B \subseteq \bigcup_{i=1}^{m} S_i$. Define $T_{ij} = \{x \in X|$ for some $s \in S_i$ we have $||x - t_j s|| < \epsilon\}$. Observe that for $a \in S$, $b \in B$ there exists i,j with $|a - t_j| < \epsilon$ and $b \in S_i$ and hence

$$||a\cdot b - t_j b|| = |a - t_j| \; ||b|| < \epsilon \text{ which yields } \quad S\cdot B \subseteq \cup T_{ij}.$$

Furthermore, for x, $y \in T_{ij}$ we have

$$||x - y|| \leq ||x - t_j s|| + ||t_j s - t_j \bar{s}|| + ||t_j \bar{s} - Y||$$

$$\leq e + |t_j|||s - \bar{s}|| + e$$

$$\leq (\sup_{t \in S}|t|)(\alpha(B) + e) + 2e.$$

Therefore $\alpha(S \cdot B) \leq (\sup_{t \in S}|t|)\alpha(B)$. ∎

Lemma 1.2. Let A,B be bounded subsets of Banach spaces X and Y respectively with $||(x,y)|| = max(||x||,||y||)$, then $\alpha(A \times B) = max(\alpha(A),\alpha(B))$.

The proof follows directly from the definitions.

The next three lemmas generalize results of Ambrosetti [1], namely the property (ix) in Theorem 1.1 to the situation of $C^1[I,X]$.

Lemma 1.3. Let $A \subseteq E = C^1[I,X]$ with A bounded, then

$$\alpha(A) \geq max[\sup_I \alpha(A(t)), \sup_I \alpha(A'(t))].$$

Proof. Let $d = \alpha(A)$ and $e > 0$. By definition of $\alpha(A)$ there exist sets $T_1,\dots,T_k \subseteq E$ such that $A \subseteq \bigcup_{i=1}^{k} T_i$ and $diam\ T_i < d + e$ for each i. For $t_0 \in I$ observe $A(t_0) \subseteq \bigcup_{i=1}^{k} T_i(t_0)$, $A'(t_0) \subseteq \bigcup_{i=1}^{k} T_i'(t_0)$ and $diam\ T_i(t_0) < d + e$, $diam\ T_i'(t_0) < d + e$. Thus $\alpha(A(t_0)) < d + e$ $\alpha(A'(t_0)) < d + e$ and the proof follows since e and t_0 are arbitrary. ∎

Lemma 1.4. Let $A \subseteq E$ with A bounded, then $\alpha(A) \geq \alpha(A(I))$ and $\alpha(A) \geq (1/2)\alpha(A'(I))$.

Proof. Let $d = \alpha(A)$ and $e > 0$, then there exist sets $S_1,\dots,S_k \subseteq E$ with $A \subseteq \bigcup_{i=1}^{k} S_i$, and $diam\ S_i < d + e$ for each i. Since A is equicontinuous, there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that for $t \in [t_j, t_{j+1}]$ and $\phi \in A$ implies $\|\phi(t_j) - \phi(t)\| < e$. Let $T_{ij} = \{x \in X \,|\, \|x - \phi(t_j)\| < e \text{ for some } \phi \in S_i\}$. Observe that if $x, y \in T_{ij}$ then

$$\|x - y\| \leq \|x - \phi(t_j)\| + \|\phi(t_j) - \psi(t_j)\|$$

$$+ \|\psi(t_j) - y\| \leq e + (d + e) + e$$

since $\phi, \psi \in S_i$ and hence $diam\ T_{ij} \leq d + 3e$. Moreover if $\phi \in A$, $t \in I$, say $t \in [t_j, t_{j+1}]$ and $\phi \in S_i$, then

$\|\phi(t) - \phi(t_j)\| < e$ and so $\phi(t) \in T_{ij}$, which implies $A(I) \subseteq \cup\, T_{ij}$.

Thus it follows $\alpha(A) \geq \alpha(A(I))$.

Suppose $d = \alpha(A)$, $e > 0$, $A \subseteq \bigcup_{i=1}^{k} S_i$ with $diam\ S_i < d + e$ for each i. Choose $\phi_i \in S_i$ and the finite set $\{\phi_1', \ldots, \phi_k'\}$ is equicontinuous so there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that if $t, s \in [t_j, t_{j+1}]$ then $\|\phi_j'(t) - \phi_j'(s)\| < e$. Define $V_{ij} = \{\phi'(t) \,|\, \phi \in S_i,\ t \in [t_j, t_{j+1}]\}$ and note if $\phi'(t), \psi'(s) \in V_{ij}$ then

$$\|\phi'(t) - \psi'(s)\| \leq \|\phi'(t) - \phi_i'(t)\| + \|\phi_i'(t) - \phi_i'(s)\| + \|\phi_i'(s) - \psi'(s)\|$$
$$\leq 2d + 3e.$$

Furthermore if $\phi \in A$, $t \in I$ then $\phi \in S_i$ for some i and $t \in [t_j, t_{j+1}]$ for some j. Hence $\phi'(t) \in V_{ij}$ and

$$\alpha(A'(I)) \leq 2\alpha(A). \blacksquare$$

Lemma 1.5. Let $A \subseteq E$ with A bounded and A' equicontinuous then

$$\alpha(A) \leq \max[\sup_I \alpha(A(t)), \sup_I \alpha(A'(t))].$$

Proof. Since A and A' are equicontinuous, for $\epsilon > 0$, there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that if $t \in I$ say $t \in [t_i, t_{i+1}]$ and $\phi \in A$ then $||\phi(t) - \phi(t_i)|| < \epsilon$ and $||\phi'(t) - \phi'(t_i)|| < \epsilon$. Let $d = \alpha\left(\bigcup_{i=0}^{n} [(A(t_i) \cup A'(t_i)]\right)$ and there exists sets $S_1, \ldots, S_k$ such that $\bigcup_{i=0}^{n} [A(t_i) \cup A'(t_i)] \subseteq \bigcup_{i=1}^{k} S_i$ and $\operatorname{diam} S_i < d + \epsilon$ for $i = 1, \ldots, k$. Define $\Phi = \{$functions from $\{0, \ldots, n\}$ into $\{1, \ldots, k\}\}$ and surely Φ is a finite family. For each $f, g \in \Phi$ define

$$T_{f,g} = \{\phi \in A | \phi(t_i) \in S_{f(i)} \text{ and } \phi'(t_i) \in S_{g(i)} \text{ for } i = 0, \ldots, n\}.$$

Note $A \subseteq \bigcup_{f,g \in \Phi} T_{f,g}$ and if $\phi, \psi \in T_{fg}$ and $t \in I$ say $t \in [t_i, t_{i+1}]$ then

$$||\phi(t) - \psi(t)|| \leq ||\phi(t_i) - \phi(t)|| + ||\psi(t_i) - \psi(t)|| + ||\psi(t_i) - \phi(t_i)||$$

$$< \epsilon + \epsilon + d + \epsilon \quad \text{since } \phi(t_i), \psi(t_i) \in S_{f(i)}$$

$$||\phi'(t) - \psi'(t)|| \leq ||\phi'(t_i) - \phi'(t)|| + ||\psi'(t_i) - \psi'(t)||$$

$$+ ||\psi'(t_i) - \phi'(t_i)|| < \epsilon + \epsilon + d + \epsilon \quad \text{since } \psi'(t_i; \phi'(t_i) \in S_{g(i)}.$$

Thus $|||\psi - \phi||| < d + 3\epsilon$ and so $\operatorname{diam} T_{fg} \leq d + 3\epsilon$ which yields

$$\alpha(A) \leq d + 3\epsilon \leq \alpha[\bigcup_{i=0}^{n} (A(t_i) \cup A'(t_i))] + 3\epsilon$$

$$\leq \max_{i=1,\ldots,n} [\alpha(A(t_i)), \alpha(A'(t_i))] + 3\epsilon$$

$$\leq \max[\sup_I \alpha(A(t)), \sup_I \alpha(A'(t))] + 3\epsilon. \blacksquare$$

2. *Existence*

Using the results of Section 1, we shall now prove an existence theorem for the boundary value problem

$$(2.1)\qquad x'' = H(t,x,x'),$$

$$(2.2)\qquad ax(0) - bx'(0) = x \quad \text{and} \quad cx(1) + dx'(1) = x\ ,$$

where $a,b,c,\ d \geq 0$ and $ad + bc > 0$. Let us denote the interval $[0,1]$ by J.

We shall be using the fixed point theorem due to Darbo [5] which we state here.

Theorem 2.1. Let S be a closed, bounded and convex subset of a Banach space X. If $T \in C[S,S]$ such that $\alpha(TA) \leq \beta\alpha(A)$ with $\beta < 1$ for each bounded subset of S, then T has a fixed point.

Let $G(t,s)$ be the Green's function associated with the scalar BVP

$$(2.3)\qquad y'' = h(t),$$

$$(2.4)\qquad ay(0) - by'(0) = 0 \quad \text{and} \quad cy(1) + dy'(1) = 0.$$

Also let ψ be the unique function satisfying

$$(2.5)\qquad \psi'' = 0,$$

$$(2.6)\qquad a\psi(0) - b\psi'(0) = x_0 \quad \text{and} \quad c\psi(1) + d\psi'(1) = x_1.$$

We let

$$(2.7)\quad \begin{cases} \max[1, \sup\limits_{J\times J} |G(t,s)|] = P, \quad \sup\limits_{J\times J} |G_t(t,s)| = R, \\ \sup\limits_{J} ||\psi(t)|| \le Q \text{ and } \sup\limits_{J} ||\psi'(t)|| \le S. \end{cases}$$

one easily verifies that

$$(2.8)\quad \begin{cases} R = 1, \quad Q = \dfrac{a+b+c+d}{ad+bc} \max[||x_0||, ||x_1||], \\ \text{and } S = \dfrac{a+c}{ad+bc} \max[||x_0||, ||x_1||]. \end{cases}$$

Clearly P is a function of only a,b,c and d.

We are now in a position to prove the following result concerning existence in the large.

Theorem 2.2. Assume that

(i) $H \in C[J \times X \times X, X]$ and for all bounded subsets A,B in X,

$$\alpha(H(J \times A \times B)) \le k \max[\alpha(A), \alpha(B)];$$

(ii) $||H(t,x,y)|| \le L$ for $(t,x,y) \in J \times X \times X$.

Then there exists a solution $x \in C[J,X]$ of the boundary value problem (2.1) and (2.2) provided $k < 1/2P$, where $P = \max[1, \sup\limits_{J\times J} |G(t,s)|]$.

Proof. Let $G(t,s)$ be the Green's function associated with the scalar BVP (2.3) and (2.4) and let $\psi(t)$ be the unique function satisfying (2.5) and (2.6). Define $T : E \to E$ where $E = C^1[J,X]$ by

$$(T\phi)(t) = \int_0^1 G(t,s)H(s,\phi(s),\phi'(s))ds + \psi(t).$$

Observe that fixed points of T are solutions of the BVP (2.1) and (2.2).

Define the set E_1 by

$$E_1 = \{\phi \in E : \sup_J ||\phi(t)|| \leq PL + Q, \ \sup_J ||\phi'(t)|| \leq L + S\}$$

clearly E_1 is a bounded, closed and convex subset of E, $T(E_1) \subseteq E_1$ and T is continuous on E_1. If we can show that $\alpha(T(A)) \leq 2Pk\alpha(A)$ for every $A \subseteq E_1$ then theorem 2.1 implies that T has a fixed point and that fixed point is a solution of the BVP (2.1) and (2.2)

Consider $A \subseteq E_1$ and observe that for $\phi \in A$

$$(T\phi)''(t) = H(t,\phi(t),\phi'(t))$$

and hence $\sup_I ||(T\phi)''(t)|| \leq L$, which means $(TA)'$ is equicontinuous. Applying Lemma 1.5 for $\epsilon > 0$ there exists $\bar{t} \in J$ or $\bar{\bar{t}} \in J$ with

(2.9) $$\alpha(T(A)) \leq \alpha(TA(\bar{t})) + \epsilon$$

or (2.10) $$\alpha(T(A)) \leq \alpha((TA)'(\bar{\bar{t}})) + \epsilon.$$

Let us first consider the case (2.9). Using the properties of α, given in Theorem 2.1, we see that

$$\begin{aligned}
\alpha(TA) &\leq \alpha(TA(\bar{t})) + \epsilon \\
&= \alpha\left(\left\{\int_0^1 G(\bar{t},s)H(s,\phi(s),\phi'(s))ds + \psi(\bar{t}) \,\middle|\, \phi \in A\right\}\right) + \epsilon \\
&= \alpha\left(\left\{\int_0^1 G(\bar{t},s)H(s,\phi(s),\phi'(s))ds \,\middle|\, \phi \in A\right\}\right) + \epsilon \\
&\leq \alpha\left(\overline{co}\ \{G(\bar{t},s)H(s,\phi(s),\phi'(s)) \,|\, \phi \in A,\ s \in I\}\right) + \epsilon \\
&= \alpha\left(\{G(\bar{t},s)H(s,\phi(s),\phi'(s)) \,|\, \phi \in A,\ s \in I\}\right) + \epsilon.
\end{aligned}$$

Now Lemma 1.1, and the assumption (i) gives

$$\alpha(TA) \leq (\max_J |G(\bar{t},s)|)\alpha(\{H(s,\phi(s),\phi'(s)) | \phi \in A,\ s \in I\}) + \epsilon$$

$$\leq P\alpha(|H(s,\phi(s),\phi'(s))| \phi \in A,\ s \in I\}) + \epsilon$$

$$\leq P\alpha(H(J \times A(J) \times A'(J))) + \epsilon$$

$$\leq Pk\ \max(\alpha(A(J)),\alpha(A'(J))) + \epsilon.$$

This implies, by Lemma 1.4

$$\alpha(TA) \leq Pk\ \max(\alpha(A),2\alpha(A)) + \epsilon$$

$$\leq 2Pk\alpha(A) + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$\alpha(TA) \leq 2kP\alpha(A).$$

Consider the alternative (2.10). Proceeding as before, we obtain

$$\alpha(TA) \leq \alpha[(TA)'(\bar{\bar{t}})] + \epsilon$$

$$= \alpha\left(\left\{ \int_0^1 G_t(\bar{\bar{t}},s)H(s,\phi(s),\phi'(s))ds + \psi'(\bar{\bar{t}}) | \phi \in A \right\}\right) + \epsilon$$

$$= \alpha\left(\left\{ \int_0^1 G_t(\bar{\bar{t}},s)H(s,\phi(s),\phi'(s))ds | \phi \in A \right\}\right) + \epsilon$$

$$\leq \alpha(\overline{co}\{G_t(\bar{\bar{t}},s)H(s,\phi(s),\phi'(s))ds | \phi \in A,\ s \in I\}) + \epsilon$$

$$= \alpha(\{G_t(\bar{\bar{t}},s)H(s,\phi(s),\phi'(s))ds | \phi \in A,\ s \in I\}) + \epsilon.$$

Lemma 1.1, the assumption (i) and the fact $\sup_{J\times J} |G_t(t,s)| \le 1$ yields

$$\alpha(TA) \le (\max_J |G_t(\bar{\bar{t}},s)|)\alpha(\{H(s,\phi(s),\phi'(s))s \in I, \phi \in A\}) + \varepsilon$$

$$\le \alpha(\{H(s,\phi(s),\phi'(s))|s \in I, \phi \in A\}) + \varepsilon$$

$$\le \alpha(H[J \times A(J) \times A'(J)]) + \varepsilon$$

$$\le k \max[\alpha(A(J)),\alpha(A'(J))] + \varepsilon.$$

In view of Lemma 1.4, this implies that

$$\alpha(TA) \le k \max[\alpha(A),2\alpha(A)] + \varepsilon.$$

Thus, as before, we get

$$\alpha(TA) \le 2k\ \alpha(A) \le 2kP\ \alpha(A).$$

Therefore, in either case, we obtain

$$\alpha(TA) \le 2kP\ \alpha(A)$$

and the proof is complete. ∎

Remark 2.1. By Lemma 1.2, we see that

$$(H(J \times A \times B)) = \max(\alpha(J),\alpha(A),\alpha(B))$$

$$= \max(\alpha(A),\alpha(B))$$

since $\alpha(J) = 0$. Consequently, it is instructive to note that the compactness-like condition assumed in Theorem 2.2 is actually equivalent to

$$\alpha(H(J \times A \times B)) \leq k\alpha(J \times A \times B).$$

Furthermore, this assumption also implies that H maps bounded sets into bounded sets.

If the assumption (ii) of Theorem 2.2 is dispensed with then we can prove a result which gives existence in the small. This is precisely the next result in which the fact that H maps bounded sets into bounded sets is fully utilized.

Theorem 2.3. Assume that the hypothesis (i) of Theorem 2.2 holds. For any given $M > 0$, let $W > 0$ be such that $||H(t,x,y)|| \leq W$ for all $(t,x,y) \in J \times \{x : ||x|| \leq M\} \times \{y : ||y|| \leq M\}$. If t_1, $t_2 \in J$ such that $0 < t_2 - t_1 < \frac{M}{2PW}$, then the BVP

(2.11) $\quad x'' = H(t,x,x'),$

(2.12) $\quad ax(t_1) - bx'(t_1) = x_0,$ and $cx(t_2) + dx'(t_2) = x_1,$

has a solution $x \in C^2[[t_1,t_2],X]$ provided $max[||x_0||,||x_1||]$ $\left(\frac{ad + bc}{a + b + c + d}\right) \frac{M}{2}$ and $k < \frac{1}{2P}$.

Proof. Let $\hat{G}(t,s)$ be the Green's function associated with the scalar BVP

$$y'' = h(t); \quad ay(t_1) - by'(t_1) = 0 \quad \text{and} \quad cy(t_2) + dy'(t_2) = 0$$

and let $\hat{\psi}(t)$ be the unique function satisfying

$$\hat{\psi}'' = 0, \quad a\hat{\psi}(t_1) - b\hat{\psi}(t_1) = x_0 \quad \text{and} \quad c\hat{\psi}(t_2) + d\hat{\psi}'(t_2) = x_1,$$

where t_1, $t_2 \in J$. It is easily verified that

$$sup[|\hat{G}(t,s)|: t,s \in [t_1, t_2]] \leq P,$$

$$sup[|\hat{G}_t(t,s)|: t,s \in [t_1, t_2]] \leq 1,$$

$$sup[||\hat{\psi}(t)|| : t \in [t_1, t_2]] \leq Q,$$

and

$$sup[||\hat{\psi}'(t)|| : t \in [t_1, t_2]] \leq S$$

where P,Q,S are the same constants given in (2.7) and (2.8).

Consider $E = C^1[[t_1,t_2],X]$ with $|||\phi||| = max\{\sup_{[t_1,t_2]} ||\phi(t)||, \sup_{[t_1,t_2]} ||\phi'(t)||\}$ and define $E_1 = \{\phi \in E \mid \sup_{[t_1,t_2]} ||\phi(t)|| \leq M, \sup_{[t_1,t_2]} ||\phi'(t)|| \leq M\}$. The operator $T : E \to E$ is defined by

$$(T\phi)(t) = \int_{t_1}^{t_2} \hat{G}(t,s)H(s,\phi(s),\phi'(s))ds + \hat{\psi}(t).$$

For $\phi \in E_1$

$$||T\phi(t)|| \leq \int_{t_1}^{t_2} |\hat{G}(t,s)|\,||H(s,\phi(s),\phi'(s))||ds + ||\hat{\psi}(t)||$$

$$\leq PW(t_2 - t_1) + Q \leq \frac{M}{2} + \left(\frac{a + b + c + d}{ad + bc}\right) \; max\{||x_0||, ||x_1||\}$$

$$\leq \frac{M}{2} + \frac{M}{2} = M,$$

and

$$||(T\phi)'(t)|| \leq \int_{t_1}^{t_2} |\hat{G}_t(t,s)|\,||H(s,\phi(s),\phi'(s))||ds + ||\psi'(t)||$$
$$\leq (t_2 - t_1)W + \left(\frac{a + c}{ad + bc}\right) max\{||x_0||, ||x_1||\} \leq \frac{M}{2} + \frac{M}{2} = M.$$

Thus $T(E_1) \subseteq E_1$. Also, E_1 is a closed, bounded, convex subset of E and T is continuous on E_1. As before, we can obtain

$\alpha(TA) \leq 2kP(t_2 - t_1)\alpha(A)$ where the factor $(t_2 - t_1)$ arises when the integral is replaced by closure of the convex hull, and so

$\alpha(TA) \leq 2kP\alpha(A)$. ∎

Remark. Since we actually get $\alpha(TA) \leq 2kP(t_2 - t_1)\alpha(A)$, it is enough to assume in Theorem 2.3 that $k < \frac{1}{2P(t_2-t_1)}$.

If we wish to dispense with the assumption (ii) of Theorem 2.2 and to prove an existence theorem in the large we have to utilize Leray-Schauder's alternative or equivalently the modified function approach. Of course, we need to impose additional assumptions when we delete the restrictive boundedness condition (ii). This technique we consider in the next section.

3. *Modified Functions Approach.*

In this Section we develop the modified function approach which allows us to remove the boundedness condition (ii) of Theorem 2.2. For this purpose, we need the following comparison theorem.

Theorem 3.1. Assume that

(iii) $f \in C[J \times R \times R, R]$, $W \in C^2[J,R]$ with $W(t) \geq 0$, $t \in J = [0,1]$

and for $t \in (0,1)$,

$$w''(t) \leq f(t, W(t), w'(t)),$$

such that $\alpha_0 W(0) - \beta_0 W'(0) \geq \gamma_0$ and $\alpha_1 W(1) + \beta_1 W'(1) \geq \gamma_1$ where $\alpha_0, \alpha_1 \geq 0$, $\beta_0, \beta_1 > 0$;

(iv) $Z \in C^2[J,R]$ with $Z(t) > 0$, $t \in J$ and for each $\lambda > 0$,

$$\lambda z'' < f(t, W(t) + \lambda z(t), W'(t) + \lambda z'(t)) - f(t, W(t), W'(t)),$$

such that $\alpha_0 z(0) - \beta_0 z'(0) > 0$ and $\alpha_1 z(1) + \beta_1 z'(1) > 0$.

(v) $u \in C^2[J,R]$ and for $t \in (0,1)$,

$$u''(t) \geq f(t, u(t), u'(t)),$$

with $\alpha_0 u(0) - \beta_0 u'(0) \leq \gamma_0$ and $\alpha_1 u(1) + \beta_1 u'(1) \leq \gamma_1$.

Then $u(t) \leq W(t)$ for $t \in J$.

For a proof of this theorem see [4,15]. In fact, in [4], we have a generalization of Theorem 3.1 for countable systems.

Let us list the following hypotheses for convenience:

(vi) the left maximal solution $r(t,1,\eta_1)$ and the right minimal solution $\rho(t,0,\eta_0)$ of $v' = \hat{f}(t,v)$ exist on J where $\hat{f}(t,v) = \min_{0 \leq u \leq B_0} f(t,u,v)$, $B_0 = \max_J w(t)$, $\eta_0 = \frac{-\gamma_0}{\beta_0}$ and $\eta_1 = \frac{\gamma_1}{\beta_1}$.

(vii) $V \in C^2[J \times X, R^+]$ such that $V(t,x) \to \infty$ as $||x|| \to \infty$ uniformly in $t \in J$;

(viii) for $0 \leq \delta \leq 1$,

$$V''_{\delta H}(t,x) \geq f(t,V(t,x),V'(t,x)) + \sigma\delta||H(t,x,x')||,$$

where $\sigma > 0$, $V''_{\delta H}(t,x) = U(t,x,x') + \delta V_x(t,x)H(t,x,x')$

and $U(t,x,x') = V_{tt}(t,x) + 2V_{tx}(t,x)x' + V_{xx}(t,x)(x',x')$.

Here we have used the known facts [9] that if $V \in C\,[J \times X, R^+]$ then $V'(t,x) = V_t(t,x) + Vx(t,x)x'$ and $V''(t,x) = U(t,x,x') + V_x(t,x)x''$, where $V_{xx}(t,x)$ is the bilinear operator mapping $X \times X$ into $L(X \times X, R)$.

(ix) the boundary conditions (2.2) imply that

$$\alpha_0 V(0,x(0)) - \beta_0 V'(0,x(0)) \leq \gamma_0 \quad \text{and} \quad \alpha_1 V(1,x(1)) + \beta_1 V'(1,x(1)) \leq \gamma_1.$$

We are now in a position to prove the following theorem concerning the existence in the large.

Theorem 3.2. Assume that (i), (iii), (iv), (vi), (vii) and (viii) hold. Then there exists a solution $x \in C^2[J,X]$ of BVP (2.1) and (2.2) provided $k < \frac{1}{2P}$.

Proof. Let $\delta(t,x,y) \in C[J \times X \times X, J]$ satisfying

$$\delta(t,x,y) = \begin{cases} 1 \text{ if } ||x|| \leq B \text{ and } ||y|| \leq M \\ 0 \text{ if } ||x|| \geq B+1 \text{ or } ||y|| \geq M+1 \end{cases},$$

where the constants B and M are to be specified later. Define the modified function $\hat{H}(t,x,y)$ by

$$\hat{H}(t,x,y) = \delta(t,x,y)H(t,x,y).$$

Surely $\hat{H} \in C[J \times X \times X, X]$. Now by assumption (i) and Lemma 1.1, we see that

(a) for A,B bounded subsets of X,

$$\alpha(\hat{H}(J \times A \times B)) \leq \alpha[\delta(J \times A \times B)H(J \times A \times B)] \leq (\max_{J\times X\times X} \delta)\alpha(H(J \times A \times B))$$

$$\leq k \max(\alpha(A),\alpha(B)).$$

(b) $||\hat{H}(t,x,y)|| = ||\delta(t,x,y)H(t,x,y)|| \leq \max\{H(t,x,y)|t \in J,\ y,\ x \in S_{M+1}(0)\}$

$\leq \Omega$, constant, since H maps bounded sets to bounded sets by (i).

Thus $\hat{H}$ satisfies the hypotheses of Theorem 2.2. Consequently there exists $x \in C^2[J,X]$ such that

$$x'' = \hat{H}(t,x(t),x'(t)),$$

and $\quad ax(0) - bx(1) = x_0, \quad cx'(0) + dx'(1) = x_1.$

Setting $m(t) = V(t,x(t))$ and using the assumption (viii) we have

(3.1) $\qquad m''(t) \geq f(t,m(t),m'(t)) + \sigma||x''(t)|| \geq f(t,m(t),m'(t)).$

By (ix), it follows that $\left\{\begin{matrix} \alpha_0 m(0) - \beta_0 m'(0) \leq \gamma_0 \\ \alpha_1 m(1) + \beta_1 m'(1) \leq \gamma_1 \end{matrix}\right\}$. Hence the comparison Theorem 3.1, shows that

(3.2) $\qquad m(t) \leq w(t) \quad$ for $\quad t \in J.$

Using (vii) we know that there exists a $B = B(B_0)$ such that $V(t,x(t)) = m(t) \leq w(t) \leq B_0$ implies $||x(t)|| \leq B$. This is the B used in the definition of δ which is independent of $x(t)$.

Consider $\hat{f}$ from (vi) and observe

$$m''(t) \geq f(t,m(t),m'(t)) \geq \hat{f}(t,m'(t))$$

Hence defining $v(t) = m'(t)$, we have

$$v' \geq \hat{f}(t,v(t)).$$

Also, notice that in view of (ix), $v(0) \geq \frac{-\gamma_0}{\beta_0} = \eta_0$ and $v(1) \leq \frac{\gamma_1}{\beta_1} = \eta_1$. From the theory of differential inequalities [ ] we therefore obtain

$$v(t) \leq r(t,1,\eta_1),$$

$$v(t) \geq \rho(t,0,\eta_0), \quad \text{for} \quad t \in J$$

where $r(t,1,\eta_1)$ and $\rho(t,0,\eta_0)$ are the left maximal and right minimal solutions of $\xi' = \hat{f}(t,\xi)$ where existence is assured by (vi).

Let $B_1 = max\{\max_J|r(t,1,\eta_1)|, \; \min_J|\rho(t,0,\eta_0)|\}$, then it follows that

$$|m'(t)| = |v(t)| \leq B_1. \quad \text{Now define}$$

$N_0 = min\{f(t,u,y)|t \in J, \; 0 \leq u \leq B_0, \; |V| \leq B_1\}$ and note, from equations (3.1), (3.2) and the definitions of B_0 and B_1, that

$$m''(t) \geq N_0 + \sigma||x''(t)||.$$

For $0 \leq s \leq t \leq 1$

$$2B_1 \geq m'(t) - m'(s) \geq \int_s^t m''(\xi)d\xi \geq N_0(t - s) + \sigma||x'(t) - x'(s)||$$

and so $2B_1 + |N_0| \geq \sigma||x'(t) - x'(s)||$. Similarly for $0 \leq t \leq s \leq 1$ we have

$$2B + |N_0| \geq \sigma||x'(t) - x'(s)||.$$

Let $t \in [0,1]$ and integrating from 0 to 1, we obtain

$$\frac{1}{\sigma}(2B_1 + |N_0|) \geq \int_0^1 ||x'(t) - x'(s)||ds \geq ||\int_0^1 (x'(t) - x'(s))ds||$$

$$= ||x'(t) - x(1) + x(0)||.$$

But $||x(t)|| \leq B$ on J and hence

$$M \equiv \frac{1}{\sigma}(2B_1 + |N_0|) + 2B \geq ||x'(t)|| \quad \text{on } J.$$

This constant M is independent of $x(t)$ and is to be used in the definition of δ. This, in view of definition of $\hat{H}$, shows $\hat{H} \equiv H$ and so $x(t)$ is actually a solution of (2.1). The proof is therefore complete. ∎

References

[1] Ambrosetti, A., *Un teorena di existenza per le equazioni differenziali segli spazi di Banach,* Rend. Sem. Math. Univ. Padua 39(1967), 349-361.

[2] Bernfeld, S., Ladde, G., and Lakshmikantham, V., *Existence of solutions of two point boundary value problems for nonlinear systems*, Jour. Diff. Eq. 18(1975),

[3] Bernfeld, S. and Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[4] Chandra, J., Lakshmikantham, V., and Leela, S., *A monotone method for infinite system of nonlinear boundary value problems*, to appear.

[5] Darbo, G., *Punti Uniti in Transformazioni a codominio non compatto*, Rend. Sem. Math. Univ. Padua 24(1955), 84-92.

[6] Hartman, P., *On boundary value problems for systems of ordinary non-linear, second order differential equations*, Trans. Amer. Math. Soc. 96(1960), 493-504.

[7] Hartman, P., *Ordinary Differential Equations*, Wiley Interscience, New York, 1964.

[8] Hartman, P., *On two point boundary value problems for nonlinear second order systems*, SIAM J. Math. Anal. 5(1974),

[9] Kantorovich, L. V. and Akilov, G. P., *Functional Analysis in Normed Linear Spaces*, MacMillan Company, 1964.

[10] Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities, I & II*, Academic Press, New York, 1969.

[11] Lasota, A and Yorke, J., *Existence of solution of two point boundary value problems for nonlinear systems*, Jour. Diff. Eq. 11(1972), 509-518.

[12] Likovets, O. A., *The method of lines*, Diff. Eq. 1(1965), 1308-1323.

[13] Martin, R. H., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York, 1976.

[14] Thompson, R. C., *Differential inequalities for infinite second order systems and an application to the method of lines*, Jour. Diff. Eq. 17(1975), 421-434.

[15] Schröder, Johann, *Upper and lower bounds for solutions of generalized two point boundary value problems*, to appear.

[16] Scorza-Dragoni, G., *Sul problema dei valori ai limiti per i systemi di equazioni differenziali del secondo ordine*, Boll. Un Mat. Ital. 14(1935), 225-230.