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Existence result and approximation of an optimal control problem for the Perona–Malik equation

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Abstract

We discuss some optimal control problem for the evolutionary Perona–Malik equations with the Neumann boundary condition. The control variable v is taken as a distributed control. The optimal control problem is to minimize the discrepancy between a given distribution $u_d \in L^2(\Omega)$ and the current system state. Since we cannot expect to have a solution of the original boundary value problem for each admissible control, we make use of a variant of its approximation using the model with fictitious control in coefficients of the principle elliptic operator. We introduce a special family of regularized optimization problems for linear parabolic equations and show that each of these problems is consistent, well-posed, and their solutions allow to attain (in the limit) an optimal solution of the original problem as the parameter of regularization tends to zero.

Keywords Perona–Malik equation · Optimal control problem · Fictitious control · Control in coefficients · Approximation approach

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1 Introduction

Recently, in the context of time interpolation of satellite multi-spectral images, the following model has been proposed (see [1])

$$u_t - \operatorname{div}\left(f\left(|\nabla u|\right)\nabla u\right) + (\nabla u, \boldsymbol{b}) = v \quad \text{in } \; Q = (0,T)\times\Omega, \tag{1}$$

$$u(0,x) = u_0(x) \quad \text{in } \; \Omega, \tag{2}$$

$$\partial_\nu u(t,x) = 0 \quad \text{on } \; \Sigma = (0,T)\times\partial\Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, $\boldsymbol{b} \in \mathfrak{B}_{ad}$ and $v \in \mathfrak{V}_{ad}$ are the control functions with

$$\mathfrak{B}_{ad} = \left\{ \boldsymbol{b} \in L^\infty(Q)^2 \cap BV(Q)^2 \;:\; \|\boldsymbol{b}\|_{L^\infty(Q)^2} \le \kappa \right\}, \tag{4}$$

$$\mathfrak{V}_{ad} = \left\{ v \in L^2(0,T; L^2(\Omega)) \right\}, \tag{5}$$

∂_ν stands for the outward normal derivative, $f \in C^{1,1}(\mathbb{R}_+)$ is a non-increasing real function such that $f(s) \to 0$ when $s \to +\infty$ and $f(s) \to 1$ when $s \to +0$. In particular,

$$f\left(|\nabla u|\right) = \frac{1}{1 + |\nabla u|^2}. \tag{6}$$

In fact, the Cauchy–Neumann problem (1)–(3) can be viewed as some improvement of the Perona–Malik model [2] that was proposed in order to avoid the blurring in images and to reduce the diffusivity at those locations which have a larger likelihood to be edges. This likelihood is measured by $|\nabla u|^2$.

It is well-known that the model (1) is an ill-posed problem from the mathematical point of view and can produce many unexpected phenomena (see [3]). To overcome this problem, many authors have been looking for some regularizations of the equation (1) which inherit its usefulness in image restoration but have better mathematical behavior (see, for instance, [4–9] and the references therein). In particular, in order to guarantee the existence and uniqueness of solution to the initial-boundary value problem (1)–(3), the authors in [1] proposed to specify the equation (1) as follows

$$u_t - \operatorname{div}\left(K(t,x)\nabla u\right) + (\nabla u, \boldsymbol{b}) = v \quad \text{in } \; Q = (0,T)\times\Omega \tag{7}$$

with $K(t,x) = f\left(|\nabla Y^*_\sigma|\right)$, where $\nabla Y^*_\sigma = \nabla G_\sigma * Y^*$ is the spatially regularized gradient of Y^*, G_σ denotes the two-dimensional Gaussian filter kernel, and $Y^* \in C([0,T]; L^2(\Omega))$ is a special function which describes the simplest model of image evolution over the interval $[0,T]$.

However, it is well-known that the Perona–Malik model with the spatially regularized gradient has several serious practical and theoretical difficulties. The first one is that the spatial regularization of gradient in the form $f(|\nabla G_\sigma * u|)$ leads to the loss of accuracy in the case when the signal is noisy, with white noise (see for instance [6]). The second drawback of the Perona–Malik model with the regularized gradient (see also the model (7), (2), (3)) is the fact that the space-invariant Gaussian smoothing inside the divergent term tends to push the edges in u away from their original locations. We refer to [10] where this issue is studied in details. This effect, known as edge dislocation, can be detrimental especially in the context of the boundary detection problem and its application to the remote sensing and monitoring.

In view of this, our prime interest in this paper is to study the equation (1) and the corresponding PDE-constrained optimization problem without the space-invariant Gaussian smoothing inside the divergent term. With that in mind we consider the following optimal control problem

$$(\mathcal{R})\quad \text{Minimize } J(v,u) = \int_{Q_T}\left|D\left(\frac{1}{1+|\nabla u|^2}\right)\right| + \frac{1}{2}\int_\Omega |u(T)-u_d|^2\,dx \\ + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u|^2\,dxdt + \frac{\gamma}{2}\int_0^T\int_\omega |v|^2\,dxdt \tag{8}$$

subject to the constraints

$$u_t - \operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right) = v\chi_\omega \quad \text{in } Q_T := (0,T)\times\Omega, \tag{9}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{10}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{11}$$

$$v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\omega)), \tag{12}$$

where $T > 0$, Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary, $N \geq 2$, ω is an open nonempty subset of Ω, $\chi_\omega = \left\{\begin{matrix} 1, & x\in\omega, \\ 0, & x\in\Omega\setminus\omega \end{matrix}\right\}$ is the characteristic function of the set ω, ∂_ν stands for the outward normal derivative, $u_0, u_d \in L^2(\Omega)$ are given functions, λ, γ are given positive constants, and $v : \omega \to \mathbb{R}$ is a control.

As was mentioned before, the operator $\operatorname{div}(f(|\nabla u|)\nabla u)$ with a function f given by (6) provides an example of a non-linear operator in divergence form with a so-called degenerate nonlinearity. Moreover, since the function $\mathbb{R}^N \ni s \mapsto \frac{s}{1+|s|^2} \in \mathbb{R}^N$ is neither monotone nor coercive, we have no existence result for the initial-boundary value problem (IBVP) (9)–(11) and its uniqueness. With that in mind, we say that (v,u) is a feasible pair to the problem (8)–(12) if

$$v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\omega)), \quad u \in L^2(0,T;H^1(\Omega)), \quad J(v,u) < +\infty, \tag{13}$$

and the following integral identity

$$\int_0^T \int_\Omega \left(-u \frac{\partial \varphi}{\partial t} + \frac{(\nabla u, \nabla \varphi)}{1 + |\nabla u|^2} \right) dxdt = \int_0^T \int_\omega v\varphi \, dxdt + \int_\Omega u_0(x)\varphi(0, x) \, dx \tag{14}$$

holds for any function $\varphi \in \Phi$, where

$$\Phi = \left\{ \varphi \in C^1(\overline{Q_T}) \ : \ \varphi(T, \cdot) = 0 \ \text{ in } \Omega \text{ and } \partial_\nu \varphi = 0 \text{ on } \ (0, T) \times \partial\Omega \right\}.$$

In order to find out in what sense the solution takes the initial value $u(0, \cdot) = u_0$, we make use of the following result.

Proposition 1 *Let (v, u) be a feasible pair to the problem (8)–(12). Then, for any $\eta \in C_0^\infty(\Omega)$, the scalar function $h(t) = \int_\Omega u(t, x)\eta(x) \, dx$ belongs to $W^{1,1}(0, T)$ and $h(0) = \int_\Omega u_0(x)\eta(x) \, dx$.*

For further convenience we denote the set of all feasible solutions to the problem (8)–(12) by Ξ. Because of the degenerate behavior of multiplier $f(|\nabla u|)$, the structure of the set Ξ and its main topological properties are unknown in general.

The main focus in this paper consists in providing an approximation framework which in spite of the technical difficulties leads to an implementable scheme, namely, to the so-called indirect approach proving the existence of optimal solutions and giving the procedure of their efficient approximation. We show that the original optimal control problem (8)–(12) can be approximated efficiently by a special family of optimal control problems for linear parabolic equations with the fictitious BV-control in the principle part of elliptic operator div $(\rho \nabla u)$.

The paper is organized as follows. In the next section, we give some preliminaries and notions that will be needed in the sequel. Section 3 contains a few technical results concerning the almost everywhere convergence of the gradients of solutions to linear parabolic equations with BV-coefficients in the main part of the elliptic operator. These results were obtained in the spirit of Bocardo and Murat approach (see Theorems 4.1 and 4.3 in [11]). In Sect. 4 we give a precise statement of the fictitious optimal control problem for linear parabolic equation with the constrained BV-control in the coefficients. The announced approximation framework is the subject of Sect. 5, where we provide an asymptotic analysis of a family of approximated optimal control problems and show that some optimal pairs to the original problem (8)–(12) can be attained (in an appropriate topology) by optimal solutions to the approximated problems.

2 Preliminaries and basic definitions

We begin with some notation. Let Ω be a given bounded open subset of R^N $(N \geq 2)$ with a sufficiently smooth boundary. For any subset $D \subset \Omega$ we denote by $|D|$ its

N-dimensional Lebesgue measure $\mathcal{L}^N(D)$. We define the characteristic function χ_D of D by $\chi_D(x) := \begin{cases} 1, & \text{for } x \in D, \\ 0, & \text{otherwise.} \end{cases}$

Let X denote a real Banach space with norm $\|\cdot\|_X$, and let X' be its dual. Let $\langle \cdot, \cdot \rangle_{X';X}$ be the duality form on $X' \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and weak* convergence in normed spaces, respectively.

We denote by $C_c^\infty(\mathbb{R}^N)$ a locally convex space of all infinitely differentiable functions with compact support. We recall here some functional spaces that will be used throughout this paper. We define the Banach space $H^1(\Omega)$ as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|y\|_{H^1(\Omega)} = \left(\int_\Omega \left(y^2 + |\nabla y|^2 \right) dx \right)^{1/2}.$$

We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$.

Let $k > 0$. We set $T_k(s) = \max\{-k, \min\{s, k\}\}$.

Theorem 2 *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $G(0) = 0$. If u belongs to $H^1(\Omega)$, then $G(u)$ belongs to $H^1(\Omega)$, $\nabla G(u) = G'(u)\nabla u$ almost everywhere in Ω, and as a result*

$$\nabla T_k(u) = \nabla u \chi_D\{|u| \le k\} \quad \textit{almost everywhere in } \Omega. \tag{15}$$

Weak and strong convergence in $L^1(\Omega)$ Let ε be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, in the case $\varepsilon \ge 0$ we also consider its limit $\varepsilon = 0$. Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$. We recall that $\{a_\varepsilon\}_{\varepsilon>0}$ is called equi-integrable if for any $\delta > 0$ there is $\tau = \tau(\delta)$ such that $\int_S |a_\varepsilon|\, dx < \delta$ for all a_ε and for every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$.

Theorem 3 (Dunford–Pettis, [12]) *Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$. Then this sequence is relatively compact with respect to the weak convergence in $L^1(\Omega)$ if and only if $\{a_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$, i.e., $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^1(\Omega)} < +\infty$, and $\{a_\varepsilon\}_{\varepsilon>0}$ is equi-integrable.*

Theorem 4 (Lebesgue–Vitali, [12]) *If a sequence $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ is equi-integrable and there exists a function $a \in L^1(\Omega)$ such that $a_\varepsilon(x) \to a(x)$ almost everywhere in Ω then $a_\varepsilon \to a$ in $L^1(\Omega)$.*

A typical application of Vitali's theorem is provided by the next simple lemmas.

Lemma 1 [12] *Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$ such that $a_\varepsilon(x) \to a(x)$ almost everywhere in Ω, and this sequence is uniformly bounded in $L^p(\Omega)$ for some $p > 1$. Then*

$$a_\varepsilon \to a \quad \textit{in } L^r(\Omega) \textit{ for all } 1 \le r < p. \tag{16}$$

Lemma 2 [12] *Let $\{a_\varepsilon\}_{\varepsilon>0}$, $\{b_\varepsilon\}_{\varepsilon>0}$, a, and b be a measurable functions such that*

$$a_\varepsilon(x) \to a(x) \ \text{a.e. in} \ \Omega, \quad \sup_{\varepsilon>0} \|a_\varepsilon\|_{L^\infty(\Omega)} < \infty, \tag{17}$$

$$b_\varepsilon \rightharpoonup b \quad \text{in} \ L^1(\Omega). \tag{18}$$

Then

$$ab \in L^1(\Omega) \quad \text{and} \quad a_\varepsilon b_\varepsilon \rightharpoonup ab \quad \text{in} \ L^1(\Omega). \tag{19}$$

Functions with bounded variation Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$\int_\Omega |Df| = \sup\Big\{ \int_\Omega f \operatorname{div} \varphi \, dx \ : \ \varphi = (\varphi_1, \dots, \varphi_N) \in C_0^1(\Omega; \mathbb{R}^N), \ |\varphi(x)| \le 1 \text{ for } x \in \Omega \Big\},$$

where $\operatorname{div} \varphi = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$.

Definition 1 A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $\int_\Omega |Df| < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation.

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df|$, $BV(\Omega)$ is a Banach space. The following compactness result for BV-functions is well-known:

Proposition 5 *The uniformly bounded sets in BV-norm are relatively compact in $L^1(\Omega)$.*

Definition 2 A sequence $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ weakly-$*$ converges to some $f \in BV(\Omega)$, and we write $f_k \overset{*}{\rightharpoonup} f$ if and only if the two following conditions hold: $f_k \to f$ strongly in $L^1(\Omega)$, and $Df_k \rightharpoonup Df$ weakly-$*$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$, where $\mathcal{M}(\Omega; \mathbb{R}^N)$ stands for the space of all vector-valued Borel measures which is, according to the Riesz theory, the dual of the space $C(\Omega; \mathbb{R}^N)$ of all continuous vector-valued functions φ vanishing at infinity.

In the proposition below we give a compactness result related to this convergence, together with the lower semicontinuity property (see [13]):

Proposition 6 *Let $\{f_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$. Then*

(i) $f \in BV(\Omega)$ and $\int_\Omega |Df| \le \liminf_{k\to\infty} \int_\Omega |Df_k|$;
(ii) $f_k \overset{*}{\rightharpoonup} f$ in $BV(\Omega)$.

3 Some auxiliaries

In this section we give a few technical results that can be viewed as some specification of the well-known results of Bocardo and Murat (see Theorems 4.1 and 4.3 in [11]). For the proof we refer to the recent paper [14].

Proposition 7 *Let $\{u_k\}_{k\in\mathbb{N}}$ be a weakly convergennt sequence in $L^2(0,T;H^1(\Omega))$, and*

$$u_k \rightharpoonup u \quad \text{weakly in } \; L^2(0,T;H^1(\Omega)). \tag{20}$$

Assume that

$$\frac{\partial u_k}{\partial t} = h_k \quad \text{in } \; \mathcal{D}'((0,T)\times\Omega) \quad \forall k \in \mathbb{N}, \tag{21}$$

where $\{h_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;H^{-1}(\Omega))$. Then

$$u_k \to u \quad \text{strongly in } \; L^2_{loc}(0,T;L^2_{loc}(\Omega)). \tag{22}$$

Proposition 8 *Let $\varepsilon \in (0,1)$ and $K \in (0,\infty)$ be given values. Assume that the sequences*

$$\begin{aligned} &\{u_k\}_{k=1}^{\infty} \subset L^2(0,T;H^1(\Omega)), \quad \{v_k\}_{k=1}^{\infty} \subset L^2(0,T;L^2(\Omega)), \\ &\text{and} \quad \{\rho_k\}_{k=1}^{\infty} \subset BV(Q_T)\cap L^{\infty}(Q_T) \end{aligned} \tag{23}$$

are bounded and such that

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{24}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{25}$$

$$\rho_k \rightharpoonup \rho \text{ weakly-} * \text{ in } BV(Q_T) \text{ and a.e. in } Q_T, \tag{26}$$

$$\rho_k \geq \varepsilon \quad \text{a.e. in} \quad Q_T, \quad \forall k \in \mathbb{N}, \tag{27}$$

$$\frac{\partial u_k}{\partial t} - \operatorname{div}(\rho_k \nabla u_k) = v_k \quad \text{in } \; \mathcal{D}'(Q_T), \quad \forall k \in \mathbb{N}. \tag{28}$$

Then

$$\nabla T_K(u_k) \to \nabla T_K(u) \text{ strongly in } L^2_{loc}(0,T;L^2_{loc}(\Omega))^N, \tag{29}$$

where $T_K : \mathbb{R} \to \mathbb{R}$ is the truncation at height K.

Theorem 9 *Let $\varepsilon \in (0,1)$ be a given value and let*

$$\begin{aligned} &\{u_k\}_{k=1}^{\infty} \subset L^2(0,T;H^1(\Omega)), \quad \{v_k\}_{k=1}^{\infty} \subset L^2(0,T;L^2(\Omega)), \\ &\text{and} \quad \{\rho_k\}_{k=1}^{\infty} \subset BV(Q_T)\cap L^{\infty}(Q_T) \end{aligned} \tag{30}$$

be bounded sequences satisfying conditions (24)–(28). *Then*

$$\nabla u_k \to \nabla u \text{ strongly in } L^q(0,T;L^q(\Omega))^N \text{ for any } q \in [1,2). \tag{31}$$

4 Regularization of the original optimal control problem

We introduce the following family of approximating control problems

$$(\mathcal{R}_\varepsilon) \quad \text{Minimize } J_\varepsilon(\rho, v, u) = \frac{1}{2}\int_\Omega |u(T)-u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u|^2\,dxdt + \frac{\gamma}{2}\int_0^T\int_\omega |v|^2\,dxdt + \int_{Q_T}|D\rho| + \frac{1}{\varepsilon}\int_0^T\int_\Omega \left|\rho - \frac{1}{1+|\nabla u|^2}\right|^2 dxdt \tag{32}$$

subject to the constraints

$$u_t - \operatorname{div}(\rho\nabla u) = v\chi_\omega \quad \text{in } Q_T := (0,T)\times\Omega, \tag{33}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{34}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{35}$$

$$v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\omega)), \tag{36}$$

$$\rho \in \mathfrak{R}_{ad} := \left\{h \in BV(Q_T)\cap L^\infty(Q_T) \;:\; 0 \le h(t,x) \le 1 \text{ a.e. in } Q_T\right\}. \tag{37}$$

We say that a tuple (ρ, v, u) is a feasible solution to the problem (32)–(37) if

$$\rho \in \mathfrak{R}_{ad}, \quad v \in \mathfrak{V}_{ad}, \quad u \in L^2(0,T;H^1(\Omega)), \tag{38}$$

$$\rho(t,x) \ge \max\left\{\frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u(t,x)|^2}\right\} \text{ a.e. in } Q_T, \tag{39}$$

and this triplet satisfies the following integral identity

$$\int_0^T\int_\Omega \left(-\varphi_t u + \rho\,(\nabla u, \nabla\varphi)\right)\,dxdt = \int_0^T\int_\omega v\varphi\,dxdt + \int_\Omega u_0\varphi(0,x)\,dx \tag{40}$$

for each $\varphi \in \Psi$, where

$$\Psi = \left\{\varphi \in C^1(\overline{Q_T}) \;:\; \varphi(T,\cdot) = 0 \text{ in } \Omega \text{ and } \partial_\nu\varphi = 0 \text{ on } (0,T)\times\partial\Omega\right\}.$$

The set of all feasible solution is denoted by Ξ_ε.

Remark 1 Arguing as in [15], it can be shown that the original IBVP has a unique solution for each $\rho \in \mathfrak{R}_{ad}$ and $v \in \mathfrak{V}_{ad}$. Moreover, in this case the integral identity (40) holds for any function $\varphi \in \Psi$ and the energy equality

$$\int_\Omega u^2(t,x)\,dx + 2\int_0^t \int_\Omega \rho|\nabla u|^2\,dxdt = 2\int_0^t \int_\omega vu\,dxdt + \int_\Omega u_0^2\,dx \tag{41}$$

is valid for all $0 \le t \le T$.

Definition 3 A sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is called bounded if

$$\sup_{k\in\mathbb{N}} \left[\|\rho_k\|_{BV(Q_T)} + \|v_k\|_{L^2(0,T;L^2(\omega))} + \|u_k\|_{L^2(0,T;H^1(\Omega))}\right] < +\infty.$$

Definition 4 We say that a bounded sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ of feasible solutions τ-converges to a triplet $(\rho, v, u) \in BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$ if conditions

$$u_k \rightharpoonup u \ \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{42}$$

$$v_k \rightharpoonup v \ \text{ weakly in } L^2(0,T;L^2(\omega)), \tag{43}$$

$$\rho_k \rightharpoonup \rho \text{ weakly} - * \ \text{ in } BV(Q_T) \text{ and a.e. in } \ Q_T \tag{44}$$

hold true.

Remark 2 As follows from Theorem 9, if $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is a τ-convergent sequence of feasible solutions and $(\rho_k, v_k, u_k) \xrightarrow{\tau} (\rho, v, u)$, then $\nabla u_k \to \nabla u$ strongly in $L^q(0,T;L^q(\Omega))^N$ for any $q \in [1,2)$ and, passing to a subsequence if necessary, we can assert that $\nabla u_k(t,x) \to \nabla u(t,x)$ a.e. in $Q_T = (0,T) \times \Omega$.

Remark 3 From (40) we deduce: if (ρ, v, u) is a feasible solution to the problem (32)–(37), then the equality

$$\frac{\partial u_k}{\partial t} - \operatorname{div}(\rho_k \nabla u_k) = \chi_\omega v_k \quad \text{in } \ \mathcal{D}'(Q_T)$$

holds in the sense of distributions for each $k \in \mathbb{N}$. Moreover, if a sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is bounded in the sense of Definition 3, then $\operatorname{div}(\rho_k \nabla u_k) + \chi_\omega v_k \in L^2(0,T;H^{-1}(\Omega))$. Therefore, $u_k \in C([0,T];L^2(\Omega))$ for all $k \in \mathbb{N}$ (see [16, Proposition III.1.2]) and due to J.L. Lions [17, Chapitre 1, Theorem 5.1] (we refer also to [18] for some generalizations), the Banach space

$$W = \left\{\varphi \ : \varphi \in L^2(0,T;H^1(\Omega)),\ \frac{\partial \varphi}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\right\}$$

is compactly embedded into $L^2(0,T;L^2(\Omega))$.

Thus, the first term in the objective functional (32) is well defined onto the set of feasible solutions. So, if $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in W and $u_k \rightharpoonup u$ weakly in

$L^2(0,T;H^1(\Omega))$, then $u_k \to u$ strongly in $L^2(0,T;L^2(\Omega))$ and, as a consequence, $u_k(T,\cdot) \to u(T,\cdot)$ strongly in $L^2(\Omega)$.

Before proceeding further, we establish the following important property.

Proposition 10 *For every $\varepsilon \in (0,1)$ the set Ξ_ε is sequentially closed with respect to the τ-convergence.*

Proof Let $\{(\rho_k, v_k, u_k)\}_{k\in\mathbb{N}} \subset \Xi_\varepsilon$ be a τ-convergent sequence of feasible solutions to the optimal control problem (32)–(37). Let (ρ, v, u) be its τ-limit. Our aim is to show that $(\rho, v, u) \in \Xi_\varepsilon$.

Since the inclusions $\chi_\omega v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\Omega))$ and $u \in L^2(0,T;H^1(\Omega))$ are obvious, let us show that the condition (27) is valid for some $\varepsilon > 0$. Indeed, in view of Remark 2, we can suppose that, up to a subsequence,

$$u_k(t,x) \to u(t,x) \quad \text{and} \quad \frac{1}{1+|\nabla u_k(t,x)|^2} \to \frac{1}{1+|\nabla u(t,x)|^2} \text{ a.e. in } Q_T.$$

Hence, in view of the definition of τ-convergence, the limit passage in the relation

$$\rho_k(t,x) \ge \max\left\{\frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u_k(t,x)|^2}\right\} \text{ a.e. in } Q_T$$

immediately leads us to the inequality (27) with $\widehat{\varepsilon} = \frac{\varepsilon^2}{1+\varepsilon^2}$. As for the inclusion $\rho \in \mathfrak{R}_{ad}$, it is a direct consequence of the weak-$*$ compactness of bounded set $\mathfrak{R}_{ad}$ in $BV(Q_T)$.

It remains to show that the limit triplet (ρ, v, u) is related by the integral identity (40). To do so, it is enough to fix an arbitrary test function $\varphi \in \Psi$ and pass to the limit in relation

$$\int_0^T \int_\Omega \left(-\varphi_t u_k + \rho_k \left(\nabla u_k, \nabla\varphi\right)\right) dxdt = \int_0^T \int_\omega v_k \varphi \, dxdt + \int_\Omega u_0 \varphi(0,x)\, dx. \tag{45}$$

Since $\rho_k \nabla u_k \to \rho\nabla u$ strongly in $L^q(Q_T)$ for $q \in [1,2)$ by Lemma 1, it follows that the limit passage in (45) leads to the integral identity (40). Thus, (ρ, v, u) is a feasible solution to optimal control problem (32)–(37). □

Theorem 11 *Let $u_d \in L^\infty(\Omega)$ be a given function, and let λ and γ be given constants. Then, for each $\varepsilon \in (0,1)$, the optimal control problem (32)–(37) admits at least one solution $(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon$.*

Proof Let $\varepsilon \in (0,1)$ be a fixed value. Then, as it was indicated in Remark 1, the optimal control problem (32)–(37) is consistent, that is, $\Xi_\varepsilon \neq \emptyset$.

Let $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ be a minimizing sequence to the problem (32)–(37). Then the relation

$$
\begin{aligned}
\inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho,v,u) = \lim_{k\to\infty}\Big[\frac{1}{2}\int_\Omega |u_k(T)-u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u_k|^2\,dxdt \\
+\frac{\gamma}{2}\int_0^T\int_\omega |v_k|^2\,dxdt \quad + \int_{Q_T}|D\rho_k| + \frac{1}{\varepsilon}\int_0^T\int_\Omega |\rho_k \\
-\frac{1}{1+|\nabla u_k|^2}|^2\,dxdt\Big] < +\infty
\end{aligned}
$$

and definition of the set $\mathfrak{R}_{ad}$ implies existence of a constant $C>0$ such that

$$
\begin{aligned}
\sup_{k\in\mathbb{N}}\|\nabla u_k\|_{L^2(0,T;L^2(\Omega)^N)} \le C, \quad \sup_{k\in\mathbb{N}}\|v_k\|_{L^2(0,T;L^2(\omega))} \le C, \\
\text{and}\ \sup_{k\in\mathbb{N}}\|\rho_k\|_{BV(Q_T)} \le C.
\end{aligned}
\tag{46}
$$

Then, from the energy equality (41), we deduce that

$$
\begin{aligned}
\int_0^T\int_\Omega u_k^2(t,x)\,dxdt &\le 2T\int_0^T\int_\omega v_k u_k\,dxdt + T\int_\Omega u_0^2\,dx \\
&\le 2T^2\int_0^T\int_\omega v_k^2\,dxdt + \frac{1}{2}\int_0^T\int_\Omega u_k^2\,dxdt + T\int_\Omega u_0^2\,dx.
\end{aligned}
$$

Hence,

$$
\sup_{k\in\mathbb{N}}\|u_k\|_{L^2(0,T;L^2(\Omega))} \le 4T^2C^2 + 2T\|u_0\|^2_{L^2(\Omega)}.
$$

Utilizing this fact together with (46), we see that the sequence $\{(\rho_k,v_k,u_k)\in\Xi_\varepsilon\}_{k\in\mathbb{N}}$ is bounded in the sense of Definition 3. As a result, there exist functions $\rho_\varepsilon^0\in BV(Q_T)$, $v_\varepsilon^0\in L^2(0,T;L^2(\omega))$, and $u_\varepsilon^0\in L^2(0,T;H^1(\Omega))$ such that, up to a subsequence, $(\rho_k,v_k,u_k)\xrightarrow{\tau}(\rho_\varepsilon^0,v_\varepsilon^0,u_\varepsilon^0)$ as $k\to\infty$. Since the set Ξ_ε is sequentially closed with respect to the τ-convergence (see Proposition 10), it follows that the τ-limit tuple $(\rho_\varepsilon^0,v_\varepsilon^0,u_\varepsilon^0)$ is a feasible solution to optimal control problem (32)–(37) (i.e., $(\rho_\varepsilon^0,v_\varepsilon^0,u_\varepsilon^0)\in\Xi_\varepsilon$). To conclude the proof, we observe that $\nabla u_k(t,x)\to\nabla u_\varepsilon^0(t,x)$ a.e. in Q_T (see Remark 2) and, therefore,

$$
\rho_k(t,x) - \frac{1}{1+|\nabla u_k(t,x)|^2} \to \rho_\varepsilon^0(t,x) - \frac{1}{1+|\nabla u_\varepsilon^0(t,x)|^2}\ \text{a.e. in}\ Q_T.
$$

Since

$$
\left\|\rho_k - \frac{1}{1+|\nabla u_k|^2}\right\|_{L^\infty(Q_T)} \le 2\ \text{for all}\ k\in\mathbb{N},
$$

it follows that the sequence $\left\{\rho_k - \dfrac{1}{1+|\nabla u_k|^2}\right\}_{k\in\mathbb{N}}$ is equi-integrable. Hence, Vitaly's theorem implies that

$$\rho_k - \frac{1}{1+|\nabla u_k|^2} \to \rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2} \quad \text{strongly in } L^2(Q_T) \tag{47}$$

(see Lemma 1). Taking this fact into account and observing that

$$\begin{aligned}
&\liminf_{k\to\infty} \int_0^T \int_\Omega |\rho_k - \frac{1}{1+|\nabla u_k|^2}|^2\,dxdt \overset{\text{by (47)}}{=} \int_0^T \int_\Omega |\rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2}|^2\,dxdt,\\
&\lim_{k\to\infty} \int_\Omega |u_k(T) - u_d|^2\,dx \overset{\text{by Remark (3)}}{\geq} \int_\Omega |u_\varepsilon^0(T) - u_d|^2\,dx,\\
&\lim_{k\to\infty} \int_0^T \int_\Omega |\nabla u_k|^2\,dxdt \overset{\text{by (42)}}{=} \int_0^T \int_\Omega |\nabla u_\varepsilon^0|^2\,dxdt,\\
&\liminf_{k\to\infty} \int_0^T \int_\omega |v_k|^2\,dxdt \overset{\text{by(43)}}{\geq} \int_0^T \int_\Omega |v_\varepsilon^0|^2\,dxdt,\\
&\liminf_{k\to\infty} \int_{Q_T} |D\rho_k| \overset{\text{by(44)}}{\geq} \int_{Q_T} |D\rho_\varepsilon^0|,
\end{aligned}$$

we see that the cost functional J_ε is sequentially lower τ-semicontinuous. Thus

$$J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \leq \liminf_{k\to\infty} J_\varepsilon(\rho_k, v_k, u_k) \leq \lim_{k\to\infty} J_\varepsilon(\rho_k, v_k, u_k) = \inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u),$$

and, therefore, $(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0)$ is an optimal triplet. □

5 Asymptotic analysis of the approximated OCP $(\mathcal{R}_\varepsilon)$

The main goal of this section is to show that the original OCP $(\mathcal{R})$ is solvable and some solutions can be attained (in an appropriate topology) by optimal solutions to the approximated problems $(\mathcal{R}_\varepsilon)$. With that in mind, we make use of the concept of variational convergence of constrained minimization problems (see [19–21]) and study the asymptotic behavior of a family of OCPs $(\mathcal{R}_\varepsilon)$ as $\varepsilon \to 0$.

Definition 5 Let $\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0} \subset BV(Q_T)\times L^2(0,T;L^2(\omega))\times L^2(0,T;H^1(\Omega))$ be an arbitrary sequence. We say that this sequence is bounded if

$$\sup_{\varepsilon>0}\left[\|\rho_\varepsilon\|_{BV(Q_T)} + \|v_\varepsilon\|_{L^2(0,T;L^2(\omega))} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right] < +\infty.$$

Definition 6 We say that a bounded sequence

$$\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0} \subset BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$$

is w-convergent as $\varepsilon \to 0$ and $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \xrightarrow{w} (\rho, v, u)$ if $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \xrightarrow{\tau} (\rho, v, u)$ as $\varepsilon \to 0$, i.e.,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0, T; H^1(\Omega)), \tag{48}$$

$$v_\varepsilon \rightharpoonup v \text{ weakly in } L^2(0, T; L^2(\omega)), \tag{49}$$

$$\rho_\varepsilon \rightharpoonup \rho \text{ weakly} - * \text{ in } BV(Q_T) \text{ and a.e. in } Q_T; \tag{50}$$

and $\nabla u_\varepsilon \to \nabla u$ strongly in $L^1(0, T; L^1(\Omega)^N)$.

The following technical result plays a significant role in the sequel.

Lemma 3 *Let $\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a τ-convergent sequence of feasible solutions to OCPs (32)–(37), and let $(\rho, v, u) \in BV(Q_T) \times L^2(0, T; L^2(\omega)) \times L^2(0, T; H^1(\Omega))$ be its τ-limit.*
Then $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \xrightarrow{w} (\rho, v, u)$ as $\varepsilon \to 0$, and (ρ, v, u) is subjected to the constrains

$$\rho \in \mathfrak{R}_{ad}, \quad v \in \mathfrak{V}_{ad}, \quad u \in L^2(0, T; H^1(\Omega)), \tag{51}$$

$$\rho(t, x) \geq \frac{1}{1 + |\nabla u(t, x)|^2} \ a.e. \ in \ \ Q_T, \tag{52}$$

$$\begin{aligned} &\int_0^T \int_\Omega (-\varphi_t u + \rho\, (\nabla u, \nabla \varphi))\, dxdt \\ &\quad = \int_0^T \int_\omega v\varphi\, dxdt + \int_\Omega u_0(x)\varphi(0, x)\, dx, \quad \forall \varphi \in \Psi. \end{aligned} \tag{53}$$

For the proof, we refer to [14].

Before we go on, we assume that the set of feasible solution Ξ to the problem (8)–(12) is non-empty. In the case when the initial state u_0 is sufficiently smooth and $\operatorname{supp}(u_0) \subset \omega$, this assumption can be easily verified. Indeed, let $\varphi \in C^\infty([0, T]; C_c^\infty(\omega))$ be an arbitrary function such that $\varphi(0, x) = u_0(x)$ in Ω. Then it is easy to check that the pair

$$(v, u) := \left(\left[\varphi_t - \operatorname{div} \left(\frac{\nabla \varphi}{1 + |\nabla \varphi|^2} \right) \right] \Big\lfloor_{x \in \omega}, \varphi \right)$$

belongs to the set Ξ. Thus, $\Xi \neq \emptyset$.

We begin with the following result that can be viewed as a direct consequence of Lemma 3 and Theorem 11.

Proposition 12 *Let $u_d \in L^\infty(\Omega)$ be a given function, and λ and γ be given constants. Let $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a bounded sequence of optimal solutions to the approximated problems (32)–(37) when the small parameter ε varies within a strictly decreasing sequence of positive numbers converging to zero. Then there is a subsequence of $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$, still denoted by the suffix ε, and distributions $\rho^0 \in \mathfrak{R}_{ad} \subset BV(Q_T)$, $v^0 \in \mathfrak{V}_{ad}$, and $u^0 \in L^2(0, T; H^1(\Omega))$ such that they satisfy conditions (52)–(53), and $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \xrightarrow{w} (\rho^0, v^0, u^0)$ as $\varepsilon \to 0$.*

The key point in Proposition 12 is the assumption that a given sequence of optimal solutions to the approximated problems (32)–(37) is bounded. Let us show that this assumption can be omitted if only the original optimal control problem is consistent, i.e. $\Xi \neq \emptyset$.

Proposition 13 *Assume that* $\Xi \neq \emptyset$. *Let* $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ *be a sequence of optimal solutions to the approximated problems* (32)–(37). *Then there exists a constant* $C > 0$ *independent of* $\varepsilon > 0$ *such that*

$$\sup_{\varepsilon>0}\left[\|\rho^0_\varepsilon\|_{BV(Q_T)} + \|v^0_\varepsilon\|_{L^2(0,T;L^2(\omega))} + \|u^0_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right] \le C. \tag{54}$$

Proof Let $(\widehat{v}, \widehat{u}) \in \Xi$ be a feasible solution to optimal control problem (8)–(12). Hence, this pair satisfies conditions (13)–(14). Setting $\widehat{\rho} := (1 + |\nabla \widehat{u}|^2)^{-1}$ in Q_T, we see that

$$0 \le \widehat{\rho}(t,x) \le 1 \text{ a.e. in } Q_T \quad \text{and} \quad \widehat{\rho} \in BV(Q_T) \cap L^\infty(Q_T),$$

and the pair $(\widehat{\rho}, \widehat{u})$ satisfies inequalities (39) for $\varepsilon > 0$ small enough. Hence, $\widehat{\rho} \in \mathfrak{R}_{ad}$ and, as a consequence, we deduce: $(\widehat{\rho}, \widehat{v}, \widehat{u}) \in \Xi_\varepsilon$ for $\varepsilon > 0$ small enough. Therefore,

$$\begin{aligned}\inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u) &= J_\varepsilon(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \le J_\varepsilon\left(\widehat{\rho}, \widehat{v}, \widehat{u}\right)\\ &= \frac{1}{2}\int_\Omega |\widehat{u}(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla \widehat{u}|^2\,dxdt\\ &\quad + \frac{\gamma}{2}\int_0^T\int_\omega |\widehat{v}|^2\,dxdt + \int_{Q_T}|D\widehat{\rho}| = C < +\infty.\end{aligned}$$

From this and definition of the set $\mathfrak{R}_{ad}$, we deduce that

$$\|\nabla u^0_\varepsilon\|^2_{L^2(0,T;L^2(\Omega)^N)} \le \frac{2}{\lambda}C, \quad \|v^0_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \le \frac{2}{\gamma}C, \tag{55}$$

$$\int_{Q_T}\left|D\rho^0_\varepsilon\right| \le C, \quad \|\rho^0_\varepsilon\|_{BV(\Omega)} \le |Q_T| + C, \tag{56}$$

$$\int_0^T\int_\Omega \left|\rho^0_\varepsilon - \frac{1}{1 + |\nabla u^0_\varepsilon|^2}\right|^2\,dxdt \le C\varepsilon \tag{57}$$

for all $\varepsilon > 0$ small enough. Then energy equality (41) implies that

$$\begin{aligned}&\int_0^T\int_\Omega \left[u^0_\varepsilon\right]^2\,dxdt \le 2T\int_0^T\int_\omega v^0_\varepsilon u^0_\varepsilon\,dxdt + T\int_\Omega u_0^2\,dx\\ &\le 2T^2\int_0^T\int_\omega\left[v^0_\varepsilon\right]^2\,dxdt + \frac{1}{2}\int_0^T\int_\Omega\left[u^0_\varepsilon\right]^2\,dxdt + T\int_\Omega u_0^2\,dx.\end{aligned}$$

Therefore,

$$\sup_{\varepsilon>0} \|u_\varepsilon^0\|_{L^2(0,T;L^2(\Omega))} \le 8T^2 \frac{C}{\gamma} + 2T\|u_0\|^2_{L^2(\Omega)}. \tag{58}$$

Thus, the sequence $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ is bounded in $BV(Q_T) \times L^2(0,T; L^2(\omega)) \times L^2(0,T; H^1(\Omega))$. □

The next step of our analysis is to show that the pair (v^0, u^0) is optimal to the original OCP $(\mathcal{R})$ provided (ρ^0, v^0, u^0) is a cluster tuple of a given sequence of optimal solutions $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$. To do so, we will utilize some hints from the recent papers [22, 23] where the so-called indirect approach to the existence problem of optimal solutions has been proposed.

Theorem 14 *Assume that $\Xi \neq \emptyset$. Let $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal solutions to the approximated problems (32)–(37). Let $(\rho^0, v^0, u^0) \in BV(Q_T) \times L^2(0,T; L^2(\omega)) \times L^2(0,T; H^1(\Omega))$ be a w-cluster tuple (in the sense of Definition 6) of a given sequence of optimal solutions Then*

$$(v^0, u^0) \in \Xi, \quad \rho^0(t,x) = \frac{1}{1 + |\nabla u^0(t,x)|^2} \ \textit{a.e. in } \ Q_T, \tag{59}$$

$$\lim_{\varepsilon\to 0} \inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u) = \lim_{\varepsilon\to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) = J(v^0, u^0) = \inf_{(v,u)\in\Xi} J(v,u). \tag{60}$$

Proof Arguing as in the proof of Proposition 13, it can be shown that there exists a constant $C > 0$ such that estimates (55)–(58) hold true. Hence, the sequence $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ is compact with respect to the τ-convergence. Moreover, in view of Proposition 12 and the Lebesgue Dominated Theorem, we can suppose that, up to a subsequence,

$$(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \xrightarrow{w} (\rho^0, v^0, u^0) \tag{61}$$

$$\frac{1}{1 + |\nabla u_\varepsilon^0|^2} \to \frac{1}{1 + |\nabla u^0|^2} \quad \text{strongly in } L^2(Q_T) \text{as } \varepsilon \to 0, \tag{62}$$

$$\rho_\varepsilon^0(t,x) - \frac{1}{1 + |\nabla u_\varepsilon^0(t,x)|^2} \to \rho^0(t,x) - \frac{1}{1 + |\nabla u^0(t,x)|^2} \ \text{a.e. in } \ Q_T, \tag{63}$$

and $\left(\rho_\varepsilon^0 - \left(1 + |\nabla u_\varepsilon^0|^2\right)^{-1}\right) \in L^\infty(\Omega)$.

Then it follows from Vitaly's theorem (see Lemma 1) that

$$\rho_\varepsilon^0 - \left(1 + |\nabla u_\varepsilon^0|^2\right)^{-1} \to \rho^0 - \frac{1}{1 + |\nabla u^0|^2} \ \text{strongly in } L^2(\Omega).$$

However, as follows from the third estimate in (57), the L^2-limit of the sequence $\left\{\rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2}\right\}_{\varepsilon>0}$ is equal to zero. Hence, we obtain

$$\rho^0(t,x) = \frac{1}{1+|\nabla u^0(t,x)|^2} \quad \text{a.e. in } Q_T.$$

Thus,

$$(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \xrightarrow{w} \left(\frac{1}{1+|\nabla u^0|^2}, v^0, u^0\right) \quad \text{as } \varepsilon \to 0.$$

Taking into account Proposition 12, we see that (v^0, u^0) is a feasible solution to the original OCP $(\mathcal{R})$. Moreover, as a direct consequence of the properties (62), we have the following estimate

$$\begin{aligned} \liminf_{\varepsilon\to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \geq \frac{1}{2}\int_\Omega |u^0(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^0|^2\,dxdt \\ + \frac{\gamma}{2}\int_0^T\int_\Omega |v^0|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1+|\nabla u^0|^2}\right)\right| = J(v^0,u^0). \end{aligned} \tag{64}$$

Let us assume for a moment that the pair (v^0, u^0) is not optimal for $(\mathcal{R})$-problem. Then there exists another pair $(v^*, u^*) \in \Xi$ such that

$$J(v^*, u^*) < J(v^0, u^0) < +\infty. \tag{65}$$

Setting $\rho^* = \left(1+|\nabla u^*|^2\right)^{-1}$, we deduce from condition $(v^*, u^*) \in \Xi$ that the tuple (ρ^*, v^*, u^*) is a feasible solution to each approximate problem $(\mathcal{R}_\varepsilon)$, i.e.,

$$\left(\rho^*, v^*, u^*\right) \in \Xi_\varepsilon, \quad \forall\, \varepsilon \in (0,1). \tag{66}$$

Taking this fact into account, we get

$$\begin{aligned} J(v^0,u^0) &= \frac{1}{2}\int_\Omega |u^0(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^0|^2\,dxdt \\ &\quad + \frac{\gamma}{2}\int_0^T\int_\Omega |v^0|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1+|\nabla u^0|^2}\right)\right| \\ &\overset{\text{by } (64)}{\leq} \liminf_{\varepsilon\to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) = \liminf_{\varepsilon\to 0} \inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u) \\ &\leq \lim_{\varepsilon\to 0} J_\varepsilon(\rho^*, v^*, u^*) \\ &= \frac{1}{2}\int_\Omega |u^*(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^*|^2\,dxdt \\ &\quad + \frac{\gamma}{2}\int_0^T\int_\Omega |v^*|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1+|\nabla u^*|^2}\right)\right| \end{aligned}$$

$$+\frac{1}{\varepsilon}\int_0^T\int_\Omega\left|\rho^* - \frac{1}{1+|\nabla u^*|^2}\right|^2 dxdt = J(v^*, u^*).$$

Thus, $J(v^0, u^0) \le J(v^*, u^*)$ and we come into a conflict with condition (65). Hence, the limit pair (v^0, u^0) is optimal for the original OCP $(\mathcal{R})$. $\square$

As follows from Theorem 14, the optimal solutions to the approximated problems $(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0)$ can be considered as a basis for the construction of suboptimal controls to the original problem $(\mathcal{R})$ (for the details we refer to [19, 24–26])

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