The Annals of Probability
2009, Vol. 37, No. 2, 428–452
DOI: 10.1214/08-AOP408

EXISTENCE OF STRONG SOLUTIONS FOR STOCHASTIC POROUS MEDIA EQUATION UNDER GENERAL MONOTONICITY CONDITIONS

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This paper addresses the existence and uniqueness of strong solutions to stochastic porous media equations $dX - \Delta\Psi(X)\,dt = B(X)\,dW(t)$ in bounded domains of $\mathbb{R}^d$ with Dirichlet boundary conditions. Here Ψ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ (possibly multivalued) with the domain and range all of $\mathbb{R}$. Compared with the existing literature on stochastic porous media equations, no growth condition on Ψ is assumed and the diffusion coefficient Ψ might be multivalued and discontinuous. The latter case is encountered in stochastic models for self-organized criticality or phase transition.

1. Introduction. This work is concerned with existence and uniqueness of solutions to stochastic porous media equations

$$
(1.1)\qquad \begin{cases} dX(t) - \Delta\Psi(X(t))\,dt = B(X(t))\,dW(t), & \text{in } (0,T)\times\mathcal{O} := Q_T,\\ \Psi(X(t)) = 0, & \text{on } (0,T)\times\partial\mathcal{O} := \Sigma_T,\\ X(0) = x, & \text{in } \mathcal{O}, \end{cases}
$$

where $\mathcal{O}$ is an open, bounded domain of $\mathbb{R}^d$, $d \ge 1$, with smooth boundary $\partial\mathcal{O}$, $W(t)$ is a cylindrical Wiener process on $L^2(\mathcal{O})$, while B is a Lipschitz continuous operator from $H := H^{-1}(\mathcal{O})$ to the space of Hilbert–Schmidt operators on $L^2(\mathcal{O})$. [See hypothesis (H_2) below.]

The function $\Psi : \mathbb{R} \to \mathbb{R}$ (or more generally the multivalued function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$) is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. (See the definition in Section 1.1 below.)

Existence results for (1.1) were obtained in [8] (see also [3, 4]) in the special case $B = \sqrt{Q}$, with Q linear nonnegative, $\operatorname{Tr} Q < +\infty$ and $\Psi \in C^1(\mathbb{R})$ satisfying

Received March 2007; revised February 2008.
[1]Supported by PN-II ID.404 (2007–2010).
[2]Supported by the research program "Equazioni di Kolmogorov" from the Italian "Ministero della Ricerca Scientifica e Tecnologica."
[3]Supported by the SFB-701 and the BIBOS-Research Center.
AMS 2000 subject classifications. 76S05, 60H15.
Key words and phrases. Stochastic porous media equation, Wiener process, convex functions, Itô's formula.

the growth condition

$$k_3 + k_1|s|^{r-1} \le \Psi'(s) \le k_2(1 + |s|^{r-1}), \qquad s \in \mathbb{R}, \tag{1.2}$$

where $k_1, k_2 > 0$, $k_3 \in \mathbb{R}$, $r > 1$.

Under these growth conditions on Ψ, (1.1) covers many important models describing the dynamics of an ideal gas in a porous medium (see, e.g., [1]) but excludes, however, other significant physical models such as plasma fast diffusion [5], which arises for $\Psi(s) = \sqrt{s}$, and phase transitions or dynamics of saturated underground water flows (the Richards equation). In the latter case, multivalued monotone graphs Ψ might appear as in [12]. Recently, in [15] (see also [14]), the existence results of [8] were extended to the case of monotone nonlinearities Ψ such that $s \mapsto s\Psi(s)$ is (comparable to) a Δ_2-regular Young function (cf. assumption (A1) in [15]) thus including the fast diffusion model. As a matter of fact, in the line of the classical work of Krylov and Rozovskii [10] the approach used in [15] is a variational one, that is, one considers the stochastic equation (1.1) in a duality setting induced by a functional triplet $V \subset H \subset V'$ and this requires one to find appropriate spaces V and H. This was done in [15] in an elaborate way even with $\mathcal{O}$ unbounded and with Δ replaced by very general (not necessarily differential) operators L.

The method we use here is quite different—essentially an L^1-approach relying on weak compacteness techniques in $L^1(Q_T)$ via the Dunford–Pettis theorem which involve minimal growth assumptions on Ψ. Restricted to single-valued continuous functions Ψ the main result, Theorem 2.2 below, gives existence and uniqueness of solutions only assuming that $\lim_{s\to+\infty} \Psi(s) = +\infty$, $\lim_{s\to-\infty} \Psi(s) = -\infty$, Ψ monotonically increasing and

$$\limsup_{|s|\to+\infty} \frac{\int_0^{-s} \Psi(t)\,dt}{\int_0^{s} \Psi(t)\,dt} < +\infty. \tag{1.3}$$

We note that the assumptions on Ψ in [15] imply our assumptions. In this sense, under assumption (H_2) below on the noise, the results of this paper extend those in [15] in case $L = \Delta$ if $\mathcal{O}$ is bounded and if the coefficients do not depend on (t, ω). The latter two were not assumed in [15]. On the other hand, a growth condition on Ψ is imposed in [15] (cf. [15], Lemma 3.2) which is not done here. Another main point of this paper is that Ψ is no longer assumed to be continuous, it might be multivalued and with exponential growth to $\pm\infty$ [for instance, of the form $\exp(a|x|^p)$]. We note that (1.3) is not a growth condition at $+\infty$ but a kind of symmetry condition about the behavior of Ψ at $\pm\infty$. If Ψ is a maximal monotone graph with potential j (i.e., $\Psi = \partial j$) then (1.3) takes the form [see hypothesis (H_3) below]

$$\limsup_{|s|\to+\infty} \frac{j(-s)}{j(s)} < +\infty.$$

Anyway this condition is automatically satisfied for even monotonically increasing functions Ψ or, for example, if a condition of the form (1.2) is satisfied. We note, however, that because of our very general conditions on Ψ the solution of (1.1) will be pathwise only weakly continuous in H. The question of pathwise strong continuity of solutions, however, remains open. The main reason is the absence of a variational setting for problem (1.1) (see [10, 14]) in the present situation. Another major technical difficulty encountered here in the proofs is that the integration by parts formula or Itô's formula (see Lemmas 3.1 and 3.2 below) cannot be applied directly because of the same reason. It should be said, however, that the L^1 approach used here, which allows us to treat very general nonlinearities, is applicable to deterministic equations as well and seems to be new also in that context. On the other hand, the existence for the deterministic part of (1.1) is an immediate consequence of the Crandall–Liggett generation theorem for nonlinear semigroups of contractions (see [2]) which is, however, not applicable to stochastic equations.

1.1. *Notation.* $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\mathcal{O}$. We set

$$Q_T = (0, T) \times \mathcal{O}, \qquad \Sigma_T = (0, T) \times \partial\mathcal{O}.$$

$L^p(\mathcal{O})$, $L^p(Q_T)$, $p \geq 1$, are standard L^p-function spaces and $H_0^1(\mathcal{O})$, $H^k(\mathcal{O})$ are Sobolev spaces on $\mathcal{O}$. By $H := H^{-1}(\mathcal{O})$ we denote the dual of $H_0^1(\mathcal{O})$ with the norm and the scalar product given by

$$|u|_{-1} := (A^{-1}u, u)^{1/2}, \qquad \langle u, v\rangle_{-1} = (A^{-1}u, v),$$

respectively, where $(\cdot, \cdot)$ is the pairing between $H_0^1(\mathcal{O})$ and H^{-1} and the scalar product of $L^2(\mathcal{O})$. Here A denotes the Laplace operator with Dirichlet homogeneous boundary conditions, that is,

$$Au = -\Delta u, \qquad u \in D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \tag{1.4}$$

Given a Hilbert space U, the norm of U will be denoted by $|\cdot|_U$ and the scalar product by $(\cdot, \cdot)_U$. By $C([0, T]; U)$ we shall denote the space of U-valued continuous functions on $[0, T]$ and by $C^w([0, T]; U)$ the space of weakly continuous functions from $[0, T]$ to U.

Given two Hilbert spaces U and V we shall denote the space of linear continuous operators from U to V by $L(U, V)$ and the space of Hilbert–Schmidt operators $F : U \to V$ by $L_{HS}(U, V)$, with norm

$$\|F\|_{L_{HS}(U,V)} := \left(\sum_{i=1}^{\infty} |Fe_i|_V^2\right)^{1/2}, \tag{1.5}$$

where $\{e_i\}$ is an orthonormal basis in U.

If $j : \mathbb{R} \to (-\infty, +\infty]$ is a lower semicontinuous convex function let $\partial j : \mathbb{R} \to 2^{\mathbb{R}}$ denote the subdifferential of j, that is,

$$\partial j(y) = \{\theta \in \mathbb{R} : j(y) \leq j(z) + \theta(y - z)\ \forall z \in \mathbb{R}\}$$

and let j^* denote the conjugate of j (the Legendre transform of j),

$$j^*(p) = \sup\{py - j(y) : y \in \mathbb{R}\}.$$

We recall that $\partial j^* = (\partial j)^{-1}$ (see, e.g., [2, 6]),

$$j(y) + j^*(p) = py \quad \text{if and only if} \quad p \in \partial j(y) \tag{1.6}$$

and

$$j(u) + j^*(p) \geq pu \qquad \text{for all } p, u \in \mathbb{R}. \tag{1.7}$$

Given a multivalued function $\Phi : \mathbb{R} \to 2^{\mathbb{R}}$, its domain is denoted by $D(\Phi) = \{u \in \mathbb{R} : \Phi(u) \neq \varnothing\}$. $R(\Phi) = \{v : v \in \Phi(u), u \in D(\Phi)\}$ is its range. The function Φ is said to be a maximal monotone graph if it is monotone, that is,

$$(y_1 - y_2)(p_1 - p_2) \geq 0 \qquad \text{for all } p_i \in \Phi(y_i), i = 1, 2$$

and $R(1 + \Phi) = \mathbb{R}$.

Given a maximal monotone graph $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$, there is a unique lower semicontinuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$ such that $\Psi := \partial j$. The function j is unique up to an additive constant and is called the *potential* of Ψ.

For the maximal monotone graph Ψ we set

$$\Psi_\lambda = \frac{1}{\lambda}\big(1 - (1 + \lambda\Psi)^{-1}\big) \in \Psi(1 + \lambda\Psi)^{-1}, \qquad \lambda > 0,$$

which is called the *Yosida* approximation of Ψ. Here 1 stands for the identity function. The function Ψ_λ is Lipschitzian and monotonically increasing.

We set $j_\lambda(u) = \int_0^u \Psi_\lambda(r)\,dr$ and recall that it is equal to the Moreau approximation of j, that is,

$$j_\lambda(u) = \min\left\{ j(v) + \frac{1}{2\lambda}|u - v|^2 : v \in \mathbb{R} \right\}. \tag{1.8}$$

We have

$$j_\lambda(u) = j\big((1 + \lambda\Psi)^{-1}u\big) + \frac{1}{2\lambda}|u - (1 + \lambda\Psi)^{-1}u|^2. \tag{1.9}$$

2. The main result.

2.1. *Hypotheses.* (H_1) $W(t)$ is a cylindrical Wiener process on $L^2(\mathcal{O})$ defined by

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \tag{2.1}$$

where $\{\beta_k\}$ is a sequence of mutually independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, with right-continuous filtration and $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$. To be more specific, $\{e_k\}$ will be chosen as the normalized sequence of eigenfunctions of the operator A, hence $e_k \in L^p(\mathcal{O})$ for all $k \in \mathbb{N}$, $p \geq 1$.

(H_2) B is Lipschitzian from $H = H^{-1}(\mathcal{O})$ to $L_{HS}(L^2(\mathcal{O}), D(A^\gamma))$, where $\gamma > d/2$.

(H_3) $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$ such that $0 \in \Psi(0)$,

$$D(\Psi) = \mathbb{R}, \qquad R(\Psi) = \mathbb{R} \tag{2.2}$$

and

$$\limsup_{|s| \to +\infty} \frac{j(-s)}{j(s)} < +\infty. \tag{2.3}$$

Here $j : \mathbb{R} \to \mathbb{R}$ is the potential of Ψ, that is, $\partial j = \Psi$, which under assumption (2.2) is a continuous convex function. Since $0 \in \Psi(0)$, by definition we have $j(0) = \inf j$. Hence subtracting $j(0)$ we can take j such that $j(0) = 0$ and $j \geq 0$ and therefore we may assume that $j^* \geq j^*(0) = 0$. We recall (see, e.g., [2, 6]) that the condition $R(\Psi) = \mathbb{R}$ is equivalent to

$$j(y) < \infty \ \forall y \in \mathbb{R} \qquad \lim_{|y| \to \infty} \frac{j(y)}{|y|} = +\infty, \tag{2.4}$$

while the condition $D(\Psi) = \mathbb{R}$ is equivalent to

$$j^*(y) < \infty \ \forall y \in \mathbb{R} \qquad \lim_{|y| \to \infty} \frac{j^*(y)}{|y|} = +\infty. \tag{2.5}$$

In particular, hypothesis (H_3) automatically holds if Ψ is a monotonically increasing, continuous function on $\mathbb{R}$ satisfying condition (1.3) and

$$\lim_{s \to +\infty} \Psi(s) = +\infty, \qquad \lim_{s \to -\infty} \Psi(s) = -\infty.$$

In particular, it is satisfied by functions Ψ satisfying (1.2) for $r > 0$ or, more generally, by those satisfying assumption (A1) in [15].

We need more notation. Given a Banach space Z,

$$C_W([0, T]; Z) = C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; Z))$$

shall denote the space of all continuous adapted stochastic processes which are mean square continuous. The space

$$L^2_W([0, T]; Z) = L^2([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P} : Z))$$

is similarly defined (see, e.g., [7, 9]).

DEFINITION 2.1. An adapted process $X \in C_W([0, T]; H) \cap L^1((0, T) \times \mathcal{O} \times \Omega)$, such that $X \in C^w([0, T], H)$, $\mathbb{P}$-a.s., is said to be a strong solution to (1.1) if there exists a process $\eta \in L^1((0,T)\times\mathcal{O}\times\Omega)$ such that

$$\eta(t,\xi) \in \Psi(X(t,\xi)) \qquad \text{a.e. } (t,\xi)\in Q_T, \mathbb{P}\text{-a.s.,} \tag{2.6}$$

$$\int_0^{\bullet} \eta(s)\,ds \in C^w([0,T]; H_0^1(\mathcal{O})), \tag{2.7}$$

$$X(t) - \Delta\int_0^t \eta(s)\,ds = x + \int_0^t B(X(s))\,dW(s) \qquad \forall t\in[0,T], \mathbb{P}\text{-a.s.,} \tag{2.8}$$

$$j(X), j^*(\eta) \in L^1\big((0,T)\times\mathcal{O}\times\Omega\big). \tag{2.9}$$

[Here $\int_0^t \eta(s)\,ds$ is initially defined as on $L^1(\mathcal{O})$-valued Bochner integral.] Of course, if Ψ is single-valued (2.6)–(2.8) reduce to

$$\int_0^{\bullet} \Psi(X(s))\,ds \in C^w([0,T]; H_0^1(\mathcal{O})) \tag{2.10}$$

and

$$X(t) - \Delta\int_0^t \Psi(X(s))\,ds = x + \int_0^t B(X(s))\,dW(s) \qquad \forall t\in[0,T], \mathbb{P}\text{-a.s.} \tag{2.11}$$

We note that X, as in Definition 2.1, is automatically predictable.

Theorem 2.2 below is the main result of this work.

THEOREM 2.2. *Under hypotheses* (H_1), (H_2), (H_3), *for each* $x\in H$ *there is a unique strong solution* $X = X(t,x)$ *to* (1.1). *Moreover, the following estimate holds*:

$$\mathbb{E}|X(t,x)-X(t,y)|_{-1}^2 \le C|x-y|_{-1}^2 \qquad \text{for all } t\ge 0, \tag{2.12}$$

where C *is independent of* $x, y \in H$.

Theorem 2.2 will be proved in Section 4 via fixed-point arguments. Before, in Section 3 we shall establish the existence of solutions for the equation

$$\begin{cases} dY(t) - \Delta\Psi(Y(t))\,dt = G(t)\,dW(t), & \text{in } Q_T,\\ \Psi(Y(t)) = 0, & \text{on } \Sigma_T,\\ Y(0) = x, & \text{in } \mathcal{O}, \end{cases} \tag{2.13}$$

where $G:[0,T]\to L_{HS}(L^2(\mathcal{O}), D(A^\gamma))$ is a predictable process such that

$$\mathbb{E}\int_0^T \|G(t)\|^2_{L_{HS}(L^2(\mathcal{O}),D(A^\gamma))}\,dt < +\infty \tag{2.14}$$

and $\gamma > d/2$. Here $G\,dW$ is given by

$$G\,dW = \sum_{k=1}^{\infty} Ge_k\,d\beta_k.$$

A solution of (2.13) is defined to be an adapted process Y satisfying along with $\eta\in L^1((0,T)\times\mathcal{O}\times\Omega)$ conditions (2.6)–(2.9) where $B(X)$ is replaced by G.

THEOREM 2.3. *Under hypotheses* (H_1), (H_3), (2.14), *for each* $x \in H$ *there is a unique strong solution* $Y = Y_G(t,x)$ *to* (2.13) *in the sense of Definition* 2.1. *Moreover, the following estimate holds*:

$$\begin{aligned}\mathbb{E}|Y_{G_1}(t,x) - Y_{G_2}(t,y)|_{-1}^2 \\ \le |x-y|_{-1}^2 + \mathbb{E}\int_0^t \|G_1(s) - G_2(s)\|^2_{L_{HS}(L^2(\mathcal{O}),H)}\,ds \qquad (2.15)\\ \text{for all } t \ge 0\end{aligned}$$

for all $x, y \in H$ *and* G_1, G_2 *satisfying* (2.14).

REMARK 2.4. It should be noted that assumption (H_2) excludes the case where the covariance operator B is of the form $B(X) = X$, that is, the case of multiplicative noise.

REMARK 2.5. Assumption (H_3), for example, allows monotonically increasing functions Ψ which are continuous from the right on $\mathbb{R}$ and have a finite number of jumps $r_1, r_2, \dots, r_N$. However in this case one must fill the jumps by replacing the function Ψ by the maximal monotone (multivalued) graph $\tilde{\Psi}(r) = \Psi(r)$ for r different from r_i and $\tilde{\Psi}(r_i) = [\Psi(r_i) - \Psi(r_{i-1} - 0)]$. Such a situation might arise in modeling underground water flows (see, e.g., [12]). In this case Ψ is the diffusivity function and (1.1) reduces to the Richards equation. It must also be said that Theorems 2.2 and 2.3 have natural extensions to equations of the form

$$dX(t) - \Delta\Psi(X(t))\,dt + \Phi(X(t))\,dt = B(X(t))\,dW(t), \qquad (2.16)$$

where Φ is a suitable monotonically increasing and continuous function (see [15]). We note also that stochastic models of self-organized criticality lead to equations of form (1.1) where $\Psi(r) = H(r - x_c)$ and H is the Heaviside function.

As in [15], one might consider the case where $\Psi = \Psi(X, \omega), \omega \in \Omega$, but we do not go into details here. We also note that assumption $D(\Psi) = \mathbb{R}$ in hypothesis (H_3) excludes a situation of the following type:

$$\Psi(s) = \begin{cases} \Psi_1(s), & \text{for } s < s_0,\ \Psi(s_0) = (\Psi_1(s_0), +\infty), \\ \varnothing, & \text{for } s > s_0, \end{cases} \qquad (2.17)$$

where Ψ_1 is a continuous monotonically increasing function and $\Psi_1(r) \to -\infty$ as $r \to -\infty$. In this case problem (1.1) reduces to a stochastic variational inequality and it is relevant in the description of saturation processes in infiltration. An analysis similar to that to be developed below shows that, in Definition 2.1, the function η is not an L^1-function but rather a bounded measure on Q_T and $\int_0^\bullet \eta(s)\,ds$ must be replaced by its distribution function. We plan to give details in a subsequent paper.

Another situation of interest covered by our assumptions (see also [15]) is that of logarithmic diffusion equations arising in plasma physics; for example, see [13]. In this case $\Psi(s) = \log(\mu + |s|)\operatorname{sign}(s)$.

3. Proof of Theorem 2.3. For every $\lambda > 0$ consider the approximating equation

$$(3.1)\qquad \begin{cases} dX_\lambda(t) - \Delta\big(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t)\big)\,dt & \\ \qquad = G(t)\,dW(t), & \text{in } (0,T)\times\mathcal{O} := Q_T,\\ \Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) = 0, & \text{on } (0,T)\times\partial\mathcal{O},\\ X_\lambda(0) = x, & \text{in } \mathcal{O}, \end{cases}$$

which has a unique solution $X_\lambda \in C_W([0,T];H)$ such that

$$X_\lambda, \Psi_\lambda(X_\lambda) \in L^2_W(0,T;H^1_0(\mathcal{O})).$$

Indeed, setting $y_\lambda(t) = X_\lambda(t) - W_G(t)$, where $W_G(t) = \int_0^t G(s)\,dW(s)$, we may rewrite (3.1) as a random equation

$$(3.2)\qquad \begin{cases} y'_\lambda(t) - \Delta\widetilde{\Psi}_\lambda\big(y_\lambda(t) + W_G(t)\big) = 0, & \mathbb{P}\text{-a.s. in } Q_T,\\ \widetilde{\Psi}_\lambda\big(y_\lambda(t) + W_G(t)\big) = 0, & \text{on } (0,T)\times\partial\mathcal{O},\\ y_\lambda(0) = x, & \text{in } \mathcal{O}, \end{cases}$$

where $\widetilde{\Psi}_\lambda(y) = \Psi_\lambda(y) + \lambda y$, $\lambda > 0$. Note that $\widetilde{\Psi}_\lambda(0) = 0$.

For each $\omega \in \Omega$ the operator $\Gamma(t): H^1_0(\mathcal{O}) \to H^{-1}(\mathcal{O})$, defined by

$$\Gamma(t)y = -\Delta\widetilde{\Psi}_\lambda\big(y + W_G(t)\big), \qquad y \in H^1_0(\mathcal{O}),$$

is continuous, monotone and coercive, that is,

$$\begin{aligned}(\Gamma(t)y, y) &\geq \lambda|y + W_G(t)|^2_{H^1_0(\mathcal{O})} - (\Gamma(t)y, W_G(t))\\ &\geq \frac{\lambda}{2}|y|^2_{H^1_0(\mathcal{O})} - C_\lambda|W_G(t)|^2_{H^1_0(\mathcal{O})}.\end{aligned}$$

Then, by classical existence theory for nonlinear equations (see, e.g., [11]), (3.2) has a unique solution

$$y_\lambda \in C([0,T];L^2(\mathcal{O})) \cap L^2(0,T;H^1_0(\mathcal{O}))$$

with $y'_\lambda \in L^2(0,T;H^{-1}(\mathcal{O}))$. The function $X_\lambda(t) = y_\lambda(t) + W_G(t)$ is, of course, an adapted process because the solution y_λ to (3.2) is a continuous function of W_G and so it satisfies the required condition.

3.1. *A priori estimates.* From now on we fix $\omega \in \Omega$ and work with the corresponding solution y_λ to (3.2). We have

$$(3.3)\qquad \begin{aligned}&\frac{1}{2}\frac{d}{dt}|y_\lambda(t)|^2_{-1} + \big(\widetilde{\Psi}_\lambda\big(y_\lambda(t) + W_G(t)\big), y_\lambda(t) + W_G(t)\big)\\ &\qquad = \big(\widetilde{\Psi}_\lambda\big(y_\lambda(t) + W_G(t)\big), W_G(t)\big),\end{aligned}$$

which is equivalent to

$$(3.4)\qquad \begin{aligned}&\frac{1}{2}\frac{d}{dt}|y_\lambda(t)|^2_{-1} + \big(\Psi_\lambda\big(y_\lambda(t) + W_G(t)\big), y_\lambda(t) + W_G(t)\big)\\ &\qquad = -\lambda\big(y_\lambda(t), y_\lambda(t) + W_G(t)\big) + \big(\Psi_\lambda\big(y_\lambda(t) + W_G(t)\big), W_G(t)\big).\end{aligned}$$

Now set $j_\lambda(u) = \int_0^u \Psi_\lambda(r)\,dr$ and let j_λ^* denote the conjugate of j_λ. Moreover, we set

$$z_\lambda = (1 + \lambda\Psi)^{-1}(y_\lambda + W_G), \qquad \eta_\lambda = \Psi_\lambda(j_\lambda + W_G). \tag{3.5}$$

The aim of this subsection is to prove the following estimate: There exists a (random) constant C_1 such that

$$\begin{aligned} &\frac{1}{2}|y_\lambda(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(j(z_\lambda) + j^*(\eta_\lambda)\big)\,d\xi\,ds \\ &\quad + \frac{1}{2\lambda}\int_0^t \int_{\mathcal{O}} (y_\lambda + W_G - z_\lambda)^2\,d\xi\,ds \le C_1(1 + |x|_{-1}^2), \qquad t \in [0, T]. \end{aligned} \tag{3.6}$$

By (1.6) we have

$$\begin{aligned} &j_\lambda^*\big(\Psi_\lambda\big(y_\lambda(t) + W_G(t)\big)\big) + j_\lambda\big(y_\lambda(t) + W_G(t)\big) \\ &\quad = \Psi_\lambda\big(y_\lambda(t) + W_G(t)\big)\big(y_\lambda(t) + W_G(t)\big). \end{aligned} \tag{3.7}$$

Substituting this identity into (3.4) yields

$$\begin{aligned} &\tfrac{1}{2}|y_\lambda(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \Big(j_\lambda\big(y_\lambda(s) + W_G(s)\big) + j_\lambda^*\big(\Psi_\lambda\big(y_\lambda(s) + W_G(s)\big)\big)\Big)\,d\xi\,ds \\ &\quad = \tfrac{1}{2}|x|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(\Psi_\lambda\big(y_\lambda(s) + W_G(s)\big)W_G(s)\big)\,d\xi\,ds \\ &\qquad - \lambda \int_0^t \int_{\mathcal{O}} y_\lambda(s)\big(y_\lambda(s) + W_G(s)\big)\,d\xi\,ds. \end{aligned} \tag{3.8}$$

Then, using (1.9), (3.5) and the fact that $j_\lambda^* \ge j^*$ for all $\lambda > 0$, by (3.8) we see that

$$\begin{aligned} &\frac{1}{2}|y_\lambda(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(j(z_\lambda(s)) + j^*(\eta_\lambda(s))\big)\,d\xi\,ds \\ &\quad + \frac{1}{2\lambda}\int_0^t \int_{\mathcal{O}} \big(y_\lambda(s) + W_G(s) - z_\lambda(s)\big)^2\,d\xi\,ds \\ &\quad \le \frac{1}{2}|x|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \eta_\lambda(s)W_G(s)\,d\xi\,ds \\ &\qquad - \lambda \int_0^t \int_{\mathcal{O}} y_\lambda(s)\big(y_\lambda(s) + W_G(s)\big)\,d\xi\,ds. \end{aligned} \tag{3.9}$$

We now estimate the first integral from the right-hand side of (3.9) as follows:

$$\left|\int_0^t \int_{\mathcal{O}} \eta_\lambda(s)W_G(s)\,d\xi\,ds\right| \le \delta \int_0^t \int_{\mathcal{O}} |\eta_\lambda(s)|\,d\xi\,ds, \tag{3.10}$$

where $\delta := \sup_{s\in[0,T]} |W_G(s)|_{L^\infty(\mathcal{O})} < +\infty$. We note that by assumption (2.14) and since $\gamma > d/2$ it follows by Sobolev embedding that $W_G(\cdot)$ has continuous sample paths in $D(A^\gamma) \subset L^\infty(\mathcal{O})$ and so δ is indeed finite.

Substituting (3.10) in (3.9) yields

$$\frac{1}{2}|y_\lambda(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(j(z_\lambda(s)) + j^*(\eta_\lambda(s))\big)\, d\xi\, ds$$
$$+ \frac{1}{2\lambda}\int_0^t \int_{\mathcal{O}} \big(y_\lambda(s) + W_G - z_\lambda(s)\big)^2\, d\xi\, ds$$
$$\leq \frac{1}{2}|x|_{-1}^2 + \delta \int_0^t \int_{\mathcal{O}} |\eta_\lambda(s)|\, d\xi\, ds - \lambda \int_0^t \int_{\mathcal{O}} y_\lambda(s)\big(y_\lambda(s) + W_G(s)\big)\, d\xi\, ds.$$

Since

$$-y_\lambda(s)\big(y_\lambda(s) + W_G(s)\big) \leq -\tfrac{1}{2}|y_\lambda(s)|^2 + \tfrac{1}{2}W_G^2(s),$$

we find

$$\begin{aligned} &\frac{1}{2}|y_\lambda(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(j(z_\lambda(s)) + j^*(\eta_\lambda(s))\big)\, d\xi\, ds \\ &\quad + \frac{\lambda}{2}\int_0^t \int_{\mathcal{O}} |y_\lambda(s)|^2\, d\xi\, ds \\ &\quad + \frac{1}{2\lambda}\int_0^t \int_{\mathcal{O}} \big(y_\lambda(s) + W_G(s) - z_\lambda(s)\big)^2\, d\xi\, ds \\ &\quad \leq \left(\frac{1}{2}|x|_{-1}^2 + \delta\int_0^t \int_{\mathcal{O}} |\eta_\lambda(s)|\, d\xi\, ds + \frac{\lambda}{2}\int_0^t \int_{\mathcal{O}} W_G^2(s)\, d\xi\, ds\right), \\ &\qquad t \in [0, T]. \end{aligned} \tag{3.11}$$

On the other hand, we recall that condition $D(\Psi) = \mathbb{R}$ is equivalent with

$$j^* < \infty \quad \text{and} \quad \lim_{|p|\to\infty} \frac{j^*(p)}{|p|} = +\infty. \tag{3.12}$$

So, there exists $N = N(\omega)$ such that

$$|\eta_\lambda(s)| > N \quad \Rightarrow \quad j^*(\eta_\lambda(s)) > 2C\delta|\eta_\lambda(s)|.$$

Consequently, for $C > |Q_T|$ we have that

$$\int_0^t \int_{\mathcal{O}} |\eta_\lambda(s)|\, d\xi\, ds = \int \int_{|\eta_\lambda(s)|>N} |\eta_\lambda(s)|\, d\xi\, ds + \int \int_{|\eta_\lambda(s)|\leq N} |\eta_\lambda(s)|\, d\xi\, ds$$
$$\leq \frac{1}{2C\delta}\int_0^t \int_{\mathcal{O}} j^*(\eta_\lambda(s))\, d\xi\, ds + NC\delta.$$

Substituting this into (3.11), since $j \geq 0$, we obtain (3.6), which in particular implies

$$\int_0^t \int_{\mathcal{O}} \big(j(z_\lambda(s)) + j^*(\eta_\lambda(s))\big)\, d\xi\, ds \leq C_1(1 + |x|_{-1}^2) \tag{3.13}$$

and

$$\int_0^t \int_{\mathcal{O}} (y_\lambda + W_G - z_\lambda)^2\, d\xi\, ds \leq 2\lambda C_1(1 + |x|_{-1}^2). \tag{3.14}$$

3.2. *Convergence for* $\lambda \to 0$. Since by (2.4) and (2.5)

$$\lim_{|u|\to\infty} j(u)/|u| = \infty, \qquad \lim_{|u|\to\infty} j^*(u)/|u| = \infty, \tag{3.15}$$

we deduce from (3.13) that the sequences $\{z_\lambda\}$ and $\{\eta_\lambda\}$ are bounded and equi-integrable in $L^1(Q_T)$. Then by the Dunford–Pettis theorem the sequences $\{z_\lambda\}$ and $\{\eta_\lambda\}$ are weakly compact in $L^1(Q_T)$. Hence along a subsequence, again denoted by λ, we have

$$z_\lambda \to z, \qquad \eta_\lambda \to \eta \qquad \text{weakly in } L^1(Q_T) \text{ as } \lambda \to 0. \tag{3.16}$$

Moreover, by (3.14) we see that $z = y + W_G$, where

$$y_\lambda \to y \qquad \text{weakly* in } L^\infty(0,T;H) \text{ and weakly in } L^1(Q_T). \tag{3.17}$$

Also note that by (3.2) we have for every $t \in [0,T]$

$$y_\lambda(t) - \Delta\left(\int_0^t \big(\eta_\lambda(s) + \lambda\big(y_\lambda(s) + W_G(s)\big)\big)\,ds\right) = x \tag{3.18}$$

and so the sequence $\{\int_0^\bullet (\eta_\lambda(s) + \lambda y_\lambda(s))\,ds\}$ is bounded in $L^\infty(0,T;H_0^1(\mathcal{O}))$. Hence, selecting a further subsequence if necessary [see (3.8)], we have

$$\lim_{\lambda\to 0}\int_0^\bullet \big(\eta_\lambda(s) + \lambda y_\lambda(s)\big)\,ds = \int_0^\bullet \eta(s)\,ds \qquad \text{weakly* in } L^\infty(0,T;H_0^1(\mathcal{O})). \tag{3.19}$$

So, by (3.18) we find

$$y(t) + A\int_0^t \eta(s)\,ds = x \qquad \text{a.e. } t \in [0,T]. \tag{3.20}$$

Since

$$\int_0^\bullet \eta(s)\,ds \in C([0,T];L^1(\mathcal{O})) \cap L^\infty(0,T;H_0^1(\mathcal{O})),$$

$t \mapsto \int_0^t \eta(s)\,ds$ is weakly continuous in $H_0^1(\mathcal{O})$ and therefore we infer that so is $t \mapsto A\int_0^t \eta(s)\,ds$ in H. Thus the function

$$\tilde{y}(t) := -A\int_0^t \eta(s)\,ds + x, \qquad t \in [0,T], \tag{3.21}$$

is an H-valued weakly continuous version of y. Furthermore, we claim that for $\lambda \to 0$

$$y_\lambda(t) \to \tilde{y}(t) \qquad \text{weakly in } H\ \forall t \in [0,T].$$

Indeed, since $\eta_\lambda \to \eta$ weakly in $L^1(Q_T)$ and $\lambda(y_\lambda + W_G) \to 0$ weakly in $L^1(Q_T)$ [since it even converges strongly in $L^2(Q_T)$ to zero by (3.11)], it follows that for every $t \in [0,T]$

$$\int_0^t \big(\eta_\lambda(s) + \lambda\big(y_\lambda(s) + W_G(s)\big)\big)\,ds \to \int_0^t \eta(s)\,ds \qquad \text{weakly in } L^1(\mathcal{O}).$$

Hence by (3.18) and the definition of $\tilde{\eta}$ we obtain that for every $t \in [0, T]$

$$(-\Delta)^{-1} y_\lambda(t) \to (-\Delta)^{-1} \tilde{y}(t) \qquad \text{weakly in } L^1(\mathcal{O}).$$

Since by (3.11) $y_\lambda(t)$, $\lambda > 0$, are bounded in H, the above immediately implies the claim.

From now on we shall consider this particular version $\tilde{y}$ of y defined in (3.21). For simplicity we denote it again by y; so we have

$$y_\lambda(t) \to y(t) \qquad \text{weakly in } H \ \forall t \in [0, T].$$

We can also rewrite (3.21) as

$$y_t(t) - \Delta \eta(t) = 0 \qquad \text{in } \mathcal{D}'(Q_T),\ y(0) = x. \tag{3.22}$$

Now we are going to show that

$$\eta(t, \xi) \in \Psi\big(y(t, \xi) + W_G(t, \xi)\big) \qquad \text{a.e. } (t, \xi) \in Q_T. \tag{3.23}$$

For this we shall need the following inequality:

$$\liminf_{\lambda \to 0} \int_{Q_T} y_\lambda \eta_\lambda \, d\xi \, dt \le \int_{Q_T} y\eta \, d\xi \, dt. \tag{3.24}$$

3.2.1. *Proof of* (3.24). We first recall (1.6), which yields

$$j_\lambda(y_\lambda + W_G) + j_\lambda^*(\eta_\lambda) = (y_\lambda + W_G)\eta_\lambda \qquad \text{a.e. in } Q_T$$

and so, by (1.9) and since $j_\lambda^* \ge j^*$, we have

$$j(y_\lambda + W_G) + j^*(\eta_\lambda) \le (y_\lambda + W_G)\eta_\lambda \qquad \text{a.e. in } Q_T,$$

which yields

$$\int_{Q_T} \big(j(y_\lambda + W_G) + j^*(\eta_\lambda)\big) \, d\xi \, dt \le \int_{Q_T} (y_\lambda + W_G)\eta_\lambda \, d\xi \, dt.$$

Since the convex functional

$$(z, \zeta) \to \int_{Q_T} \big(j(z) + j^*(\zeta)\big) \, d\xi \, dt$$

is lower semicontinuous on $L^1(Q_T) \times L^1(Q_T)$ (and consequently weakly lower semicontinuous on this space) we obtain that

$$\begin{aligned} &\int_{Q_T} \big(j(y + W_G) + j^*(\eta)\big) \, d\xi \, dt \\ &\qquad \le \liminf_{\lambda \to 0} \int_{Q_T} y_\lambda \eta_\lambda \, d\xi \, dt + \int_{Q_T} W_G \eta \, d\xi \, dt. \end{aligned} \tag{3.25}$$

Furthermore, by (3.6) and again by the weak lower semicontinuity of convex integrals in $L^1(Q_T)$, it follows that

$$j(y + W_G), \qquad j^*(\eta) \in L^1(Q_T). \tag{3.26}$$

On the other hand, since $j(u) + j^*(p) \geq up$ for all $u, p \in \mathbb{R}$ [see (1.7)], we have

$$(W_G + y)\eta \leq j(y + W_G) + j^*(\eta) \qquad \text{a.e. in } Q_T. \tag{3.27}$$

Moreover, by assumption (2.3) we see that for every $M > 0$ there exists $R = R(M) \geq 0$, such that

$$j(-y - W_G) \leq Mj(y + W_G) \qquad \text{on } Q^R,$$

where

$$Q^R = \{(t, \xi) \in Q_T : |y(t, \xi) + W_G(t, \xi)| \geq R\}.$$

Since $j(y + W_G) \in L^1(Q_T)$ we have, by continuity of j,

$$j(-y - W_G) \leq h \qquad \text{a.e. in } Q_T, \tag{3.28}$$

where $h \in L^1(Q_T)$. On the other hand, since j is bounded from below we have

$$j(-y - W_G) \in L^1(Q_T). \tag{3.29}$$

Taking into account that by virtue of the same inequality (1.7), besides (3.27), we have that

$$-(y + W_G)\eta \leq j(-y - W_G) + j^*(\eta) \qquad \text{a.e. in } Q_T, \tag{3.30}$$

by (3.27) and (3.28) it follows that a.e. in Q_T we have

$$|(W_G + y)\eta| \leq \max\{j(y + W_G) + j^*(\eta), j(-y - W_G) + j^*(\eta)\} \in L^1(Q_T)$$

and therefore $y\eta \in L^1(Q_T)$ as claimed [recall that $W_G \in L^\infty(Q_T)$].

Now we come back to (3.4), which by integration yields

$$\tfrac{1}{2}\big(|y_\lambda(T)|_{-1}^2 - |x|_{-1}^2\big) + \int_{Q_T} y_\lambda \eta_\lambda \, d\xi \, dt + \lambda \int_{Q_T} y_\lambda (y_\lambda + W_G) \, d\xi \, dt = 0. \tag{3.31}$$

Taking into account that

$$y_\lambda(T) \to y(T) \qquad \text{weakly in } H, \tag{3.32}$$

by (3.31) we have that

$$\liminf_{\lambda \to 0} \int_{Q_T} y_\lambda \eta_\lambda \, d\xi \, dt \leq -\tfrac{1}{2}\big(|y(T)|_{-1}^2 - |x|_{-1}^2\big). \tag{3.33}$$

In order to complete the proof one needs an integration by parts formula in (3.21) or (3.22), obtained by multiplying the equation by y and integrating on Q_T. Formally this is possible because $y\eta \in L^1(Q_T)$ and $y(t) \in H^{-1}(\mathcal{O})$ for all $t \in [0, T]$. But, in order to prove it rigorously, one must give sense to $(y'(t), y(t))$. Lemma 3.1 below answers this question positively and by (3.33) also proves (3.24).

We first note that since j, j^* are nonnegative and convex and $j(0) = 0 = j^*(0)$, we have for all measurable $f : Q_T \to \mathbb{R}$ and $\alpha \in [0, 1]$,

$$j(f) \in L^1(Q_T) \quad \Rightarrow \quad j(\alpha f) \in L^1(Q_T)$$

and

$$j^*(f) \in L^1(Q_T) \quad \Rightarrow \quad j^*(\alpha f) \in L^1(Q_T).$$

Furthermore, as in the proof of (3.28) by (2.3) we obtain that

$$j(f) \in L^1(Q_T) \quad \Rightarrow \quad j(-f) \in L^1(Q_T).$$

By (2.3) we see that the latter is also true for j^*, if $f \in L^1(Q_T)$ and α is small enough. Indeed by (2.3) there are $M, R > 0$ such that

$$j(-s) \le Mj(s), \qquad \text{if } |s| \ge R,$$

hence replacing s by $(-s)$ we get

$$\frac{1}{M} j(s) \le j(-s), \qquad \text{if } |s| \ge R.$$

Now an elementary calculation implies that for all $p \in \mathbb{R}$

$$j^*(-p) \le R|p| + \frac{1}{M} j^*(Mp).$$

Hence

$$j^*(-p/M) \le \frac{R}{M}|p| + \frac{1}{M} j^*(p).$$

Therefore for $\alpha := 1/M$ we have

$$0 \le j^*(-\alpha f) \le \frac{R}{M}|f| + \frac{1}{M} j^*(f) \in L^1(Q_T).$$

Hence, y and η constructed above fulfill all conditions in the following lemma since $W_G \in L^\infty(Q_T)$.

LEMMA 3.1. *Let $y \in C^w([0,T]; H^{-1}(\mathcal{O})) \cap L^1(Q_T)$ and $\eta \in L^1(Q_T) \cap L^\infty(0,T; H^1(\mathcal{O}))$ satisfy*

$$y(t) + A\int_0^t \eta(s)\,ds = x, \qquad t \in [0,T]. \tag{3.34}$$

Furthermore, assume that for some $\alpha > 0$, $j(\alpha y), j^*(\alpha\eta) \in L^1(Q_T)$. Then $y\eta \in L^1(Q_T)$,

$$\int_{Q_T} y\eta\,d\xi\,dt = -\tfrac{1}{2}\big(|y(T)|_{-1}^2 - |x|_{-1}^2\big) \tag{3.35}$$

and

$$Y_\varepsilon \Sigma_\varepsilon \to y\eta \qquad \text{in } L^1(Q_T),$$

where Y_ε, Σ_ε are defined in (3.36) below.

PROOF. We set for $\varepsilon > 0$

$$Y_\varepsilon = (1+\varepsilon A)^{-m} y, \qquad \Sigma_\varepsilon = (1+\varepsilon A)^{-m}\eta, \tag{3.36}$$

where $m \in \mathbb{N}$ is such that $m > \max\{2, (d+2)/4\}$. Then

$$Y_\varepsilon \in C^w\big([0,T]; H_0^1(\mathcal{O}) \cap H^{2m-1}(\mathcal{O})\big) \subset C^w\big([0,T]; H_0^1(\mathcal{O}) \cap C(\overline{\mathcal{O}})\big)$$

and

$$\Sigma_\varepsilon \in L^1(0,T; W^{2,q}(\mathcal{O})), \qquad 1 < q < \frac{d}{d-1}.$$

Hence $Y_\varepsilon \Sigma_\varepsilon \in L^1(Q_T)$ and for $\varepsilon \to 0$

$$\begin{cases} Y_\varepsilon(t) \to y(t), & \text{strongly in } H^{-1}(\mathcal{O})\ \forall t \in [0,T], \\ Y_\varepsilon \to y, & \text{strongly in } L^1(Q_T), \\ \Sigma_\varepsilon \to \eta, & \text{strongly in } L^1(Q_T), \\ \displaystyle\int_0^t \Sigma_\varepsilon(s)\,ds \to \int_0^t \eta(s)\,ds, & \text{strongly in } H_0^1(\mathcal{O})\ \forall t \in [0,T]. \end{cases} \tag{3.37}$$

Here we note that the last fact follows because (3.34) implies that $\int_0^\bullet \eta(s)\,ds \in C^w([0,T]; H_0^1(\mathcal{O}))$. We have also by (3.34)

$$Y_\varepsilon(t) + A\int_0^t \Sigma_\varepsilon(s)\,ds = (1+\varepsilon A)^{-m} x \qquad \forall t \in [0,T],$$

which implies

$$\frac{d}{dt} Y_\varepsilon(t) + A\Sigma_\varepsilon(t) = 0$$

and, taking the inner product in $H^{-1}(\mathcal{O})$ with $Y_\varepsilon(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}|Y_\varepsilon(t)|_{-1}^2 + \int_{\mathcal{O}} \Sigma_\varepsilon(t) Y_\varepsilon(t)\,d\xi = 0 \qquad \text{a.e. } t \in [0,T].$$

Hence

$$\lim_{\varepsilon \to 0} \int_{Q_T} \Sigma_\varepsilon(t) Y_\varepsilon(t)\,d\xi\,dt = -\tfrac{1}{2}\big(|y(T)|_{-1}^2 - |x|_{-1}^2\big) \tag{3.38}$$

and by (3.37) we may assume that for $\varepsilon \to 0$

$$Y_\varepsilon \to y, \qquad \Sigma_\varepsilon \to \eta \qquad \text{a.e. in } Q_T. \tag{3.39}$$

Moreover by (1.7) we have

$$\alpha^2 \Sigma_\varepsilon Y_\varepsilon \le j(\alpha Y_\varepsilon) + j^*(\alpha \Sigma_\varepsilon), \qquad -\alpha^2 \Sigma_\varepsilon Y_\varepsilon \le j(-\alpha Y_\varepsilon) + j^*(\alpha \Sigma_\varepsilon) \qquad \text{a.e. in } Q_T. \tag{3.40}$$

Now we claim that for $\varepsilon \to 0$

$$j(\alpha Y_\varepsilon) \to j(\alpha y), \qquad j^*(\alpha\Sigma_\varepsilon) \to j^*(\alpha\eta),$$
$$j(-\alpha Y_\varepsilon) \to j(-\alpha y) \qquad \text{in } L^1(Q_T). \tag{3.41}$$

By (3.39) these convergences hold a.e. in Q_T. So, in order to prove (3.41) it suffices to show that $\{j(\alpha Y_\varepsilon)\}, \{j^*(\alpha\Sigma_\varepsilon)\}, \{j(-\alpha Y_\varepsilon)\}$ are equi-integrable on Q_T and that they are weakly compact in $L^1(Q_T)$. To this end let $y \in L^1(\mathcal{O})$ and let $Y_\varepsilon := (1+\varepsilon A)^{-1}y$, that is, Y_ε is the solution to the Dirichlet problem

$$\begin{cases} Y_\varepsilon - \varepsilon\Delta Y_\varepsilon = y, & \text{in } \mathcal{O}, \\ Y_\varepsilon = 0, & \text{on } \partial\mathcal{O}. \end{cases} \tag{3.42}$$

It may be represented as

$$Y_\varepsilon(\xi) = \int_{\mathcal{O}} G(\xi,\xi_1) y(\xi_1)\, d\xi_1 \qquad \forall \xi \in \mathcal{O}, \tag{3.43}$$

where G is the associated Green function. It is well known that $\int_{\mathcal{O}} G(\xi,\xi_1)\, d\xi_1$ is the solution to (3.42) with $y = 1$ so that by the maximum principle we have $0 < \int_{\mathcal{O}} G(\xi,\xi_1)\, d\xi_1 \le 1$ for all $\xi \in \mathcal{O}$.

We may rewrite Y_ε as

$$Y_\varepsilon(\xi) = \int_{\mathcal{O}} G(\xi,\xi_2)\, d\xi_2 \int_{\mathcal{O}} \tilde{G}(\xi,\xi_1) y(\xi_1)\, d\xi_1 \qquad \forall \xi \in \mathcal{O},$$

where

$$\tilde{G}(\xi,\xi_1) = \frac{G(\xi,\xi_1)}{\int_{\mathcal{O}} G(\xi,\xi_2)\, d\xi_2}$$

and so $\int_{\mathcal{O}} \tilde{G}(\xi,\xi_1)\, d\xi_1 = 1$ for all $\xi \in \mathcal{O}$.

Then, if $j(y) \in L^1(\mathcal{O})$, $j(0) = 0$ by Jensen's inequality we have

$$\begin{aligned} j(Y_\varepsilon(\xi)) &\le \int_{\mathcal{O}} G(\xi,\xi_2)\, d\xi_2 \int_{\mathcal{O}} \tilde{G}(\xi,\xi_1) j(y(\xi_1))\, d\xi_1 \\ &= \int_{\mathcal{O}} G(\xi,\xi_1) j(y(\xi_1))\, d\xi_1 \qquad \forall \xi \in \mathcal{O}. \end{aligned}$$

Hence we have shown that for any $y \in L^1(\mathcal{O})$ with $j(y) \in L^1(\mathcal{O})$,

$$j\big((1+\varepsilon A)^{-1}y\big) \le (1+\varepsilon A)^{-1} j(y).$$

Iterating and using the fact that $(1+\varepsilon A)^{-1}$ preserves positivity we get for all $m \in \mathbb{N}$

$$j\big((1+\varepsilon A)^{-m}y\big) \le (1+\varepsilon A)^{-m} j(y) \qquad \text{a.e. in } \mathcal{O}. \tag{3.44}$$

Now let y be as in the assertion of the lemma and Y_ε as in (3.36). Integrating over Q_T, since $(1+\varepsilon A)^{-m}$ is a contraction on $L^1(\mathcal{O})$, (3.44) applied to αy implies

$$\int_{Q_T} j(\alpha Y_\varepsilon(\xi,t))\, d\xi\, dt \le \int_{Q_T} j(\alpha y(\xi_2,t))\, d\xi_2\, dt.$$

Taking into account that $j(\alpha y) \in L^1(Q_T)$ we infer that $\{j(\alpha Y_\varepsilon)\}$ is equi-integrable on Q_T. The same argument applies to $\{j^*(\alpha \Sigma_\varepsilon)\}, \{j(-\alpha Y_\varepsilon)\}$.

Then (3.40) implies that the sequence $\{\Sigma_\varepsilon Y_\varepsilon\}$ is equi-integrable on Q_T and consequently by the Dunford–Pettis theorem, weakly compact in $L^1(Q_T)$. Since $\{\Sigma_\varepsilon Y_\varepsilon\}$ is a.e. convergent to $y\eta$ we infer that for $\varepsilon \to 0$

$$\Sigma_\varepsilon Y_\varepsilon \to y\eta \qquad \text{strongly in } L^1(Q_T), \tag{3.45}$$

which, combined with (3.38), implies (3.35) as desired. □

3.2.2. *Proof of* (3.23). We have

$$j(z_\lambda) - j(u) \le \eta_\lambda (z_\lambda - u) \qquad \forall u \in \mathbb{R} \text{ a.e. in } Q_T.$$

Integrating over Q_T yields

$$\int_{Q_T} j(z_\lambda)\, d\xi\, dt \le \int_{Q_T} j(u)\, d\xi\, dt + \int_{Q_T} \eta_\lambda (z_\lambda - u)\, d\xi\, dt \qquad \forall u \in L^\infty(Q_T).$$

Note that by the definition of Ψ_λ we have

$$z_\lambda = -\lambda \eta_\lambda + y_\lambda + W_G.$$

Therefore, since $z = y + W_G$, by (3.24) and Fatou's lemma we can let $\lambda \to 0$ to obtain

$$\int_{Q_T} j(z)\, d\xi\, dt - \int_{Q_T} j(u)\, d\xi\, dt \le \int_{Q_T} \eta(z - u)\, d\xi\, dt \qquad \forall u \in L^\infty(Q_T).$$

Now by Lusin's theorem for each $\epsilon > 0$ there is a compact subset $Q_\epsilon \subset Q_T$ such that $(d\xi \otimes dt)(Q_T \setminus Q_\epsilon) \le \epsilon$ and y, η are continuous on Q_ϵ. Let (t_0, x_0) be a Lebesgue point for y, η and $y\eta$ and let B_r be the ball of center (t_0, x_0) and radius r. We take

$$u(t, \xi) = \begin{cases} z(t, \xi), & \text{if } (t, \xi) \in Q_\epsilon \cap B_r^c, \\ v, & \text{if } (t, \xi) \in (Q_\epsilon \cap B_r) \cup (Q_T \setminus Q_\epsilon). \end{cases}$$

Here v is arbitrary in $\mathbb{R}$. Since u is bounded we can substitute into the above inequality to get

$$\int_{B_r \cap Q_\epsilon} \big(j(z) - j(v) - \eta(z - v)\big)\, d\xi\, dt \le \int_{Q_T \setminus Q_\epsilon} \big(\eta(z - v) + j(v) - j(z)\big)\, d\xi\, dt.$$

Letting $\epsilon \to 0$ we obtain that

$$\int_{B_r} \big(j(z) - j(v) - \eta(z - v)\big)\, d\xi\, dt \le 0 \qquad \forall v \in \mathbb{R}, r > 0.$$

This yields

$$j(z(t_0, x_0)) \le j(v) + \eta(t_0, x_0)\big(z(t_0, x_0) - v\big) \qquad \forall v \in \mathbb{R}$$

and therefore $\eta(t_0, x_0) \in \partial j(z(t_0, x_0)) = \Psi(z(t_0, x_0))$. Since almost all points of Q_T are Lebesgue points we get (3.23) as claimed.

3.3. *Completion of proof of Theorem* 2.3. Let us first summarize what we have proved for the pair $(y,\eta)\in L^1(Q_T)\times L^1(Q_T)$. We have

$$y\in C^w([0,T];H),\qquad \int_0^\bullet \eta(s)\,ds\in C^w([0,T];H_0^1(\mathcal{O})),$$

$$\eta(t,\xi)\in\Psi(y(t,\xi))\qquad \text{for a.e. } (t,\xi)\in Q^T,$$

$$y(t)+A\int_0^t\eta(s)\,ds=x,\qquad t\in[0,T],$$

$$j(\alpha y),\ j^*(\alpha y)\in L^1(Q_T)\qquad \text{for some } \alpha\in(0,1].$$

We claim that (y,η) is the only such pair. Indeed, if $(\tilde{y},\tilde{\eta})$ is another, then

$$j\biggl(\frac{\alpha}{2}(y-\tilde{y})\biggr)\le\frac{1}{2}j(\alpha y)+\frac{1}{2}j(-\alpha\tilde{y})$$

and

$$j^*\biggl(\frac{\alpha}{2}(y-\tilde{y})\biggr)\le\frac{1}{2}j^*(\alpha y)+\frac{1}{2}j^*(-\alpha\tilde{y}).$$

But as we have seen before Lemma 3.1 the right-hand sides are in $L^1(Q_T)$. Hence $y-\tilde{y}$, $\eta-\tilde{\eta}$ fulfill all conditions of Lemma 3.1 and adopting the notation from there we have for $\varepsilon>0$

$$\begin{aligned}Y_\varepsilon(t)-\tilde{Y}_\varepsilon(t)&=\Delta\int_0^t\bigl(\Sigma_\varepsilon(s)-\tilde{\Sigma}_\varepsilon(s)\bigr)\,ds\\&=\int_0^t\Delta\bigl(\Sigma_\varepsilon(s)-\tilde{\Sigma}_\varepsilon(s)\bigr)\,ds,\qquad t\in[0,T].\end{aligned}$$

Differentiating and subsequently taking the inner product in H with $Y_\varepsilon(t)-\tilde{Y}_\varepsilon(t)$ and integrating again we arrive at

$$\begin{aligned}&\tfrac{1}{2}\bigl|(1+\varepsilon A)^{-m}\bigl(Y_\varepsilon(t)-\tilde{Y}_\varepsilon(t)\bigr)\bigr|_{-1}^2\\&\quad=\int_0^t\int_{\mathcal{O}}\bigl(Y_\varepsilon(s)-\tilde{Y}_\varepsilon(s)\bigr)\bigl(\Sigma_\varepsilon(s)-\tilde{\Sigma}_\varepsilon(s)\bigr)\,d\xi\,ds\\&\quad=\int_0^t\int_{\mathcal{O}}(1+\varepsilon A)^{-m}\bigl(y(s)-\tilde{y}(s)\bigr)(1+\varepsilon A)^{-m}\bigl(\eta(s)-\tilde{\eta}(s)\bigr)\,d\xi\,ds,\\&\hspace{20em} t\in[0,T].\end{aligned}$$

Letting $\varepsilon\to0$ and applying Lemma 3.1 we obtain that for $t\in[0,T]$

$$\tfrac{1}{2}|y(t)-\tilde{y}(t)|_{-1}^2=\int_0^t\int_{\mathcal{O}}\bigl(y(s)-\tilde{y}(s)\bigr)\bigl(\eta(s)-\tilde{\eta}(s)\bigr)\,d\xi\,ds\le0$$

by the monotonicity of Ψ.

Now let us consider the ω-dependence of y and η. By (3.21), (3.23) we know that $y = y(t, \xi, \omega)$ is the solution to the equation

$$\begin{cases} y'(t) - \Delta\Psi\big(y(t) + W_G(t)(\omega)\big) = 0, & \text{a.e. } t \in [0, T], \\ y(0) = x \end{cases} \tag{3.46}$$

and as seen earlier for $\eta = \eta(t, \xi, \omega)$ as in (3.16)

$$\begin{aligned} &y \in C^w([0, T]; H) \cap L^1(Q_T), \qquad \eta \in L^1(Q_T), \\ &\int_0^\bullet \eta(s)\, ds \in C^w([0, T]; H_0^1(\mathcal{O})) \end{aligned} \tag{3.47}$$

and

$$\eta(t, \xi, \omega) \in \Psi(y(t, \xi, \omega)) + W_G(t, \xi, \omega) \qquad \text{a.e. } (t, \xi, \omega) \in Q_T \times \Omega. \tag{3.48}$$

By the above uniqueness of (y, η), it follows that for any sequence $\lambda \to \infty$ we have $\mathbb{P}$-a.s.

$$\begin{aligned} y_\lambda(t) &\to y(t) && \text{weakly in } H = H^{-1}(\mathcal{O})\ \forall t \in [0, T], \\ y_\lambda &\to y && \text{weakly in } L^1(Q_T), \\ \int_0^t \eta_\lambda(s)\, ds &\to \int_0^t \eta(s)\, ds && \text{weakly in } L^1(\mathcal{O})\ \forall t \in [0, T] \\ & && \text{and weakly in } H_0^1(\mathcal{O}) \text{ a.e. } t \in [0, T], \\ \eta_\lambda &\to \eta && \text{weakly in } L^1(Q_T). \end{aligned}$$

Therefore y and η are strong $L^1(Q_T)$-limits of a sequence of convex combinations of y_λ, η_λ, respectively, and since y_λ and η_λ are predictable processes, it follows that so are y and η. In particular, this means that $Y(t) = y(t) + W_G(t)$ is an H-valued weakly continuous adapted process and that the following equation is satisfied:

$$Y(t) - \Delta \int_0^t \eta(s)\, ds = x + \int_0^t G(s)\, dW(s), \qquad t \in [0, T]. \tag{3.49}$$

Equivalently

$$\begin{cases} dY(t) - \Delta\Psi(Y(t))\, dt = G(t)\, dW(t), \\ Y(0) = x. \end{cases} \tag{3.50}$$

In order to prove that Y is a solution of (3.50) in the sense of Definition 2.1 with $G(t)$ replacing $B(X(t))$ and to prove uniqueness and some energy estimates for solutions to (3.50) we need an Itô's formula-type result. As in the case of Lemma 3.1 the difficulty is that the integral

$$\int_{Q_T} \Psi(Y) Y\, d\xi\, dt$$

might not be (in general) well defined, taking into account that $\Psi(Y), Y \in L^1(Q_T)$ only. We have, however, the following.

LEMMA 3.2. *Let Y be an H-valued weakly continuous adapted process satisfying* (3.49). *Then the following equality holds*:

$$\begin{aligned}\tfrac{1}{2}|Y(t)|_{-1}^2 = \tfrac{1}{2}|x|_{-1}^2 &- \int_0^t \int_{\mathcal{O}} \eta(s)Y(s)\,d\xi\,ds \\ &+ \int_0^t \langle Y(s), G(s)\,dW(s)\rangle_{-1} \\ &+ \tfrac{1}{2}\int_0^t \|G(s)\|_{L_{HS}(L^2(\mathcal{O}),H)}^2\,ds, \qquad \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.51}$$

Furthermore, $Y \in C_W([0,T];H) \cap L^1((0,T)\times\mathcal{O}\times\Omega)$, *and* $\eta \in L^1((0,T)\times\mathcal{O}\times\Omega)$ *and all conditions* (2.6)–(2.9) *are satisfied.*

PROOF. By Lemma 3.1 we have that $Y\eta \in L^1(Q_T)$. Next we introduce the sequences (see the proof of Lemma 3.1) for $m \in \mathbb{N}$

$$Y_\varepsilon = (1+\varepsilon A)^{-m}Y, \qquad \Sigma_\varepsilon = (1+\varepsilon A)^{-m}\eta.$$

For large enough m we can apply Itô's formula to the problem

$$\begin{cases} dY_\varepsilon(t) + A\Sigma_\varepsilon(t) = (1+\varepsilon A)^{-m}G\,dW(t), \\ Y_\varepsilon(0) = (1+\varepsilon A)^{-m}x = x_\varepsilon. \end{cases} \tag{3.52}$$

We have

$$\begin{aligned}\tfrac{1}{2}|Y_\varepsilon(t)|_{-1}^2 = \tfrac{1}{2}|x_\varepsilon|_{-1}^2 &- \int_0^t \int_{\mathcal{O}} \Sigma_\varepsilon(s)Y_\varepsilon(s)\,d\xi\,ds \\ &+ \int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s)\,dW(s)\rangle_{-1} \\ &+ \tfrac{1}{2}\int_0^t \|G_\varepsilon(s)\|_{L_{HS}(L^2(\mathcal{O}),H)}^2\,ds, \qquad t\in[0,T],\end{aligned} \tag{3.53}$$

where $G_\varepsilon = (1+\varepsilon A)^{-m}G$. Letting $\varepsilon \to 0$ [since $W_G \in L^\infty(Q_T)$] we get by (3.45)

$$\int_{Q_T} Y_\varepsilon \Sigma_\varepsilon\,d\xi\,ds \to \int_{Q_T} Y\eta\,d\xi\,ds, \qquad \mathbb{P}\text{-a.s.}$$

Furthermore

$$Y_\varepsilon(t) \to Y(t) \qquad \text{strongly in } H^{-1}(\mathcal{O})\ \forall t\in[0,T],$$

which by virtue of (3.53) yields (3.51), if we prove first that for $t \in [0,T]$

$$\mathbb{P} - \lim_{\varepsilon\to 0}\int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s)\,dW(s)\rangle = \int_0^t \langle Y(s), G(s)\,dW(s)\rangle. \tag{3.54}$$

We shall even show that this convergence in probability is locally uniform in t. We have by a standard consequence of the Burkholder–Davis–Gundy inequality for

$p = 1$ (see, e.g., [14], Corollary D-0.2) that for $\bar{Y}_\varepsilon := (1+\varepsilon A)^{-2m} Y$ and $\delta_1, \delta_2 > 0$

$$\begin{aligned}(3.55)\quad &\mathbb{P}\biggl[\sup_{t\in[0,T]}\biggl|\int_0^t \langle Y(s), G(s)\,dW(s)\rangle - \int_0^t \langle Y_\varepsilon(s), G_\varepsilon(s)\,dW(s)\rangle\biggr| \geq \delta_1\biggr]\\ &\quad \leq \frac{3\delta_2}{\delta_1} + \mathbb{P}\biggl[\int_0^T \|G(s)\|^2_{L_{HS}(L^2(\mathcal{O}),H)} |Y(s) - \overline{Y}_\varepsilon(s)|^2_{-1}\,ds \geq \delta_2\biggr].\end{aligned}$$

Since $Y \in C^w([0,T]; H)$, $\mathbb{P}$-a.s. and $(1+\varepsilon A)^{-1}$ is a contraction on H we have

$$\sup_{s\in[0,T]} |Y(s) - \overline{Y}_\varepsilon(s)|_{-1} \leq 2 \sup_{s\in[0,T]} |Y(s)|^2_{-1}, \qquad \mathbb{P}\text{-a.s.}$$

Hence by (2.14) the second term on the right-hand side of (3.55) converges to zero as $\varepsilon \to 0$. Taking subsequently $\delta_2 \to 0$, (3.55) implies (3.54). We emphasize that, since the left-hand size of (3.51) is not continuous $\mathbb{P}$-a.s. (though all terms on the right-hand side are), the $\mathbb{P}$-zero set of $\omega \in \Omega$ for which (3.51) does not hold might depend on t.

Next we want to prove that

$$(3.56)\qquad \mathbb{E}\biggl[\sup_{t\in[0,T]} |Y(t)|^2_{-1}\biggr] < \infty.$$

To this end first note that by (3.48) and (1.6) we have

$$(3.57)\qquad \eta(s)Y(s) = j(Y(s)) + j^*(\eta(s)) \geq 0,$$

hence (3.51) implies that for every $t \in [0,T]$

$$(3.58)\quad |Y(t)|^2_{-1} \leq |x|^2_{-1} + N_t + \int_0^t \|G(s)\|^2_{L_{HS}(L^2(\mathcal{O}),H)}\,ds, \qquad \mathbb{P}\text{-a.s.},$$

where

$$N_t := \int_0^t \langle Y(s), G(s)\,dW(s)\rangle_{-1}, \qquad t \geq 0,$$

is a continuous local martingale such that

$$\langle N\rangle_t = 2\int_0^t |G^*(s)Y(s)|^2_{L^2(\mathcal{O})}\,ds, \qquad t \geq 0,$$

where $G^*(s)$ is the adjoint of $G(s) : L^2(\mathcal{O}) \to H$. We shall prove that

$$(3.59)\qquad \mathbb{E}\biggl[\sup_{t\in[0,T]} |N_t|\biggr] < +\infty.$$

By the Burkholder–Davis–Gundy inequality for $p = 1$ applied to the stopping times

$$\tau_N := \inf\{t \geq 0 : |N_t| \geq N\} \wedge T, \qquad N \in \mathbb{N},$$

we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,\tau_N]}|N_t|\Big]$$

$$\le 3E\Big[\sup_{s\in[0,\tau_N]}|Y(s)|_{-1}\Big(4\int_0^{\tau_N}\|G(s)\|^2_{L_{HS}(L^2(\mathcal{O});H)}\,ds\Big)^{1/2}\Big] \tag{3.60}$$

$$\le 6C\Big(\mathbb{E}\Big[\sup_{s\in[0,\tau_N]}|Y(s)|^2_{-1}\Big]\Big)^{1/2},$$

where

$$C := \Big(\mathbb{E}\Big[\int_0^T \|G(s)\|^2_{L_{HS}(L^2(\mathcal{O});H)}\,ds\Big]\Big)^{1/2} < \infty.$$

Since $Y \in C^w([0,T];H)$, we know that $s \mapsto |Y(s)|^2_{-1}$ is lower semicontinuous. Therefore by (3.58)

$$\sup_{s\in[0,\tau_N]}|Y(s)|^2_{-1} = \sup_{s\in[0,\tau_N]\cap\mathbb{Q}}|Y(s)|^2_{-1}$$

$$\le |x|^2_{-1} + \sup_{s\in[0,\tau_N]}|N_s| + \int_0^T \|G(s)\|^2_{L_{HS}(L^2(\mathcal{O});H)}\,ds, \qquad \mathbb{P}\text{-a.s.}$$

So (3.60) implies that for all $N \in \mathbb{N}$

$$\Big(\mathbb{E}\Big[\sup_{t\in[0,\tau_N]}|N_t|\Big]\Big)^2 \le 36C^2\Big[|x|^2_{-1} + \mathbb{E}\Big[\sup_{s\in[0,\tau_N]}|N_s|\Big] + C^2\Big],$$

which entails that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,\tau_N]}|N_t|\Big] < \infty.$$

By monotone convergence this implies (3.59), since N_t has continuous sample paths and $\tau_N \uparrow T$ as $N \to \infty$. Now (3.58) implies that (4.11) holds.

By (3.59), (3.57) and (3.51) it follows that

$$\eta Y \in L^1\big((0,T)\times\mathcal{O}\times\Omega\big). \tag{3.61}$$

Hence by (3.57)

$$j(Y),\, j^*(\eta) \in L^1\big((0,T)\times\mathcal{O}\times\Omega\big)$$

and therefore

$$Y, \eta \in L^1\big((0,T)\times\mathcal{O}\times\Omega\big).$$

Taking the expectation in (3.51) we see that $t \mapsto \mathbb{E}[|Y(t)|^2_{-1}]$ is continuous. Since $Y \in C^w([0,T];H)$, $\mathbb{P}$-a.s., (3.61) then also implies that $Y \in C_W([0,T];H)$. This in turn, together with (3.49), also implies that (2.7) holds. □

Now we come back to the proof of Theorem 2.3. We first note that Lemma 3.2 also implies the uniqueness of the solution Y and estimate (2.15). Indeed by (3.55) and the monotonicity of Ψ we have for YG_i, $i = 1, 2$, the estimate (2.15). This concludes the proof of Theorem 2.3.

4. Proof of Theorem 2.2. Consider the space

$$\mathcal{K} = \Big\{ X \in C_W([0,T]; H) \cap L^1\big((0,T) \times \mathcal{O} \times \Omega\big) : X \text{ predictable}, \sup_{t \in [0,T]} \mathbb{E}[e^{-2\alpha t} |X(t)|_{-1}^2] \le M_1^2, \mathbb{E} \int_{Q_T} j(X(s))\, d\xi\, ds \le M_2 \Big\}, \tag{4.1}$$

where $\alpha > 0$, $M_1 > 0$ and $M_2 > 0$ will be specified later.

The space $\mathcal{K}$ is endowed with the metric induced by the norm

$$\|X\|_\alpha = \Big(\sup_{t \in [0,T]} \mathbb{E}[e^{-2\alpha t} |X(t)|_{-1}^2] \Big)^{1/2}.$$

Note that $\mathcal{K}$ is closed in the norm $\|\cdot\|_\alpha$. Indeed, if $X_n \to X$ in $\|\cdot\|_\alpha$ then since

$$\mathbb{E} \int_{Q_T} j(X_n(s))\, d\xi\, ds \le M_2 \qquad \forall n \in \mathbb{N},$$

(3.15) implies that

$$X_n \to X \qquad \text{in } L^1\big((0,T) \times \mathcal{O} \times \Omega\big)$$

and by Fatou's lemma we get

$$\mathbb{E} \int_{Q_T} j(X(s))\, d\xi\, ds \le M_2$$

as claimed. Now consider the mapping $\Gamma : \mathcal{K} \to C_W([0,T]; H) \cap L^1((0,T) \times \mathcal{O} \times \Omega)$ defined by

$$Y = \Gamma(X), \tag{4.2}$$

where $Y \in C_W([0,T]; H) \cap L^1((0,T) \times \mathcal{O} \times \Omega)$ is the solution in the sense of Definition 2.1 of the problem

$$\begin{cases} dY(t) - \Delta\Psi(Y(t))\, dt = B(X(t))\, dW(t), & \text{in } Q_T, \\ \Psi(Y(t)) = 0, & \text{on } \Sigma_T, \\ Y(0) = x, & \text{in } \mathcal{O}. \end{cases} \tag{4.3}$$

We shall prove that for α, M_1, M_2 suitably chosen, Γ maps $\mathcal{K}$ into itself and it is a contraction in the norm $\|\cdot\|_\alpha$.

Indeed by (3.51) and (1.6) for any solution Y to (4.3) we have that

$$\tfrac{1}{2}|Y(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} \big(j(Y(s)) + j^*(\eta(s))\big)\, d\xi\, ds$$

$$= \int_0^t \langle Y(s), B(X(s))\, dW(s)\rangle_{-1}$$

$$+ \tfrac{1}{2}\int_0^t \|B(X(s))\|^2_{L_{HS}(L^2(\mathcal{O}),H)}\, ds + \tfrac{1}{2}|x|_{-1}^2, \qquad t \in [0,T], \mathbb{P}\text{-a.s.}$$

By hypothesis (H_2) we have

$$\frac{1}{2}\sup_{t\in[0,T]} \mathbb{E}[e^{-2\alpha t}|Y(t)|_{-1}^2] + e^{-2\alpha t}\mathbb{E}\int_0^t \int_{\mathcal{O}} \big(j(Y(s)) + j^*(\eta(s))\big)\, d\xi\, ds$$

$$\le \frac{1}{2}|x|_{-1}^2 + \frac{L^2}{2}\sup_{t\in[0,T]}\bigg[e^{-2\alpha t}\int_0^t \mathbb{E}|X(s)|_{-1}^2\, ds\bigg]$$

$$\le \frac{1}{2}|x|_{-1}^2 + \frac{L^2}{2}\sup_{t\in[0,T]}\int_0^t e^{-2\alpha(t-s)}\mathbb{E}e^{-2\alpha s}|X(s)|_{-1}^2\, ds \le \frac{1}{2}|x|_{-1}^2 + \frac{L^2M_1^2}{4\alpha}.$$

Hence

$$\sup_{t\in[0,T]} \mathbb{E}[e^{-2\alpha t}|Y(t)|_{-1}^2] \le \frac{L^2M_1^2}{2\alpha} + |x|_{-1}^2$$

and

$$\mathbb{E}\int_{Q_T} \big(j(Y(s)) + j^*(\eta(s))\big)\, d\xi \le \bigg(\frac{L^2M_1^2}{2\alpha} + |x|_{-1}^2\bigg)e^{2\alpha T}.$$

Hence for $\alpha > L^2$, $M_1^2 > 2|x|_{-1}^2$ and $M_2 \ge M_1^2 e^{2\alpha T}$ we have that $Y \in \mathcal{K}$ and the operator Γ maps $\mathcal{K}$ into itself. By a similar computation involving hypothesis (H_2) we see that for M_1, M_2 and α suitably chosen we have

$$\|Y_1 - Y_2\|_\alpha \le \frac{C}{\sqrt{\alpha}}\|X_1 - X_2\|_\alpha, \tag{4.4}$$

where $Y_i = \Gamma X_i$, $i = 1, 2$. Hence for a suitable α, Γ is a contraction and so equation $X = \Gamma(X)$ has a unique solution in Γ. This completes the proof.

Acknowledgments. The authors are indebted to the anonymous referee for a number of useful suggestions which led to an improved presentation of this paper.

A part of this work was done during a very pleasant stay of the last named author at the Rumanian Academy of Sciences in Iasi whose warm hospitality is gratefully acknowledged.

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