J. Korean Math. Soc. **49** (2012), No. 1, pp. 113–138
http://dx.doi.org/10.4134/JKMS.2012.49.1.113

# EXISTENCE OF STRONG MILD SOLUTION OF THE NAVIER-STOKES EQUATIONS IN THE HALF SPACE WITH NONDECAYING INITIAL DATA

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Abstract. We construct a mild solutions of the Navier-Stokes equations in half spaces for nondecaying initial velocities. We also obtain the uniform bound of the velocity field and its derivatives.

## 1. Introduction

Consider the Navier-Stokes equations in the $n$-dimensional half space $\mathbb{R}^n_+$, $n \geq 2$;

$$(1.1)\qquad \begin{cases} \frac{\partial}{\partial t}u - \Delta u + \nabla p = -\mathrm{div}(u \otimes u) & \text{for } x \in \mathbb{R}^n_+, t > 0, \\ \nabla \cdot u = 0 & \text{for } x \in \mathbb{R}^n_+, t > 0, \\ u|_{t=0} = a & \text{for } x \in \mathbb{R}^n_+, \\ u|_{x_n=0} = 0 & \text{for } t > 0. \end{cases}$$

Here, $u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ represent the unknown velocity vector field and the pressure, respectively, of the fluid, and $a$ is a given initial velocity vector field, with $\nabla \cdot a = 0$ and $a|_{x_n=0} = 0$, which may not decay at infinity.

In this paper, we concentrate the existence and the uniqueness of solutions for nondecaying initial data $a$.

There are several results in the whole space $\mathbb{R}^n$ for such a problem. It is well-known that for $a \in L^\infty(\mathbb{R}^n)$ there is a unique local in time solution $u$ for the Navier-Stokes equations with

$$(1.2)\qquad p = \sum_{i,j=1}^{n} R_i R_j u_i u_j,$$

Received August 30, 2010.
2010 *Mathematics Subject Classification.* 35Q30, 76D05.
*Key words and phrases.* Navier-Stokes, nondecaying data, existence, uniqueness.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology 2009-0072196.

where $R_j = (-\Delta)^{-1/2}\frac{\partial}{\partial x_j}$ is the Riesz operator. Refer to [1] and [7] for survey and references. In $\mathbb{R}^2$, this solution can be extended globally in time [7].

In [10], it is shown that for linearly growing initial data of the form $a = b - Mx$, the Navier-Stokes equations admit a unique local in time solution in $L^p(\mathbb{R}^n)$, where $M$ is an $n \times n$ matrix with constant real entries and $b \in L^p(\mathbb{R}^n)$.

As mentioned in [11], for $u = g(t)$, then $(u, p)$ always solves (1.1) in $\mathbb{R}^n$ with $p = -g'(t) \cdot x$, no matter what function $g$ is. Therefore, solution $u$ with a constant initial data is not unique without assuming (1.2). In [11], it is shown that for initial data $a \in L^\infty(\mathbb{R}^n)$ with $\nabla \cdot a = 0$, if $(u, p)$ is a solution with

$$u \in L^\infty((0,T) \times \mathbb{R}^n), \quad p \in L^1_{\text{loc}}([0,T); \text{BMO}),$$

then $(u, \nabla p)$ is uniquely determined by $a$, and $\nabla p = \sum \nabla R_i R_j u_i u_j$ in the distributional sense. Recently, in [14], the BMO space for the pressure is replaced by the generalized Campanato space.

Up to now, in the half space case, the local in time existence of mild solution of the Navier-Stokes problems is provided in [16] with continuous, bounded initial data. In this article, we also work on the existence and the uniqueness of (strong) mild solutions for the Navier-Stokes equations with nondecaying initial data in $\mathbb{R}^n_+$. See also [9] for the rotating Navier-Stokes equations with nondecaying data.

Recently, we have an information of P. Maremonti [12]'s existence result on the Navier-Stokes equations with bounded and continuous initial data. However, our result is different in the sense that our initial data is assumed to be just bounded, and our estimate of the linear and nonlinear operator is from different technique.

We transform (1.1) into the abstract differential equations

$$\frac{\partial}{\partial t}u + Au = -\mathbb{P}\text{div}(u \otimes u), \tag{1.3}$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator and $\mathbb{P}$ is the Leray projection operator, which is represented as

$$\mathbb{P}f(x) \equiv f(x) + \nabla_x \int_{\mathbb{R}^n_+} \nabla_y \mathcal{G}(x,y) \cdot f(y)dy,$$

when $f_n|_{x_n=0} = 0$. Here,

$$\mathcal{G}(x,y) \equiv N(x-y) + N(x-y^*), \tag{1.4}$$

where $y^* \equiv (y_1, \ldots, y_{n-1}, -y_n)$, $N(x) \equiv \frac{1}{n(2-n)\omega_n}|x|^{2-n}$, if $n \geq 3$, and $N(x) \equiv \frac{1}{2\pi}\ln|x|$ if $n = 2$, and $\omega_n$ denotes the surface area of the unit sphere in $\mathbb{R}^n$.

Using the solution operator of the Stokes equations in $\mathbb{R}^n_+$, the solution of (1.3) is formally expressed in the integral form

$$u(t) = e^{-At}a - \int_0^t e^{-A(t-s)}\mathbb{P}\text{div}(u \otimes u)(s)ds. \tag{1.5}$$

We define a bilinear operator $B$ by

$$B(u,v)(t) \equiv -\int_0^t e^{-A(t-s)}\mathbb{P}\mathrm{div}(u\otimes v)(s)ds.$$

A mild solution of the Navier-Stokes equations is defined as following.

**Definition 1.1.** Let $a \in S'$. A measurable function $u \in L^\infty_{\mathrm{loc}}((0,T)\times\mathbb{R}^n_+)$ is called a mild solution of Navier-Stokes equations (1.1) on $(0,T)$, if $u$ satisfies

$$u(t) = e^{-tA}a + B(u,u)(t) \text{ on } (0,T)$$

in the sense of distribution. Here, $S'$ is the set of distributions.

Solonnikov [16] has expressed the solution operator of the Stokes equations in $\mathbb{R}^n_+$ in the integral form

$$e^{-At}a \equiv \int_{\mathbb{R}^n_+} G(x,y,t)\cdot a(y)dy,$$

where $G = (G_{ij})_{i,j=1,\dots,n}$ is defined by

$$\begin{aligned} G_{ij}(x,y,t) &\equiv \delta_{ij}(\Gamma(x-y,t) - \Gamma(x-y^*,t)) \\ &\quad + 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_i}N(x-z)\Gamma(z-y^*,t)dz. \end{aligned} \tag{1.6}$$

The function $\Gamma(x,t)$ is the $n$ dimensional Gaussian kernel defined by $\Gamma(x,t) \equiv \Gamma_t(x) \equiv \frac{1}{(2\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}$.

Stokes solutions in $\mathbb{R}^n_+$ have also been derived in [17]. This solution formula have been used in the $L^q$ framework, mainly for $1 < q < \infty$ (see [4, 6], see also [2, 3, 8, 15] for the $L^1$ or $L^\infty$ estimates of the Stokes flow or its gradient). The solution formula in [16] has been used mainly for $L^\infty$ framework (see [5, 13, 16]).

The goal of this paper is to construct strong mild solution of the Navier-Stokes equations in $\mathbb{R}^n_+$ when the initial data is in $L^\infty(\mathbb{R}^n_+)$. For the construction of the solutions, the following estimate plays important role: for $\mathcal{F}$ with 0 on $x_n = 0$,

$$\|e^{-At}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le C\|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}, \tag{1.7}$$

$$\|\nabla e^{-At}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le Ct^{-\frac{1}{2}}\|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}, \tag{1.8}$$

$$\|e^{-At}\mathbb{P}\mathrm{div}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le Ct^{-\frac{1}{2}}\|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}, \tag{1.9}$$

$$\|\nabla e^{-At}\mathbb{P}\mathrm{div}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le Ct^{-\frac{1}{2}}\|\nabla\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}. \tag{1.10}$$

Especially, the estimate (1.10) is used for the gradient of the velocity.

The followings are the main theorems.

**Theorem 1.2.** *Let $a \in L^\infty(\mathbb{R}^n_+)$ be the given vector field* $\mathrm{div} a = 0$ *and* $a|_{x_n=0} = 0$. *Then there is a positive time $T > 0$, and there is a mild solution $u \in$*

$L^\infty((0,T)\times\mathbb{R}^n_+)$ *which satisfies* (1.5). *Moreover, it holds that for* $0<t<T$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n_+)} + t^{\frac{1}{2}}\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n_+)} \le C\|a\|_{L^\infty(\mathbb{R}^n_+)}.$$

**Theorem 1.3.** *Suppose that* $u, v \in L^\infty_{\rm loc}((0,T)\times\mathbb{R}^n_+)$ *are mild solutions of the Navier-Stokes equations* (1.5) *with the same initial data. If both $u$ and $v$ satisfy the following conditions*

$$\sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(t)\|_{L^\infty} < \infty \text{ and } \sup_{0<t<T} t^{\frac{\alpha}{2}}\|v(t)\|_{L^\infty} < \infty \text{ for some } 0\le\alpha<1,$$

*then* $u\equiv v$ *on* $(0,T)\times\mathbb{R}^n_+$.

We compare our results with those in [16] with several points of view. In [16], to get the short time existence of mild solution, the estimates (1.7)–(1.9) are provided, but (1.10) is not.

In this paper as well as the boundedness of the velocity, we obtain a higher regularity, and hence our solution is not only mild solution but also strong solution for a short time. For the estimate of the gradient of the velocity, it is necessary to have the pointwise estimate of the kernels of the operator $\nabla e^{-tA}\mathbb{P}{\rm div}$. Unfortunately, $\nabla$ and $e^{-At}\mathbb{P}$ are not commutative, while $\nabla_{\bar x}$ and $e^{-At}\mathbb{P}$ are commutative. To obtain (1.10), we first treat the case $\nabla_{\bar x}e^{-At}\mathbb{P}{\rm div}\mathcal{F}$, then $\frac{\partial}{\partial x_n}(e^{-At}\mathbb{P}{\rm div}\mathcal{F})_n$, and finally $\frac{\partial}{\partial x_n}\overline{e^{-At}\mathbb{P}{\rm div}\,\mathcal{F}}$, separately. Here $\nabla_{\bar x} = (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}})$, $\bar f = (f_1,\dots,f_{n-1})$.

We also compare our result with those in [12], where the classical solution $u$ is constructed when the initial velocity $a$ is bounded and continuous and the pointwise estimate of pressure function is also given. For the pointwise estimate of the pressure, the continuity assumption of the given initial data has been necessary.

**Acknowledgement.** We want to express our thanks to TongKeun Chang for valuable discussions, especially for the estimate of the Stokes operator and bilinear operators.

## 2. Preliminaries

Let us consider the Stokes problems in $\mathbb{R}^n_+$, $n\ge 2$:

$$\begin{cases} {\rm div} u = 0 & \text{for } x\in\mathbb{R}^n_+, t>0,\\ u_t - \Delta u + \nabla p = {\rm div}\mathcal{F} & \text{for } x\in\mathbb{R}^n_+, t>0,\\ u|_{t=0} = a & \text{for } x\in\mathbb{R}^n_+,\\ u|_{x_n=0} = 0 & \text{for } t>0. \end{cases} \tag{2.11}$$

In [16] solution $u$ of the Stokes problems (2.11) is formally given by

$$u(x,t) = \int_{\mathbb{R}^n_+} G(x,y,t)\cdot a(y)dy + \int_0^t d\tau\int_{\mathbb{R}^n_+} G(x,y,t-\tau)\cdot \mathbb{P}{\rm div}\mathcal{F}(y,\tau)dy.$$

For later usage, we use the notations:

$$\Gamma(x,t) \equiv \Gamma_t(x) \equiv \bar{\Gamma}_t(\bar{x})\Gamma_t^n(x_n) \equiv \Big((2\pi t)^{-\frac{n-1}{2}} e^{-\frac{|\bar{x}|^2}{4t}}\Big)\Big((2\pi t)^{-\frac{1}{2}} e^{-\frac{|x_n|^2}{4t}}\Big),$$

$\|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty(\mathbb{R}^n_+)}$, $\quad \partial_{x_i} \equiv \dfrac{\partial}{\partial x_i}$ for short.

$\nabla_{\bar{x}} \equiv (\partial_{x_1}, \dots, \partial_{x_{n-1}})$, $\quad \bar{x} = (x_1, \dots, x_{n-1})$.

$1_{\mathbb{R}^n_+}$ denotes the characteristic function which is 1 on $\mathbb{R}^n_+$, and otherwise 0.

$\tau_a$ a translation operator defined by $\tau_a f(x) \equiv f(a-x)$.

*Remark* 2.1. Suppose that $\operatorname{div} f = 0$ with $f_n|_{x_n=0} = 0$. Then direct computation shows that

$$\operatorname{div} e^{-tA} f = 0.$$

**Proposition 2.1.** *If $k = 1, \dots, n-1$, then we have*

$$\frac{\partial}{\partial x_k} G_{ij}(x,y,t) = -\frac{\partial}{\partial y_k} G_{ij}(x,y,t). \tag{2.12}$$

*And we have*

$$\begin{aligned}\frac{\partial}{\partial x_n} G_{ij}(x,y,t) &= \frac{\partial}{\partial y_n} G_{ij}(x,y,t) - 2\delta_{ij}\frac{\partial}{\partial y_n}\Gamma(x-y,t)\\ &\quad - 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(\bar{x}-\bar{z}, x_n)\Gamma(\bar{z}-\bar{y}, y_n, t)d\bar{z}.\end{aligned}$$

*Proof.* Let us consider the case $k \neq n$. Differentiate $G_{ij}$ in terms of $x_k$ variable, then we have

$$\begin{aligned}\partial_{x_k} G_{ij}(x,y,t) &= \delta_{ij}\partial_{x_k}\big[\Gamma_t(x-y^*) - \Gamma_t(x-y^*)\big]\\ &\quad + 4(1-\delta_{jn})\partial_{x_k}\partial_{x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z)\Gamma_t(z-y^*)dz\\ &= \delta_{ij}\partial_{x_k}\big[\Gamma_t(x-y^*) - \Gamma_t(x-y^*)\big]\\ &\quad + 4(1-\delta_{jn})\partial_{x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z)\partial_{z_k}\Gamma_t(z-y^*)dz.\end{aligned}$$

The last term of the right hand side of the above identity comes from the fact that

$$\partial_{x_k}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z)g(z)dz = \int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z)\partial_{z_k} g(z)dz.$$

Note that $\partial_{y_k}\Gamma_t(x-y) = -\partial_{x_k}\Gamma_t(x-y)$ and $\partial_{y_k}\Gamma_t(x-y^*) = -\partial_{x_k}\Gamma(x-y^*)$ if $k \neq n$. Hence we obtain the identity (2.12).

Now consider the case $k = n$. Notice that

$$\begin{aligned}&\partial_{x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x-z)g(z)dz\\ &= \int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_n} N(x-z)g(z)dz\end{aligned}$$

$$+\frac{\delta_{in}}{2n}g(x)+\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},0)g(\bar{z},x_n)d\bar{z}$$

and

$$\begin{aligned}
&-\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}\partial_{x_n}N(x-z)g(z)dz\\
&=\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}\partial_{z_n}N(x-z)g(z)dz\\
&=-\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(x-z)\partial_{z_n}g(z)dz\\
&\quad+\frac{\delta_{in}}{2n}g(x)+\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},0)g(\bar{z},x_n)d\bar{z}\\
&\quad-\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},z_n)g(\bar{z},0)d\bar{z}.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&\partial_{x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(x-z)\Gamma(z-y^*,t)dz\\
&=\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(x-z)\partial_{z_n}\Gamma(z-y^*,t)dz\\
&\quad+\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},x_n)\Gamma(\bar{z}-\bar{y},y_n,t)d\bar{z}.
\end{aligned}$$

Note that $\partial_{y_n}\Gamma_t(x-y)=-\partial_{x_n}\Gamma_t(x-y)$ and $\partial_{y_n}\Gamma_t(x-y^*)=\partial_{x_n}\Gamma(x-y^*)$. Therefore if we differentiate $G_{ij}$ in terms of $x_n$ variables, then we have

$$\begin{aligned}
\partial_{x_n}G_{ij}(x,y,t)=&-\delta_{ij}\partial_{y_n}\big[(\Gamma(x-y,t)+\Gamma(x-y^*,t)\big]\\
&+4(1-\delta_{jn})\partial_{y_n}\partial_{x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(x-z)\Gamma(z-y^*,t)dz\\
&+4(1-\delta_{jn})\partial_{x_j}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},x_n)\Gamma(\bar{z}-\bar{y},y_n,t)d\bar{z}\\
=&\,\partial_{y_n}G_{ij}(x,y,t)-2\delta_{ij}\partial_{y_n}\Gamma(x-y,t)\\
&+4(1-\delta_{jn})\partial_{x_j}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar{x}-\bar{z},x_n)\Gamma(\bar{z}-\bar{y},y_n,t)d\bar{z}.
\end{aligned}$$ □

**Proposition 2.2.** *Let $x\in\mathbb{R}^n_+$. Let $f$ be a Hölder continuous function with the exponent $0<\alpha<1$,*

$$[f]_\alpha=\sup_{x,z}\frac{|f(x)-f(z)|}{|x-z|^\alpha}<\infty.$$

*Then, for $i,j\neq n$ or $i,j=n$, we have*

$$\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_i}N(x-z)f(z)dz$$

$$= -\frac{\delta_{ij}}{2n} f(x) + \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z) dz$$
$$+ \delta_{jn} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(\bar{x} - \bar{z}) f(\bar{z}, x_n) d\bar{z}.$$

*For $i = n, j \neq n$ or $j = n, i \neq n$, we have*

$$\frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(x-z) f(z) dz = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z) dz.$$

*Proof.* The proof is given in Appendix A. □

Now we define the Hardy space $\mathcal{H}^1$. Define $\mathcal{N}g(x) \equiv \sup_{s>0} |g * \Gamma_s(x)|$. Let $\mathcal{H}^1(\mathbb{R}^n)$ be the space of functions $g$ so that $\mathcal{N}g \in L^1(\mathbb{R}^n)$ with the norm $\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} \equiv \|\mathcal{N}g\|_{L^1(\mathbb{R}^n)}$. The dual space of $\mathcal{H}^1(\mathbb{R}^n)$ is defined by BMO$(\mathbb{R}^n)$. Define $\mathcal{H}^1(\mathbb{R}^n_+)$ be the space of functions $f$ so that there is $\tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)$ with $\tilde{f}|_{\mathbb{R}^n_+} = f$ with the norm $\|f\|_{\mathcal{H}^1(\mathbb{R}^n_+)} \equiv \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)} : \tilde{f}|_{\mathbb{R}^n_+} = f\}$, and define BMO$(\mathbb{R}^n_+)$ be the space of functions $f$ so that there is $\tilde{f} \in$ BMO$(\mathbb{R}^n)$ with $\tilde{f}|_{\mathbb{R}^n_+} = f$ with the norm $\|f\|_{\mathrm{BMO}(\mathbb{R}^n_+)} = \inf\{\|\tilde{f}\|_{\mathrm{BMO}(\mathbb{R}^n)} : \tilde{f}|_{\mathbb{R}^n_+} = f\}$. It is well known fact that $\nabla\Gamma_t \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover, in [8] it is shown that $1_{\mathbb{R}^n_+} \partial_{x_k} \Gamma_t \in \mathcal{H}^1(\mathbb{R}^n)$ for $k \neq n$.

**Proposition 2.3.** *Fix $a \in \mathbb{R}^n$. Then $\nabla(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$ with*

$$\|\nabla(\tau_a \Gamma_t)\|_{\mathcal{H}^1} \leq C t^{-\frac{1}{2}}.$$

The above proposition is well known and we omit the proof. Instead, we give the proof that $1_{\mathbb{R}^n_+} \nabla_{\bar{x}}(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$.

**Lemma 2.4.** *Fix $a \in \mathbb{R}^n$. Then*

$$1_{\mathbb{R}^n_+} \nabla_{\bar{x}}(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$$

*with*

$$\|1_{\mathbb{R}^n_+} \nabla_{\bar{x}}(\tau_a \Gamma_t)\|_{\mathcal{H}^1} \leq C t^{-\frac{1}{2}}.$$

*Proof.* The proof is given in Appendix B, where we follow the idea in [8]. □

Let $a \in \mathbb{R}^n_+$. Then $\tau_{a^*}\Gamma_t(x) = \Gamma_t(a^* - x) = \bar{\Gamma}_t(\bar{a} - \bar{x}) \Gamma^n_t(a_n + x_n)$. We observe that

$$\begin{aligned}\int_{-\infty}^{x_n} \Gamma^n_t(a_n + x_n - y_n) \Gamma^n_s(y_n) dy_n &\leq e^{-\frac{a_n^2}{4t}} \int_{-\infty}^{x_n} \Gamma^n_t(x_n - y_n) \Gamma^n_s(y_n) dy_n \\ &\leq e^{-\frac{a_n^2}{4t}} \int_{-\infty}^{\infty} \Gamma^n_t(x_n - y_n) \Gamma^n_s(y_n) dy_n \\ &= e^{-\frac{a_n^2}{4t}} \Gamma^n_{s+t}(x_n).\end{aligned}$$

Then, we also have the following estimate.

**Corollary 2.5.** *Fix $a = (a_1, \dots, a_n) \in \mathbb{R}^n_+$. Then we have*

$$\|1_{\mathbb{R}^n_+} \nabla_{\bar{x}} (\tau_{a^*} \Gamma_t)\|_{\mathcal{H}^1} \le C t^{-\frac{1}{2}} e^{-\frac{a_n^2}{4t}}.$$

**Lemma 2.6.** *Fix $a = (\bar{a}, a_n), b = (\bar{b}, a_n) \in \mathbb{R}^n_+$. Let $0 < \alpha < 1$. Then we have*

$$\left\|1_{\mathbb{R}^n_+} \left[\tau_{a^*} \Gamma_t - \tau_{b^*} \Gamma_t\right]\right\|_{\mathcal{H}^1} \le C t^{-\alpha/2} e^{-\frac{a_n^2}{4t}} |\bar{a} - \bar{b}|^\alpha.$$

*Proof.* The proof is given in Appendix C. □

*Remark* 2.2. In Lemma 2.4, Corollary 2.5, and Lemma 2.6, the results hold true even if we replace $1_{\mathbb{R}^n_+}$ by $\mathbb{R}^{n-1} \times K = \{x_n \in K\}$ for any one dimensional open set $K \subset \mathbb{R}$. Moreover, making use of the properties of the Gaussian kernels, we have

$$\|1_{\{x_n \in K\}} \nabla_x^k \nabla_{\bar{x}} (\tau_a \Gamma_t)\|_{\mathcal{H}^1} \le C t^{-\frac{k+1}{2}},$$

$$\|1_{\{x_n \in K\}} \nabla_x^k \nabla_{\bar{x}} (\tau_{a^*} \Gamma_t)\|_{\mathcal{H}^1} \le C t^{-\frac{k+1}{2}} e^{-\frac{a_n^2}{4t}},$$

and

$$\left\|1_{\{x_n \in K\}} \nabla_x^k \left[\tau_{a^*} \Gamma_t - \tau_{b^*} \Gamma_t\right]\right\|_{\mathcal{H}^1} \le C t^{-\frac{k+\alpha}{2}} e^{-\frac{a_n^2}{4t}} |\bar{a} - \bar{b}|^\alpha.$$

**Lemma 2.7.** *Let $j = 1, \dots, n-1$ and $i = 1, \dots, n$. Then we have*

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \Big| \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial z_i \partial z_j} N(x-z) f(z,y) dz \Big| dy \\ &\le C \sup_{x \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} |f(x,y)| dy \\ &\quad + C x_n^\alpha \sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n_+} |f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)| dy}{|\bar{x} - \bar{z}|^\alpha}. \end{aligned}$$

*Proof.* The proof is given in Appendix D. □

## 3. Estimate of $\nabla e^{-tA} \mathbb{P} \nabla \cdot \mathcal{F}$.

The following is well known (see Solonnikov [16] and Shimizu [15]).

**Proposition 3.1.**

$$\|\nabla^k e^{-At} f\|_{L^\infty(\mathbb{R}^n_+)} \le C t^{-\frac{k}{2}} \|f\|_{L^\infty(\mathbb{R}^n_+)}, \tag{3.13}$$

$$\|e^{-At} \mathbb{P} \mathrm{div} \mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le C t^{-\frac{1}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}.$$

In this section we would like to show that

$$\|\nabla e^{-At} \mathbb{P} \mathrm{div} \mathcal{F}\|_\infty \le C t^{-\frac{1}{2}} \|\nabla \mathcal{F}\|_\infty. \tag{3.14}$$

Recall Proposition 2.1 that

$$\nabla_{\bar{x}} e^{-tA} \mathbb{P} \nabla \cdot \mathcal{F} = e^{-tA} \nabla_{\bar{x}} \mathbb{P} \nabla \cdot \mathcal{F}.$$

We have the identity

$$\nabla_{\bar{x}} \mathbb{P} \nabla \cdot \mathcal{F} = \mathbb{P} \nabla \cdot [\nabla_{\bar{x}} \mathcal{F}],$$

since for $k \neq n$

$$\partial_{x_k}(\mathbb{P}f)_j(x) = \partial_{x_k} f_j(x) + \partial_{x_k}\partial_{x_j}\int_{\mathbb{R}^n_+} \nabla_y \mathcal{G}(x,y)\cdot f(y)dy$$
$$= \partial_{x_k} f_j(x) + \partial_{x_j}\int_{\mathbb{R}^n_+} \nabla_y \mathcal{G}(x,y)\cdot \partial_{y_k} f(y)dy.$$

Applying the known result (3.14) of Proposition 3.1, we have the following estimate.

**Lemma 3.2.**

$$\|\nabla_{\bar{x}} e^{-At}\mathbb{P}\mathrm{div}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} = \|e^{-At}\mathbb{P}\mathrm{div}\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \le ct^{-\frac{1}{2}}\|\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}.$$

From the divergence free property of $e^{-At}\mathbb{P}\mathrm{div}\mathcal{F}$, we have that

$$\partial_{x_n}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_n = -\sum_{j=1}^{n-1}\partial_{x_j}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_j.$$

Hence we have the following estimate.

**Lemma 3.3.**

$$\|\partial_{x_n}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_n\|_{L^\infty(\mathbb{R}^n_+)} \le ct^{-\frac{1}{2}}\|\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}.$$

Now let us consider the case $i \neq n$. Denote by $H = (H_{ijk})^n_{i,j,k=1}$ the kernel tensors of the operator $e^{-tA}\mathbb{P}\mathrm{div}$. Then

$$[e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}]_i(x) = \int_{\mathbb{R}^n_+} H_{ijk}(x,y,t)F_{jk}(y)dy.$$

Direct computation shows that

$$H_{ijk}(x,y,t) = -\partial_{y_k}G_{ij}(x,y,t) + \sum_{l=1}^{n}\partial_{y_j}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz.$$

Moreover, since $G_{in} = 0$ if $i \neq n$, we have

(3.15)
$$H_{ijk}(x,y,t) = -\partial_{y_k}G_{ij}(x,y,t) + \sum_{l=1}^{n-1}\partial_{y_j}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz.$$

From Proposition 2.1, we have

$$\partial_{x_n}G_{il}(x,z,t) = \partial_{z_n}G_{il}(x,z,t) - 2\delta_{il}\partial_{z_n}\Gamma(x-z,t)$$
$$- 4(1-\delta_{ln})\partial_{x_l}\int_{\mathbb{R}^{n-1}}(\partial_{x_i}N)(\bar{x}-\bar{w},x_n)\Gamma(\bar{w}-\bar{z},z_n,t)d\bar{w}.$$

Apply Proposition 2.2 for $i, l \neq n$, then we have

$$G_{il}(x,z,t) = \delta_{il}(\Gamma(x-z,t) - \Gamma(x-z^*,t)) - \frac{2\delta_{il}}{n}\Gamma(x-z^*,t)$$

$$+ 4\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_l} N(x-w)\Gamma(w-z^*,t)dw.$$

Hence we have that for $i, l \neq n$,

$$\begin{aligned}
&\partial_{x_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz\\
&= \int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}\partial_{x_n}G_{il}(x,z,t)dz\\
&= 4\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_n}\partial_{z_l}\Big[\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_l}N(x-w)\Gamma(w-z^*,t)dw\Big]dz\\
&\quad - \delta_{il}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}\partial_{z_n}\big[\Gamma(x-z,t) + (1+\frac{2}{n})\Gamma(x-z^*,t)\big]dz\\
&\quad - 4\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}\Big[\partial_{x_l}\int_{\mathbb{R}^{n-1}} \partial_{x_i}N(\bar{x}-\bar{w},x_n)\Gamma(\bar{w}-\bar{z},z_n,t)d\bar{w}\Big]dz.
\end{aligned}$$

Note that $\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_n}f(z)dz = \partial_{y_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}^-(z,y)f(z)dz$, where

$$\mathcal{G}^-(z,y) \equiv N(z-y) - N(z-y^*),$$

since $\partial_{x_n}[\Gamma(x-z,t) - \Gamma(x-z^*,t)] = -\partial_{z_n}[\Gamma(x-z,t) + \Gamma(x-z^*,t)]$. Thus, we obtain that for $i, l \neq n$,

(3.16)

$$\begin{aligned}
&\partial_{x_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz\\
&= 4\partial_{y_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}^-(z,y)\Big[\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_l}N(x-w)\partial_{w_l}-(w-z^*,t)dw\Big]dz\\
&\quad - \delta_{il}\partial_{y_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}^-(z,y)\partial_{z_l}[\Gamma(x-z,t) + (1+\frac{2}{n})\Gamma(x-z^*,t)]dz\\
&\quad - 4\partial_{y_l}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}(z,y)\Big[\partial_{x_l}\int_{\mathbb{R}^{n-1}} \partial_{x_i}N(\bar{x}-\bar{w},x_n)\Gamma(\bar{w}-\bar{z},z_n,t)d\bar{w}\Big]dz.
\end{aligned}$$

Combining the above estimates (3.15) and (3.16), we obtain

$$\begin{aligned}
\partial_{x_n}H_{ijk}(x,y,t) = &-\partial_{x_n}\partial_{y_k}G_{ij}(x,y,t)\\
&+\sum_{l=1}^{n-1}\partial_{y_j}(\tilde{H}^1_{ijk,l}(x,z,t) + \tilde{H}^2_{ijk,l}(x,y,t) + \tilde{H}^3_{ijk,l}(x,z,t)),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{H}^1_{ijk,l}(x,y,t)\\
&\equiv -\delta_{il}\partial_{y_n}\int_{\mathbb{R}^n_+} \partial_{y_k}\mathcal{G}^-(z,y)\partial_{z_l}\Big[\Gamma(x-z,t) + (1+\frac{2}{n})\Gamma(x-z^*,t)\Big]dz,
\end{aligned}$$

$$\begin{aligned}
&\tilde H^2_{ijk,l}(x,y,t)\\
\equiv\ & -4\partial_{y_n}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}^-(z,y)\partial_{z_l}\Big[\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}\partial_{x_l}N(x-w)\Gamma(w-z^*,t)dw\Big]dz,\\
&\tilde H^3_{ijk,l}(x,y,t)\\
\equiv\ & -4\partial_{y_l}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}(z,y)\Big[\partial_{x_l}\int_{\mathbb{R}^{n-1}}\partial_{x_i}N(\bar x-\bar w,x_n)\Gamma(\bar w-\bar z,z_n,t)d\bar w\Big]dz.
\end{aligned}$$

If $\mathcal{F}=(F_{jk})=0$ on $x_n=0$, taking integrations by parts, $\partial_{x_n}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_i$ can be written by

$$\begin{aligned}
&\partial_{x_n}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_i(x,t)\\
=\ &\int_{\mathbb{R}^n_+}\partial_{x_n}H_{ijk}(x,y,t)F_{jk}(y)dy\\
=\ &\int_{\mathbb{R}^n_+}\partial_{x_n}G_{ij}(x,y,t)\partial_{y_k}F_{jk}(y)dy\\
&-\sum_{l=1}^{n-1}\int_{\mathbb{R}^n_+}(\tilde H^1_{ijk,l}+\tilde H^2_{ijk,l}+\tilde H^3_{ijk,l})(x,y,t)\partial_{y_j}F_{jk}(y)dy\\
=\ &\partial_{x_n}(e^{-tA}\mathrm{div}\mathcal{F})_i(x)-\sum_{l=1}^{n-1}\int_{\mathbb{R}^n_+}(\tilde H^1_{ijk,l}+\tilde H^2_{ijk,l}+\tilde H^3_{ijk,l})(x,y,t)\partial_{y_j}F_{jk}(y)dy.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big\|\partial_{x_n}(e^{-At}\mathbb{P}\mathrm{div}\mathcal{F})_i\Big\|_{L^\infty}\\
&\le C\|\partial_{x_n}e^{-At}\mathrm{div}\mathcal{F}\|_{L^\infty}+C\max_{1\le j,k\le n}\int_{\mathbb{R}^n_+}\big(|\tilde H^1_{ijk}(x,y,t)|+|\tilde H^2_{ijk}(x,y,t)|\\
&\qquad+|\tilde H^3_{ijk}(x,y,t)|\big)dy\Big\|\partial_{y_k}F_{jk}\Big\|_{L^\infty}.
\end{aligned}$$

From the known result (3.13) of Proposition 3.1, we have

$$\|\partial_{x_n}e^{-At}\mathrm{div}\mathcal{F}\|_{L^\infty}\le Ct^{-1/2}\|\mathrm{div}\mathcal{F}\|_{L^\infty}.$$

Hence, we have only to estimate

$$\int_{\mathbb{R}^n_+}(|\tilde H^1_{ijk}(x,y,t)|+|\tilde H^2_{ijk}(x,y,t)|+|\tilde H^3_{ijk}(x,y,t)|)dy$$

for $i,l\neq n$. For the estimation, we will make use of the Hardy space theory.

First let us estimate $\int_{\mathbb{R}^n_+}|\tilde H^1_{ijk}(x,y,t)|dy$. Note that $\tilde H^1_{ijk}(x,y,t)$ is the linear sum of the terms

$$\begin{aligned}
&\partial_{y_n}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\mathbb{R}^n_+}\partial_{z_l}\Gamma(x-z,t)dz,\\
&\partial_{y_n}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\mathbb{R}^n_-}\partial_{z_l}\Gamma(x-z,t)dz,
\end{aligned}$$

$$\partial_{y_n}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\mathbb{R}^n_+}\partial_{z_l}\Gamma(x-z^*,t)dz,$$

and

$$\partial_{y_n}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\mathbb{R}^n_-}\partial_{z_l}\Gamma(x-z^*,t)dz.$$

Recall the well known result that $1_{\{z_n>0\}}\partial_{z_l}\Gamma(x-z,t)$, $1_{\{z_n<0\}}\partial_{z_l}\Gamma(x-z,t)$ are in Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for any fixed $x\in\mathbb{R}^n$. By the boundedness of the Calderon-Zygmund type transforms in Hardy spaces, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\tilde{H}^1_{ijk}(x,y,t)|dy &\le C\Big\|1_{\mathbb{R}^n_+}\partial_{z_i}\Gamma(x-\cdot,t)\Big\|_{\mathcal{H}^1}+C\Big\|1_{\mathbb{R}^n_-}\partial_{z_i}\Gamma(x^*-\cdot,t)\Big\|_{\mathcal{H}^1}\\ &\quad+C\Big\|1_{\mathbb{R}^n_+}\partial_{z_i}\Gamma(x^*-\cdot,t)\Big\|_{\mathcal{H}^1}+C\Big\|1_{\mathbb{R}^n_-}\partial_{z_i}\Gamma(x-\cdot,t)\Big\|_{\mathcal{H}^1}\\ &\le Ct^{-1/2}.\end{aligned}$$

We rewrite $\tilde{H}^2_{ijk,l}$ by

$$\begin{aligned}&\tilde{H}^2_{ijk,l}(x,y,t)\\ &=\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial^2}{\partial x_n\partial x_i}N(x-w)\Big[\partial_{y_l}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}(z,y)\partial_{w_l}\Gamma(w-z^*,t)dz\Big]dw.\end{aligned}$$

Set

$$J_{kl}(w,y,t)\equiv\partial_{y_l}\int_{\mathbb{R}^n_+}\partial_{y_k}\mathcal{G}(z,y)\partial_{w_l}\Gamma(w-z^*,t)dz.$$

By Lemma 2.7 we have

$$\begin{aligned}&\int_{\mathbb{R}^n_+}|\tilde{H}^2_{ijk,l}(x,y,t)|dy \qquad (3.17)\\ &\le C\sup_{y\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|J_{kl}(w,y,t)|dy\\ &\quad+Cx_n^\alpha\sup_{x_n/2<z_n<x_n}\sup_{\bar{w}\in\mathbb{R}^{n-1}}\frac{\int_{\mathbb{R}^n_+}|J_{kl}(w,y,t)-J_{kl}(\bar{x},w_n,y,t)|dy}{|\bar{x}-\bar{w}|^\alpha}.\end{aligned}$$

Note that

$$\begin{aligned}J_{kl}(w,y,t)&=\partial_{y_l}\int_{\mathbb{R}^n_+}\partial_{y_k}[N(z-y)+N(z-y^*)]\partial_{w_l}\Gamma(w-z^*,t)dz\\ &=\partial_{y_l}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\{z_n>0\}}\partial_{w_l}\Gamma(w-z^*,t)dz\\ &\quad+\partial_{y_l}\int_{\mathbb{R}^n}\partial_{y_k}N(z-y)1_{\{z_n<0\}}\partial_{w_l}\Gamma(w-z,t)dz.\end{aligned}$$

Recall the well known result that $1_{\mathbb{R}^n_+}\partial_{w_l}\Gamma(w-z^*,t)$, $1_{\mathbb{R}^n_-}\partial_{w_l}\Gamma(w-z,t)$ are in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for any fixed $w\in\mathbb{R}^n$. Since Calderon-Zygmund type transforms are bounded in Hardy spaces, we obtain that

$$\int_{\mathbb{R}^n_+} |J_{kl}(w,y,t)|dy$$

$$\leq C\Big\|1_{\mathbb{R}^n_+}\partial_{w_l}\Gamma(w^*-\cdot,t)\Big\|_{\mathcal{H}^1} + \Big\|1_{\mathbb{R}^n_-}\partial_{w_l}\Gamma(w-\cdot,t)\Big\|_{\mathcal{H}^1} \leq Ct^{-1/2}. \tag{3.18}$$

Note that

$$J_{kl}(w,y,t) - J_{kl}(\bar{x},w_n,y,t)$$

$$= \partial_{y_l}\int_{\mathbb{R}^n_+} \partial_{y_k}[N(z-y)+N(z-y^*)]\partial_{w_l}[\Gamma(w-z^*,t)-\Gamma(\bar{x}-\bar{z},w_n+z_n,t)]dz.$$

Recall Lemma 2.6 that for any $a=(\bar{w},w_n)$, $b=(\bar{x},w_n)\in\mathbb{R}^n_+$,

$$1_{\mathbb{R}^n_+}\Big[\partial_{z_l}(\tau_{a^*}\Gamma_t)-\partial_{z_l}(\tau_{b^*}\Gamma_t)\Big]\in\mathcal{H}^1(\mathbb{R}^n),$$

of which norm is bounded by $Ct^{-(1+\alpha)/2}e^{-\frac{w_n^2}{4t}}|\bar{w}-\bar{x}|^\alpha$ for $0<\alpha<1$. Hence by the same reasoning for $J_{kl}$, we conclude that

$$\int_{\mathbb{R}^n_+} |J_{kl}(w,y,t)-J_{kl}(\bar{x},w_n,y,t)|dy \leq Ct^{-(1+\alpha)/2}e^{-\frac{w_n^2}{4t}}|\bar{w}-\bar{x}|^\alpha. \tag{3.19}$$

Apply the estimates (3.18) and (3.19) to (3.17) we have

$$\int_{\mathbb{R}^n_+} |\tilde{H}^2_{ijk,l}(x,y,t)|dy \leq Ct^{-1/2} + Cx_n^\alpha t^{-(1+\alpha)/2}e^{-\frac{x_n^2}{4t}} \leq Ct^{-1/2}.$$

Divide the domain of integration into two parts that

$$\begin{aligned}\tilde{H}^3_{ijk,l}(x,y,t) &= p.v.\int_{|\bar{x}-\bar{w}|\leq\sqrt{t}} \partial_{x_i}N(\bar{x}-\bar{w},x_n)J_{kl}((\bar{w},0),y,t)d\bar{w}\\ &\quad + p.v.\int_{|\bar{x}-\bar{w}|>\sqrt{t}} \partial_{x_i}N(\bar{x}-\bar{w},x_n)J_{kl}((\bar{w},0),y,t)d\bar{w}\\ &\equiv I+II.\end{aligned}$$

By integrations by parts

$$\begin{aligned}I = &-p.v.\int_{|\bar{x}-\bar{w}|\leq\sqrt{t}} N(\bar{x}-\bar{w},x_n)\partial_{y_i}J_{kl}((\bar{w},0),y,t)d\bar{w}\\ &+ J_{kl}((\bar{x},0),y,t)\lim_{\epsilon\to0}\int_{|\bar{x}-\bar{w}|=\epsilon}\frac{x_i-y_i}{|(\bar{x}-\bar{w})^2+x_n^2|^{(n-1)/2}}dS_{\bar{w}}\\ &-\int_{|\bar{x}-\bar{w}|=\sqrt{t}}\frac{x_i-y_i}{|\bar{x}-\bar{w}|^{n-1}}J_{kl}((\bar{w},0),y,t)dS_{\bar{w}}.\end{aligned}$$

By the anti-symmetry, the second term is zero, hence we have

$$I = -p.v.\int_{|\bar{x}-\bar{w}|\leq\sqrt{t}} N(\bar{x}-\bar{w},x_n)\partial_{y_i}J_{kl}((\bar{w},0),y,t)d\bar{w}$$

$$- \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{x_i - y_i}{|(\bar{x}-\bar{w})^2 + x_n^2|^{(n-1)/2}} J_{kl}((\bar{w},0),y,t) dS_{\bar{w}}$$
$$\equiv I_1 + I_2.$$

Recall the fact that $\partial_{y_i} J_{kl}$ is in Hardy space whose norm is bounded by $Ct^{-1}$. If we integrate $I$ over $\mathbb{R}^n_+$ in terms of $y$ variables, then we have

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |I_1(x,y,t)|dy &\le \int_{|\bar{x}-\bar{w}|\le\sqrt{t}} \frac{d\bar{w}}{|\bar{x}-\bar{w}|^{n-2}} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_+} |\partial_{y_i} J_{kl}((\bar{w},0),y,t)|dy \\
&\le C\sqrt{t} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_+} |\partial_{y_i} J_{kl}((\bar{w},0),y,t)|dy \le Ct^{-1/2},
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |I_2(x,y,t)|dy &\le \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{dS_{\bar{w}}}{|\bar{x}-\bar{w}|^{n-2}} \Big( \sup_{\bar{w}\in\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_+} |J_{kl}((\bar{w},0),y,t)|dy \Big) \\
&\le C \sup_{\bar{w}\in\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_+} |J_{kl}((\bar{w},0),y,t)|dy \le Ct^{-1/2}.
\end{aligned}$$

Set

$$\tilde{J}_{kl}(\bar{w},y,t) \equiv \partial_{y_l} \int_{\mathbb{R}^n_+} \partial_{y_k} \mathcal{G}(z,y)\Gamma((\bar{w}-\bar{z},0),t)dz.$$

Note that $J_{kl}((\bar{w},0),y,t) = \partial_{w_l} \tilde{J}_{kl}(\bar{w},y,t)$. Since $\partial_{w_l} \tilde{J}_{kl}(\bar{x},y,t) = 0$, by integration by parts, we obtain that

$$\begin{aligned}
II = p.v.&\int_{|\bar{x}-\bar{w}|>\sqrt{t}} \partial_{x_i}\partial_{x_l} N(\bar{x}-\bar{w},x_n)[\tilde{J}_{kl}(\bar{w},y,t) - \tilde{J}_{kl}(\bar{x},y,t)]d\bar{w} \\
&- \lim_{R\to\infty} \int_{|\bar{x}-\bar{w}|=R} \frac{x_i-y_i}{||\bar{x}-\bar{w}|^2 + x_n^2|^{(n-1)/2}} \frac{y_l - x_l}{|\bar{x}-\bar{y}|} [\tilde{J}_{kl}(\bar{w},y,t) - \tilde{J}_{kl}(\bar{x},y,t)]dS_{\bar{w}} \\
&+ \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{x_i-y_i}{||\bar{x}-\bar{w}|^2 + x_n^2|^{(n-1)/2}} \frac{y_l - x_l}{|\bar{x}-\bar{y}|} [\tilde{J}_{kl}(\bar{w},y,t) - \tilde{J}_{kl}(\bar{x},y,t)]dS_{\bar{w}}.
\end{aligned}$$

Since

$$\lim_{R\to\infty} \int_{|\bar{x}-\bar{w}|=R} \frac{R}{||\bar{x}-\bar{w}|^2 + x_n^2|^n} dS_{\bar{w}} \le \lim_{R\to\infty} \frac{1}{R^{n-1}} \int_{|\bar{x}-\bar{w}|=R} dS_{\bar{w}} = 0,$$

we have

$$\begin{aligned}
II = p.v.&\int_{|\bar{x}-\bar{w}|\ge\sqrt{t}} \partial_{x_i}\partial_{x_l} N(\bar{x}-\bar{w},x_n)[\tilde{J}_{kl}(\bar{w},y,t) - \tilde{J}_{kl}(\bar{x},y,t)]d\bar{w} \\
&+ \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{(x_i-y_i)(x_l-y_l)}{|(\bar{x}-\bar{w})^2 + x_n^2|^{(n+1)/2}} [\tilde{J}_{kl}(\bar{w},y,t) - \tilde{J}_{kl}(\bar{x},y,t)]dS_{\bar{w}} \\
\equiv{}& II_1 + II_2.
\end{aligned}$$

If we integrate $II$ over $\mathbb{R}^n_+$ in terms of $y$ variables, then we have

$$\int_{\mathbb{R}^n_+} |II_1(x,y,t)|dy$$

$$
\begin{aligned}
&\leq \int_{|\bar{x}-\bar{w}|\geq\sqrt{t}} \frac{d\bar{w}}{|\bar{x}-\bar{w}|^{n-\alpha}} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n_+} |\tilde{J}_{kl}(\bar{w},y,t)-\tilde{J}_{kl}(\bar{x},y,t)|dy}{|\bar{x}-\bar{w}|^{\alpha}}\\
&\leq C(\sqrt{t}+x_n)^{-1+\alpha} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n_+} |\tilde{J}_{kl}(\bar{w},y,t)-\tilde{J}_{kl}(\bar{x},y,t)|dy}{|\bar{x}-\bar{w}|^{\alpha}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} |II_2(x,y,t)|dy\\
&\leq \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{dS_{\bar{w}}}{|\bar{x}-\bar{w}|^{n-1-\alpha}} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n_+} |\tilde{J}_{kl}(\bar{w},y,t)-\tilde{J}_{kl}(\bar{x},y,t)|dy}{|x-\bar{w}|^{\alpha}}\\
&\leq C(\sqrt{t}+x_n)^{-1+\alpha} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^n_+} |\tilde{J}_{kl}(\bar{w},y,t)-\tilde{J}_{kl}(\bar{x},y,t)|dy}{|x-\bar{w}|^{\alpha}}.
\end{aligned}
$$

Now recall that $\Gamma(\bar{w}-\bar{z},z_n,t)-\Gamma(\bar{x}-\bar{z},z_n,t)\in\mathcal{H}^1(\mathbb{R}^n_+)$ for any fixed $\bar{w},\bar{x}\in\mathbb{R}^{n-1}$ whose norm is bounded by $Ct^{-\alpha/2}$ and $\tilde{J}_{kl}$ is linear sum of the terms

$$
\begin{aligned}
&\partial_{y_n}\int_{\mathbb{R}^n} \partial_{y_k}N(z-y)1_{\mathbb{R}^n_+}\partial_{z_l}[\Gamma(\bar{w}-\bar{z},z_n,t)-\Gamma(\bar{x}-\bar{z},z_n,t)]dz,\\
&\partial_{y_n}\int_{\mathbb{R}^n} \partial_{y_k}N(z-y)1_{\mathbb{R}^n_-}[\Gamma(\bar{w}-\bar{z},z_n,t)-\Gamma(\bar{x}-\bar{z},z_n,t)]dz.
\end{aligned}
$$

By the boundedness of Calderon-Zygmund type transforms in Hardy spaces, we obtain that

$$
\int_{\mathbb{R}^n_+} |\tilde{J}_{kl}(\bar{w},y,t)-\tilde{J}_{kl}(\bar{x},y,t)|dy \leq Ct^{-\alpha/2}|\bar{x}-\bar{w}|^{\alpha}, \qquad 0<\alpha<1.
$$

Hence we have

$$
\int_{\mathbb{R}^n_+} |II(x,y,t)|dy \leq Ct^{-1/2}.
$$

Therefore we conclude that:

**Lemma 3.4.**

$$
\|\partial_{x_n}\overline{e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-1/2}\|\partial_{x_n}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}.
$$

Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have the following estimate for $\nabla e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}$.

**Theorem 3.5.**

$$
\|\nabla e^{-tA}\mathbb{P}\mathrm{div}\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-1/2}\|\nabla\mathcal{F}\|_{L^\infty(\mathbb{R}^n_+)}.
$$

## 4. Proof of Theorem 1.2

Set $u_1 = e^{-tA}a$ and $u_0 = 0$. For given $u_{n-1}$, define $u_n$ recursively by

$$u_n = e^{-tA}a + B(u_{n-1}, u_{n-1}).$$

In [16], it is well known that

$$\|e^{-tA}a\|_\infty \le C_0\|a\|_\infty,$$

and from Proposition 3.1 we have the estimate

$$\|e^{-(t-s)A}\mathbb{P}\mathrm{div}(u_{n-1}\otimes u_{n-1})\|_\infty \le C_1(t-s)^{-\frac{1}{2}}\|(u_{n-1}\otimes u_{n-1})\|_\infty.$$

Hence we have the following estimate for $\|u_n\|_\infty$.

$$\|u_n\|_\infty \le C_0\|u_0\|_\infty + C_1\int_0^t (t-s)^{-\frac{1}{2}}\|u_{n-1}(s)\|_\infty^2 ds. \tag{4.20}$$

Take large $M \ge 2C_0\|a\|_{L^\infty}$. Suppose $\sup_{0<t<T}\|u_{n-1}(t)\|_\infty \le M$, then from (4.20) we have

$$\begin{aligned}\|u_n(t)\|_\infty &\le C_0\|u_0\|_\infty + C_1\Big[\int_0^t (t-s)^{-\frac{1}{2}}ds\Big]\big[\sup_{0<s<t}\|u_{n-1}(t)\|_\infty\big]^2\\ &\le \frac{M}{2} + C_1C_2\sqrt{t}M^2,\end{aligned}$$

where $\int_0^t (t-s)^{-\frac{1}{2}}ds = C_2\sqrt{t}$. Hence we have the inequality

$$\sup_{0<t<T}\|u_n(t)\|_\infty \le \frac{M}{2} + C_1C_2\sqrt{T}M^2.$$

Take $T>0$ small enough that $2C_1C_2\sqrt{T}M < 1$, then we have

$$\sup_{0<t<T}\|u_n(t)\|_\infty \le M.$$

By (3.13) of Proposition 3.1, we have the estimate

$$\|\nabla e^{-tA}a\|_\infty \le C_3 t^{-1/2}\|a\|_\infty,$$

and by Theorem 3.5, we have the estimate

$$\|\nabla e^{-(t-s)A}\mathbb{P}\mathrm{div}(u_{n-1}\otimes u_{n-1})\|_\infty \le C_4(t-s)^{-\frac{1}{2}}\|\nabla(u_{n-1}\otimes u_{n-1})\|_\infty.$$

Hence we have the following estimate for $\|\nabla u_n\|_{L^\infty(\mathbb{R}^n_+)}$.

(4.21)

$$\|\nabla u_n\|_\infty \le C_3 t^{-1/2}\|u_0\|_\infty + \int_0^t (t-s)^{-\frac{1}{2}}C_4\|u_{n-1}(s)\|_\infty\|\nabla u_{n-1}(s)\|_\infty ds.$$

Take large $M_1 \ge 2C_3\|a\|_\infty$. Suppose $\sup_{0<t<T} t^{1/2}\|\nabla u_{n-1}(t)\|_\infty \le M_1$, then from (4.21) we have

$$\|\nabla u_n(t)\|_\infty \le C_3 t^{-1/2}\|u_0\|_\infty$$

$$+ C_4\Big[\int_0^t (t-s)^{-\frac{1}{2}} s^{-1/2} ds\Big]\Big[\sup_{0<s<t} \|u_{n-1}(s)\|_\infty\Big]\Big[\sup_{0<s<t} s^{1/2}\|\nabla u_{n-1}(s)\|_\infty\Big]$$

$$\leq t^{-1/2}\frac{M_1}{2} + C_4C_5MM_1,$$

where $\int_0^t (t-s)^{-\frac{1}{2}} s^{-1/2} ds = C_5$. Hence we have the inequality

$$\sup_{0<t<T} t^{1/2}\|\nabla u_n\|_\infty \leq \frac{M_1}{2} + C_4C_5MM_1\sqrt{T}.$$

Take $T>0$ small enough that $4C_4C_5M\sqrt{T}<1$, then we have

$$\sup_{0<t<T} t^{1/2}\|\nabla u_n(t)\|_\infty \leq M_1.$$

This implies that $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty((0,T);W^{1,\infty}(\mathbb{R}^n_+))$, if $T$ is so small that

$$2C_1C_2\sqrt{T}M<1, \quad 4C_4C_5M\sqrt{T}<1,$$

where $M = 2C_0\|a\|_\infty$ and $M_1 = 2C_3\|a\|_\infty$.

Set $w_n = u_{n+1} - u_n$, $n = 1, 2, \ldots$. Note that

$$\sup_{0<t<T} \|w_0(t)\|_\infty = \sup_{0<t<T} \|e^{-tA}u_0\|_\infty \leq C_0\|a\|_\infty.$$

Note

$$B(u_n,u_n) - B(u_{n-1},u_{n-1}) = B(u_n, u_n - u_{n-1}) + B(u_n - u_{n-1}, u_{n-1}).$$

Then we also have the following estimate for $\|w_n\|_\infty$.

$$\|w_n(t)\|_\infty \leq C_1\int_0^t (t-s)^{-\frac{1}{2}}\big(\|u_n(s)\|_\infty + \|u_{n-1}(s)\|_\infty\big)\|w_{n-1}(s)\|_\infty ds. \tag{4.22}$$

Recall that if $M \geq 2C_0\|a\|_{L^\infty(\mathbb{R}^n_+)}$ and $2C_1C_2M\sqrt{T}<1$, then

$$\sup_{0<t<T} \|u_n\|_{L^\infty(\mathbb{R}^n_+)} \leq M$$

for all $n = 1, 2, \ldots$. From (4.22) we have

$$\sup_{0<t<T} \|w_n(t)\|_\infty \leq 2C_1C_2M\sqrt{T}\sup_{0<t<T} \|w_{n-1}(t)\|_\infty$$
$$\leq (2C_1C_2M\sqrt{T})^n C_0\|a\|_\infty.$$

Since $2C_1C_2M\sqrt{T} < 1$, the series $\sum_{n=1}^\infty w_n(t)$ converges to some $u \in L^\infty((0,T)\times\mathbb{R}^n_+)$. This implies $u_n$ converges to some $u$ in $L^\infty((0,T)\times\mathbb{R}^n_+)$, since $\sum_{k=0}^{n-1} w_n(t) = u_n(t)$. By the lower semi-continuity, we have

$$\sup_{0<t<T} \|u(t)\|_\infty \leq M, \qquad \sup_{0<t<T} t^{1/2}\|\nabla u(t)\|_\infty \leq M_1.$$

## 5. Proof of Theorem 1.3

Let $u$ and $v$ be the mild solution of the Navier-Stokes equations in the class $L^\infty_{\rm loc}(0,T;L^\infty(\mathbb{R}^n_+))$ with the property

$$\sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(t)\|_\infty = M_1 \text{ and } \sup_{0<t<T} t^{\frac{\alpha}{2}}\|u(t)\|_\infty = M_2.$$

Here, $0<\alpha<1$.

Set $w=u-v$, then $w$ satisfies the integral equation

$$w = B(u,u)-B(v,v) = B(w,u)+B(v,w).$$

Apply Proposition 3.1, then we obtain the following Gronwall inequality

$$\begin{aligned}\|w(t)\|_\infty &\le C_1\int_0^t (t-s)^{-\frac12}\big(\|u(s)\|_\infty + \|v(s)\|_\infty\big)\|w(s)\|_\infty ds\\ &\le C_1(M_1+M_2)\int_0^t (t-s)^{-\frac12}s^{-\frac{\alpha}{2}}\|w(s)\|_\infty ds. \quad (5.23)\end{aligned}$$

Set $X(t)=\|w(t)\|_\infty$. Then $X$ satisfies inequality

$$X(t)\le C_1(M_1+M_2)\left[\int_0^t (t-s)^{-\frac12}s^{-\frac{\alpha}{2}}X(s)ds\right]. \quad (5.24)$$

Note that $\int_0^t (t-s)^{-\frac12}s^{-\alpha}ds = C_2t^{\frac12-\alpha}$ if $0\le\alpha<1$. Then

$$\sup_{0<t\le t_0} t^{\frac{\alpha}{2}}X(t)\le C_1C_2(M_1+M_2)t_0^{\frac12-\frac{\alpha}{2}}\Big(\sup_{0<s\le t_0} s^{\frac{\alpha}{2}}X(s)\Big). \quad (5.25)$$

Choose $t_0$ small enough that $C_1C_2(M_1+M_2)t_0^{\frac12-\frac{\alpha}{2}}<1$, then (5.25) implies that

$$\sup_{0<t\le t_0} t^{\frac{\alpha}{2}}X(t)=0.$$

This again implies that $X(t)=0$ for all $0<t\le t_0$. Then from (5.24) we have

$$\begin{aligned}X(t)&\le C_1(M_1+M_2)\Big[\int_{t_0}^t (t-s)^{-\frac12}s^{-\frac{\alpha}{2}}X(s)ds\Big]\\ &\le C_1(M_1+M_2)t_0^{-\frac{\alpha}{2}}\int_{t_0}^t (t-s)^{-\frac12}X(s)ds. \quad (5.26)\end{aligned}$$

Iterating the estimate (5.26), we obtain

$$\begin{aligned}X(t)&\le C_1^2(M_1+M_2)^2t_0^{-\alpha}\int_{t_0}^t (t-s)^{-\frac12}\Big[\int_{t_0}^s (s-\tau)^{-\frac12}X(\tau)d\tau\Big]ds\\ &= C_1^2(M_1+M_2)^2t_0^{-\alpha}\int_{t_0}^t \Big[\int_{\tau}^t (t-s)^{-\frac12}(s-\tau)^{-\frac12}ds\Big]X(\tau)d\tau\\ &\le C_1^2C_3(M_1+M_2)^2t_0^{-\alpha}\int_{t_0}^t X(\tau)d\tau. \quad (5.27)\end{aligned}$$

Here, we noted that $\int_\tau^t (t-s)^{-\frac12}(s-\tau)^{-\frac12}ds = C_3<\infty$.

Set $Y(t) = \int_{t_0}^{t} X(\tau)d\tau$. Then from (5.27) we have the Gronwall inequality

$$Y'(t) \le C_1^2 C_3 (M_1 + M_2)^2 t_0^{-\alpha} Y(t) \quad \text{and} \quad Y(t_0) = 0.$$

Solving the above Gronwall inequality we conclude that $Y(t) \equiv 0$ for $t_0 < t < T$. Applying this result to (5.27), then we have $X(t) \le C_1^2 C_3 (M_1+M_2)^2 t_0^{-\alpha} Y(t) = 0$ for $t_0 < t < T$.

Therefore we conclude that $\|w(t)\|_\infty = X(t) = 0$ for all $0 < t < T$, that is, $u(t) = v(t)$ for all $0 < t < T$.

## Appendix A. Proof of Proposition 2.2.

Let $x \in \mathbb{R}^n_+$. Choose a nonnegative smooth function $\phi \in C_0^\infty([0,2))$ with $\phi(t) = 1$ for $0 \le t \le 1$ and $\int_0^2 \phi(t)dt = 1$. For $\epsilon < \frac{x_n}{2}$, define a cut-off function $\phi_{\epsilon,x}$ by $\phi_{\epsilon,x}(z) = \phi(\frac{|x-z|}{\epsilon})$. Then $\phi_{\epsilon,x}$ is compactly supported in $\mathbb{R}^n_+$.

Define a function $v_\epsilon$ by

$$v_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\epsilon,x}(z)\partial_{z_i} N(x-z) f(z)dz.$$

We differentiate $v_\epsilon$ by $x_j$ variable. If $j \ne n$, we have

$$\partial_{x_j} v_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \Big[\partial_{x_j}\phi_{\epsilon,x}(z)\partial_{z_i} N(x-z) + \phi_{\epsilon,x}(z)\partial_{z_i}\partial_{x_j} N(x-z)\Big] f(z)dz.$$

If $j = n$, we have

$$\begin{aligned}\partial_{x_n} v_\epsilon(x) = &\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \Big[\partial_{x_n}\phi_{\epsilon,x}(z)\partial_{z_i} N(x-z) + \phi_{\epsilon,x}(z)\partial_{z_i}\partial_{x_n} N(x-z)\Big] f(z)dz \\ &+ \int_{\mathbb{R}^{n-1}} \phi_{\epsilon,x}(\bar z, x_n)\partial_{x_i} N(\bar x - \bar z, 0) f(\bar z, x_n)d\bar z.\end{aligned}$$

Set

$$I_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j}\phi_{\epsilon,x}(z)\partial_{x_i} N(x-z) f(z)dz, \qquad \text{and}$$

$$II_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\epsilon,x}(z)\frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z)dz.$$

Obviously,

$$II_\epsilon(x) \to \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z)dz \qquad \text{as } \epsilon \to 0.$$

Divide $I_\epsilon$ by $I_\epsilon^1$ and $I_\epsilon^2$ defined by

$$I_\epsilon^1(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j}\phi_{\epsilon,x}(z)\partial_{x_i} N(x-z)\big[f(z) - f(x)\big]dz$$

and

$$I_\epsilon^2(x) = f(x)\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j}\phi_{\epsilon,x}(z)\partial_{x_i} N(x-z)dz.$$

Note that $I^1_\epsilon(x) \to 0$ as $\epsilon \to 0$, since

$$|I^1_\epsilon(x)| \le \int_{|x-z|\le\epsilon} \left|\partial_{x_j}\phi_{\epsilon,x}(z)\right| \left|\partial_{x_i}N(x-z)\right| |f(z)-f(x)|dz$$
$$\le \int_{|x-z|\le\epsilon} \frac{1}{\epsilon}\frac{1}{|x-z|^{n-1}}|x-z|^\alpha dz\,[f]_\alpha \le \epsilon^\alpha[f]_\alpha,$$

where

$$[f]_\alpha = \sup_{x,z,|x-z|\le\epsilon} \frac{|f(x)-f(z)|}{|x-z|^\alpha}.$$

Now it remains to show that

$$\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_j}\phi_{\epsilon,x}(z)\partial_{x_i}N(x-z)dz = \begin{cases} -\frac{\delta_{ij}}{2n} \text{ if } i,j\ne n \text{ or } i=j=n \\ 0 \text{ if } i=n, j\ne n \text{ or } j=n, i\ne n. \end{cases}$$

Taking change of variables and sending $\epsilon$ to the zero, we obtain

$$\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_j}\phi_{\epsilon,x}(z)\partial_{x_i}N(x-z)dz$$
$$= \frac{1}{\epsilon}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{(z_i-x_j)(z_j-x_j)}{n\omega_n|x-z|^{n+1}}\phi'(\frac{|x-z|}{\epsilon})dz$$
$$\to \int_0^\infty\int_{\mathbb{R}^{n-1}} \frac{w_iw_j}{n\omega_n|w|^{n+1}}\phi'(|w|)dw.$$

If $i,j\ne n$ or $i=j=n$, then by the symmetry of $\frac{w_iw_j}{n\omega_n|w|^{n+1}}\phi'(|w|)$ in terms of $w_n$ variables, we have

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} \frac{w_iw_j}{n\omega_n|w|^{n+1}}\phi'(|w|)dw = \frac{1}{2}\int_{\mathbb{R}^n}\frac{w_iw_j}{n\omega_n|w|^n}\phi'(|w|)dw$$
$$= \frac{1}{2n\omega_n}\int_0^2\int_{s^{n-1}} rw_iw_j\phi'(r)dS_wdr = -\frac{1}{2n}\delta_{ij}\int_0^2\phi(r)dr = -\frac{1}{2n}\delta_{ij}.$$

If $i=n, j\ne n$ or $j=n, i\ne n$, then by the anti-symmetry of $\frac{w_iw_j}{n\omega_n|w|^{n+1}}\phi'(|w|)$ in terms of $w_n$ variables, we have

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} \frac{w_iw_j}{n\omega_n|w|^{n+1}}\phi'(|w|)dw = 0.$$

## Appendix B. Proof of Lemma 2.4

Let $k=1,\ldots,n-1$. Set $f_t(x) = f(x,t) = 1_{\mathbb{R}^n_+}\partial_{x_k}(\tau_a\Gamma_t)$. We would like to show that $\mathcal{N}f_t(x)\in L^1(\mathbb{R}^n)$. For $s>0$, we have

$$f_t * \Gamma_s(x)$$
$$= \int_{\mathbb{R}^n} f(x-y,t)\Gamma_s(y)dy$$
$$= \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{x_n} \partial_{x_k}\Gamma(a-x+y,t)\Gamma(y,s)dy$$

$$= \Big[ \int_{\mathbb{R}^{n-1}} \partial_{x_k} \bar{\Gamma}(\bar{a} - \bar{x} + \bar{y}, t) \bar{\Gamma}(\bar{y}, s) d\bar{y} \Big] \Big[ \int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \Big]$$

$$= \Big(\partial_{x_k} \bar{\Gamma}_{s+t}\Big)(\bar{a} - \bar{x}) \Big[ \int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \Big].$$

Note that

$$\Big|\partial_{x_k} \bar{\Gamma}_{s+t}(\bar{a} - \bar{x})\Big| \le C(s+t)^{-\frac{1}{2}} \bar{\Gamma}_{2(s+t)}(\bar{a} - \bar{x}),$$

and

$$\int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \le \int_{-\infty}^{\infty} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n = \Gamma_{s+t}^n(a_n - x_n).$$

Hence we obtain that

$$|f_t * \Gamma_s(x)| \le C(s+t)^{-\frac{1}{2}} \Gamma_{2(t+s)}(a - x).$$

Note that $\Gamma_{2(t+s)}(a-x) \le C(s+t)^{-n/2}$ and $\Gamma_{2(t+s)}(a-x) \le C(s+t)^{\frac{1}{2}}|a - x|^{-n-1}$. Hence we have that

$$\mathcal{N} f_t(x) = \sup_{s>0} |\Gamma_s * f_t(x)| \le \sup_{s>0} \min\{(s+t)^{-\frac{n+1}{2}}, |x-a|^{-n-1}\} \le \min\{t^{-\frac{n+1}{2}}, |x-a|^{-n-1}\}.$$

Now we estimate the $L^1$ norm of $\mathcal{N} f_t$.

$$\int_{\mathbb{R}^n} \mathcal{N} f_t(x) dx = \int_{|x-a| \ge \sqrt{t}} \mathcal{N} f_t(x) dx + \int_{|x-a| \le \sqrt{t}} \mathcal{N} f_t(x) dx \le \int_{|x-a| \ge \sqrt{t}} |x-a|^{-n-1} dx + \int_{|x-a| \le \sqrt{t}} t^{-\frac{n+1}{2}} dx \le C t^{-\frac{1}{2}}.$$

This leads to the conclusion that

$$f_t = 1_{\mathbb{R}^n_+} \tau_a \partial_{y_k} \Gamma_t \in \mathcal{H}^1(\mathbb{R}^n).$$

## Appendix C. Proof of Lemma 2.6

Let $a = (\bar{a}, a_n)$ and $b = (\bar{b}, a_n)$. Set

$$f_t(x) = f(x,t) = 1_{\mathbb{R}^n_+} \Big[ (\tau_{a^*} \Gamma_t)(x) - (\tau_{b^*} \Gamma_t)(x) \Big].$$

We would like to estimate that the $L^1$ norm of $\mathcal{N} f_t(x) = \sup_{s>0} |f_t * \Gamma_s(x)|$. In the proof of Lemma 2.4, we recall that

$$(1_{\mathbb{R}^n_+} \Gamma_t^{a^*}) * \Gamma_s(x) = \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) \Big[ \int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n) \Gamma_s^n(y_n) dy_n \Big].$$

Hence we have that

$$f_t * \Gamma_s(x) = \Big[ \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) - \bar{\Gamma}_{t+s}(\bar{b} - \bar{x}) \Big] \Big[ \int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n) \Gamma_s^n(y_n) dy_n \Big].$$

Recall that

$$\int_{-\infty}^{x_n} \Gamma^n_t(a_n + x_n - y_n)\Gamma^n_s(y_n)dy_n \le Ce^{-\frac{a_n^2}{4t}}\Gamma^n_{s+t}(x_n).$$

If $|\bar a - \bar b| \le \frac{|\bar x - \bar a|}{2}$, then by the mean value theorem we have that

$$\begin{aligned}&\Big|\bar\Gamma_{t+s}(\bar a - \bar x) - \bar\Gamma_{t+s}(\bar b - \bar x)\Big|\\ &\le |\bar b - \bar a||\nabla_{\bar x}\bar\Gamma_{s+t}(\bar a - \bar x + \theta(\bar b - \bar a)|\\ &\le C(s+t)^{-1/2}|\bar b - \bar a|\bar\Gamma_{4(s+t)}(\bar a - \bar x).\end{aligned}$$

Therefore, if $|\bar a - \bar b| \le \frac{|\bar x - \bar a|}{2}$, then

$$|f_t * \Gamma_s(x)| \le C(s+t)^{-1/2}|\bar b - \bar a|\bar\Gamma_{4(s+t)}(\bar a - \bar x)e^{-\frac{a_n^2}{4t}}\Gamma^n_{s+t}(x_n).$$

Choose $0 < \alpha < 1$. Recall that

$$\bar\Gamma_{4(t+s)}(\bar a - \bar x) \le C(s+t)^{(1-\alpha)/2}|\bar a - \bar x|^{-n+\alpha},$$

and

$$\Gamma^n_{s+t}(x_n) \le \min\{C(s+t)^{-\frac12},\, C(s+t)^{\frac{\alpha}{2}}x_n^{-1-\alpha}\}.$$

Hence if $|\bar a - \bar b| \le \frac{|\bar x - \bar a|}{2}$, then

$$|f_t * \Gamma_s(x)| \le C|\bar b - \bar a|e^{-\frac{a_n^2}{4t}}\begin{cases}(s+t)^{-\frac12-\frac{\alpha}{2}}|\bar a - \bar x|^{-n+\alpha},\\ |\bar a - \bar x|^{-n+\alpha}x_n^{-1-\alpha},\end{cases}$$

which implies that

$$\mathcal{N}f_t(x) \le C|\bar b - \bar a|e^{-\frac{a_n^2}{4t}}\begin{cases}t^{-\frac12+\frac{\alpha}{2}}|\bar a - \bar x|^{-n+\alpha},\\ |\bar a - \bar x|^{-n+\alpha}x_n^{-1-\alpha}.\end{cases} \tag{C.28}$$

We also note that

$$\begin{aligned}|f_t * \Gamma_s(x)| &\le C\Big|\bar\Gamma_{t+s}(\bar a - \bar x) - \bar\Gamma_{t+s}(\bar b - \bar x)\Big|e^{-\frac{a_n^2}{4t}}\Gamma^n_{s+t}(x_n)\\ &\le C\Big(\Big|\bar\Gamma_{t+s}(\bar a - \bar x)\Big| + \Big|\bar\Gamma_{t+s}(\bar b - \bar x)\Big|\Big)e^{-\frac{a_n^2}{4t}}\Gamma^n_{s+t}(x_n)\\ &\le C\big[\bar\Gamma_{2(s+t)}(\bar a - \bar x) + \bar\Gamma_{2(s+t)}(\bar b - \bar x)\big]e^{-\frac{a_n^2}{4t}}\Gamma^n_{s+t}(x_n).\end{aligned}$$

Note that

$$\bar\Gamma_{2(t+s)}(\bar a - \bar x) \le C(s+t)^{-\frac{\alpha}{2}}|\bar a - \bar x|^{-n+1+\alpha}$$

and

$$\Gamma^n_{t+s}(x_n) \le \min\{C(s+t)^{-\frac12}, C(t+s)^{\frac{\alpha}{2}}x_n^{-1-\alpha}\}.$$

Hence, we also have

$$|f_t * \Gamma_s(x)| \le Ce^{-\frac{a_n^2}{4t}}\begin{cases}(s+t)^{-\frac12-\frac{\alpha}{2}}|\bar a - \bar x|^{-n+1+\alpha}\\ |\bar a - \bar x|^{-n+1+\alpha}x_n^{-1-\alpha},\end{cases}$$

which implies that

$$\mathcal{N}f_t(x) \leq Ce^{-\frac{a_n^2}{4t}} \begin{cases} t^{-\frac{\alpha}{2}}|\bar{a}-\bar{x}|^{-n+1+\alpha} \\ |\bar{a}-\bar{x}|^{-n+1+\alpha}x_n^{-1-\alpha}. \end{cases} \tag{C.29}$$

Now we estimate the $L^1$ norm of $\mathcal{N}f_t$. For $|\bar{a}-\bar{b}| \leq \frac{|\bar{x}-\bar{a}|}{2}$, we make use of the estimate (C.28) and for $|\bar{a}-\bar{b}| \geq \frac{|\bar{x}-\bar{a}|}{2}$ we make use of the estimate (C.29).

Decompose the domain $\mathbb{R}^n_+$ into four parts $D_i, i=1,\dots,4$, where

$$\begin{aligned} D_1 &= \{(\bar{z},x_n):\ |\bar{x}-\bar{a}| \geq 2|\bar{a}-\bar{b}|, |x_n| \geq \sqrt{t}\}, \\ D_2 &= \{(\bar{z},x_n):\ |\bar{x}-\bar{a}| \geq 2|\bar{a}-\bar{b}|, |x_n| \leq \sqrt{t}\}, \\ D_3 &= \{(\bar{z},x_n):\ |\bar{x}-\bar{a}| \leq 2|\bar{a}-\bar{b}|, |x_n| \geq \sqrt{t}\}, \\ D_4 &= \{(\bar{z},x_n):\ |\bar{x}-\bar{a}| \leq 2|\bar{a}-\bar{b}|, |x_n| \leq \sqrt{t}\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{N}f_t(x)dx &= \sum_{i=1}^4 \int_{D_i} \mathcal{N}f_t(x)dx \\ &\leq Ce^{-\frac{a_n^2}{4t}} \Big[ \int_{D_1} |\bar{a}-\bar{b}||\bar{a}-\bar{x}|^{-n+\alpha}x_n^{-1-\alpha}dx \\ &\qquad + \int_{D_2} |\bar{a}-\bar{b}|t^{-\frac{1}{2}-\frac{\alpha}{2}}|\bar{a}-\bar{x}|^{-n+\alpha}dx \\ &\qquad + \int_{D_3} |\bar{a}-\bar{x}|^{-n+1+\alpha}x_n^{-1-\alpha}dx \\ &\qquad + \int_{D_4} t^{-\frac{1}{2}-\frac{\alpha}{2}}|\bar{a}-\bar{x}|^{-n+1+\alpha}dx \Big] \\ &\leq Ce^{-\frac{a_n^2}{4t}}|\bar{a}-\bar{b}|^{\alpha}t^{-\frac{\alpha}{2}}. \end{aligned}$$

Therefore, we conclude that

$$f_t = 1_{\mathbb{R}^n_+}\Big[\tau_{a^*}\Gamma_t - \tau_{b^*}\Gamma_t\Big] \in \mathcal{H}^1(\mathbb{R}^n).$$

## Appendix D. Proof of Lemma 2.7.

Set

$$\begin{aligned} (a) &= \int_0^{x_n/2}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_j}N(x-z)f(z,y)dz, \\ (b) &= \int_{x_n/2}^{x_n}\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_j}N(x-z)\big[f(z,y)-f(\bar{x},z_n,y)\big]dz, \quad \text{and} \\ (c) &= \int_{x_n/2}^{x_n} f(\bar{x},z_n,y)\int_{\mathbb{R}^{n-1}} \partial_{x_i}\partial_{x_j}N(x-z)d\bar{z}dz_n. \end{aligned}$$

Then

$$\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\partial_{x_i}\partial_{x_j}N(x-z)f(z,y)dz=(a)+(b)+(c).$$

Since $|\partial_{x_i}\partial_{x_j}N(x-z)|\le C|x-z|^{-n}$ and $\int_{\mathbb{R}^{n-1}}|x-z|^{-n}d\bar z=\frac{C}{|x_n-z_n|}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}(a)dy&\le C\Big(\int_0^{x_n/2}\frac{1}{x_n-z_n}dz_n\Big)\Big(\sup_{z\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|f(z,y)|dy\Big)\\&\le C\sup_{z\in\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}|f(z,y)|dy.\end{aligned}$$

Since, for $0<\alpha<1$,

$$\begin{aligned}&\big|\partial_{x_i}\partial_{x_j}N(x-z)[f(z,y)-f(\bar x,z_n,y)]\big|\\&\le C|x-z|^{\alpha}|x-z|^{-n}\frac{|f(\bar z,z_n,y)-f(\bar x,z_n,y)|}{|x-z|^{\alpha}}\\&\le C|x_n-z_n|^{\alpha}|x-z|^{-n}\frac{|f(\bar z,z_n,y)-f(\bar x,z_n,y)|}{|\bar x-\bar z|^{\alpha}},\end{aligned}$$

we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n_+}(b)dy\\&\le C\int_{\frac{x_n}{2}}^{x_n}(x_n-z_n)^{\alpha}\int_{\mathbb{R}^{n-1}}|x-z|^{-n}\Big[\sup_{\bar z\in\mathbb{R}^{n-1}}\int_{\mathbb{R}^n_+}\frac{|f(\bar z,z_n,y)-f(\bar x,z_n,y)|}{|\bar x-\bar z|^{\alpha}}dy\Big]d\bar z dz_n\\&\le Cx_n^{\alpha+1}\Big(\int_{\mathbb{R}^{n-1}}|x-z|^{-n}d\bar z\Big)\Big(\sup_{x_n/2<z_n<x_n}\sup_{\bar z\in\mathbb{R}^{n-1}}\int_{\mathbb{R}^n_+}\frac{|f(\bar z,z_n,y)-f(\bar x,z_n,y)|}{|\bar x-\bar z|^{\alpha}}dy\Big)\\&\le Cx_n^{\alpha}\Big(\sup_{x_n/2<z_n<x_n}\sup_{\bar z\in\mathbb{R}^{n-1}}\int_{\mathbb{R}^n_+}\frac{|f(\bar z,z_n,y)-f(\bar x,z_n,y)|}{|\bar x-\bar z|^{\alpha}}dy\Big).\end{aligned}$$

Now we would like to show that $(c)=0$. Note that

$$p.v.\int_{\mathbb{R}^{n-1}}\frac{\partial^2}{\partial y_i\partial y_j}N(x-y)d\bar y=\lim_{\epsilon\to 0,R\to\infty}\int_{\epsilon\le|\bar x-\bar y|\le R}\frac{\partial^2}{\partial y_i\partial y_j}N(x-y)dS_{\bar y}$$

and

$$\begin{aligned}&\int_{\epsilon\le|\bar x-\bar y|\le R}\frac{\partial^2}{\partial y_i\partial y_j}N(x-y)d\bar y\\&=\int_{S_R(\bar x)}\partial_{y_i}N(x-y)n_jdS_{\bar y}-\int_{S_\epsilon(\bar x)}\partial_{y_i}N(x-y)n_jdS_{\bar y}.\end{aligned}$$

Here $S_R(\bar x)=\{\bar y\in\mathbb{R}^{n-1}:|\bar x-\bar y|=R\}$ and $n_j=\frac{y_j-x_j}{|\bar y-\bar x|}$ is the $j$-th component of the unit outer normal vector. If $i=1,\dots,n-1$, then

$$\Big|\int_{S_R(\bar x)}\partial_{y_i}N(x-y)n_jdS_{\bar y}\Big|=\Big|\int_{S_R(\bar x)}\frac{y_i-x_i}{|x-y|^n}\frac{y_j-x_j}{|\bar y-\bar x|}dS_{\bar y}\Big|$$

$$= \Big| \int_{S_{n-2}} \frac{w_i w_j}{n\omega_n (R^2 + (x_n - y_n)^2)^{n/2}} R^{n-1} dS_w \Big| \le \frac{C}{R} \to 0 \quad \text{as } R \to \infty,$$

where $S_{n-2}$ is the unit sphere in $\mathbb{R}^{n-1}$, $w_i = \frac{y_i}{|\bar{y}|}$ is the outward unit normal vector to $S_{n-2}$, and

$$\Big| \int_{S_\epsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \Big|$$

$$= \Big| \int_{S_{n-2}} \frac{w_i w_j}{n\omega_n (\epsilon^2 + (x_n - y_n)^2)^{n/2}} \epsilon^{n-1} dS_{\bar{y}} \Big| \le C \frac{\epsilon^{n-1}}{|x_n - y_n|^n} \to 0 \quad \text{as } \epsilon \to 0.$$

If $i = n$, then

$$\int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = 0,$$

since

$$\int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = \int_{S_{n-2}} \frac{(x_n - y_n) w_j}{n\omega_n (R^2 + (x_n - y_n)^2)^{n/2}} R^{n-2} dS_{\bar{y}}$$

$$= \frac{(x_n - y_n) R^{n-2}}{n\omega_n (R^2 + (x_n - y_n)^2)^{n/2}} \int_{S_{n-2}} w_j dS_{\bar{y}} = 0.$$

Likewise,

$$\int_{S_\epsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = 0.$$

This implies that

$$\int_{\mathbb{R}^{n-1}} \partial_{y_i} \partial_{y_j} N(x-y) d\bar{y} = 0,$$

and this again implies that $(c) = 0$.

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