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Existence of weak solution of Navier-Stokes-Fourier System with a new successive approximation method

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Abstract. In this paper we prove the existence of a weak solution of the complete compressible Navier-Stokes system. We follow an previous work where we added an artificial viscosity in the continuity equation and then rewrite the system in hyperbolic and symmetric form. Our study is based on the symmetric hyperbolic theory. We use for this aim a successive approximation in time to show the existence of the hyperbolic system solution and by the fixed point theorem the compacity property of some appropriate sobolev spaces and some established a priori estimates we can pass to several limits to prove our result. As state law, we use the Stiffened gas law.

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1. Introduction

The compressible Navier-Stokes equations describe a viscous fluid flow in a bounded tridimensional domain, they are given by:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \dfrac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\triangle\mathbf{u} - (\dfrac{\mu}{3}+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \nabla P = \mathbf{f}, \\ \dfrac{\partial \rho E}{\partial t} + \nabla\cdot\Big(\big(\rho E + P\big)\mathbf{u}\Big) = -\nabla\cdot\Big(\mathbf{q} - [\mu\big(\nabla\mathbf{u}+\nabla\mathbf{u}^T\big) + \big(\lambda - \dfrac{2}{3}\mu\big)\mathbf{I_d}\nabla\cdot\mathbf{u}]\mathbf{u}\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{u}\cdot\mathbf{f}. \end{cases} \tag{1}$$

Where ρ is the density, $\mathbf{u} = (u_1, u_2, u_3)$ are the velocity components, P is the pressure, E is the tolal energy, $\mathbf{f}$ is the external force and $\mathbf{q}$ is the heat flux.

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The mathematical analysis of these equations was made only recently with the works of [16], in particular in the case of barotropic fluids $P(\rho) = a\rho^{\gamma}$. The author had established a regularity result of weak solution under certain conditions related to γ. Studying a more general case, [9] had looked the case of non-monotone pressure-law. [7, 8] was interested in the system in dimension two where he refined the conditions on the initial data. The author established a regularity result with periodic boundary conditions and especially in prevision of possible appearance of the vacuum. In the case of the complete Navier-Stokes equation, with heat conduction, [16] had sketched the existence of global weak solution by compactness arguments and some a priori estimates. However, the author indicated that these a priori estimates are very difficult to be establish and sometimes unavailable except in a very restrictive cases. More recently, [19] made a major contribution in the development of the general mathematical theory of this system without any limitation on the data size. The authors proved the existence of a variational solution using a series of approximations: artificial pressure, relaxation in the continuity equation and finally a regularized thermal energy equation. Passing to the limit they obtain their variational solution of the Navier-Stokes equations. Although the theory is very nice, the question of the physical sense of these approximations could possibly arise. With another approach, [9, 10] justify the existence of a variational solution using in addition some a priori estimates.
In this work, we establish the existence of a weak solution for the compressible Navier-Stokes system using the theory of the symmetric hyperbolic system [11, 13, 17]. For that purpose, we add an artificial viscosity term $-\varepsilon\triangle\rho$ in the continuity equation. Classically this artificial viscosity is added to regularize the solution, but here we add it in the aim of rewriting and obtaining a hyperbolic system from compressible Navier-Stokes equations. After adding $-\varepsilon\triangle\rho$, we make a change of variable similar to that used by [5]. With this change of variable, we come down to a system of order 1.
A semi-discretization in time and a linearization of the obtained system allow us to use the results of [13]. By a fixed point theorem we show the existence of the solution of the stationary, nonlinear system. Then by some a priori estimates we can pass to limit in time ($\triangle t \to 0$) and after when $\varepsilon \to 0$ we justify the existence of Navier-Stokes weak solution.

The outline of this paper is organized as follow. In Section 2 we will present the added system and our main results. In section 3 we will prove that the density in the added system remain strictly positive when its initial value is positive. In section 4 we show how we obtain the hyperbolic system and we will present the successive approximations which allows us to construct and prove the existence of the weak solution of the obtained system. Finally in Section 5 we pass to limit in the artificial viscosity and then we prove the existence of the weak solution for the compressible Navier-Stokes system.

2. Preliminaries and main results

Let us consider a bounded domain Ω of $\mathbb{R}^3$ with the boundary $\partial\Omega$ is supposed to be enough regular, $[0,T]$ be a time interval with $T > 0$. We assume that heat flux $\mathbf{q}$ is determined by the Fourier's law ($\mathbf{q} = -k\nabla\theta$, with k is the thermal conductivity of the fluid) and the pressure P, the internal energy e are described by the following state laws [1, 3, 18, 20]:

$$P(\rho, e) = (\gamma - 1)\rho e - \gamma P_\infty \qquad \text{and} \qquad e(\rho, \theta) = \frac{P_\infty}{\rho} + C_v\theta. \tag{2}$$

Where γ and P_∞ are given constants depending on the fluid, C_v is the specific heat at constant volume and θ is the temperature. The relationship between the total energy E and the internal energy is: $E = \rho e + \frac{\|\mathbf{u}\|^2}{2}$.
We add an artificial viscosity term $-\varepsilon\triangle\rho$ in the continuity equation of the system (1), we obtain: (see [4]):

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \nabla\cdot(\rho_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon\triangle\rho_\varepsilon = 0, \\ \rho_\varepsilon \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t} + \rho_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon - \mu\triangle\mathbf{u}_\varepsilon - (\dfrac{\mu}{3} + \lambda)\nabla(\nabla\cdot\mathbf{u}_\varepsilon) + C_v(\gamma-1)\theta\nabla\rho_\varepsilon \\ \qquad\qquad + (\gamma-1)\rho_\varepsilon\nabla\theta_\varepsilon = \mathbf{f}, \\ \rho_\varepsilon C_v \dfrac{\partial \theta_\varepsilon}{\partial t} + \rho_\varepsilon C_v \mathbf{u}_\varepsilon\cdot\nabla\theta_\varepsilon - k\triangle\theta_\varepsilon - \big(\lambda - \dfrac{2}{3}\mu\big)|\nabla\cdot\mathbf{u}_\varepsilon|^2 - \dfrac{\mu}{2}|\nabla\mathbf{u}_\varepsilon + \nabla\mathbf{u}_\varepsilon^T|^2 \\ \qquad\qquad + \rho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla(\dfrac{\mathbf{u}_\varepsilon\cdot\mathbf{u}_\varepsilon}{2}) + (\gamma-1)\rho_\varepsilon C_v\theta_\varepsilon\nabla\cdot\mathbf{u}_\varepsilon = 0, \\ \rho_\varepsilon|_{t=0} = \rho_0, \quad \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0, \quad \theta_\varepsilon|_{t=0} = \theta_0, \\ \mathbf{u}_\varepsilon|_{\partial\Omega} = \mathbf{u}_b, \quad \theta_\varepsilon|_{\partial\Omega} = \theta_b. \end{cases} \tag{3}$$

To close the system (3), we need an additional boundary condition on ρ_ε given by:

$$\varepsilon\frac{\partial\rho_\varepsilon}{\partial \mathrm{n}} = 0. \tag{4}$$

In all the sequel we assume that the following assumptions are hold:

$$(\mathbf{H1}) \quad \begin{cases} \text{the viscosities verify} : 0 < \lambda + \dfrac{2}{3}\mu \leqslant \dfrac{4}{3}\mu, \\ s \in \mathbb{R} \text{ such that } s \geq \frac{5}{2}, \end{cases}$$

$$(\mathbf{H2}) \quad \begin{cases} \mathbf{f} \in (L^\infty(0,T;H^s(\Omega)^3))^3, \\ \mathbf{u}_b \in (W^{1,\infty}(0,T;H^{s+\frac{3}{2}}(\partial\Omega)^3))^3, \\ \theta_b \in W^{1,\infty}(0,T;H^{s+\frac{3}{2}}(\partial\Omega)), \end{cases} \qquad (\mathbf{H3}) \begin{cases} \rho_0 \in H^s(\Omega)\cap W^{1,\infty}(\Omega), \\ \mathbf{u}_0 \in (H^s(\Omega))^3 \cap (W^{1,\infty}(\Omega))^3, \\ \theta_0 \in H^s(\Omega)\cap W^{1,\infty}(\Omega), \\ \rho_0 > \bar\rho \text{ with } \bar\rho \in \mathbb{R}_*^+. \end{cases}$$

We denote $L^2(\cdot)$, $L^\infty(\cdot)$, $H^s(\cdot)$ and $W^{1,\infty}(\cdot)$ the usual Lebesgue and Sobolev spaces and $||\cdot||_{L^2}$, $||\cdot||_{L^\infty}$, $||\cdot||_{H^s}$ and $||\cdot||_{W^{1,\infty}}$ their corresponding norms (see [2]). In all the following we will adopte the following notation:
$||\cdot||_{L^2\left(0,T;H^s(\Omega)\right)} = ||\cdot||_{2,s}$, $||\cdot||_{L^\infty\left(0,T;H^s(\Omega)\right)} = ||\cdot||_{\infty,s}$, $||\cdot||_{L^\infty\left(0,T;L^\infty(\Omega)\right)} = ||\cdot||_{\infty,\infty}$

We are now able to announce our main results:

Theorem 1. *Under assumptions* (**H1**), (**H2**) *and* (**H3**), *suppose that the state law is given by (2). Then for any fixed time* $T > 0$ *the system (3) has a solution* $(\rho_\varepsilon,\ \mathbf{u}_\varepsilon,\ \theta_\varepsilon)$ *such that:*

$$(\rho_\varepsilon,\ \theta_\varepsilon) \in \big(L^\infty(0,T;H^s(\Omega))\big)^2 \ \textit{and}\ (\frac{\partial \rho_\varepsilon}{\partial t}, \frac{\partial \theta_\varepsilon}{\partial t}) \in \big(L^2(0,T;H^s(\Omega))\big)^2$$
$$\mathbf{u}_\varepsilon \in \big(L^\infty\big(0,T;H^s(\Omega)\big)\big)^3 \ \textit{and}\ \frac{\partial \mathbf{u}_\varepsilon}{\partial t} \in \big(L^2\big(0,T;H^s(\Omega)\big)\big)^3.$$

In addition we have:

$$\rho_\varepsilon > \bar{\rho}\exp^{-KT} \ \ \textit{and} \ \ \|\mathbf{u}_\varepsilon\|_{L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)} \le K,$$

where K *is a constant independent of* ε.

Theorem 2. *Let* $(\rho_\varepsilon,\ \mathbf{u}_\varepsilon,\ \theta_\varepsilon)$ *be a solution given by the theorem 1. Then, up to extracting subsequence, we have the following convergence when* ε *goes to zero:*

$$\begin{array}{lll}
\rho_\varepsilon & \rightharpoonup & \rho \ \textit{weakly-}* \textit{ in } \ L^\infty\big(0,T;H^s(\Omega)\big), \\
\dfrac{\partial \rho_\varepsilon}{\partial t} & \rightharpoonup & \dfrac{\partial \rho}{\partial t} \ \textit{weakly in } \ L^2\big(0,T;L^2(\Omega)\big), \\
\theta_\varepsilon & \rightharpoonup & \theta \ \textit{weakly-}* \textit{ in } \ L^\infty\big(0,T;H^s(\Omega)\big), \\
\dfrac{\partial \theta_\varepsilon}{\partial t} & \rightharpoonup & \dfrac{\partial \theta}{\partial t} \ \textit{weakly in } \ L^2\big(0,T;L^2(\Omega)\big), \\
\mathbf{u}_\varepsilon & \rightharpoonup & \mathbf{u} \ \textit{weakly-}* \textit{ in } \ \big(L^\infty\big(0,T;H^s(\Omega)\big)\big)^3, \\
\dfrac{\partial \mathbf{u}_\varepsilon}{\partial t} & \rightharpoonup & \dfrac{\partial \mathbf{u}}{\partial t} \ \textit{weakly in } \ \big(L^2\big(0,T;L^2(\Omega)\big)\big)^3,
\end{array}$$

where $(\rho,\ \mathbf{u},\ \theta)$ *is a solution of the system (1).*

The proof of these two theorems will be done in several steps. In section 4 we shortly recall how to rewrite the system (3) in the hyperbolic form, his symmetrization and the study of deduced operators that we be used in the demonstrations (see [4] for details). The proof of Theorem 1 will be given in Section 4 and for the Theorem 2 in section 5.

3. Positivity of the density

Before we begin our demonstrations, we have to verify that the density ρ_ε remains strictly positive in time if its initial value is greater than a positive quantity. For this, we take the first equation of the system (3):

$$\frac{\partial \rho_\varepsilon}{\partial t} + \nabla \cdot (\rho_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon \triangle \rho_\varepsilon = 0. \tag{5}$$

An implicit semi-discretization in time of the equation (5) gives:

$$\frac{1}{\triangle t}\rho_\varepsilon^{n+1} + \nabla \cdot (\rho_\varepsilon^{n+1}\mathbf{u}_\varepsilon^{n+1}) - \varepsilon \triangle \rho_\varepsilon^{n+1} = \frac{1}{\triangle t}{\rho_\varepsilon}^n. \tag{6}$$

To justify this strict positivity, we considered a more general convection-diffusion equation. So, we have the following results:

Lemma 1. *Let* $\mathbf{v} \in (\mathbf{W}^{1,\infty}(\Omega))^3$*,* $h \in L^\infty(\Omega)$*,* K *a constant as* $\|\mathbf{v}\|_{\mathbf{W}^{1,\infty}} \leqslant K$*. Then the solution* ρ *of the problem:*

$$\begin{cases} \sigma\rho + \nabla\cdot(\rho\mathbf{v}) - \varepsilon\triangle\rho = \sigma h & in \quad \Omega, \\ \varepsilon\dfrac{\partial\rho}{\partial n} = 0, \;\; \mathbf{v} = 0 & on \quad \partial\Omega, \end{cases} \tag{7}$$

verify: for all $\bar\rho > 0$ *such as* $h > \bar\rho$ *we have* $\rho > \bar\rho \exp^{-\frac{K}{\sigma}}$.

Proof. Using the same idea like in [6], we set: $\tilde\rho = \rho - \bar\rho_\infty$, with $\bar\rho_\infty = \bar\rho\exp^{-\frac{K}{\sigma}}$. $\tilde\rho$ is then solution of:

$$\sigma\tilde\rho + \nabla\cdot(\rho\mathbf{v}) - \varepsilon\triangle\tilde\rho = \sigma\big(h - \bar\rho_\infty\big). \tag{8}$$

Using the same decomposition like in [19]: $\tilde\rho = \tilde\rho^+ - \tilde\rho^+$, where:

$$\tilde\rho^+ = \begin{cases} \tilde\rho \;\text{ if }\; \tilde\rho > 0, \\ 0 \;\text{ if not .} \end{cases} \qquad \tilde\rho^- = \begin{cases} -\tilde\rho \;\text{ if }\; \tilde\rho < 0, \\ 0 \;\text{ if not .} \end{cases} \tag{9}$$

$$\partial_j\tilde\rho^+ = \begin{cases} \partial_j\tilde\rho \;\text{ if }\; \tilde\rho > 0, \\ 0 \;\text{ if not .} \end{cases} \qquad \partial_j\tilde\rho^- = \begin{cases} -\partial_j\tilde\rho \;\text{ if }\; \tilde\rho < 0, \\ 0 \;\text{ if not .} \end{cases} \tag{10}$$

Multiplying (8) by the test function $\tilde\eta^- = -(\tilde\rho^- + l)^\beta$ with $\beta \in [0,1]$, one obtain:

$$-\int_\Omega \sigma\tilde\rho(\tilde\rho^- + l)^\beta - \int_\Omega \varepsilon\nabla\tilde\rho\nabla(\tilde\rho^- + l)^\beta + \int_\Omega \sigma(h - \bar\rho_\infty)(\tilde\rho^- + l)^\beta = -\int_\Omega \rho\mathbf{v}\nabla(\tilde\rho^- + l)^\beta \tag{11}$$

The different terms of equality (11) are estimated by:

$$\begin{array}{rcl} \varepsilon\int_\Omega \nabla\tilde\rho\nabla(\tilde\rho^- + l)^\beta & = & -\varepsilon\int_\Omega \nabla\tilde\rho^-\nabla(\tilde\rho^- + l)^\beta \\ & = & -\varepsilon\int_\Omega \nabla(\tilde\rho^- + l)\nabla(\tilde\rho^- + l)^\beta \\ & = & -\varepsilon\int_\Omega \beta(\tilde\rho^- + l)^{\beta-1}|\nabla(\tilde\rho^- + l)|^2. \end{array} \tag{12}$$

Setting $\rho = (\tilde\rho + l) + (\bar\rho_\infty - l)$ one can found:

$$\begin{array}{rl} \big|\int_\Omega \rho\mathbf{v}\nabla(\tilde\rho^- + l)^\beta\big| = & \big|\int_\Omega \mathbf{v}(\tilde\rho^- + l)\nabla(\tilde\rho^- + l)^\beta - (\bar\rho_\infty - l)\int_\Omega(\nabla\cdot\mathbf{v})(\tilde\rho^- + l)^\beta\big| \\ \leq & K\beta\int_\Omega(\tilde\rho^- + l)^\beta\nabla(\tilde\rho^- + l) + K(\bar\rho_\infty + l)\int_\Omega(\tilde\rho^- + l)^\beta \\ \leq & K\int_\Omega \beta(\tilde\rho^- + l)^{\frac{\beta+1}{2}}(\tilde\rho^- + l)^{\frac{\beta-1}{2}}\nabla(\tilde\rho^- + l) + K(\bar\rho_\infty + l)\int_\Omega(\tilde\rho^- + l)^\beta \\ \leq & K\beta\|\tilde\rho^- + l\|_{L^{\beta+1}}^{\frac{\beta+1}{2}}\sqrt{\int_\Omega(\tilde\rho^- + l)^{\beta-1}|\nabla(\tilde\rho^- + l)|^2} + K(\bar\rho_\infty + l)\int_\Omega(\tilde\rho^- + l)^\beta \\ \leq & \frac{\beta K^2}{2\varepsilon}\|\tilde\rho^- + l\|_{L^{\beta+1}}^{\beta+1} + \frac{\varepsilon\beta}{2}\int_\Omega(\tilde\rho^- + l)^{\beta-1}|\nabla(\tilde\rho^- + l)|^2 + K(\bar\rho_\infty + l)\int_\Omega(\tilde\rho^- + l)^\beta. \end{array} \tag{13}$$

Since the left terms in (11) being positive, using (12) and (13) we get:

$$\begin{aligned} -\int_\Omega \sigma\tilde{\rho}(\tilde{\rho}^- + l)^\beta &\leq \tfrac{\beta K^2}{2\varepsilon}\|\tilde{\rho}^- + l\|_{L^{\beta+1}}^{\beta+1} + K(l+\bar{\rho}_\infty)\int_\Omega(\tilde{\rho}^- + l)^\beta - \sigma\int_\Omega(h-\bar{\rho}_\infty)(\tilde{\rho}^- + l)^\beta \\ &\leq \tfrac{\beta K^2}{2\varepsilon}\|\tilde{\rho}^- + l\|_{L^{\beta+1}}^{\beta+1} - \sigma\int_\Omega\big(h - \bar{\rho}_\infty(1+\tfrac{K}{\sigma})\big)\big(\tilde{\rho}^- + l\big)^\beta + Kl\int_\Omega(\tilde{\rho}^- + l)^\beta. \end{aligned} \tag{14}$$

With the Taylor polynomial of order 1 one obtain:

$$\bar{\rho} \geqslant \bar{\rho}_\infty \exp(\frac{K}{\sigma}) \geqslant \bar{\rho}_\infty(1+\frac{K}{\sigma}),$$

then we have:

$$0 < h - \bar{\rho} \leqslant h - \bar{\rho}_\infty(1+\frac{K}{\sigma}).$$

Then, the inequality (14) becomes:

$$-\int_\Omega \sigma\tilde{\rho}(\tilde{\rho}^- + l)^\beta \leq \frac{\beta K^2}{2\varepsilon}\|\tilde{\rho}^- + l\|_{L^{\beta+1}}^{\beta+1} + Kl\int_\Omega(\tilde{\rho}^- + l)^\beta.$$

If $l \longrightarrow 0$, we get:

$$\big(\sigma - \frac{\beta K^2}{2\varepsilon}\big)\|\tilde{\rho}^-\|_{L^{\beta+1}}^{\beta+1} \leq 0.$$

Choosing $\beta < \inf(1, \dfrac{2\sigma\varepsilon}{K^2})$, one obtain $\tilde{\rho}^- = 0$ which means that $\rho > \bar{\rho}\exp^{-\frac{K}{\sigma}}$. □

Lemma 2. *For* $s \geq \frac{5}{2}$*, let* $\mathbf{v} \in (H^{s+2}(\Omega))^3$*,* $h \in L^\infty(\Omega)$*,* $\mathbf{v}^* \in (H^{s+\frac{3}{2}}(\partial\Omega))^3$*. For all* $\sigma > \dfrac{K^2}{\varepsilon} + K$*, where* K *is a constant such as* $\|\mathbf{v}\|_{H^{s+2}} \leqslant K$*. Then, the solution* ρ *of the problem:*

$$\begin{cases} \sigma\rho + \nabla\cdot(\rho\mathbf{v}) - \varepsilon\triangle\rho = \sigma h & in \quad \Omega, \\ \varepsilon\dfrac{\partial\rho}{\partial n} = 0, \ \ \mathbf{v} = \mathbf{v}^* & on \quad \partial\Omega, \end{cases} \tag{15}$$

verify the following properties:

i) if $h > 0$ *then* $\rho > 0$*,*

ii) let $\bar{\rho} > 0$*, if* $h > \bar{\rho}$ *then* $\rho > \bar{\rho}\exp^{-\frac{K}{\sigma}}$ *.*

Proof. *i)*: We have $\mathbf{v}^* \in (H^{s+\frac{3}{2}}(\partial\Omega))^3$, then there exist $\tilde{\mathbf{v}}^* \in (H^{s+2}(\Omega))^3$ such that $\tilde{\mathbf{v}}^*|_{\partial\Omega} = \mathbf{v}^*$; let set $\tilde{\mathbf{v}} = \mathbf{v} - \tilde{\mathbf{v}}^*$. According to the Sobolev embendding theorem (see [2]) and the fact that $s \geqslant \frac{5}{2} > \frac{3}{2} - 1$; we have $\tilde{\mathbf{v}} \in (W^{1,\infty}(\Omega))^3$, so replacing $\mathbf{v} = \tilde{\mathbf{v}} + \tilde{\mathbf{v}}^*$ in (15), ρ is then solution of the following system:

$$\begin{cases} \sigma\rho + \nabla\cdot(\rho\tilde{\mathbf{v}}) + \varepsilon\triangle\rho = \sigma h - \nabla\cdot(\rho\tilde{\mathbf{v}}^*) & \text{in} \quad \Omega, \\ \varepsilon\dfrac{\partial\rho}{\partial n} = 0, \ \ \tilde{\mathbf{v}} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{16}$$

Considering the same test function $\eta^- = -(\rho^- + l)^\beta$ and also use the same decomposition of the density $\rho = \rho^+ + \rho^-$, we get:

$$\begin{aligned} -\sigma \int_\Omega \rho(\rho^- + l)^\beta + \varepsilon \int_\Omega \nabla\rho\nabla(\rho^- + l)^\beta + \sigma \int_\Omega h(\rho^- + l)^\beta = \\ -\int_\Omega \rho\tilde{\mathbf{v}}\nabla(\rho^- + l)^\beta + \int_\Omega \rho(\nabla\cdot\tilde{\mathbf{v}}^*)(\rho^- + l)^\beta + \int_\Omega \tilde{\mathbf{v}}^*(\nabla\rho)(\rho^- + l)^\beta. \end{aligned} \tag{17}$$

The differents terms in the equality (17) can be estimated as follow:

$$\begin{aligned} |\int_\Omega \rho\tilde{\mathbf{v}}\nabla(\rho^- + l)^\beta| &\leq K\beta \textstyle\int_\Omega (\rho^- + l)^\beta \nabla(\rho^- + l) + l|\int_\Omega \tilde{\mathbf{v}}\nabla(\rho^- + l)^\beta| \\ &\leq \tfrac{\beta K^2}{2\varepsilon}\|\rho^- + l\|^{\beta+1}_{L^{\beta+1}} + \tfrac{\beta\varepsilon}{2}\textstyle\int_\Omega (\rho^- + l)^{\beta-1}|\nabla(\rho^- + l)|^2 + \\ &\quad lK|\Omega|^{\frac{1}{\beta+1}}\|\rho^- + l\|^\beta_{L^{\beta+1}}, \end{aligned} \tag{18}$$

$$\begin{aligned} |\int_\Omega \rho(\nabla\cdot\tilde{\mathbf{v}}^*)(\rho^- + l)^\beta| &= |\int_\Omega (\rho^- + l)(\nabla\cdot\tilde{\mathbf{v}}^*)(\rho^- + l)^\beta - l\int_\Omega (\nabla\cdot\tilde{\mathbf{v}}^*)(\rho^- + l)^\beta| \\ &\leq K\textstyle\int_\Omega (\rho^- + l)^{\beta+1} + lK\int_\Omega (\rho^- + l)^\beta \\ &\leq K\|\rho^- + l\|^{\beta+1}_{L^{\beta+1}} + lK|\Omega|^{\frac{1}{\beta+1}}\|\rho^- + l\|^\beta_{L^{\beta+1}}, \end{aligned} \tag{19}$$

$$\begin{aligned} |\int_\Omega \tilde{\mathbf{v}}^*(\nabla\rho)(\rho^- + l)^\beta| &\leq K\textstyle\int_\Omega (\nabla(\rho^- + l))(\rho^- + l)^{\frac{\beta-1}{2}}(\rho^- + l)^{\frac{\beta+1}{2}} \\ &\leq \tfrac{\varepsilon\beta}{2}\textstyle\int_\Omega |\nabla(\rho^- + l)|^2(\rho^- + l)^{\beta-1} + \tfrac{K^2}{2\varepsilon\beta}\|\rho^- + l\|^{\beta+1}_{L^{\beta+1}}. \end{aligned} \tag{20}$$

Using inequalities (18) to (20) into (17) and h being positive we get:

$$-\sigma\int_\Omega \rho(\rho^- + l)^\beta \leq \left(\tfrac{\beta K^2}{2\varepsilon} + \tfrac{K^2}{2\varepsilon\beta} + K\right)\|\rho^- + l\|^{\beta+1}_{L^{\beta+1}} + 2lK|\Omega|^{\frac{1}{\beta+1}}\|\rho^- + l\|^\beta_{L^{\beta+1}}. \tag{21}$$

So when $l \longrightarrow 0$, we obtain:

$$\left(\sigma - K^2\frac{\beta^2+1}{2\beta\varepsilon} - K\right)\|\rho^-\|^{\beta+1}_{L^{\beta+1}} \leq 0. \tag{22}$$

let set:

$$\sigma - K^2\tfrac{\beta^2+1}{2\beta\varepsilon} - K = \tfrac{f(\beta)}{2\varepsilon\beta}, \tag{23}$$

where, $f(\beta) = -K^2\beta^2 + 2\varepsilon\beta(\sigma - K) - K^2$.

Since $\sigma > \dfrac{K^2}{\varepsilon} + K$, we have $f'(\beta) = -2K^2\beta + 2\varepsilon(\sigma - K)$ is strictly positive on $[0, 1]$ and

$f(0) = -K^2 < 0$, $f(1) = -2K^2 + 2\varepsilon(\sigma - K) > 0$, then there exist an unique $\beta_1 \in]0,1[$ such as $f(\beta_1) = 0$. Choosing $\beta \in]\beta_1, 1[$, we have $\sigma - K^2\frac{\beta^2+1}{2\beta\varepsilon} - K > 0$. We conclude that $\rho^- = 0$, so $\rho > 0$.
$ii)$: By setting $\tilde{\rho} = \rho - \bar{\rho}_\infty$; $\tilde{\rho}$ is then solution of:

$$\begin{cases} \sigma\tilde{\rho} + \nabla\cdot(\rho\tilde{\mathbf{v}}) - \varepsilon\triangle\tilde{\rho} = \sigma(h - \bar{\rho}_\infty) - \nabla\cdot(\rho\tilde{\mathbf{v}}^*), \\ \varepsilon\dfrac{\partial\tilde{\rho}}{\partial n} = 0, \ \ \tilde{\mathbf{v}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{24}$$

We use the decomposition (9) and (10). So multiplying (24) by the test function $\tilde{\eta}^- = -(\tilde{\rho}^- + l)^\beta$, we obtain after integration by parts:

$$\begin{aligned} -\sigma\int_\Omega \tilde{\rho}(\tilde{\rho}^- + l)^\beta - \varepsilon\int_\Omega \nabla\tilde{\rho}\nabla(\tilde{\rho}^- + l)^\beta + \sigma\int_\Omega (h - \bar{\rho}_\infty)(\tilde{\rho}^- + l)^\beta = -\int_\Omega \rho\tilde{\mathbf{v}}\nabla(\tilde{\rho}^- + l)^\beta \\ + \int_\Omega \rho(\nabla\cdot\tilde{\mathbf{v}}^*)(\tilde{\rho}^- + l)^\beta + \int_\Omega \tilde{\mathbf{v}}^*(\nabla\rho)(\tilde{\rho}^- + l)^\beta \end{aligned} \tag{25}$$

By taking the modulus, we follow the same procedure as above to justify that $\tilde{\rho}^- = 0$. This allows us to conclude. □

4. Hyperbolization, demonstration of the theorem 1

the rewriting of the system (3) in hyperbolic form was presented in [4] and inspired by [5]. Its consist to set:

$$\mathbf{W}_\varepsilon = (\mathbf{U}_\varepsilon,\ \mathcal{D}_1\mathbf{U}_\varepsilon,\ \mathcal{D}_2\mathbf{U}_\varepsilon,\ \mathcal{D}_3\mathbf{U}_\varepsilon), \tag{26}$$

with

$$\mathbf{U}_\varepsilon = (\rho_\varepsilon,\ \mathbf{u}_\varepsilon,\ \theta_\varepsilon) \qquad \text{and} \qquad \mathcal{D}_i = \frac{\partial}{\partial x_i} \text{ for } i = 1,2,3. \tag{27}$$

Let set

$$\alpha = \lambda + \frac{4}{3}\mu \ \ , \quad \beta = \lambda + \frac{\mu}{3} \ \text{ and } \quad \delta = \lambda - \frac{2}{3}\mu. \tag{28}$$

One can found in [4] the proof of the following result:

Proposition 1. *([4]) Under assumption* (**H1**) *and using the change of variables (26)-(27), the system (3) can be rewrite as:*

$$\begin{cases} \mathbf{A}_0(\mathbf{W}_\varepsilon)\dfrac{\partial\mathbf{W}_\varepsilon}{\partial t} + \displaystyle\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\dfrac{\partial\mathbf{W}_\varepsilon}{\partial x_i} + \mathbf{K}(\mathbf{W}_\varepsilon)\mathbf{W}_\varepsilon = \mathbf{F}, \\ \mathbf{W}_\varepsilon|_{t=0} = (\rho_0,\ \mathbf{u}_0,\ \theta_0,\ 0,\ \ldots,\ 0), \\ \mathbf{W}_\varepsilon|_{\partial\Omega} = (0,\ \mathbf{u}_b,\ \theta_b,\ 0,\ \ldots,\ 0), \\ \Big((\mathbf{W}_\varepsilon)_6 n_1 + (\mathbf{W}_\varepsilon)_{11} n_2 + (\mathbf{W}_\varepsilon)_{16} n_3\Big)\Big|_{\partial\Omega} = 0. \end{cases} \tag{29}$$

Where $\mathbf{n} = (n_1, n_2, n_3)^T$ *is the outward normal vector,* $\mathbf{A}_0(\mathbf{W}_\varepsilon)$*,* $\mathbf{A}_{i\varepsilon}$ *(i=1..3),* $\mathbf{K}_\varepsilon(\mathbf{W}_\varepsilon)$ *are matrix in* $\mathcal{M}_{20}(\mathbb{R})$ *and* $\mathbf{K}_\varepsilon(\mathbf{W}_\varepsilon)$ *contains all non-linear terms.*
In addition there exist a positive definte matrix $\mathcal{S}$ *which symmetrizes the system (29). The corresponding boundary conditions are also rewrite like in [13]. we obtain:*

$$\begin{cases} (\mathbf{SA}_0(\mathbf{W}_\varepsilon))\dfrac{\partial \mathbf{W}_\varepsilon}{\partial t} + \displaystyle\sum_{i=1}^{3}(\mathcal{S}\mathbf{A}_{i\varepsilon})\dfrac{\partial \mathbf{W}_\varepsilon}{\partial x_i} + (\mathcal{S}\mathbf{K}_\varepsilon)(\mathbf{W}_\varepsilon)\mathbf{W}_\varepsilon = \mathcal{S}\mathbf{F}, \\ \\ (\mathbf{B}_\varepsilon - \mathbf{M}_\varepsilon)\mathbf{W}_\varepsilon = \mathbf{g}, \end{cases} \tag{30}$$

with $\mathbf{B}_\varepsilon = \displaystyle\sum_{i=1}^{3} \mathbf{n_i}(\mathcal{S}\mathbf{A_{i\varepsilon}})$*. the matrix* $\mathbf{M}_\varepsilon \in \mathcal{M}_{20}(\mathbb{R})$ *and the vector* $\mathbf{g} \in \mathbb{R}^{20}$ *are obtained in accordance with the change of variables and:*

$$\begin{aligned}
&g_i = 0 \textit{ for } i = 1, 6, \quad g_{11} = g_{16} = 0, \\
&g_7 = -2(\alpha n_1 u_b^1 + \beta n_2 u_b^2 + \beta n_3 u_b^3), \quad g_8 = -2\mu n_1 u_b^2, \\
&g_9 = -2\mu n_1 u_b^3, \quad g_{10} = -2k n_1 \theta_b, \\
&g_{12} = -2\mu n_2 u_b^1, \quad g_{13} = -2(\beta n_1 u_b^1 + \alpha n_2 u_b^2 + \beta n_3 u_b^3), \\
&g_{14} = -2\mu n_2 u_b^3, \quad g_{15} = -2k n_2 \theta_b, \\
&g_{17} = -2\mu n_3 u_b^1, \quad g_{18} = -2\mu n_3 u_b^2, \\
&g_{19} = -2(\beta n_1 u_b^1 + \beta n_2 u_b^2 + \alpha n_3 u_b^3), \quad g_{20} = -2k n_3 \theta_b.
\end{aligned}$$

Notation: For simplicity, in all the following we will denote by:

1) $(\mathcal{S}\mathbf{A}_0(\mathbf{W})) = \mathbf{A}_0(\mathbf{W}), \qquad (\mathcal{S}\mathbf{A}_{i\varepsilon}) = \mathbf{A}_{i\varepsilon}, \qquad (\mathcal{S}\mathbf{K}_\varepsilon) = \mathbf{K}_\varepsilon$

2) $\mathcal{A}_0 = \begin{pmatrix} \mathbf{Id}_5 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{Id}_5$ is an identity matrix in $\mathcal{M}_5(\mathbb{R})$.

Before we present our approach for the construction of the weak solution for the system (30), note that by the assumption (**H2**) we have $\mathbf{g} \in \mathbf{W}^{1,\infty}(0, T; H^{s+\frac{3}{2}}(\partial\Omega))^{20}$, and by the trace theorem (see [15]), for $\mathbf{u_b} \in \mathbf{W}^{1,\infty}(0, T; H^{s+\frac{3}{2}}(\partial\Omega)^3)$ and $\theta_b \in \mathbf{W}^{1,\infty}(0, T; H^{s+\frac{3}{2}}(\partial\Omega))$ there exist $(\mathbf{W}_g)^2$, $(\mathbf{W}_g)^3$, $(\mathbf{W}_g)^4$, $(\mathbf{W}_g)^5$ in $\mathbf{W}^{1,\infty}(0, T; H^{s+2}(\Omega))$ such as:

$$(\mathbf{W}_g)^2|_{\partial\Omega} = \mathbf{u}_b^1, \quad (\mathbf{W}_g)^3|_{\partial\Omega} = \mathbf{u}_b^2, \quad (\mathbf{W}_g)^4|_{\partial\Omega} = \mathbf{u}_b^3, \quad (\mathbf{W}_g)^5|_{\partial\Omega} = \theta_b.$$

Setting: $\mathbf{Z}^g = (0, (\mathbf{W}_g)^2, (\mathbf{W}_g)^3, (\mathbf{W}_g)^4, (\mathbf{W}_g)^5)$, we obtain:

$$\mathbf{W}_g = \big(\mathbf{Z}^g, \mathcal{D}_1\mathbf{Z}^g, \mathcal{D}_2\mathbf{Z}^g, \mathcal{D}_3\mathbf{Z}^g\big) \in \mathbf{L}^\infty(0, T; \mathbf{H}^{s+1}(\Omega)^{20}) \tag{31}$$

4.1. Construction of successive approximations

For the existence study of a weak solution of the system (30), we use a process of successive construction of a solution. One can found the used of this approach in [5, 21], or in the continuous version in [22]. Thus in the first step, we perform a implicite semi-discretization in time. So, let N be a given integer, we subdivide $[0, T]$ into N intervals with the time step is $\triangle t = \frac{T}{N}$, we denote $\mathbf{W}^n(\mathbf{x}) = \mathbf{W}(t^n, \mathbf{x})$.

The essential idea of our study is based on use of the following algorithm:

- Initialization: $\mathbf{W}^n = \mathbf{W}_0$
- By using the semi-discretization in time of the system (30), we construct the sequence $\mathbf{W}^{n+1}$ solution of:

$$\begin{cases} \dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W}^{n+1} + \displaystyle\sum_{i=1}^{3} \mathbf{A}_{i\varepsilon} \dfrac{\partial \mathbf{W}^{n+1}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{n+1} = \mathbf{F}^{n+1} + \dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W}^n, \\ (\mathbf{B}_\varepsilon - \mathbf{M}_\varepsilon)\mathbf{W}^{n+1} = \mathbf{g}^{n+1}. \end{cases} \tag{32}$$

- By linearizing le system (32) we construct the sequence $\mathbf{W}^{n+1}_{k+1}$ solution of:

$$\begin{cases} \dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W}^{n+1}_{k+1} + \displaystyle\sum_{i=1}^{3} \mathbf{A}_{i\varepsilon} \dfrac{\partial \mathbf{W}^{n+1}_{k+1}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^{n+1}_k)\mathbf{W}^{n+1}_{k+1} = \mathbf{F}^{n+1} + \dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W}^n, \\ (\mathbf{B}_\varepsilon - \mathbf{M}_\varepsilon)\mathbf{W}^{n+1}_{k+1} = \mathbf{g}^{n+1}. \end{cases} \tag{33}$$

- Convergences
  - With the fixed point theorem, we prove:
    $\mathbf{W}^{n+1}_{k+1} \to \mathbf{W}^{n+1}$, when $k \to +\infty$ and where $\mathbf{W}^{n+1}$ is a solution of (32).
  - By some a priori estimations, we prove:
    $\mathbf{W}^{n+1} \to \mathbf{W}_\varepsilon$, when $n \to +\infty$ and where $\mathbf{W}_\varepsilon$ is a solution of (30).

4.1.1. Existence of $\mathbf{W}^1_1$

To prove the existence of the sequence $(\mathbf{W}^1_k)$, we begin with the existence of $\mathbf{W}^1_1$ by initializing with $\mathbf{W}^1_0 = \mathbf{W}^0$. Thus $\mathbf{W}^1_1$ will be solution of:

$$\begin{cases} \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^1_1 + \displaystyle\sum_{i=1}^{3} \mathbf{A}_{i\varepsilon} \dfrac{\partial \mathbf{W}^1_1}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^0)\mathbf{W}^1_1 = \mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^0, \\ (\mathbf{B}_\varepsilon - \mathbf{M}_\varepsilon)\mathbf{W}^1_1 = \mathbf{g}^1. \end{cases} \tag{34}$$

Definition 1. $\mathbf{W}_1^1$ *is called a weak solution of (34) if* $\mathbf{W}_1^1 \in L^2(\Omega)^{20}$ *and for all test function* $V \in H_0^1(\Omega)^{20}$ *we have*

$$< \mathbf{W}_1^1, \Psi^* V >=< \mathbf{G}, V >, \tag{35}$$

where $\mathbf{G} = \mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^0$ *and* Ψ^* *is the adjoint operator of*

$$\Psi(\cdot) = \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}(\cdot) + \sum_{i=1}^{3} \mathbf{A}_{i\varepsilon}\frac{\partial(\cdot)}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^0)(\cdot).$$

The existence of $\mathbf{W}_1^1$ is given by following result:

Proposition 2. *Let that assumptions* (**H2**) *and* (**H3**) *hold. Then the system (34) admit a weak solution* $\mathbf{W}_1^1 \in L^2(\Omega)^{20}$. *In addition if* $\mathbf{W}_1^1 \in H^1(\Omega)^{20}$ *we have uniqueness.*

Moreover if $\mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^0 \in H^s(\Omega)^{20}$, $\mathbf{g}^1 \in H^{s+\frac{3}{2}}(\partial\Omega)^{20}$, *then the system (34) admit a weak solution* $\mathbf{W}_1^1 \in H^s(\Omega)^{20}$.
In addition we have $(\mathbf{W}_1^1)^1 > \bar{\rho}\exp(-\triangle t K)$, *with* K *given by (49).*

Proof. By the assumptions (**H2**) and (**H3**), we have $\mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^\mathbf{0} \in L^2(\Omega)^{20}$. In the aim to homogeneous the boundary conditions of the system (34), let us set:

$$\mathbf{W}_1^{1*} = \mathbf{W}_1^1 - \mathbf{W}_g^1, \tag{36}$$

$\mathbf{W}_1^{1*}$ is then solution of:

$$\begin{cases} \left(\dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t} + \mathbf{K}_\varepsilon(\mathbf{W}^\mathbf{0})\right)\mathbf{W}_1^{1*} + \displaystyle\sum_{i=1}^{3} \mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}_\mathbf{1}^{\mathbf{1}*}}{\partial x_i} = \mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^0 \\ \qquad\qquad -\left(\dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t} + \mathbf{K}_\varepsilon(\mathbf{W}^\mathbf{0})\right)\mathbf{W}_g^1 - \displaystyle\sum_{i=1}^{3} \mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}_g^1}{\partial x_i}, \\ (\mathbf{B} - \mathbf{M})\mathbf{W}_\mathbf{1}^{\mathbf{1}*} = \mathbf{0}. \end{cases} \tag{37}$$

Let define the operator:

$$\Re(\mathbf{W}^0, \mathbf{W}^0)(\cdot) = \left(\frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t} + \mathbf{K}_\varepsilon(\mathbf{W}^\mathbf{0})\right)(\cdot) + \sum_{i=1}^{d} \mathbf{A}_{i\varepsilon}\frac{\partial(\cdot)}{\partial x_i}. \tag{38}$$

$(\Re + \Re^*)$ is positive (see for instance [4], Proposition 2). Do to [13], we can deduce the existence of a weak solution in $L^2(\Omega)^{20}$. Moreover if the solution is in $H^1(\Omega)^{20}$ we have uniqueness and the following estimate:

$$\|\mathbf{W}_1^{1*}\|_{L^2} \leq \frac{2}{\Upsilon(\varepsilon,\triangle t)}\Big(\|\mathbf{F}^1\|_{L^2} + \frac{1}{\triangle t}\|\mathbf{A}_0(\mathbf{W}^0)\mathbf{W}^0\|_{L^2} + \frac{1}{\triangle t}\|\mathbf{A}_0(\mathbf{W}^0)\mathbf{W}_g^1\|_{L^2}$$
$$+\|\mathbf{K}_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1 + \sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}_g^1}{\partial x_i}\|_{L^2}\Big)$$
$$\leq \frac{2}{\Upsilon(\varepsilon,\triangle t)}\Big(\|\mathbf{F}^1\|_{L^2} + \frac{C_{03}(\mathbf{W}^0)}{\triangle t}\|\mathcal{A}_0\mathbf{W}^0\|_{L^2} + \big(1+\frac{1}{\triangle t}\big)C_1(\mathbf{W}^0)\|\mathbf{W}_g^1\|_{L^2} + \tilde{C}\|\mathbf{W}_g^1\|_{H^1}\Big)\cdot$$

$\Upsilon(\varepsilon,\triangle t)$ is the constant positivity of $(\Re+\Re^*)$. Using (36), we obtain:

$$\|\mathbf{W}_\mathbf{1}^\mathbf{1}\|_{L^2} \leq \frac{2}{\Upsilon(\varepsilon,\triangle t)}\Big(\|\mathbf{F}^1\|_{L^2} + \frac{C_{03}(\mathbf{W}^0)}{\triangle t}\|\mathcal{A}_0\mathbf{W}^0\|_{L^2} + \big(1+\frac{1}{\triangle t}\big)C_1(\mathbf{W}^0)\|\mathbf{W}_g^1\|_{L^2}$$
$$+\tilde{C}\|\mathbf{W}_g^1\|_{H^1}\Big) + \|\mathbf{W}_g^1\|_{L^2}$$

where

$$C_{03}(\mathbf{W}^0) = \sup(1, \|\mathbf{W}^0\|_{L^\infty}, C_v\|\mathbf{W}^0\|_{L^\infty})$$

$$C_{04}(\mathbf{W}^0) = \max_{1\leqslant j\leqslant 20}\sum_{i=1}^{5}|(K_\varepsilon)_{ij}(\mathbf{W}^0)|\ , \qquad C_1(\mathbf{W}^0) = \sup(C_{03}(\mathbf{W}^0), C_{04}(\mathbf{W}^0))$$

$\tilde{C} = \sup(\mu,\beta,\alpha,k)$ a constant independente to ε. By construction of $\mathbf{W}_g$ (equality (31)), have: $(\mathbf{W}_g)^1$, $(\mathbf{W}_g)^6$, $(\mathbf{W}_g)^{11}$, and $(\mathbf{W}_g)^{16}$ are all equal to zero.

For the second part of the proposition, let us choose a test function test $\mathbf{V}\in\mathcal{V}$ with

$$\mathcal{V} = \big\{\mathbf{V}\in\mathbf{D}(\Omega)^{20}/\mathbf{V}_{i+5} = \mathcal{D}_1\mathbf{V}_i,\ \mathbf{V}_{i+10} = \mathcal{D}_2\mathbf{V}_i,\ \mathbf{V}_{i+15} = \mathcal{D}_3\mathbf{V}_i,\ \text{for } i=1,5\big\}.$$

One can remark that: $<\mathbf{A}_{i\varepsilon}\mathbf{W}_1^{1*},\mathbf{V}>=0$ for $i=1\cdots3$. Thus, for $\mathbf{V}\in\mathcal{V}$ we obtain

$$< \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}_1^{1*}, V> + <\mathbf{K}_\varepsilon(\mathbf{W}^0)\mathbf{W}_1^{1*}, V> = <\mathbf{F}^1 + \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{0*}, V>$$
$$+ < H_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1, V> - < \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}(\mathbf{W}_g^1 - \mathbf{W}_g^0), V> \tag{39}$$

where

$$H_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1 = \mathbf{K}_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1 + \sum_{i=1}^{d}\mathbf{A}_{i\varepsilon}\frac{\partial\mathbf{W}_g^1}{\partial x_i}\cdot \tag{40}$$

Using the fact that in the expression of $H_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1$ only elements at the five first lines are not equal to zero. We decompose the matrix $\mathbf{K}_\varepsilon(\mathbf{W}^0)$ as follow:

$$\mathbf{K}_\varepsilon(\mathbf{W}^0) = \mathbf{D} - \mathcal{A}_\mathbf{0} + \mathbf{R}_1(\mathbf{W}^0) + \mathbf{R}_2(\mathbf{W}^0), \tag{41}$$

where: $$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{K^{22}} & \mathbf{K^{23}} & \mathbf{K^{24}} \\ 0 & \mathbf{K^{32}} & \mathbf{K^{33}} & \mathbf{K^{34}} \\ 0 & \mathbf{K^{42}} & \mathbf{K^{43}} & \mathbf{K^{44}} \end{pmatrix},$$

$$\mathbf{R}_1 = \begin{pmatrix} \mathbf{K^{11}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} 0 & \mathbf{K^{12}} & \mathbf{K^{13}} & \mathbf{K^{14}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\mathbf{K^{11}} = \begin{pmatrix} w_7 + w_{13} + w_{19} & w_6 & w_{11} & w_{16} & 0 \\ (\gamma-1)C_v w_{10} & w_1 w_7 & w_1 w_{12} & w_1 w_{17} & (\gamma-1)c_v w_6 \\ (\gamma-1)C_v w_{15} & w_1 w_8 & w_1 w_{13} & w_1 w_{18} & (\gamma-1)C_v w_{11} \\ (\gamma-1)C_v w_{20} & w_1 w_9 & w_1 w_{14} & w_1 w_{19} & (\gamma-1)C_v w_{16} \\ T_1 & T_2 & T_3 & T_4 & T_5 \end{pmatrix}$$

and

$$\begin{aligned} T_1 &= C_v(w_2 w_{10} + w_3 w_{15} + w_4 w_{20}), \\ T_2 &= w_1(w_2 w_7 + w_3 w_8 + w_4 w_9), \\ T_3 &= w_1(w_2 w_{12} + w_3 w_{13} + w_4 w_{14}), \\ T_4 &= w_1(w_2 w_{17} + w_3 w_{18} + w_4 w_{19}), \\ T_5 &= C_v w_1(w_7 + w_{13} + w_{19}). \end{aligned}$$

$$(\mathbf{K^{12}})_{ij} = \begin{cases} -\delta(w_7 + w_{13} + w_{19}) - \mu w_7 & \text{if } (i,j) = (5,2) , \\ -\frac{\mu}{2}(w_8 + w_{12}) & \text{if } (i,j) = (5,3) , \\ -\frac{\mu}{2}(w_9 + w_{17}) & \text{if } (i,j) = (5,4) , \\ 0 & \text{if not } . \end{cases}$$

$$(\mathbf{K^{13}})_{ij} = \begin{cases} -\frac{\mu}{2}(w_8 + w_{12}) & \text{if } (i,j) = (5,2) , \\ -\delta(w_7 + w_{13} + w_{19}) - \mu w_{13} & \text{if } (i,j) = (5,3) , \\ -\frac{\mu}{2}(w_{14} + w_{18}) & \text{if } (i,j) = (5,4) , \\ 0 & \text{if not } . \end{cases}$$

$$(\mathbf{K^{14}})_{ij} = \begin{cases} -\frac{\mu}{2}(w_9 + w_{17}) & \text{if } (i,j) = (5,2) , \\ -\frac{\mu}{2}(w_8 + w_{14}) & \text{if } (i,j) = (5,3) , \\ -\delta(w_7 + w_{13} + w_{19}) - \mu w_{19} & \text{if } (i,j) = (5,4) , \\ 0 & \text{if not } . \end{cases}$$

$$\mathbf{K^{22}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix}, \quad \mathbf{K^{33}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix}, \quad \mathbf{K^{44}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix},$$

$$(\mathbf{K^{23}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(2,3)\ , \\ 0 & \text{if not .} \end{cases} \quad (\mathbf{K^{32}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(3,2)\ , \\ 0 & \text{if not .} \end{cases}$$

$$(\mathbf{K^{42}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(4,2)\ , \\ 0 & \text{if not .} \end{cases} \quad (\mathbf{K^{24}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(2,4)\ , \\ 0 & \text{if not .} \end{cases}$$

$$(\mathbf{K^{34}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(3,4)\ , \\ 0 & \text{if not .} \end{cases} \quad (\mathbf{K^{43}})_{ij} = \begin{cases} \beta & \text{if } (i,j)=(4,3)\ , \\ 0 & \text{if not .} \end{cases}$$

Multiplying the equality (39) by $\triangle t$ and using the decomposition (41) we have:

$$\begin{aligned} < \mathbf{A_0}(\mathbf{W^0})\mathbf{W}_1^{1*}, \mathcal{A}_\mathbf{0} V > +\triangle t < \mathbf{D}\mathbf{W_1^{1*}}, V > +\triangle t < \mathbf{R_2}(\mathbf{W^0})\mathbf{W_1^{1*}}, V >= \triangle t < \mathbf{F}^1, \mathcal{A}_\mathbf{0} V > \\ + < \mathbf{A_0}(\mathbf{W^0})\mathbf{W}^{0*}, \mathcal{A}_\mathbf{0} V > +\triangle t < (\mathcal{A}_\mathbf{0} - \mathbf{R_1}(\mathbf{W^0}))\mathbf{W}_1^{1*}, \mathcal{A}_\mathbf{0} V > \\ -\triangle t < H_\varepsilon(\mathbf{W}^0)\mathbf{W}_g^1, \mathcal{A}_\mathbf{0} V > - < \mathbf{A_0}(\mathbf{W^0})(\mathbf{W}_g^1 - \mathbf{W}_g^0), \mathcal{A}_\mathbf{0} V > \end{aligned}$$

One can note that all elements of $\mathbf{R}_2(\cdot)$ are equal to zero except somes of the fifth line, then we get

$$\begin{aligned} \sup_{V\in\mathcal{V}} \frac{|< \mathbf{A_0}(\mathbf{W^0})\mathbf{W}_1^{1*}, \mathcal{A}_\mathbf{0} V >|}{\|\mathcal{A}_\mathbf{0} V\|_{H^{-s}}} \quad & \leqslant \sup_{V\in\mathcal{V}} \Bigg| \frac{< \mathbf{A_0}(\mathbf{W^0})\mathbf{W}_1^{1*}, \mathcal{A}_\mathbf{0} V >}{\|\mathcal{A}_\mathbf{0} V\|_{H^{-s}}} + \triangle t \frac{< \mathbf{D}\mathbf{W}_1^{1*}, V >}{\|V\|_{H^{-s+1}}} \\ & \qquad + \triangle t \frac{< \mathbf{R}_2(\mathbf{W}^0)\mathbf{W}_1^{1*}, \mathcal{A}_\mathbf{0} V >}{\|\mathcal{A}_\mathbf{0} V\|_{H^{-s}}} \Bigg| \end{aligned}$$

$$\begin{aligned} \leqslant \triangle t\|\mathbf{F}^1\|_{H^s} + C_{03}(\mathbf{W}^0)\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \quad & \triangle t(C + C\|\mathbf{W}^0\|_{L^\infty}^3)\|\mathcal{A}_\mathbf{0}\mathbf{W}_1^{1*}\|_{H^s} \\ +\triangle t(C + C\|\mathbf{W}^0\|_{L^\infty}^3)\|\mathbf{W}_g^1\|_{H^{s+1}} + \quad & C_{03}(\mathbf{W}^0)\|\mathcal{A}_\mathbf{0}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s} \end{aligned}$$

where

$$C = \sup\big(3, 3C_v, (\gamma-1)C_v, \mu, \beta, \alpha, k\big)$$

By definition of the H^s norm given in [12], we finally deduce:

$$\begin{aligned} \|\mathcal{A}_\mathbf{0}\mathbf{W}_1^{1*}\|_{H^s} \leq \frac{\triangle t}{C_{01}(\bar\rho)}\|\mathbf{F}^1\|_{H^s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \triangle t\frac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}_1^{1*}\|_{H^s} \\ +\triangle t\frac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar\rho)}\|\mathbf{W}_g^1\|_{H^{s+1}} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s}, \end{aligned} \tag{42}$$

with $C_{01}(\bar\rho)$ is a strict positive constant independent to ε given by:

$$C_{01}(\bar\rho_\infty) = \inf(1, \bar\rho_\infty, C_v\bar\rho_\infty)\cdot \tag{43}$$

The discrete Grönwall's inequality allows us to deduct:ä

$$\|\mathcal{A}_{\mathbf{0}}\mathbf{W}_1^{1*}\|_{H^s} \leq M \exp^{\triangle t \dfrac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}}, \tag{44}$$

where

$$M = \frac{\triangle t}{C_{01}(\bar{\rho})}\|\mathbf{F}^1\|_{H^s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_{\mathbf{0}}\mathbf{W}^{0*}\|_{H^s} + \triangle t \frac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}\|\mathbf{W}_g^1\|_{H^{s+1}} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_{\mathbf{0}}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s}.$$

From inequality (44) we obtain:

$$\|\mathcal{A}_{\mathbf{0}}\mathbf{W}_1^{1}\|_{H^s} \leq M \exp^{\triangle t \dfrac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}} + \|\mathcal{A}_{\mathbf{0}}\mathbf{W}_g\|_{H^s}. \tag{45}$$

So with the compactly embedded of H^{s-1} into L^∞, we obtain:

$$\|\mathbf{W}_1^{1*}\|_{L^\infty} \leq \|\mathcal{A}_{\mathbf{0}}\mathbf{W}_1^{1*}\|_{H^s} \leq M_T \exp^{T \dfrac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}}, \tag{46}$$

where M_T is a constant independent of ε and of $\triangle t$, given by:

$$\begin{aligned} M_T \;=\; & \frac{T}{C_{01}(\bar{\rho})}\|\mathbf{F}\|_{\infty,s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_{\mathbf{0}}\mathbf{W}^{0*}\|_{H^s} + T\frac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}\|\mathbf{W}_g\|_{\infty,s+1} \\ & + \frac{2C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_{\mathbf{0}}\mathbf{W}_g\|_{\infty,s} \end{aligned} \tag{47}$$

Finally we have the estimate:

$$\|\mathbf{W}_1^1\|_{L^\infty} \leq K \tag{48}$$

with

$$K = M_T \exp^{T \dfrac{C + C\|\mathbf{W}^0\|_{L^\infty}^3}{C_{01}(\bar{\rho})}} + \|\mathbf{W}^g\|_{\infty,s}. \qquad \square \tag{49}$$

4.1.2. Existence of $\mathbf{W}_{k+1}^1$

In this part, we will establish the existence of $\mathbf{W}_{k+1}^1$, weak solution of the following system:

$$\begin{cases} \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}_{\mathbf{k+1}}^{\mathbf{1}} + \displaystyle\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}_{\mathbf{k+1}}^{\mathbf{1}}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}_{\mathbf{k}}^{\mathbf{1}})\mathbf{W}_{\mathbf{k+1}}^{\mathbf{1}} = \mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{\mathbf{0}}, \\ (\mathbf{B} - \mathbf{M})\mathbf{W}_{\mathbf{k+1}}^{\mathbf{1}} = \mathbf{g}^{\mathbf{1}}. \end{cases} \tag{50}$$

In fact we have the following result:

Proposition 3. *Let the assumptions* (**H2**) *and* (**H3**) *hold, then the system (50) admit a weak solution* $\mathbf{W^1_{k+1}} \in L^2(\Omega)^{20}$. *In addition if* $\mathbf{W^1_{k+1}} \in H^1(\Omega)^{20}$ *this solution is unique.*
If $\mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W^0} \in H^s(\Omega)^{20}$, $\mathbf{g}^1 \in H^{s+\frac{3}{2}}(\partial\Omega)^{20}$, *then the weak solution of the system (34):* $\mathbf{W^1_{k+1}} \in H^s(\Omega)^{20}$.
We have $(\mathbf{W}^1_{k+1})^1 > \bar{\rho}\exp(-K\triangle t)$.

Proof. We use recurrence and follow the same approach as in the Proposition 2. So in the second step we will construct $\mathbf{W}^1_2$ starting from $\mathbf{W}^1_1$ given by the Proposition 2. Then we will get the following estimates:

$$\begin{aligned}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{1*}_2\|_{H^s} \leq \Bigg(&\frac{\triangle t}{C_{01}(\bar{\rho})}\|\mathbf{F}^1\|_{H^s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \triangle t\frac{C + C\|\mathbf{W}^0\|^3_{L^\infty}}{C_{01}(\bar{\rho})}\|\mathbf{W}^1_g\|_{H^{s+1}} \\ &+\frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}(\mathbf{W}^1_g - \mathbf{W}^0_g)\|_{H^s}\Bigg)\exp^{\triangle t\frac{C + C\|\mathbf{W}^{1*}_1\|^3_{L^\infty}}{C_{01}(\bar{\rho})}},\end{aligned} \tag{51}$$

$$\begin{aligned}\|\mathbf{W}^{1*}_2\|_{L^\infty} \leq \Bigg(&\frac{T}{C_{01}(\bar{\rho})}\|\mathbf{F}\|_{\infty,s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + T\frac{C + C\|\mathbf{W}^0\|^3_{L^\infty}}{C_{01}(\bar{\rho})}\|\mathbf{W}_g\|_{\infty,s+1} \\ &+\frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}(\mathbf{W}^1_g - \mathbf{W}^0_g)\|_{\infty,s}\Bigg)\exp^{T\frac{C + C\|\mathbf{W}^{1*}_1\|^3_{L^\infty}}{C_{01}(\bar{\rho})}}.\end{aligned} \tag{52}$$

Note here that we cannot continue the recurrence only if these estimates are uniform in the sense that $T\dfrac{C + C\|\mathbf{W}^{1*}_1\|^3_{L^\infty}}{C_{01}(\bar{\rho})}$ remains lower than a quantity independent of the iteration. For this aim let us consider the term $\mathcal{M}$ defined by:

$$\mathcal{M} = \tilde{T}\frac{C + C\|\mathbf{W}^0\|^3_{L^\infty}}{C_{01}(\bar{\rho})}, \tag{53}$$

where $\tilde{T}$ is an enough time to have $T\dfrac{C + C\|\mathbf{W}^{1*}_1\|^3_{L^\infty}}{C_{01}(\bar{\rho})} \leqslant \mathcal{M}$. We will note in the sequel $C_{01}(\bar{\rho})$ as:

$$C_{01}(\bar{\rho}) = \inf\big(1, \bar{\rho}\exp(-\tilde{T}\tilde{K}), C_v\bar{\rho}\exp(-\tilde{T}\tilde{K})\big), \tag{54}$$

with

$$\tilde{K} = M_{\tilde{T}}\exp^{\tilde{T}\mathcal{M}} + \|\mathbf{W}^g\|_{L^\infty(0,\tilde{T};L^\infty(\Omega)^{20})}. \tag{55}$$

Using the estimation (46) and the equality (47) we get:

$$\frac{T}{C_{01}(\bar{\rho})}\big(C + C\|\mathbf{W}_1^{1*}\|_{L^\infty}^3\big) \leqslant \frac{T}{C_{01}(\bar{\rho})}\Big(C + C\mathcal{K}^3\Big)$$

$$\leqslant \frac{T}{C_{01}(\bar{\rho})}\Bigg[C + C\Bigg(\frac{T}{C_{01}(\bar{\rho})}\|\mathbf{F}\|_{L^\infty(0,\tilde{T};H^s(\Omega)^{20}} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} \tag{56}$$

$$+\mathcal{M}\|\mathbf{W}_g\|_{L^\infty(0,\tilde{T};H^{s+1}(\Omega)^{20})} + \frac{2C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}_g\|_{L^\infty(0,\tilde{T};H^s(\Omega)^{20})}\Bigg)^3 \exp^{3\mathcal{M}}\Bigg],$$

where we have set:

$$\mathcal{K} = M_T \exp^{\mathcal{M}}. \tag{57}$$

We wish that the right term in the estimate (56) be majoreted by $\mathcal{M}$. Thus, we rewrite the difference between the right term of (56) and $\mathcal{M}$ as a polynomial function in T:

$$g_{\mathcal{M}}(T) = \frac{a_1^3}{C_{01}(\bar{\rho})}\exp^{3\mathcal{M}} T^4 + 3\frac{a_1^2 a_2}{C_{01}(\bar{\rho})}\exp^{3\mathcal{M}} T^3 + 3\frac{a_1 a_2^2}{C_{01}(\bar{\rho})}\exp^{3\mathcal{M}} T^2 + \frac{C + a_2^3\exp^{3\mathcal{M}}}{C_{01}(\bar{\rho})}T - \mathcal{M},$$

where:

$$a_1 = C\frac{\|\mathbf{F}\|_{L^\infty(0,\tilde{T};H^s(\Omega))}}{C_{01}(\bar{\rho})},$$

$$a_2 = C\Bigg(\frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \mathcal{M}\|\mathbf{W}_g\|_{L^\infty(0,\tilde{T};H^{s+1}(\Omega)^{20})} + \frac{2C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}_g\|_{L^\infty(0,\tilde{T};H^s(\Omega)^{20})}\Bigg).$$

We expect to have:

$$g_{\mathcal{M}}(T) \leqslant 0\cdot$$

Let $\mathbf{E}$ be a subset of $\mathbb{R}^+$ containing the zeros of the function $g_{\mathcal{M}}$:

$$\mathbf{E} = \Big\{T \in \mathbb{R}^+ \text{ such as } g_{\mathcal{M}}(T) = 0\Big\}.$$

First one can note that the $\mathbf{E}$ is not empty, indeed $g_{\mathcal{M}}(T)$ is a polynomial function of degree four which is negative at $T = 0$, and is tends towards $+\infty$ when $T \to +\infty$. Thus there exists at least one $T \in \mathbb{R}$ such as $g_{\mathcal{M}}(T) = 0$. Thus we takes T_0 as follows:

$$T_0 = \inf_{T\in[0,+\infty[} \mathbf{E} \tag{58}$$

Consequently, for any $T \in [0, T_0[$ we have $g_{\mathcal{M}}(T) \leqslant 0$. We then obtain the following uniform estimate:

$$\|\mathcal{A}_\mathbf{0}\mathbf{W}_2^{1*}\|_{H^s} \leq \Bigg(\frac{\triangle t}{C_{01}(\bar{\rho})}\|\mathbf{F}^1\|_{H^s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \mathcal{M}\|\mathbf{W}_g^1\|_{H^{s+1}} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar{\rho})}\|\mathcal{A}_\mathbf{0}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s}\Bigg)\exp^{\mathcal{M}} \tag{59}$$

and

$$\|\mathbf{W}_2^1\|_{L^\infty} \leq \left(\frac{T}{C_{01}(\bar\rho)}T\|\mathbf{F}\|_{\infty,s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \mathcal{M}\|\mathbf{W}_g\|_{\infty,s+1} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s}\right)\exp^{\mathcal{M}} + \|\mathbf{W}_g\|_{\infty,\infty}. \tag{60}$$

Finally we have by recurrence:

$$\|\mathcal{A}_\mathbf{0}\mathbf{W}_k^1\|_{H^s} \leq \left(\frac{\triangle t}{C_{01}(\bar\rho)}\|\mathbf{F}^1\|_{H^s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \mathcal{M}\|\mathbf{W}_g^1\|_{H^{s+1}} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}(\mathbf{W}_g^1 - \mathbf{W}_g^0)\|_{H^s}\right)\exp^{\mathcal{M}} + \|\mathcal{A}_\mathbf{0}\mathbf{W}_g^1\|_{H^s} \tag{61}$$

$$\|\mathbf{W}_k^1\|_{L^\infty} \leq \left(\frac{T}{C_{01}(\bar\rho)}\|\mathbf{F}\|_{\infty,s} + \frac{C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}^{0*}\|_{H^s} + \mathcal{M}\|\mathbf{W}_g\|_{\infty,s+1} + \frac{2C_{03}(\mathbf{W}^0)}{C_{01}(\bar\rho)}\|\mathcal{A}_\mathbf{0}\mathbf{W}_g\|_{\infty,s}\right)\exp^{\mathcal{M}} + \|\mathbf{W}_g\|_{\infty,\infty}. \qquad \square \tag{62}$$

4.1.3. Passing to the limit $k \to \infty$

We have shown in the paragraphs 4.1.1 and 4.1.2 the existence of the sequence $(\mathbf{W}_{k+1}^1)$, here we will show that this sequence converge and its limit is solution of the following system:

$$\begin{cases} \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^\mathbf{1} + \displaystyle\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}^\mathbf{1}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^\mathbf{1})\mathbf{W}^\mathbf{1} = \mathbf{F}^1 + \dfrac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^\mathbf{0}, \\ (\mathbf{B} - \mathbf{M})\mathbf{W}^\mathbf{1} = \mathbf{g}^\mathbf{1}. \end{cases} \tag{63}$$

Let us set:

$$R_s = M_T \exp^{\mathcal{M}} + \|\mathbf{W}_g\|_{\infty,\infty} \tag{64}$$

Lemma 3. *Let* $\mathbf{B}$ *be a ball of* $H^s(\Omega)^{20}$ *with radius* R_s *given by (64). Then the function* $\mathcal{G}$ *defined by:*

$$\mathcal{G} : \begin{pmatrix} \mathbf{B} \longrightarrow \mathbf{B} \\ \mathbf{W}_k^1 \longrightarrow \mathcal{G}(\mathbf{W}_k^1) = \mathbf{W}_{k+1}^1 \end{pmatrix}$$

admits an unique fixed point $\mathbf{W}^1$ *which is solution of the system (63). In addition we have* $\mathbf{W}^1 \in L^\infty(\Omega)^{20}$ *with* $(\mathbf{W}^1)_1 > \bar\rho\exp(-\triangle t R_s)$ ·

Proof. Let $\mathbf{W}_k^1$ and $\mathbf{W}_{k+1}^1$ be two successive solutions of (50) then we have:

$$\frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} + \sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k}})\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} = \mathbf{F}^1 + \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{\mathbf{0}}, \tag{65}$$

$$\frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{\mathbf{1}}_{\mathbf{k}} + \sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}}{\partial x_i} + \mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}})\mathbf{W}^{\mathbf{1}}_{\mathbf{k}} = \mathbf{F}^1 + \frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t}\mathbf{W}^{\mathbf{0}}. \tag{66}$$

The difference between (65) and (66) gives:

$$\begin{aligned}\Big[\frac{\mathbf{A}_0(\mathbf{W}^0)}{\triangle t} + \mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k}})\Big]\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big) + \sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial}{\partial x_i}\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big)\\ = \big[\mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}) - \mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k}})\big](\mathbf{W}^{\mathbf{1}}_{\mathbf{k}}).\end{aligned}$$

A scalar product with $\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}$ and using the fact that: the matrix $\mathbf{A}_{i\varepsilon}$ are symmetric, $\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}} = \mathbf{W}^{\mathbf{1}*}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}*}_{\mathbf{k}}$ and the operator $\Re + \Re^*$ is positive we obtain:

$$< (\Re+\Re^*)(\mathbf{W}^0, \mathbf{W}_k^1)(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}), \mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}} > \geqslant 2\ \Upsilon(\varepsilon, \triangle t)\|\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\|^2_{L^2} \tag{67}$$

and

$$< (\Re+\Re^*)(\mathbf{W}^0, \mathbf{W}_k^1)(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}), \mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}} > \geqslant \frac{C_{01}(\bar\rho)}{\triangle t}\|\mathcal{A}_{\mathbf{0}}\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big)\|^2_{L^2}. \tag{68}$$

By the obtained result in [4] (lemma 2) we have:

$$< \big[\mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}) - \mathbf{K}_\varepsilon(\mathbf{W}^{\mathbf{1}}_{\mathbf{k}})\big]\mathbf{W}^{\mathbf{1}}_{\mathbf{k}}, \mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}} > \leqslant C^{\mathbf{W}}_{1,\infty}\|\mathbf{W}_k^1 - \mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}\|_{L^2}\|\mathcal{A}_{\mathbf{0}}\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big)\|_{L^2}, \tag{69}$$

where the constant $C^{\mathbf{W}}_{1,\infty}$ is defined as follow:

$$C^{\mathbf{W}}_{1,\infty} = \max_{1\leqslant j\leqslant 20}\sum_{i=1}^{20}\sup_{\substack{\|\mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}\|_{L^\infty}\leqslant R_s\\ \|\mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\|_{L^\infty}\leqslant R_s}}\left|\Big(\mathcal{N}[\mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}, \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}, \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}]\Big)_{ij}\right|. \tag{70}$$

Using the inequality (69) into (67) we get:

$$2\Upsilon(\varepsilon, \triangle t)\|\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\|^2_{L^2} \leqslant C(R_s)\|\mathbf{W}_k^1 - \mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}\|_{\mathbf{L^2}}\|\mathcal{A}_{\mathbf{0}}\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big)\|_{L^2}. \tag{71}$$

The inequality (68) allows us to deduce:

$$\|\mathcal{A}_{\mathbf{0}}\big(\mathbf{W}^{\mathbf{1}}_{\mathbf{k+1}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k}}\big)\|_{L^2} \leqslant C(R_s)\frac{\triangle t}{C_{01}(\bar\rho)}\|\mathbf{W}^{\mathbf{1}}_{\mathbf{k}} - \mathbf{W}^{\mathbf{1}}_{\mathbf{k-1}}\|_{L^2}. \tag{72}$$

The time step $\triangle t$ being intended to goes to zero before ε, taking (72) into (71) it follow:

$$\|\mathbf{W}_{k+1}^1 - \mathbf{W}_k^1\|_{L^2}^2 \leq \frac{\triangle t C^2(R_s)}{4\varepsilon C_{01}(\bar{\rho})}\|\mathbf{W}_{\mathbf{k}}^{\mathbf{1}} - \mathbf{W}_{\mathbf{k-1}}^{\mathbf{1}}\|_{L^2}^2. \tag{73}$$

Choosing the time step such that $\triangle t < \frac{4\varepsilon C_{01}(\bar{\rho})}{C^2(R_s)}$, we deduce that $\mathcal{G}$ is strictly L^2-contraction. Hence there exist an unique $\mathbf{W}^1 \in L^2(\Omega)^{20}$ such as $\mathcal{G}(\mathbf{W}^1) = \mathbf{W}^1$.
It remains to prove that $\mathbf{W}^1 \in H^s(\Omega)$. By Sobolev spaces interpolation:

$$\begin{aligned}
&\text{for all reels} \quad s, s' \quad \text{with} \quad 0 < s' < s, \exists \ C(s,\Omega), \ \text{such that} \\
\|\mathbf{W}_k^1 - \mathbf{W}_l^1\|_{H^{s'}} &\leqslant \ C(s,\Omega)\|\mathbf{W}_k^1 - \mathbf{W}_l^1\|_{L^2}^{1-\frac{s'}{s}}\|\mathbf{W}_k^1 - \mathbf{W}_l^1\|_{H^s}^{\frac{s'}{s}} \\
&\leqslant \ 2R_s^{\frac{s'}{s}}C(s,\Omega)\|\mathbf{W}_k^1 - \mathbf{W}_l^1\|_{L^2}^{1-\frac{s'}{s}}.
\end{aligned} \tag{74}$$

According to (73), $(\mathbf{W}_k^1)$ is a Cauchy sequence in $H^{s'}(\Omega)$. So choosing $s' \geq \frac{5}{2}$ we have $\mathbf{W}_k^1 \longrightarrow \mathbf{W}^1$ in $H^{s'}(\Omega)$. By duality product of Sobolev spaces we have:

$$< \Phi, \mathbf{W}_k^1 > \longrightarrow < \Phi, \mathbf{W}^1 > \quad \forall\ \Phi \in H^{-s'}(\Omega),$$

and by density of $H^{-s'}(\Omega)$ in $H^{-s}(\Omega)$, we have:

$$< \Phi, \mathbf{W}_k^1 > \longrightarrow < \Phi, \mathbf{W}^1 > \quad \forall\ \Phi \in H^{-s}(\Omega). \qquad \square$$

Now we are able to pass to the limit in the system (50). By term by term convergence we obtain that $\mathbf{W}^1$ is a solution of the system (63). By construction of the matrix $\mathbf{A}_0$, $\mathbf{A}_{i\varepsilon}(i = \ldots 3)$ and $\mathbf{K}_{i\varepsilon}$, the last fifteen lines of the system (63) gives that $\mathbf{W}^1$ have exactly the following form:

$$\begin{gathered}
\mathbf{W}^1 = (\mathbf{U}^1,\ \mathcal{D}_1\mathbf{U}^1,\ \mathcal{D}_2\mathbf{U}^1,\ \mathcal{D}_3\mathbf{U}^1) \\
\text{with } \mathbf{U}^1 = \Big((\mathbf{W}^1)_1, \mathbf{W}^1)_2,\ (\mathbf{W}^1)_3,\ (\mathbf{W}^1)_4,\ \mathbf{W}^1)_5\Big) \text{ and } \mathcal{D}_i = \frac{\partial}{\partial x_i} \text{ for } i = 1,2,3
\end{gathered} \tag{75}$$

The estimate (48) being true for the sequence $(\mathbf{W}_{k+1}^1)$, the limit also verify the same estimate. Then we have $\mathbf{v} = \big((\mathbf{W}^1)_2,\ (\mathbf{W}^1)_3,\ (\mathbf{W}^1)_4\big) \in \mathbf{W}^{1,\infty}(\Omega)^3$ and according to lemma 2 we deduce that $(\mathbf{W}^1)_1 > \bar{\rho}\exp(-\triangle t R_s)$.

4.1.4. Existence of $\mathbf{W}_{k+1}^{n+1}$ and passing to limit $k \to \infty$

In the paragraph 4.1.2, we have proved the existence of $(\mathbf{W}_{k+1}^1)$ solution of (50), whose limit $\mathbf{W}^{\mathbf{1}}$ is solution of the system (63) (paragraph 4.1.3).
Now we construct the sequence $(\mathbf{W}_{k+1}^2)$ whose first term $\mathbf{W}_0^2 = \mathbf{W}^1$. According to the lemma 3 and the positivity of the operator $\Re + \Re^*$, one obtain the existence of $(\mathbf{W}_1^2)$, and moreover we have $(\mathbf{W}_1^2)_1 > \bar{\rho}\exp(-2\triangle t R_s)$. Then, by recurrence we have the existence of $(\mathbf{W}_{k+1}^2)$. We repeat the process in the same way, so successively we construct the sequence $(\mathbf{W}_{\mathbf{k+1}}^{\mathbf{n+1}})$ solution of the sytem (33) and whose existence is given by the following result:

Proposition 4. *Suppose that* $\mathbf{F}^{n+1}+\dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W^n}\in L^2(\Omega)^{20}$ *and* $\mathbf{g}^{n+1}\in H^{s+\frac{3}{2}}(\partial\Omega)^{20}$, *then the system (33) admit a weak solution* $\mathbf{W^{n+1}_{k+1}}\in L^2(\Omega)^{20}$. *In addition if* $\mathbf{W^{n+1}_{k+1}}\in H^1(\Omega)^{20}$, *then this solution is unique.*
If $\mathbf{F}^{n+1}+\dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}\mathbf{W^n}\in H^s(\Omega)^{20}$, $\mathbf{g}^{n+1}\in H^{s+\frac{3}{2}}(\partial\Omega)^{20}$ *then the system (33) admit a weak solution* $\mathbf{W^{n+1}_{k+1}}\in H^s(\Omega)^{20}$. *In addition, we have* $(\mathbf{W^{n+1}_{k+1}})_1>\bar{\rho}\exp(-(n+1)\triangle tR_s)$

Proof. Use the same demarche as in the existence of $\mathbf{W^1_1}$. □

At the limit with the same argument of the fixed point theorem one can get $\mathbf{W^{n+1}}$ solution of the system (32).

4.2. A priori estimations

Lemma 4. *Under assumptions* (**H2**) *and* (**H3**), *the following estimations hold:*

$$\begin{aligned}
&i)\ \ \|\mathcal{A}_\mathbf{0}\mathbf{W}^n\|_{H^s}\le C(T,\mathbf{F},\mathbf{W}_g),\\
&ii)\ \ \triangle t\sum_{i=0}^{n}\|\mathbf{A_0}(\mathbf{W}^i)\frac{\mathbf{W}^{i+1}-\mathbf{W}^i}{\triangle t}\|^2_{L^2}\le C(T,\mathbf{F},\mathbf{W}_g),\\
&iii)\ \ \triangle t\sum_{i=0}^{n}\|\frac{1}{\triangle t}\mathcal{A}_\mathbf{0}\big(\mathbf{W}^{i+1}-\mathbf{W}^i\big)\|^2_{L^2}\le C(T,\mathbf{F},\mathbf{W}_g),
\end{aligned}\tag{76}$$

with $C(T,\mathbf{F},\mathbf{W}_g)$ *is a constante independente of* ε *and* $\triangle t$.

Proof. *i*): Let $\mathbf{W^{n+1}}$ be the solution of the following system:

$$\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n)+\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial\mathbf{W}^{(n+1)*}}{\partial x_i}+\mathbf{K}_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{(n+1)*}=\mathbf{F}^{n+1}-H_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{n+1}_g,$$

with

$$H_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{n+1}_g=\mathbf{K}_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{n+1}_g+\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial\mathbf{W}^{n+1}_g}{\partial x_i}.$$

Using the same decomposition of $\mathbf{K}_\varepsilon$ as previously, we get:

$$\begin{aligned}
&<\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n),\mathcal{A}_\mathbf{0}V>+<\mathbf{D}\mathbf{W}^{n+1},V>+<\mathbf{R}_2(\mathbf{W}^{n+1})\mathbf{W}^{n+1},\mathcal{A}_\mathbf{0}V>\\
&=<\mathbf{F}^{n+1},\mathcal{A}_\mathbf{0}V>+<\big[\mathcal{A}_\mathbf{0}-\mathbf{R}_1(\mathbf{W}^{n+1})\big]\mathbf{W}^{(n+1)*},\mathcal{A}_\mathbf{0}V>+<H_\varepsilon(\mathbf{W}^{n+1})\mathbf{W}^{n+1}_g,\mathcal{A}_\mathbf{0}V>.
\end{aligned}\tag{77}$$

It follows that:

$$\begin{aligned}
\sup_{V\in\mathcal{V}}\frac{|<\dfrac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n),\mathcal{A}_\mathbf{0}V>|}{\|\mathcal{A}_\mathbf{0}V\|_{H^{-s}}}&\leqslant\|\mathbf{F}^{n+1}\|_{H^s}+(C+CR_s^3)\|\mathcal{A}_0\mathbf{W}^{n+1}\|_{H^s}\\
&+(C+CR_s^3)\|\mathbf{W}^{n+1}_g\|_{H^s}+\tilde{C}\|\mathbf{W}^{n+1}_g\|_{H^{s+1}},
\end{aligned}$$

then,

$$\|\mathbf{A_0}(\mathbf{W^n})(\mathbf{W}^{n+1}-\mathbf{W}^n)\|_{H^s} \leqslant \triangle t\|\mathbf{F}^{n+1}\|_{H^s} + (C+CR_s^3)\Big(\triangle t\|\mathcal{A}_0\mathbf{W}^{n+1}\|_{H^s} + \triangle t\|\mathbf{W}_g^{n+1}\|_{H^s}\Big) + \tilde{C}\triangle t\|\mathbf{W}_g^{n+1}\|_{H^{s+1}}. \tag{78}$$

Now we set:

$$C_{01}^{R_s}(\bar\rho) = \inf\big(1, \bar\rho\exp(-T\ R_s), C_v\bar\rho\exp(-T\ R_s)\big) \tag{79}$$

we get finally:

$$\|\mathcal{A}_0\mathbf{W}^{n+1}\|_{H^s} \leqslant \frac{\triangle t}{C_{01}^{R_s}(\bar\rho)}\|\mathbf{F}^{n+1}\|_{H^s} + \triangle t\frac{C+CR_s^3}{C_{01}^{R_s}(\bar\rho)}\|\mathcal{A}_0\mathbf{W}^{n+1}\|_{H^s} + \triangle t\frac{C+CR_s^3}{C_{01}^{R_s}(\bar\rho)}\|\mathbf{W}_g^{n+1}\|_{H^s} + \triangle t\frac{\tilde{C}}{C_{01}^{R_s}(\bar\rho)}\|\mathbf{W}_g^{n+1}\|_{H^{s+1}} + \|\mathcal{A}_0\mathbf{W}^n\|_{H^s}. \tag{80}$$

By summation we have:

$$\|\mathcal{A}_0\mathbf{W}^{n+1}\|_{H^s} \leqslant \frac{\triangle t}{C_{01}^{R_s}(\bar\rho)}\sum_{i=1}^{n}\|\mathbf{F}^{i+1}\|_{H^s} + \triangle t\frac{C+CR_s^3}{C_{01}^{R_s}(\bar\rho)}\sum_{i=1}^{n}\|\mathcal{A}_0\mathbf{W}^{i+1}\|_{H^s} + \triangle t\frac{C+CR_s^3}{C_{01}^{R_s}(\bar\rho)}\sum_{i=1}^{n}\|\mathbf{W}_g^{i+1}\|_{H^s} + \triangle t\frac{\tilde{C}}{C_{01}^{R_s}(\bar\rho)}\sum_{i=1}^{n}\|\mathbf{W}_g^{i+1}\|_{H^{s+1}} + \|\mathcal{A}_0\mathbf{W}^0\|_{H^s}. \tag{81}$$

By the Grönwall's inequality we deduce:

$$\|\mathcal{A}_0\mathbf{W}^n\|_{H^s} \quad \leqslant \quad \left(\frac{T}{C_{01}^{R_s}(\bar\rho)}\|\mathbf{F}\|_{\infty,s} + CT\|\mathbf{W}_g\|_{\infty,s+1} + \|\mathcal{A}_0\mathbf{W}^0\|_{H^s}\right)\exp^{\mathcal{M}}, \tag{82}$$

where $\mathcal{M}$ is given by (53). By assumptions (**H2**) and (**H3**), we deduce i).

For the proof of ii), let us consider two successive solutions $\mathbf{W}^{n+1}$ and $\mathbf{W}^n$ of the system (32), using the same change of variable like (36) and setting: $(\partial\mathbf{W})_n^{n+1} = \mathbf{W}^{(n+1)*} - \mathbf{W}^{n*}$, we get:

$$\begin{aligned}\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\partial\mathbf{W})_n^{n+1} + \sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial}{\partial x_i}((\partial\mathbf{W})_n^{n+1}) + \mathbf{K}_\varepsilon(\mathbf{W}^{n+1})(\partial\mathbf{W})_n^{n+1} \\ +\big(\mathbf{K}_\varepsilon(\mathbf{W}^{n+1}) - \mathbf{K}_\varepsilon(\mathbf{W}^n)\big)\mathbf{W}^{n*} \\ = \mathbf{F}^{n+1} - \mathbf{F}^n + \frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\partial\mathbf{W})_{n-1}^n - \big(H_\varepsilon(\mathbf{W}^{n+1}) - H_\varepsilon(\mathbf{W}^n)\big)\mathbf{W}_g^{n+1} \\ -H_\varepsilon(\mathbf{W}^n)(\mathbf{W}_g^{n+1} - \mathbf{W}_g^n).\end{aligned} \tag{83}$$

By the lemma 2, proved in [4], we rewrite (83) as follow:

$$\left[\frac{\mathbf{A_0}(\mathbf{W}^n)}{\triangle t}+\mathbf{K}_\varepsilon(\mathbf{W}^{n+1})+\mathcal{N}[\mathbf{W}^{n+1},\mathbf{W}^n,\mathbf{W}^{n*}]\right](\partial\mathbf{W})_n^{n+1}+\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial}{\partial x_i}((\partial\mathbf{W})_n^{n+1})$$
$$=\mathbf{F}^{n+1}-\mathbf{F}^n+\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\partial\mathbf{W})_{n-1}^n-\mathbf{K}_\varepsilon(\mathbf{W}^{n+1})(\mathbf{W}_g^{n+1} \quad (84)$$
$$-\mathbf{W}_g^n)-\big(\mathbf{K}_\varepsilon(\mathbf{W}^{n+1})-\mathbf{K}_\varepsilon(\mathbf{W}^n)\big)\mathbf{W}_g^n-\sum_{i=1}^{3}\mathbf{A}_{i\varepsilon}\frac{\partial}{\partial x_i}(\mathbf{W}_g^{n+1}-\mathbf{W}_g^n)$$

Applying the Proposition 3 of [4] to the system (84) we have:

$$\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n)\|_{L^2}\leqslant 2\|\mathbf{F}^{n+1}-\mathbf{F}^n\|_{L^2}+2\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}^n-\mathbf{W}^{n-1})\|_{L^2}$$
$$+2\|\mathbf{K}_\varepsilon(\mathbf{W}^n)\|_{L^\infty}\|\mathbf{W}_g^{n+1}-\mathbf{W}_g^n\|_{L^2}+2\|\mathcal{N}[\mathbf{W}^{n+1},\mathbf{W}^n,\mathbf{W}^{n*}]\|_{L^\infty}\|\mathbf{W}_g^{n+1}-\mathbf{W}_g^n\|_{L^2}$$
$$+2\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}_g^n-\mathbf{W}_g^{n-1})\|_{L^2}+\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}_g^{n+1}-\mathbf{W}_g^n)\|_{L^2}$$

or

$$\left\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n)\right\|_{L^2}\leqslant 2\|\mathbf{F}^{n+1}-\mathbf{F}^n\|_{L^2}+2\left\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}^n-\mathbf{W}^{n-1})\right\|_{L^2}$$
$$+2\left\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}_g^n-\mathbf{W}_g^{n-1})\right\|_{L^2}+\left\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}_g^{n+1}-\mathbf{W}_g^n)\right\|_{L^2}$$
$$+C(R_s)\|\mathbf{W}_g^{n+1}-\mathbf{W}_g^n\|_{L^2}$$

Squared, one obtain:

$$\left\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}^{n+1}-\mathbf{W}^n)\right\|_{L^2}^2\leqslant C\|\mathbf{F}^{n+1}-\mathbf{F}^n\|_{L^2}^2+\frac{1}{2}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}^n-\mathbf{W}^{n-1})\right\|_{L^2}^2$$
$$+C\left\|\frac{\mathbf{A}_0(\mathbf{W}^{n-1})}{\triangle t}(\mathbf{W}_g^n-\mathbf{W}_g^{n-1})\right\|_{L^2}^2+C\left\|\frac{\mathbf{A}_0(\mathbf{W}^n)}{\triangle t}(\mathbf{W}_g^{n+1}-\mathbf{W}_g^n)\right\|_{L^2}^2 \quad (85)$$
$$+C(R_s)^2\|\mathbf{W}_g^{n+1}-\mathbf{W}_g^n\|_{L^2}^2$$

with C a positive constant. The inequality (85) remains true for $\mathbf{i}=1,2,\cdots n$, by summation, we get:

$$\sum_{i=1}^{n}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i-1})}{\triangle t}(\mathbf{W}^{i+1}-\mathbf{W}^i)\right\|_{L^2}^2\leqslant C\sum_i^n\|\mathbf{F}^{i+1}-\mathbf{F}^i\|_{L^2}^2+C\sum_i^n\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i-1})}{\triangle t}(\mathbf{W}_g^i-\mathbf{W}_g^{i-1})\right\|_{L^2}^2$$
$$+C\sum_i^n\left\|\frac{\mathbf{A}_0(\mathbf{W}^i)}{\triangle t}(\mathbf{W}_g^{i+1}-\mathbf{W}_g^i)\right\|_{L^2}^2+C(R_s)^2\sum_i^n\|\mathbf{W}_g^{i+1}-\mathbf{W}_g^i\|_{L^2}^2. \quad (86)$$

Multiplying by $\triangle t$ we have:

$$\triangle t\sum_{i=1}^{n}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i-1})}{\triangle t}(\mathbf{W}^{i+1}-\mathbf{W}^{i})\right\|_{L^2}^2 \leqslant \triangle tC\sum_{i}^{n}\left\|\mathbf{F}^{i+1}-\mathbf{F}^{i}\right\|_{L^2}^2$$
$$+\triangle t\sum_{i}^{n}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i-1})}{\triangle t}(\mathbf{W}_g^{i+1}-\mathbf{W}_g^{i})\right\|_{L^2}^2+\triangle t\sum_{i}^{n}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i})}{\triangle t}(\mathbf{W}_g^{i+1}-\mathbf{W}_g^{i})\right\|_{L^2}^2$$
$$+2\triangle tC(R_s)^2\sum_{i}^{n}\left\|\mathbf{W}_g^{i+1}\right\|_{L^2}^2+\left\|\mathbf{W}_g^{i}\right\|_{L^2}^2.$$

Finally we obtain $ii)$, with:

$$\triangle t\sum_{i=1}^{n}\left\|\frac{\mathbf{A}_0(\mathbf{W}^{i-1})}{\triangle t}(\mathbf{W}^{i+1}-\mathbf{W}^{i})\right\|_{L^2}^2 \leqslant TC\Big(\|\mathbf{F}\|_{\infty,2}^2+\|\mathcal{A}_{\mathbf{0}}\mathbf{W}_g\|_{\mathbf{W}^{1,\infty}(0,T;L^2(\Omega)^{20})}^2+\|\mathbf{W}_g\|_{\infty,2}^2\Big)$$

For $iii)$, we multiply (86) by $\triangle t$, which implies

$$\triangle t\sum_{i=1}^{n}\left\|\frac{\mathcal{A}_{\mathbf{0}}}{\triangle t}(\mathbf{W}^{i+1}-\mathbf{W}^{i})\right\|_{L^2}^2 \leqslant \triangle t\frac{C}{C_{01}}\sum_{i}^{n}\left\|\mathbf{F}^{i+1}-\mathbf{F}^{i}\right\|_{L^2}^2+2\triangle t\frac{C_{03}}{C_{01}}\sum_{i}^{n}\left\|\frac{\mathcal{A}_{\mathbf{0}}}{\triangle t}(\mathbf{W}_g^{i+1}-\mathbf{W}_g^{i})\right\|_{L^2}^2$$
$$+2\triangle t\frac{C(R_s)^2}{C_{01}}\sum_{i}^{n}\left\|\mathbf{W}_g^{i+1}\right\|_{L^2}^2+\left\|\mathbf{W}_g^{i}\right\|_{L^2}^2,$$

and finally we obtain:

$$\triangle t\sum_{i=1}^{n}\left\|\frac{\mathcal{A}_{\mathbf{0}}}{\triangle t}(\mathbf{W}^{i+1}-\mathbf{W}^{i})\right\|_{L^2}^2 \leqslant TC\Big(\|\mathbf{F}\|_{\infty,2}^2+\|\mathcal{A}_{\mathbf{0}}\mathbf{W}_g\|_{\mathbf{W}^{1,\infty}(0,T;L^2(\Omega)^{20})}^2+\|\mathbf{W}_g\|_{\infty,2}^2\Big). \qquad \square$$

4.2.1. Passing to limit $m \to \infty$

The a priori estimates given by the lemma 4 allow us to pass to the limit in non-linear terms. But to be able to do it, we introduce the approximate functions on $[0,T]$ and which are defined as follows:

$$\begin{aligned}&\mathbf{V}_m^*(t)=\mathbf{W}^n+\frac{1}{k}(\mathbf{W}^{n+1}-\mathbf{W}^n)(t-nk) \qquad \forall t\in[nk,(n+1)k[,\\&\mathbf{V}^m(t)=\mathbf{W}^n \qquad \forall t\in[nk,(n+1)k]\end{aligned} \tag{87}$$

with $k=\frac{T}{m}$. So we have the following result:

Lemma 5. *Let suppose that the assumptions* (**H2**) *and* (**H3**) *hold, then up to extract sub sequences of* $\mathbf{V}_m^*$ *and* $\mathbf{V}^m$*, if necessary (which we will note in the same maner) we have the following convergences:*

a) $\mathcal{A}_0\mathbf{V}_m^* \rightharpoonup \mathcal{A}_0\mathbf{V}$ *weakly in* $L^2(0,T;H^s(\Omega)^{20})$,

b) $\mathcal{A}_0\mathbf{V}^m \rightharpoonup \mathcal{A}_0\mathbf{V}$ *weakly in* $L^2(0,T;H^s(\Omega)^{20})$,

c) $\mathbf{V}_m^*$ *and* $\mathbf{V}^m \rightharpoonup \mathbf{V}$ *weakly-$*$ in* $L^\infty(0,T;H^{s-1}(\Omega)^{20})$,

d) $\dfrac{d}{dt}(\mathcal{A}_0\mathbf{V}_m^*) \rightharpoonup \dfrac{d}{dt}(\mathcal{A}_0\mathbf{V})$ *weakly in* $L^2(0,T;L^2(\Omega)^{20})$.

Proof. By construction of the functions $\mathbf{V}_m^*$, $\mathbf{V}^m$ and the estimation $i)$ of the lemma 4, we have $\mathcal{A}_\mathbf{0}\mathbf{V}_m^*$ and $\mathcal{A}_\mathbf{0}\mathbf{V}^m$ are bounded in $L^\infty(0,T;H^s(\Omega)^{20})$ thus we have the weak-$*$ convergence of the two sequences in $L^\infty(0,T;H^s(\Omega))^{20}$. By compact embedded we get $a)$ and $b)$.
Notice that $\mathcal{A}_\mathbf{0}\mathbf{V}_m^*$ and $\mathcal{A}_\mathbf{0}\mathbf{V}^m$ represent the first five variables of $\mathbf{V}_m^*$ and $\mathbf{V}^m$ thus, $\mathbf{V}_m^*$ et $\mathbf{V}^m$ are bounded in $L^\infty(0,T;H^{s-1}(\Omega)^{20})$. Then we have $c)$.
Finally $d)$ come from $iii)$ of the lemma 4. It remains to prove that $\mathbf{V}_m^*$ and $\mathbf{V}^m$ have the same limit when $m \to \infty$. For that, one can remark that:

$$\mathcal{A}_\mathbf{0}\mathbf{V}_m^*(t) - \mathcal{A}_\mathbf{0}\mathbf{V}^m(t) = \frac{t-nk}{k}(\mathcal{A}_\mathbf{0}\mathbf{W}^{n+1} - \mathcal{A}_\mathbf{0}\mathbf{W}^n) \quad \forall t \in [nk,(n+1)k]$$

$$\begin{aligned}\int_{nk}^{(n+1)k} |\mathcal{A}_\mathbf{0}\mathbf{V}_m^*(t) - \mathcal{A}_\mathbf{0}\mathbf{V}^m(t)|^2\, dt &= |\mathcal{A}_\mathbf{0}\mathbf{W}^{n+1} - \mathcal{A}_\mathbf{0}\mathbf{W}^n|^2 \int_{nk}^{(n+1)k} (\frac{\tau-nk}{k})^2\, d\tau \\ &= \frac{k^2}{3}|\mathcal{A}_\mathbf{0}\mathbf{W}^{n+1} - \mathcal{A}_\mathbf{0}\mathbf{W}^n|^2,\end{aligned}$$

so, by summation, we obtain:

$$\|\mathcal{A}_\mathbf{0}\mathbf{V}_m^* - \mathcal{A}_\mathbf{0}\mathbf{V}^m\|_{2,2} \leq \frac{Ck}{3}.$$

Passing to the limit $m \to +\infty$ which means $k \to 0$, we have: $\mathcal{A}_\mathbf{0}\mathbf{V}^* = \mathcal{A}_\mathbf{0}\mathbf{V}$ and consequently $\mathbf{V}^* = \mathbf{V}$. □

Now we can announce the result which allows us to pass to the limit in the nonlinear term.

Lemma 6. *Let us suppose that the assumptions* (**H1**), (**H2**) *and* (**H3**) *hold, then up to subsequence, we have:*

1) $\mathbf{V}_1^m \rightharpoonup \mathbf{W}_1$ *weakly-$*$ in* $L^\infty(0,T;L^2(\Omega))$,

2) $\mathbf{V}_i^m \to \mathbf{W}_i$ *strongly in* $L^2(0,T;L^4(\Omega))$,

3) $\mathbf{V}_i^m \to \mathbf{W}_i$ *strongly in* $L^2(0,T;L^6(\Omega))$,

4) $\mathbf{V}_i^m\mathbf{V}_j^m \rightharpoonup \mathbf{W}_i\mathbf{W}_j$ *weakly* $L^2(0,T;L^2(\Omega))$, *5)* $\mathbf{V}_i^m\mathbf{V}_j^m\mathbf{V}_r^m \rightharpoonup \mathbf{W}_i\mathbf{W}_j\mathbf{W}_r$ *weakly in* $L^2(0,T;L^2(\Omega))$,

6) $\mathbf{V}_1^m\mathbf{V}_i^m\mathbf{V}_j^m\mathbf{V}_r^m \rightharpoonup \mathbf{W}_1\mathbf{W}_i\mathbf{W}_j\mathbf{W}_r$ *weakly in* $L^2(0,T;L^2(\Omega))$,

with $1 \leq i,j,r \leq 20$.

Proof. The point 1) is a direct deduction of the point *c)* of the lemma 5.
Form the apriori estimates, we have $(\mathbf{V}_i^m)$ is bounded in $L^2(0,T;H^{s-1}(\Omega))$, thus $(\mathbf{V}_i^m)$ converge weakly in $L^2(0,T;H^{s-1}(\Omega))$. For $s \geq \frac{3}{2}+1$, we have the compact embendding of $H^{s-1}(\Omega)$ in $L^4(\Omega)$ (see [2]), so by the compacity theorem do to [14], we have that $(\mathbf{V}_i^m)$ converge strongly in $L^2(0,T;L^4(\Omega))$. So we have the point 2). By the same argument one can obtain the point 3).
For the point 4), we have $(\mathbf{V}_i^m)$ and $(\mathbf{V}_j^m)$ are bounded in $L^2(0,T;L^4(\Omega))$, thus $(\mathbf{V}_i^m\mathbf{V}_j^m)$ is bounded in $L^2(0,T;L^2(\Omega))$ hence it converges weakly towards a limit that we note $\mathbf{\Psi}_{ij}$. To justify the equality $\mathbf{\Psi}_{ij} = \mathbf{W}_i\mathbf{W}_j$, we takes $\mathcal{Q} = [0,T[\times\Omega$, thus we have:

$$< \mathbf{V}_i^m\mathbf{V}_j^m, \phi > = < \mathbf{V}_i^m, \mathbf{V}_j^m\phi >, \quad \forall \ \ \phi \in \mathcal{D}(\mathcal{Q})$$

By the point 2), we have $\mathbf{V}_i^m$ is bounded in $L^2(0,T;L^2(\Omega))$, then converge weakly, on the other hand we have the strong convergence of $\mathbf{V}_j^m\phi$ in $L^2(0,T;L^2(\Omega))$. It result that:

$$\int_{\mathcal{Q}} \mathbf{V}_i^m\mathbf{V}_j^m\phi \, d\mathbf{x}dt \longrightarrow \int_{\mathcal{Q}} \mathbf{W}_i\mathbf{W}_j\phi \, d\mathbf{x}dt \quad \forall \ \ \phi \in \mathcal{D}(\mathcal{Q}).$$

For the point 5), given that $(\frac{3}{2},3)$ are conjugate, we have the following estimate:

$$\begin{aligned} \|(\mathbf{V}_i^m\mathbf{V}_j^m\mathbf{V}_r^m)(t)\|_{L^2}^2 &\leq \|(\mathbf{V}_i^m\mathbf{V}_j^m)(t)\|_{L^3}^2\|\mathbf{V}_r(t)\|_{L^6}^2 \\ &\leq \left(\|\mathbf{V}_i^m(t)\|_{L^6}\|\mathbf{V}_j^m(t)\|_{L^6}\|\mathbf{V}_r^m(t)\|_{L^6}\right)^2. \end{aligned}$$

The sequence $(\mathbf{V}_i^m\mathbf{V}_j^m\mathbf{V}_r^m)$ is then bounded in $L^2(0,T;L^2(\Omega))$ so it converges weakly towards $\mathbf{\Phi}_{ijr}$. For the equality $\mathbf{\Phi}_{ijr} = \mathbf{W}_i\mathbf{W}_j\mathbf{W}_r$, we juste follow the same way as previously.
The point 6) is a deduction of 1) and 5). □

Lemma 7. *Under the same assumptions like in the lemma 5, the following convergence hold:*

$$\mathcal{A}_0\mathbf{V}^m\frac{d}{dt}(\mathcal{A}_0\mathbf{V}_m^*) \rightharpoonup \mathcal{A}_0\mathbf{V}\frac{d}{dt}(\mathcal{A}_0\mathbf{V}) \text{ weakly in } L^2(0,T;L^2(\Omega))^{20}.$$

Proof. From the estimation *ii)* of the lemma 4 we deduce that $\mathcal{A}_0\mathbf{V}^m\frac{d}{dt}(\mathcal{A}_0\mathbf{V}_m^*)$ is bounded in $L^2(0,T;L^2(\Omega)^{20})$, hence it follow the weak converge to Φ in the same space. To justify the equality $\Phi = \mathcal{A}_0\mathbf{V}\frac{d}{dt}(\mathcal{A}_0\mathbf{V})$, we have: firstly the estimate *i)* of lemma 4 guarantees the weak convergence of $\mathcal{A}_0\mathbf{V}^m$ in $L^\infty(0,T;H^s(\Omega))$ and due to the compact embendding $H^s(\Omega) \hookrightarrow L^2(\Omega)$, we have the strong convergence of $\mathcal{A}_0\mathbf{V}^m$ in $L^\infty(0,T;L^2(\Omega))$, and as $T < \infty$, the strong convergence also holds in $L^2(0,T;L^2(\Omega))$. And secondly, the point $d)$ of lemma 7 gives us the weak convergence of $\dfrac{d}{dt}(\mathcal{A}_0\mathbf{V}^*_m)$ in $L^2(0,T;L^2(\Omega))^5$. Thus, equality is obtained in the same way as previously. $\square$

Lemma 8. $\mathbf{K}_\varepsilon(\mathbf{V^m})\mathbf{V^m} \rightharpoonup \mathbf{K}_\varepsilon(\mathbf{W})\mathbf{W}$ *weakly in* $L^2(0,T;L^2(\Omega))^{20}$ *when* $m \to \infty$

Proof. Performing the vector-matrix product $\mathbf{K}_\varepsilon(\mathbf{V^m})\mathbf{V^m}$, the result is a consequence of the lemma 6. $\square$

We now be able to pass to the limit in the weak sense in the following system:

$$< \mathbf{A}_0(\mathbf{V}^m)\frac{d}{dt}\big(\mathcal{A}_0(\mathbf{V}^*_m)\big), \phi > + < \sum_{i=1}^{3} \mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{V}^*_m}{\partial x_i}, \phi > + < \mathbf{K}_\varepsilon(\mathbf{V}^m)\mathbf{V}^*_m, \phi > = < \mathbf{F}, \phi >, \quad \forall \phi \in \mathcal{D}(\mathcal{Q})^{20}$$

with $\mathcal{Q} = [0,T[\times\Omega$.
Applying the lemmas 7 and 8 we get: for all test function $\phi \in \mathcal{D}(\mathcal{Q})^{20}$:

$$< \mathbf{A}_0(\mathbf{W}^\varepsilon)\frac{\partial}{\partial t}\big(\mathcal{A}_0(\mathbf{W}^\varepsilon)\big), \phi > + < \sum_{i=1}^{3} \mathbf{A}_{i\varepsilon}\frac{\partial \mathbf{W}^\varepsilon}{\partial x_i}, \phi > + < \mathbf{K}_\varepsilon(\mathbf{W}^\varepsilon)\mathbf{W}^\varepsilon, \phi > = < \mathbf{F}, \phi >,$$

hence $\mathcal{A}_0\mathbf{W}^\varepsilon$ is a weak solution of the system (3). $\square$

5. Passing to limit $\varepsilon \longrightarrow 0$, démonstration of the theorem 2

In this section, we will pass to the limit $\varepsilon \to 0$ in $\mathbf{W}^\varepsilon$ solution of (3). By definition of the matrix $\mathcal{A}_0$, we have: $\mathcal{A}_0\mathbf{W}^\varepsilon = (\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon, 0, \ldots, 0)^T$. So by the theorem 1 we have:

$$\|\mathcal{A}_0\mathbf{W}^\varepsilon\|_{\infty,s} \leqslant C \qquad \text{and} \qquad \|\mathcal{D}_0\mathbf{W}^\varepsilon\|_{2,s-1} \leqslant C$$

where C is a positive constant independente to ε and $\mathcal{D}_0 \in \mathcal{M}(\mathbb{R}^{20})$ a matrix given by:

$$(\mathcal{D}_0)_{ij} = \begin{cases} 1 & \text{if } i = j \text{ with } i \notin (6, 11, 16) \\ 0 & \text{if not} \end{cases} \tag{88}$$

Then, extracting subsequence if necessary, we have the convergence of $\mathbf{W}^\varepsilon$ to $\mathbf{W}$ in $L^\infty(0,T,H^s(\Omega))^{20}$. It remains now to verify that the limit $\mathcal{A}_0\mathbf{W}$ is solution of the system (1) which means to show that:

$$\frac{\partial \mathbf{W}_i}{\partial \mathbf{x}_j} = \mathbf{W}_{i+5j} \qquad \text{for} \qquad i = 1, \ldots, 5 \quad \text{and} \quad j = 1, \ldots, 3 \tag{89}$$

We have $(\mathbf{W}^\varepsilon)_1 \in L^2(0,T;H^s(\Omega))^{20}$, which gives:

$$\begin{aligned} \left\|\frac{\partial \mathbf{W}^\varepsilon_1}{\partial \mathbf{x}_1}\right\|_{H^{s-1}} &\leqslant \|\mathbf{W}^\varepsilon_1\|_{H^s} \leqslant C \\ \left\|\frac{\partial \mathbf{W}^\varepsilon_1}{\partial \mathbf{x}_1}\right\|_{L^2} &\leqslant \|\mathbf{W}^\varepsilon_1\|_{H^s} \leqslant C \end{aligned}$$

$(\mathbf{W}^\varepsilon)_1$ being bounded in $L^2(0,T;H^s(\Omega))$, one can deduce that, $\dfrac{\partial \mathbf{W}_1^\varepsilon}{\partial \mathbf{x}_1} = (\mathbf{W}^\varepsilon)_6$ is bounded in $L^2(0,T;H^{s-1}(\Omega))$. Consequently their exist subsequence noted again $\dfrac{\partial \mathbf{W}_1^\varepsilon}{\partial \mathbf{x}_1}$ which converges weakly to $\overline{\overline{\overline{\mathbf{W}}}}$ in $L^2(0,T;H^{s-1}(\Omega))$. On the other hand we have $\mathbf{W}_1^\varepsilon$ converge strongly to $\longrightarrow \mathbf{W}_1$ in $L^2(0,T;H^s(\Omega))$, which gives $\overline{\overline{\overline{\mathbf{W}}}} = \dfrac{\partial \mathbf{W}_1}{\partial \mathbf{x}_1}$. This allows us to deduce the following convergences:

$$\begin{array}{ccc} \dfrac{\partial (\mathbf{W}^\varepsilon)_1}{\partial \mathbf{x}_1} & \longrightarrow & \dfrac{\partial \mathbf{W}_1}{\partial \mathbf{x}_1} \quad \text{in} \quad L^2(0,T;H^{s-1}(\Omega)) \\ (\mathbf{W}^\varepsilon)_6 & \longrightarrow & \mathbf{W}_6 \quad \text{in} \quad L^2(0,T;H^{s-1}(\Omega)) \end{array}$$

By equality of sequences and their limits, we have: $\dfrac{\partial \mathbf{W}_1}{\partial \mathbf{x}_1} = \mathbf{W}_6$.

We repeat the same calculation for $\dfrac{\partial \mathbf{W}_1}{\partial \mathbf{x}_2} = \mathbf{W}_{11}$, which finally gives:

$$\begin{array}{rcl} \mathcal{A}_0\mathbf{W} & \in & L^\infty(0,T;H^s(\Omega)) \\ \dfrac{\partial}{\partial t}(\mathcal{A}_0\mathbf{W}) & \in & L^2(0,T;L^2(\Omega)) \end{array}$$

The verification of $\mathcal{A}_0\mathbf{W}$ solution of the compressible Navier-Stokes system (1) is done in the same way as before.

6. Conclusion

In this work we have proved the existence of a weak solution of the Navier-Stokes-Fourier system if the second member remains bounded in time and with a certain regularity in spaces. The justification is based on a reduction of the system order due to the add of a diffusion in the continuity equation. However, we prove that this addition preserves the positivity of the density as long as there is not a vacuum at the initial time. The treatment of the nonlinearity of the Navier-Stokes-Fourier system is done by a successive approximation and the passage to the limit was possible thanks to some a priori estimations.

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