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Existence of solutions of non-autonomous fractional differential equations with integral impulse condition

Ashish Kumar[1], Harsh Vardhan Singh Chauhan[2], Chokkalingam Ravichandran[3],
Kottakkaran Sooppy Nisar[4*] and Dumitru Baleanu[5,6,7]

*Correspondence:
n.sooppy@psau.edu.sa
[4]Department of Mathematics,
College of Arts and Sciences, Prince
Sattam bin Abdulaziz University,
Wadi Aldawasir 11991, Saudi Arabia
Full list of author information is
available at the end of the article

Abstract

In this paper, we investigate the existence of solution of non-autonomous fractional differential equations with integral impulse condition by the measure of non-compactness (MNC), fixed point theorems, and k-set contraction. The obtained results are verified via a supporting example.

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1 Introduction

Fractional calculus is a generalization of the standard integer calculus. The advantage of fractional calculus over integer-order calculus is that it provides a great deal for the kind of thought and hereditary characteristics of diversified materials and methods. From the past two decades, fractional calculus has attracted research attention towards itself due to its importance in several parts of science, like physics, fluid mechanics, heat conduction [1, 19, 21, 24, 26, 27, 32, 33, 37–42]. We can relate to the monographs [2, 22, 30, 36] for the fundamentals and to [4, 44, 45] for the current developments in the field of fractional calculus.

In recent years, non-autonomous differential equations of integer order, as well as fractional order, have been studied by many researchers. One can see the references [5–8, 10–13, 15–17, 20, 25, 35] for more details. In [14], Chen et al. discussed the existence of mild solutions as well as approximate controllability for a class of non-autonomous evolution systems of parabolic type with nonlocal conditions in Banach spaces by using the Schauder's fixed-point theorem as well as the theory of an evolution family. In the same year, Chen et al. [9] explored the existence of mild solutions for the initial value problem to a new class of abstract evolution equations with non-instantaneous impulses on ordered Banach spaces by using a perturbation technique and by dropping the compactness condition on the semigroup. Malik et al. [28] used the Rothe's fixed point theorem to study the

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controllability of non-autonomous nonlinear differential system with non-instantaneous impulses in the space $\mathbb{R}^n$. By using Krasnoselskii's fixed point theorem, Wang et al. [43] formed a set of sufficient conditions for the existence and stability for a class of impulsive non-autonomous differential equations. Kucche [23] investigated the existence and uniqueness of mild solutions for impulsive delay integrodifferential equations with integral impulses in Banach spaces by using Krasnoselskii–Schaefer fixed point theorem.

Motivated by the above, we discuss the non-autonomous fractional differential system with integral impulses in the following form:

$$^{C}D^{q}u(s) + \mathbb{A}(s)u(s) = f(s, u(s)), \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

$$\Delta u(s_k) = I_k\left(\int_{s_k - \tau_k}^{s_k - \theta_k} (G, u(t)) \, dt\right), \tag{1.3}$$

where $^{C}D^{q}$ denotes the CFD of order $0 < q \le 1$, $J = [0, S]$, $S \in \mathbb{R}$ is a positive constant. Suppose that F is a Banach space and $\{\mathbb{A}(s)\}_{s \in J}$ is a family of closed linear operator from F to F. The domain of $\{\mathbb{A}(s)\}$ is $D(\mathbb{A})$ which is independent of s and dense in F; $f : J \times F \to F$ is continuous and with $G : J \times F \to F$ are given functions; $0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = S$, $I_k \in C(F, F)$ are bounded functions, $0 \le \theta_k \le \tau_k \le s_k - s_{k-1}$ for $k = 1, 2, \ldots, m$, $\Delta u(s) = u(s_k^+) - u(s_k^-)$, $u(s_k^+) = \lim_{\epsilon \to 0} u(s_k + \epsilon)$, $u(s_k^-) = \lim_{\epsilon \to 0} u(s_k - \epsilon)$.

2 Preliminaries

Let F be a Banach space with norm $\|\cdot\|$. Further $C(J, F) = \{u : u : J \to F$ is continuous$\}$ denotes a Banach space with norm $\|u\|_C = \sup_{s \in J} \|u(s)\|$ and $\mathcal{L}(F)$ denotes the Banach space of all bounded linear operators in F with the operator norm topology. Let $L^1(J, F)$ be the Banach space of all Bochner integrable functions with the norm $\|u\|_1 = \int_0^S \|u(s)\| \, ds$.

Now, we recall some definitions of fractional derivatives and integral.

Definition 2.1 ([22]) The fractional integral operator (in Riemann–Liouville sense) of order $q > 0$ of a function u is defined as

$$I^{q}u(s) = \frac{1}{\Gamma(q)} \int_0^s (s - t)^{q-1} u(t) \, dt,$$

here $\Gamma(\cdot)$ denotes the Euler Gamma function.

Definition 2.2 ([22]) We define the derivative of u of the fractional order $q > 0$ in the Caputo sense as

$$^{C}D^{q}u(s) = \frac{1}{\Gamma(1-q)} \int_0^s (s - t)^{-q} u'(t) \, dt, \tag{2.1}$$

here $0 < q \le 1$ and $u'(t) = \frac{du(t)}{dt}$.

A measurable function $f : [0, \infty) \to F$ is called Bochner integrable if $\|f\|$ is Lebesgue integrable. The integrals which appear in (2.1) and (2.2) are taken in Bochner's sense. Let the operator $-\mathbb{A}(s)$ satisfies the following conditions:

(H_1)　$\mathbb{A}(s)$ is a closed operator, the domain of $\mathbb{A}(s)$ is independent of s, and dense in F.

(H_2)　For any $\lambda \geq 0$, the operator $\lambda I + \mathbb{A}(s)$ has a bounded inverse operator $[\lambda I + \mathbb{A}(s)]^{-1}$ in $\mathcal{L}(F)$ and

$$\left\| \left[\lambda I + \mathbb{A}(s) \right]^{-1} \right\| \leq \frac{C}{|\lambda| + 1},$$

where C is a positive constant independent of s and λ.

(H_3)　For any $s, \tau, t \in J$, there is a constant $p \in (0, 1]$ such that

$$\left\| \left[\mathbb{A}(s) - \mathbb{A}(\tau) \right] \mathbb{A}^{-1}(t) \right\| \leq C|s - \tau|^p,$$

where the constants p and $C > 0$ are independent of s, τ, and t.

Following Pazy [34], (H_1) means that for each $t \in J$, $-A(t)$ generates an analytic semigroup $e^{-s\mathbb{A}(t)}$ $(s > 0)$, and there is a $C > 0$ independent of both s and t such that $\| \mathbb{A}^n(t) e^{-s\mathbb{A}(t)} \| \leq \frac{C}{s^n}$, where $n = 0, 1$, $s > 0$, $t \in J$.

By [18], we can give the definition of operators $\tilde{\Psi}(s, t)$, $\tilde{\Phi}(s, \sigma)$, and $\tilde{U}(s)$:

$$\tilde{\Psi}(s, t) = q \int_0^\infty \theta s^{q-1} \xi_q e^{-s^q \theta \mathbb{A}(t)} \, d\theta, \tag{2.2}$$

$$\tilde{\Phi}(s, \sigma) = \sum_{k=1}^\infty \tilde{\Phi}_k(s, \sigma), \tag{2.3}$$

$$\tilde{U}(s) = -\mathbb{A}(s)\mathbb{A}^{-1}(0) - \int_0^s \tilde{\Phi}(s, t)\mathbb{A}(t)\mathbb{A}^{-1}(0) \, dt, \tag{2.4}$$

where ξ_q is probability density function defined on $[0, \infty)$ such that it's Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \xi_q(\theta) \, d\theta = \sum_{i=0}^\infty \frac{(-x)^i}{\Gamma(1 + qi)}, \quad 0 < q \leq 1, z > 0,$$

$$\tilde{\Phi}_1(s, \sigma) = \left[\mathbb{A}(s) - \mathbb{A}(\sigma) \right] \tilde{\Psi}(s - \sigma, \sigma),$$

$$\tilde{\Phi}_{k+1}(s, \sigma) = \int_\sigma^s \tilde{\Phi}_k(s, t) \tilde{\Phi}_1(t, \sigma) \, d\sigma, \quad k = 1, 2, \ldots.$$

Using the above facts, we define the mild solution of problem (1.1)–(1.3).

Definition 2.3 ([29]) A function $u \in C(J, F)$ is called a mild solution of (1.1)–(1.3) if it satisfies the integral equation

$$u(s) = u_0 + \int_0^s \tilde{\Psi}(s - \sigma, \sigma)\tilde{U}(\sigma)\mathbb{A}(0)u_0 \, d\sigma$$

$$+ \int_0^s \tilde{\Psi}(s - \sigma, \sigma)f(\sigma, u(\sigma)) \, d\sigma$$

$$+ \int_0^s \int_0^\sigma \tilde{\Psi}(s - \sigma, \sigma)\tilde{\Phi}(\sigma, t)f(t, u(t)) \, dt \, d\sigma$$

$$+ \sum_{0 < s_k < s} \tilde{\Psi}(s_k - \sigma, \sigma)I_k \left(\int_{s_k - \tau_k}^{s_k - \theta_k} G(t, u(t)) \, dt \right).$$

The following lemma gives some properties of $\tilde{\Psi}(s,t)$, $\tilde{\Phi}(s,\sigma)$, and $\tilde{U}(s)$ that are required to prove our main result.

Lemma 2.1 ([18]) *Functions $\tilde{\Psi}(s-\sigma,\sigma)$ and $\mathbb{A}(s)\tilde{\Psi}(s-\sigma,\sigma)$ are continuous in the uniform topology in s and σ, where $s \in J$, $0 \leq \sigma \leq s - \epsilon$ for any $\epsilon > 0$ and*

$$\left\| \tilde{\Psi}(s-\sigma,\sigma) \right\| \leq (s-\sigma)^{q-1}, \tag{2.5}$$

where $C > 0$ is independent of both s and σ. Moreover,

$$\left\| \tilde{\Phi}(s,\sigma) \right\| \leq (s-\sigma)^{p-1} \tag{2.6}$$

and

$$\left\| \tilde{U}(s) \right\| \leq C\left(1 + t^p\right). \tag{2.7}$$

Next, we define the MNC, which is required in our results.

Definition 2.4 ([3]) Assume that G is a bounded set of F. Then the Kuratowski MNC $\mu(\cdot)$ defined on G is

$$\mu(G) = \inf\left\{ \delta > 0 : G = \bigcup_{k=1}^{n} G_k \text{ and } \operatorname{diam}(G_k) \leq \delta \text{ for } k = 1, 2, \ldots, n \right\}.$$

Some properties of $\mu(\cdot)$ are given in the following lemma.

Lemma 2.2 ([3]) *Let Z, W be bounded subsets of F and $\lambda \in \mathbb{R}$. Then*
  1. *$\mu(Z) = 0$ if and only if $\overline{Z}$ is compact, where $\overline{Z}$ denotes the closure of Z;*
  2. *$Z \subseteq W$ implies $\mu(Z) \leq \mu(W)$;*
  3. *$\mu(\overline{Z}) = \mu(\operatorname{conv} Z) = \mu(Z)$, where $\operatorname{conv} Z$ denotes the convex hull of Z;*
  4. *$\mu(Z \cup W) = \max\{\mu(Z), \mu(W)\}$;*
  5. *$\mu(\lambda Z) = |\lambda|\mu(Z)$, where $\lambda Z = \{\lambda z : z \in Z\}$;*
  6. *$\mu(Z + W) \leq \mu(Z) + \mu(W)$, where $Z + W = \{z + w : z \in Z, w \in W\}$;*
  7. *If G is any bounded subset of $D(P)$, and if $P : D(P) \subset F \to F$ is a Lipschitz-continuous mapping with constant k, then $\mu(P(G)) \leq k\mu(G)$.*

In this article, MNC on the set F and $C(J, F)$ is denoted by $\mu(\cdot)$ and $\mu_C(\cdot)$, respectively. For any $s \in J$ and $B \subset C(J, F)$, we denote $B(s) = \{u(s) : u \in B\}$, then $B(s) \subset F$. The boundedness of $B(s) \subset C(J, F)$ implies the boundedness of $B(s)$ in F and $\mu(B(s)) \leq \mu_C(B)$.

The prove our main result, the following lemmas are required.

Lemma 2.3 ([3]) *Assume that B is a bounded subset of F. Then there is a countable subset B_0 of B such that $\mu(B) \leq 2\mu(B_0)$.*

Lemma 2.4 ([31]) *If $B = \{u_n\}_1^\infty \subset C([0, S], F)$ is a countable set in a Banach space F and there is a function $m \in L^1([0, S], \mathbb{R}^+)$ such that for every $n \in \mathbb{N}$,*

$$\left\| u_n(s) \right\| \leq m(s), \quad a.e. \ s \in [0, S],$$

then $\mu(B(s))$ is Lebesgue integrable on $[0, S]$ and

$$\mu\left(\left\{\int_0^S \mu_n(s)\,ds : n \in \mathbb{N}\right\}\right) \le 2 \int_0^S \mu\big(B(s)\big)\,ds.$$

Lemma 2.5 ([2]) *If $B \subset C(J, F)$ is equicontinuous and bounded in a Banach space F then $\mu(B(s))$ is continuous on $[0, S]$ and $\mu_C(B) = \max_{s \in [0,S]} \mu(B(s))$.*

Definition 2.5 ([3]) If G is a nonempty subset of F, then $P : G \to F$ is called to k-set contractive if it is continuous and there is a constant $k \in [0, 1)$ such that for every bounded subset D of G, we have

$$q\big(P(D)\big) \le kq(D).$$

Lemma 2.6 ([3]) *Assuming that D is a closed, bounded, and convex subset of F, $P : D \to D$ has at least one fixed point in D if it is k-set contractive.*

3 Main results

(F_1) For $r > 0$ there are constants $0 < q_1 < \min(q, p)$, $\rho > 0$ and a function $h_r \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that for any $s \in J$ and $u \in F$ with $\|u\| \le r$, $\|f(s, u)\| \le h_r(s)$ and

$$\liminf_{r \to \infty} \frac{\|h_r\|_{L^{\frac{1}{q_1}}[0,S]}}{r} = \rho \le \infty;$$

(F_2) There is an $L > 0$ such that for every bounded, countable, and equicontinuous set $D \subset F$,

$$\mu\big(f(s, D)\big) \le L\mu(D) \quad \text{for all } s \in J.$$

(F_3) There are constants L_N, M such that

$$\begin{aligned} \big\|N(s, v_1) - N(s, v_2)\big\| &\le L_N \|v_1 - v_2\|, \\ M &= \sup_{s \in [0,S]} \big\|N(s, 0)\big\|; \end{aligned} \tag{3.1}$$

(F_4) There are constants D_k $(k = 1, 2, 3, \ldots, m)$ such that

$$\big\|I_k(w_1) - I_k(w_2)\big\| \le D_k \|w_1 - w_2\|; \quad k = 1, 2, \ldots, m \text{ for } w_1, w_2 \in F. \tag{3.2}$$

In this study, $\mathbf{B}(q, p) = \int_0^1 s^{q-1}(1-s)^{p-1}\,ds$ is the Beta function.

Theorem 3.1 *If $f : J \times F \to F$ is continuous and satisfies conditions (F_1)–(F_4) then there is at least one mild solution of (1.1)–(1.3) in $C(J, F)$ provided that*

$$\frac{CS^{q-q_1}\rho}{(1 - C^* S^{q+1})}\left[\left(\frac{1 - q_1}{q - q^1}\right)^{1-q_1} + C\mathbf{B}(q, p)S^p\left(\frac{1 - q_1}{q + p - q_1}\right)^{1-q_1}\right] < 1, \tag{3.3}$$

and

$$CS^q\left[\frac{L_1}{q} + \frac{2CL\mathbf{B}(q,p)S^p}{q+p} + 2L_N S\right] \leq \frac{1}{4} \qquad (3.4)$$

Proof: Let the operator $\mathcal{P} : C(J,F) \to C(J,F)$ be defined by

$$(\mathcal{P}u)(s) = u_0 + \int_0^s \tilde{\Psi}(s-\sigma,\sigma)\tilde{U}(n)\mathbb{A}(0)u_0\,d\sigma$$

$$+ \int_0^s \tilde{\Psi}(s-\sigma,\sigma)f(\sigma,u(\sigma))\,d\sigma$$

$$+ \int_0^s \int_0^\sigma \tilde{\Psi}(s-\sigma,\sigma)\tilde{\Phi}(\sigma,t)f(t,u(t))\,dt\,d\sigma$$

$$+ \sum_{0<s_k<s} \tilde{\Psi}(s_k-\sigma,\sigma)I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k} N(t,u(t))\,dt\right). \qquad (3.5)$$

First, we show that $\mathcal{P}$ maps G_R to G_R which is a bounded, closed and convex set, where R is a positive constant such that $G_R = \{u \in C(J,F) : \|u(s)\| \leq R \text{ for } \forall s \in J\}$. If this were not true, then there would exist $s^r \in J$ and $u^r \in G_R$ such that $\|(\mathcal{P}u^r)(s^r)\| > r$ for each $r > 0$. Now by using Hölder inequality, (F_1) and Lemma 2.1, we get

$$r < \left\|\left(\mathcal{P}u^r\right)\left(s^r\right)\right\|$$

$$\leq \|u_0\| + \left\|\int_0^{s^r} \tilde{\Psi}\left(s^r-\sigma,\sigma\right)\tilde{U}(\sigma)\mathbb{A}(0)u_0\,d\sigma\right\|$$

$$+ \left\|\int_0^{s^r} \tilde{\Psi}\left(s^r-\sigma,\sigma\right)f\left(\sigma,u^r(\sigma)\right)d\sigma\right\|$$

$$+ \left\|\int_0^{s^r} \int_0^\sigma \tilde{\Psi}\left(s^r-\sigma,\sigma\right)\tilde{\Phi}(\sigma,t)f\left(t,u^r(t)\right)dt\,d\sigma\right\|$$

$$+ \left\|\sum_{0<s_k<s} \tilde{\Psi}\left(s_k^r-\sigma,\sigma\right)I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k} N\left(t,u^r(t)\right)dt\right)\right\|$$

$$\leq \|u_0\| + C^2\left\|\int_0^{s^r} \left(s^r-\sigma\right)^{q-1}\left(1+\sigma^p\right)d\sigma\right\|\|\mathbb{A}(0)u_0\|$$

$$+ C\int_0^{s^r} \left(s^r-\sigma\right)^{q-1}h_r(\sigma)\,d\sigma$$

$$+ C^2\int_0^{s^r} \int_0^\sigma \left(s^r-\sigma\right)^{q-1}(\sigma-t)^{p-1}h_r(t)\,dt\,d\sigma$$

$$+ \sum_{0<s_k<s} CL_N D_k\left(s_k^r-\sigma\right)^{q-1}\left(\int_{s_k-\tau_k}^{s_k-\theta_k} \left[\|u^r\| + \|N(t,0)\|\right]dt\right)$$

$$\leq \|u_0\| + C^2\left(s^r\right)^q\left[\frac{1}{q} + \left(s^r\right)^p B(q,p+1)\right]\|\mathbb{A}(0)u_0\|$$

$$+ C\int_0^{s^r} \left(s^r-\sigma\right)^{q-1}h_r(\sigma)\,d\sigma + C^2\mathbf{B}(q,p)\int_0^{s^r} \left(s^r-\sigma\right)^{q+p-1}h_\sigma(t)\,dt\,d\sigma$$

$$+ \sum_{0<s_k<s} CL_N D_k S^p(r+M)2S$$

$$\leq \|u_0\| + C^2 S^q \left[\frac{1}{q} + (S)^p B(q, p+1)\right] \|\mathbb{A}(0) u_0\|$$

$$+ C \left(\int_0^{s^r} (s^r - \sigma)^{\frac{q-1}{1-q_1}} \, d\sigma\right)^{1-q} \left(\int_0^{s^r} h_r^{\frac{1}{q_1}}(\sigma) \, d\sigma\right)^{q_1}$$

$$+ C^2 \mathbf{B}(q, p) \left(\int_0^{s^r} (s^r - \sigma)^{\frac{q+p-1}{1-q_1}} \, d\sigma\right)^{1-q} \left(\int_0^{s^r} h_r^{\frac{1}{q_1}}(\sigma) \, d\sigma\right)^{q_1}$$

$$+ \sum_{0 < s_k < s} 2 C L_N D_k S^{q+1}(r + M)$$

$$\leq \|u_0\| + C^2 S^q \left[\frac{1}{q} + (S)^p B(q, p+1)\right] \|\mathbb{A}(0) u_0\|$$

$$+ C S^{q-q_1} \left(\frac{1-q_1}{q-q^1}\right)^{1-q_1} \|h_r\|_{L^{\frac{1}{q_1}}[0,S]}$$

$$+ C^2 \mathbf{B}(q, p) S^{q+p-q_1} \left(\frac{1-q_1}{q+p-q_1}\right)^{1-q_1} \|h_r\|_{L^{\frac{1}{q_1}}[0,S]}$$

$$+ r C^* S^{q+1} + M C^* S^{q+1},$$

which gives

$$r(1 - C^* S^{q+1}) \leq \|u_0\| + C^2 S^q \left[\frac{1}{q} + (S)^p B(q, p+1)\right] \|\mathbb{A}(0) u_0\|$$

$$+ C S^{q-q_1} \left(\frac{1-q_1}{q-q^1}\right)^{1-q_1} \|h_r\|_{L^{\frac{1}{q_1}}[0,S]}$$

$$+ C^2 \mathbf{B}(q, p) S^{q+p-q_1} \left(\frac{1-q_1}{q+p-q_1}\right)^{1-q_1} \|h_r\|_{L^{\frac{1}{q_1}}[0,S]}$$

$$+ M C^* S^{q+1}. \tag{3.6}$$

Dividing both sides of (3.6) by $r(1 - C^* S^{q+1})$, using (F_1), and taking the limit as $r \to \infty$, we get

$$1 \leq \frac{C S^{q-q_1} \rho}{(1 - C^* S^{q+1})} \left[\left(\frac{1-q_1}{q-q^1}\right)^{1-q_1} + C \mathbf{B}(q, p) S^p \left(\frac{1-q_1}{q+p-q_1}\right)^{1-q_1}\right] < 1,$$

which is a contradiction. Therefore $\mathcal{P} : G_R \to G_R$.

Now, we prove that $\mathcal{P} : G_R \to G_R$ is a continuous operator. Consider $\{u_n\}_{n=1}^{\infty} \subset G_R$ such that $\lim_{n \to \infty} u_n = u$ in G_R. Since the function f is continuous in the second variable, for any $s \in J$, we get

$$\lim_{n \to \infty} \|f(s, u_n(s)) - f(s, u(s))\| = 0. \tag{3.7}$$

From (3.5) and Lemma 2.1, we get

$$\|(\mathcal{P} u_n)(s) - (\mathcal{P} u)(s)\| = \left\|\int_0^s \tilde{\Psi}(s - \sigma, \sigma)[f(\sigma, u_n(\sigma)) - f(\sigma, u(\sigma))] \, d\sigma\right\|$$

$$+ \left\|\int_0^s \int_0^\sigma \tilde{\Psi}(s - \sigma, \sigma) \tilde{\Phi}(\sigma, t)[f(t, u_n(t)) - f(t, u(t))] \, dt \, d\sigma\right\|$$

$$+ \left\| \sum_{0<s_k<s} \tilde{\Psi}(s_k - \sigma, \sigma) I_k \left(\int_{s_k-\tau_k}^{s_k-\theta_k} \left[G(t, u_n(t)) - G(t, u(t)) \right] dt \right) \right\|$$

$$\leq C \int_0^s (s-\sigma)^{q-1} \left\| f(\sigma, u_n(\sigma)) - f(\sigma, u(\sigma)) \right\| d\sigma$$

$$+ C^2 \int_0^s \int_0^\sigma (s-\sigma)^{q-1} (\sigma - t)^{p-1} \left\| f(t, u_n(t)) - f(t, u(t)) \right\| dt\, d\sigma$$

$$+ 2SCL_N \|u_n - u\| \sum_{0<s_k<s} (s_k - \sigma)^{q-1} D_k,$$

which gives that, for every $s \in J$, $\|(\mathcal{P}u_n)(s) - (\mathcal{P}u)(s)\| \to 0$ as $n \to \infty$,

Therefore, $\mathcal{P} : G_R :\to G_R$ is a continuous operator. It remains to prove that $\{\mathcal{P}u : u \in G_R\}$ is an equicontinuous function set. For any $u \in S_R$ and $s_1, s_2 \in [0, S]$, $s_1 < s_2$, we get

$$\left\| (\mathcal{P}u(s_2) - (\mathcal{P}u)(s_1) \right\|$$

$$\leq \left\| \int_0^{s_1} \left[\tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] \tilde{U}(\sigma) \mathbb{A}(0) u_0 \, d\sigma \right\|$$

$$+ \left\| \int_0^{s_1} \left[\tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] f(\sigma, u(\sigma)) \, d\sigma \right\|$$

$$+ \left\| \int_0^{s_1} \int_0^\sigma \left[\tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] \tilde{\Phi}(\sigma, t) f(t, u(t)) \, dt\, d\sigma \right\|$$

$$+ \left\| \int_{s_1}^{s_2} \tilde{\Psi}(s_2 - \sigma, \sigma) \tilde{U}(\sigma) A(0) u_0 \, d\sigma \right\|$$

$$+ \left\| \int_{s_1}^{s_2} \tilde{\Psi}(s_2 - \sigma, \sigma) f(\sigma, u(\sigma)) \, d\sigma \right\|$$

$$+ \left\| \int_{s_1}^{s_2} \int_0^\sigma \tilde{\Psi}(s_2 - \sigma, \sigma) \tilde{\Phi}(\sigma, t) f(t, u(t)) \, dt\, d\sigma \right\|$$

$$+ \left\| \sum_{0<s_k<s_2-s_1} \tilde{\Psi}(s_k - \sigma, \sigma) I_k \left(\int_{s_k-\tau_k}^{s_k-\theta_k} N(t, u(t)) \, dt \right) \right\|$$

$$\leq I^1 + I^2 + I^3 + I^4 + I^5 + I^6 + I^t,$$

where

$$I^1 = \int_0^{s_1} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] \tilde{U}(\sigma) \mathbb{A}(0) u_0 \right\| d\sigma,$$

$$I^2 = \int_0^{s_1} \left\| \left[\tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] \right\| h_R(\sigma) \, d\sigma,$$

$$I^3 = \int_0^{s_1} \int_0^\sigma \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right] \tilde{\Phi}(\sigma, t) \right\| h_R(t) \, dt\, d\sigma,$$

$$I^4 = \int_{s_1}^{s_2} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) \tilde{U}(\sigma) A(0) u_0 \right\| d\sigma,$$

$$I^5 = \int_{s_1}^{s_2} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) \right\| h_R(\sigma) \, d\sigma,$$

$$I^6 = \int_{s_1}^{s_2} \int_0^\sigma \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) \tilde{\Phi}(\sigma, t) \right\| h_R(t) \, dt\, d\sigma,$$

$$I^t = \sum_{0<s_k<s_2-s_1} \left\| \tilde{\Psi}(s_k - \sigma, \sigma) I_k \left(\int_{s_k-\tau_k}^{s_k-\theta_k} N(t, u(t)) \, dt \right) \right\|.$$

Now, we prove that $I^j \to 0$ independently of $u \in S_R$ as $s_2 - s_1 \to 0$ for $j = 1, 2, 3, 4, 5, 6, t$.

First, we prove $I^1 = 0$. For $\epsilon > 0$ and $s_1 > 0$, by the continuity of $\tilde{\Psi}(s - \sigma, \sigma)$ in the uniform topology in s and σ, for $0 \le s \le S$ and $0 \le \sigma \le s - \epsilon$, by Lemma 2.1, we have

$$
\begin{aligned}
I^1 &\le \sup_{\sigma \in [0, s_1 - \epsilon]} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right\| . C \left\| \mathbb{A}(0) u_0 \right\| \int_0^{s_1 - \epsilon} \left(1 + \sigma^p \right) d\sigma \\
&\quad + C^2 \left\| \mathbb{A}(0) u_0 \right\| \int_{s_1 - \epsilon}^{s_1} \left[(s_2 - \sigma)^{q-1} - (s_1 - \sigma)^{q-1} \right] \left(1 + \sigma^p \right) d\sigma \\
&\to 0 \quad \text{as } \epsilon \to 0 \text{ and } s_2 \to s_1.
\end{aligned}
$$

Obviously, $I^2 = 0$. For $\epsilon > 0$ and $s_1 > 0$, using the continuity of $\tilde{\Psi}(s - \sigma, \sigma)$ in the uniform topology in t and σ, for $0 \le s \le S$ and $0 \le \sigma \le s - \epsilon$, by Lemma 2.1, we get

$$
\begin{aligned}
I^2 &\le (s_1 - \epsilon)^{1-q} \| h_R \|_{L^{\frac{1}{q_1}}[0,S]} \sup_{\sigma \in [0, s_1 - \epsilon]} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right\| \\
&\quad + C \int_{s_1 - \epsilon}^{s_1} \left[(s_2 - \sigma)^{q-1} - (s_1 - \sigma)^{q-1} \right] h_R(\sigma) d\sigma \\
&\to 0 \quad \text{as } \epsilon \to 0 \text{ and } s_2 \to s_1.
\end{aligned}
$$

Obviously, $I^3 = 0$, for $s_1 = 0$ and $0 < s_2 \le S$. For $s_1 > 0$ and $\epsilon > 0$, by Lemma 2.1, (F_1), and since the functions $\sigma \to (s_2 - \sigma)^{q-1} I_\sigma^p h_R(\sigma)$ and $\sigma \to (s_1 - \sigma)^{q-1} I_\sigma^p h_R(\sigma)$ are Lebesgue integrable, as well as $\tilde{\Psi}(s - \sigma, \sigma)$ is continuous in the uniform operator topology in s and σ, for $0 \le s \le S$ and $0 \le \sigma \le s - \epsilon$, we get

$$
\begin{aligned}
I^3 &\le \sup_{\sigma \in [0, s_1 - \epsilon]} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right\| . C \int_0^{s_1 - \epsilon} \int_0^{\sigma} (\sigma - t)^{p-1} \, dt \, d\sigma \\
&\quad + C^2 \int_{s_1 - \epsilon}^{s_1} \int_0^{\sigma} \left[(s_2 - \sigma)^{q-1} - (s_1 - \sigma)^{q-1} \right] (\sigma - t)^{p-1} h_R(t) \, dt \, d\sigma \\
&\le \left(\frac{1 - q_1}{q - q_1} \right)^{1 - q_1} \frac{C s_1^p \| h_R \|_{L^{\frac{1}{q_1}}[0,S]}}{p} \sup_{\sigma \in [0, s_1 - \epsilon]} \left\| \tilde{\Psi}(s_2 - \sigma, \sigma) - \tilde{\Psi}(s_1 - \sigma, \sigma) \right\| \\
&\quad + C^2 p(p) \int_{s_1 - \epsilon}^{s_1} \left[(s_2 - \sigma)^{q-1} I_\sigma^p h_R(\sigma) - (s_1 - \sigma)^{q-1} I_\sigma^p h_R(\sigma) \right] d\sigma \\
&\to 0 \quad \text{as } \epsilon \to 0 \text{ and } s_2 \to s_1.
\end{aligned}
$$

For I^4, by Lemma 2.1, we see that

$$
I^4 \le C^2 \left\| A(0) u_0 \right\| \int_{s_1}^{s_2} (s_2 - \sigma)^{q-1} \left(1 + \sigma^p \right) d\sigma \to 0 \quad \text{as } s_2 \to s_1.
$$

For I^5, using Lemma 2.1, (F_1), and Hölder inequality, we have

$$
\begin{aligned}
I^5 &\le C \int_{s_1}^{s_2} (s_2 - \sigma)^{q-1} h_R(\sigma) d\sigma \\
&\le C \left(\int_{s_1}^{s_2} (s_2 - \sigma)^{\frac{q-1}{1-q_1}} \right)^{1-q_1} \left(\int_{s_1}^{s_2} h_R^{\frac{1}{q_1}}(\sigma) d\sigma \right)^{q_1}
\end{aligned}
$$

$$\leq C\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} \|h_R\|_{L^{\frac{1}{q_1}}[0,S]} (s_2-s_1)^{q-q_1}$$

$$\to 0 \quad \text{as } s_2 \to s_1.$$

For I^6, using Lemma 2.1, (F_1), and the fact that the function $\sigma \to (s_1-\sigma)^{q-1}I_\sigma^p h_R(\sigma)$ is Lebesgue integrable, we get

$$I^6 \leq C^2 \int_{s_1}^{s_2} \int_0^\sigma (s_2-\sigma)^{q-1}(\sigma-t)^{p-1} h_R(t)\, dt\, d\sigma$$

$$\leq C^2 p(p) \int_{s_1}^{s_2} (s_2-\sigma)^{q-1} I_\sigma^p h_R(\sigma)\, d\sigma$$

$$\to 0 \quad \text{as } s_2 \to s_1.$$

Also $I^t = 0$ as $s_2 \to s_1$.

Hence $\|(\mathcal{P}u)(s_2) - (\mathcal{P}u)(s_1)\|$ tends to 0 independently of $u \in G_R$ as $s_2 \to s_1$, which means that the operator $\mathcal{P}: G_R \to G_R$ is equicontinuous.

Let $D = \overline{\text{co}}\, \mathcal{P}(G_R)$, where $\overline{\text{co}}$ denotes the closure of the convex hull. Then it can be easily seen that the operator $\mathcal{P}: D \to D$ $(D \subset C(J,F))$ is equicontinuous.

Now, we show that $\mathcal{P}: D \to D$ is a condensing operator. For any $B \subset D$, by Lemma 2.3, there exists a countable set $B_0 = \{u_n\} \subset B$ such that

$$\mu_C(\mathcal{P}(B)) \leq 2(\mathcal{P}(B_0)). \tag{3.8}$$

From the equicontinuity of B, $B_0 \subset B$ is also equicontinuous. Consequently, from Lemma 2.4 and (F_2), we get

$$\mu(\mathcal{P}(B_0(s))) = \mu\left(u_0 + \int_0^s \tilde{\Psi}(s-\sigma,\sigma)\tilde{U}(\sigma)\mathbb{A}(0)u_0\, d\sigma\right)$$

$$+ \mu\left(\int_0^s \tilde{\Psi}(s-\sigma,\sigma)f(\sigma,u_n(\sigma))\, d\sigma\right)$$

$$+ \mu\left(\int_0^s \int_0^\sigma \tilde{\Psi}(s-\sigma,\sigma)\tilde{\Phi}(\sigma,t)f(t,u_n(t))\, dt\, d\sigma\right)$$

$$+ \mu\left(\sum_{0<s_k<s} \tilde{\Psi}(s_k-\sigma,\sigma)I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k}(N,u_n(t))\, dt\right)\right)$$

$$\leq 2C \int_0^s (s-\sigma)^{q-1}\mu(f(\sigma,u_n(\sigma)))\, d\sigma$$

$$+ 4C^2 \int_0^s \int_0^\sigma (s-\sigma)^{q-1}(\sigma-t)^{p-1}\mu(f(t,u_n(t)))\, dt\, d\sigma$$

$$+ 2C\left(\sum_{0<s_k<s}(s-\sigma)^{q-1}I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k}\mu(N,u_n(t))\, dt\right)\right)$$

$$\leq 2C \int_0^s (s-\sigma)^{q-1}L\mu(B_0(\sigma))\, d\sigma$$

$$+ 4C^2 \int_0^s \int_0^\sigma (s-\sigma)^{q-1}(\sigma-t)^{p-1}L\mu(B_0(t))\, dt\, d\sigma$$

$$+ 2C\left(\sum_{0<s_k<s}(s-\sigma)^{q-1}I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k}L_N\mu\bigl(B_0(t)\bigr)\,dt\right)\right)$$

$$\le 2CL\int_0^s(s-\sigma)^{q-1}\,d\sigma\,\mu_C(B)$$

$$+ 4C^2L\mathbf{B}(q,p)\int_0^s(s-\sigma)^{q+p-1}\,d\sigma\,\mu_C(B)$$

$$+ 2CL_N\left(\sum_{0<s_k<s}(s-\sigma)^{q-1}I_k\left(\int_{s_k-\tau_k}^{s_k-\theta_k}\mu_C(B)\,ds\right)\right)$$

$$\le 2CS^q\left[\frac{L}{q}+\frac{2CL\mathbf{B}(q,p)T^p}{q+p}+2L_NS\right]\mu_C(B).$$

From Lemma 2.5 and since $\mathcal{P}(B_0)\subset B$ is bounded and equicontinuous, we have

$$\mu_C\bigl(\mathcal{P}(B_0)\bigr)=\max_{s\in[0,S]}\mu\bigl(\mathcal{P}(B_0)(s)\bigr). \tag{3.9}$$

Therefore, by (2.5)–(2.6), we have

$$\mu_C\bigl(\mathcal{P}(B)\bigr)\le 4CS^q\left[\frac{L}{q}+\frac{2CL\mathbf{B}(q,p)T^p}{q+p}+2L_NS\right]\mu_C(B). \tag{3.10}$$

By combining (3.10), (3.4) and Definition 2.5, we know that $\mathcal{P}:G_R\to G_R$ is a k-set contractive operator. By Lemma 2.6, $\mathcal{P}$ has at least one fixed point $u\in G_R$. Therefore $\mathcal{P}$ is a mild solution of (1.1)–(1.3).

4 Example

Consider the following nonlinear reaction–diffusion equation with integral impulse condition:

$$^CD^q u(z,s)-b(s)\Delta u(z,s)=f\bigl(z,s,u(z,s)\bigr), \tag{4.1}$$

$$u(z,s)=0,\quad z\in\partial G,s\in J, \tag{4.2}$$

$$\Delta^*u(z,s_1)=I_1\left(\int_{s_1-\tau_1}^{s_1-\theta_1}N\bigl(z,t,u(z,t)\bigr)\,dt\right), \tag{4.3}$$

where $^CD^q$ is a CFD of order $0<q\le 1$, $b(s)$ is diffusion coefficient which is continuous on $J=[0,S]$, and $|b(s_2)-b(s_1)|\le C|s_2-s_1|^p$, $s_1,s_2\in J$, $0<p\le 1$, C is a positive constant independent of s_1 and s_2, Δ is Laplace operator; $S\subset\mathbb{R}^m$ is a bounded domain with a sufficiently smooth boundary ∂G; $f,N:G\times J\times\mathbb{R}\to\mathbb{R}$ are nonlinear functions.

Let $\|\cdot\|$ denote the L^2-norm on the Banach space $F=L^2(G)$. In F we define a linear operator by

$$\mathbb{A}(s)u=b(s)\Delta u \tag{4.4}$$

with the domain

$$D(A)=H^2(G)\cap H_0^1(G), \tag{4.5}$$

where $H^2(G)$ is the completion of $C^2(G)$ with respect to the norm

$$\|u\|_{H^2(G)} = \left(\int_G \sum_{|v| \leq 2} \left| D^v u(z) \right|^2 dz \right)^{\frac{1}{2}},$$

$H_0^1(G)$ is the completion of $C^1(G)$ with respect to the norm $\|u\|_{H^1(G)}$, and $C_0^1(G)$ is the set of all functions $u \in C^1(G)$ with compact support on the domain G. From [34], we know that $-\mathbb{A}(t)$ generates an analytic semigroup $e^{-s\mathbb{A}(t)}$ in E satisfying (H_3) and (H_4). Let $u(s) = u(\cdot, s)$, $f(s, u(s)) = f(\cdot, s, u(\cdot, s))$, $N(s, u(s)) = N(\cdot, s, u(\cdot, s))$. Then system (4.1)–(4.3) can be modified into system (1.1)–(1.3).

Let the function f satisfy the following condition:

(i) There is a bounded function $h_r(s)$ such that for any $s \in [0, S]$, $z \in G$, and $u \in L^2(G)$ satisfying $(\int_G |u(z)|^2 \, dz)^{\frac{1}{2}} \leq r$ for some $r > 0$,

$$\left(\int_G \left| f\big(s, u(z,s)\big) \right|^2 dz \right)^{\frac{1}{2}} \leq h_r(s).$$

Theorem 4.1 *Consider the nonlinear function $f(z, s, u(z,s)) = \frac{\cos(z, s, u(z,s))}{e^s}$ and the function $I_1 : \mathbb{R} \to \mathbb{R}$ defined by $I_1(z) = \cos z$. Then for the choices $\theta_1 = 0$, $\tau_1 = 1$, problem (4.1)–(4.3) has at least one mild solution.*

Proof: By the definition of f, condition (F_1) is clearly satisfied. Also I_1 and N are Lipschitz functions. Condition (3.4) is satisfied with $q = \frac{1}{2}$ and $0 \leq q_1 < \frac{1}{2}$, $h_r(s) = \frac{\sqrt{\text{meas}(G)}}{e^s}$, $\rho = 0$. So all the assumptions of Theorem 3.4 are satisfied. Hence the initial boundary value problem to the nonlinear reaction–diffusion equation with integral impulse condition has at least one mild solution due to Theorem 3.1.

5 Conclusions

The existence of solutions of non-autonomous fractional differential equations with integral impulse condition by using the measure of non-compactness has been discussed in this article. One can extend this work for impulsive non-autonomous fractional differential equations with integral impulse condition by using the fixed point theorems.

Author details
[1] Department of Mathematics, Government PG College, Bina, Sagar 470113, Madhya Pradesh, India. [2] Department of Mathematics, University of Delhi, Delhi 110007, India. [3] Post Graduate and Research Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore 641 029, Tamil Nadu, India. [4] Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawasir 11991, Saudi Arabia. [5] Department of Mathematics, Cankaya University, Ankara, Turkey. [6] Institute of Space Sciences, Magurele-Bucharest, Romania. [7] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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