J. Korean Math. Soc. **45** (2008), No. 3, pp. 645–681

EXISTENCE RESULT FOR HEAT-CONDUCTING VISCOUS INCOMPRESSIBLE FLUIDS WITH VACUUM

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Abstract. The Navier-Stokes system for heat-conducting incompressible fluids is studied in a domain $\Omega \subset \mathbf{R}^3$. The viscosity, heat conduction coefficients and specific heat at constant volume are allowed to depend smoothly on density and temperature. We prove local existence of the unique strong solution, provided the initial data satisfy a natural compatibility condition. For the strong regularity, we do not assume the positivity of initial density; it may vanish in an open subset (vacuum) of Ω or decay at infinity when Ω is unbounded.

1. Introduction

The governing system of equations for a heat-conducting viscous incompressible fluid is the following Navier-Stokes system of the scalar or vector fields $\rho(t,x), u(t,x)$ and $\theta(t,x)$ for $(t,x) \in (0,T) \times \Omega \subset \mathbf{R}_+ \times \mathbf{R}^3$ (see P. L. Lion's book [19]):

$$\operatorname{div} u = 0, \tag{1}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{2}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu \mathrm{d}u) + \nabla p = \rho f, \tag{3}$$

$$c_v\left((\rho\theta)_t + \operatorname{div}(\rho u\theta)\right) - \operatorname{div}(\kappa\nabla\theta) = 2\mu|\mathrm{d}u|^2 + \rho h. \tag{4}$$

We consider the system (1)-(4) supplemented with the initial and boundary value conditions:

$$\begin{gathered} (\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0) \quad \text{in} \quad \Omega, \\ (u,\theta) = (0,0) \quad \text{on} \quad \partial\Omega \times [0,T], \\ (\rho(t,x), u(t,x), \theta(t,x)) \to (0,0,0) \ \text{as} \ |x| \to \infty, \quad (t,x) \in (0,T) \times \Omega. \end{gathered} \tag{5}$$

Here we denote by ρ, u, p and θ the unknown density, velocity, pressure and temperature fields for the fluid, respectively. The equations (1), (2), and (3)

Received September 2, 2006.

2000 *Mathematics Subject Classification.* Primary 35A05, 76D03.

Key words and phrases. heat-conducting incompressible Navier-Stokes equations, strong solutions, vacuum.

*Supported by Sogang University Research Grant 20061015.

imply the incompressibility, mass conservation, balance of momentum, respectively, for the fluid, while (4) is derived from the balance of energy

$$(\rho e)_t + \operatorname{div}(\rho e u) - \operatorname{div}(\kappa \nabla \theta) = 2\mu |du|^2 + \rho h$$

by using the relationship $c_v = \frac{\partial e}{\partial \theta}$, where $e = e(\rho, \theta)$ is the internal energy. We also denote by du the deformation tensor $\frac{1}{2}(\nabla u + \nabla^t u)$, where ∇u is the gradient matrix $\left(\frac{\partial u^j}{\partial x_i}\right)$ of u and $\nabla^t u$ its transpose. We assume that the viscosity coefficient $\mu = \mu(\rho, \theta)$, specific heat at constant volume $c_v = c_v(\rho, \theta)$ and heat conductivity $\kappa = \kappa(\rho, \theta)$ are *positive* functions of ρ and θ. The known fields f and h denote a given external force and heat source per unit mass. Finally, $(0, T) \times \Omega$ is the time-space domain for the evolution of the fluid, where T is a finite positive number and Ω is either a bounded domain in $\mathbf{R}^3$ with smooth boundary or an unbounded domain such as the whole space $\mathbf{R}^3$ and an exterior domain with smooth boundary.

Throughout this paper, we adopt the following simplified notations for homogeneous and inhomogeneous Sobolev spaces in Ω:

$$\begin{aligned}
L^r &= L^r(\Omega), \quad D^{k,r} = \{v \in L^1_{loc}(\Omega) : |v|_{D^{k,r}} < \infty\}, \quad |v|_{D^{k,r}} = |\nabla^k v|_{L^r}, \\
D^1_0 &= \{v \in L^6(\Omega) : |v|_{D^1_0} < \infty \text{ and } v = 0 \text{ on } \partial\Omega\}, \quad |v|_{D^1_0} = |\nabla v|_{L^2}, \\
D^1_{0,\sigma} &= \{v \in D^1_0 : \operatorname{div} v = 0 \text{ in } \Omega\}, \quad D^k = D^{k,2}, \quad W^{k,r} = L^r \cap D^{k,r}, \\
H^k &= W^{k,2}, \quad H^1_0 = L^2 \cap D^1_0 \quad \text{and} \quad H^1_{0,\sigma} = L^2 \cap D^1_{0,\sigma}.
\end{aligned}$$

We also denote by H^{-1}_σ and D^{-1}_σ the dual space of $H^1_{0,\sigma}$ and $D^1_{0,\sigma}$, respectively, with $\langle \cdot, \cdot \rangle$ being the corresponding dual paring. A detailed study of homogeneous Sobolev spaces may be found in the book [10] by G. Galdi.

In this paper, we assume that the fluid flow governed by (1)-(4) may have non-negative initial density. That is to say, we consider the incompressible fluid with an initial *vacuum*, which is a spatial domain whose interior is non-empty and in which the density of fluid vanishes identically. Thus from the continuity equation (2), we can observe that the vacuum may evolve as time goes on and then the parabolicity of the momentum equation (3) comes disappeared at any vacuum state. As a consequence, we cannot expect directly any high regularity or uniqueness from (3). To avoid this difficulty, it is necessary to introduce a suitable compatibility condition which shows how the initial fluid should behave near the vacuum.

For global existence of weak solution to the problem (1)-(3) and (5), one may assume the condition $\sqrt{\rho_0} u_0 \in L^2$. See [15, 23, 25] for the case of constant viscosity coefficient and [19] for density-dependent case. But the uniqueness is still open even in two dimensional case.

As for strong solutions to the problem (1)-(3) and (5), H. Kim and H. J. Choe [8] used the condition

$$(6) \qquad -\mu \Delta u_0 + \nabla p_0 = \rho_0^{\frac{1}{2}} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \qquad \text{in} \quad \Omega$$

for some $(p_0, g) \in H^1 \times L^2$ and they proved existence of unique local strong solutions for the case of constant viscosity coefficient in bounded domains or the whole space. In case of density-dependent viscosity, Y. Cho and H. Kim [6] used the condition

$$-\operatorname{div}(2\mu(\rho_0)\mathrm{d}u_0) + \nabla p_0 = \rho_0^{\frac{1}{2}} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \quad \text{in } \Omega \tag{7}$$

for some $(p_0, g) \in H^1 \times L^2$. They also obtained the uniqueness and strong solvability for two or three-dimensional bounded domains. It should be noted that (6) or (7) turns out to be necessary and sufficient for the strong regularity. For other related topic to the vacuum, one can refer to [9, 21, 22], etc. Regarding the initial density with a positive lower bound, we refer the readers to [4, 13, 14, 18, 20] and the references therein.

If the temperature equation (4) is under consideration, the situation is more complicated. The coupling between (3) and (4) is deepened by the vacuum even in the case of constant coefficients. Thus we have to consider the relation between initial density and temperature as well as velocity near the vacuum. The authors in [7] first treated the problems with temperature equation for the polytropic compressible fluid with constant coefficients and vacuum, and suggested a compatibility condition similar to (8) below to prove existence of unique strong solutions. However, up to now, there have been few results on the uniqueness and regularity of heat-conducting incompressible or compressible Navier-Stokes equations with nonconstant coefficients. For the weak solvability to the problem (1)-(5), see the result of P.-L. Lions in Chapter 3 of [19]. For another related topic, we refer the readers to [2] in which unique solvability is studied under a small data condition and uniform density condition but with coefficients depending on $\mathrm{d}u$ and θ.

In this paper, we study strong solutions to the initial boundary value problem (1)-(5) with initial vacuum and with slightly modified coefficients. We also consider minimal regularity of strong solutions in Sobolev sense. To do these, we first assume the natural extension of compatibility condition (7) to the heat-conducting case as follows

$$\left.\begin{aligned} -\operatorname{div}(2\mu_0 \mathrm{d}u_0) + \nabla p_0 &= \rho_0^{\frac{1}{2}} g_1, \\ -\operatorname{div}(\kappa_0 \nabla\theta_0) - 2\mu_0|\mathrm{d}u_0|^2 &= \rho_0^{\frac{1}{2}} g_2 \end{aligned}\right\} \quad \text{in} \quad \Omega \tag{8}$$

for some $p_0 \in H^1$ and $(g_1, g_2) \in L^2$. Furthermore we assume that

$$\begin{gathered} 0 < \mu, c_v, \kappa \in C^1(\mathbf{R}^2), \\ \mu = \mu(\rho, \rho\theta), \quad c_v = c_v(\rho, \rho\theta), \quad \kappa = \kappa(\rho, \rho\theta) \end{gathered} \tag{9}$$

for nonconstant coefficients. The following is our main result.

Theorem 1.1. *Assume that the data $(\rho_0, u_0, \theta_0, h, f)$ satisfies the regularity condition*

$$\rho_0 \geq 0, \quad \rho_0 \in L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}, \quad (u_0, \theta_0) \in D_0^1 \cap D^2, \quad \operatorname{div} u_0 = 0,$$
$$(h, f) \in C([0,T]; L^2) \cap L^2([0,T]; L^q) \quad \text{and} \quad (h_t, f_t) \in L^2(0,T; H^{-1})$$

for some $3 < q \leq 6$. Further assume the compatibility condition (8) *and coefficient condition* (9). *Then there exist a small time $T_* > 0$ and a unique strong solution (ρ, u, p, θ) to the initial boundary value problem for* (1)-(5) *such that*

$$\begin{aligned} &\rho \in C([0,T_*]; L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T_*]; L^{\frac{3}{2}} \cap L^q), \\ &(u, \theta) \in C([0,T_*]; D_0^1 \cap D^2) \cap L^2(0,T_*; D^{2,q}), \\ &p \in C([0,T_*]; H^1) \cap L^2(0,T_*, W^{1,q}) \\ &(u_t, \theta_t) \in L^2(0,T_*; D_0^1) \quad \text{and} \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T_*; L^2). \end{aligned} \tag{10}$$

Adopting the arguments used in [5, 6, 8], one can easily show that the compatibility condition (8) also turns out to be necessary and sufficient for the strong regularity (10). The condition (9) is used to avoid some technical difficulties in estimating $\int_0^t |\mu_t|_{L^3}^2 \, ds$ and $|\mu - \mu_0|_{L^\infty}$, which appear in the basic energy estimates and are crucial in finding the local existence time. See Section 3 below. Until now, we don't know whether the condition (9) is inevitable or not. It seems to be hard to remove the condition (9) in the presence of vacuum but worth while to try. But in case when ρ has a positive lower bound, one can assume that the coefficients are functions of ρ and θ not of $\rho\theta$.

The condition $L^{\frac{3}{2}} \cap H^1$ for initial density is necessary to prove the existence and uniqueness in case of unbounded domains. See Section 4 below. In particular, the $L^{\frac{3}{2}}$-condition is used for the regularity of the solution to the Stokes problem

$$-\operatorname{div}(2\mu d u) + \nabla p = F, \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega,$$

where $F = \rho f - \rho u_t - \rho u \cdot \nabla u$ (see Section 2 below). For the regularity of (u, p), we need $F \in D_\sigma^{-1}$ the dual of $D_{0,\sigma}^1$ and hence $\rho u_t \in D_\sigma^{-1}$. Since we assume the initial vacuum in an unbounded domain, we cannot expect $u_t \in L^2$ but in general $u_t \in D_{0,\sigma}^1$. Therefore $L^{\frac{3}{2}}$-condition is necessary. Furthermore, for higher regularity of solution of the above Stokes system (at least D^2 regularity for (u, θ) is necessary), it is inevitable to show that the density is $W^{1,q}$ and hence $W^{1,q}$ condition for initial density is imposed. For the details, see the proof of Proposition 2.8 below.

This paper is organized as follows. In Section 2, we show a new Sobolev embedding result and some regularity results for the solutions of linear transport equation, stationary Stokes and elliptic equations. In Section 3, we prove a priori estimates for linearized problems under the assumptions that the initial density has positive lower bound and the domain is bounded. We show that the estimates are independent of the lower bound and the size of domain. In Section 4, we prove Theorem 1.1 by using the standard domain expansion and iteration of solutions to the linearized problems as mentioned in Section 2, which are possible thanks to the uniformity of a priori estimates on the lower bound of initial density and the size of domain.

The methods mentioned above can be applied directly to the compressible model with momentum (3) and energy (4) equations replaced by

$$\begin{aligned} &p = p(\rho, \rho\theta), \\ &(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu \mathrm{d}u) - \nabla(\lambda \operatorname{div} u) + \nabla p = \rho f, \\ &c_v\left((\rho\theta)_t + \operatorname{div}(\rho u \theta)\right) + (\gamma c_v \rho\theta - p)\operatorname{div} u - \operatorname{div}(\kappa \nabla\theta) \\ &= 2\mu|\mathrm{d}u|^2 + \lambda(\operatorname{div} u)^2 + \rho h. \end{aligned}$$

The factor $\gamma c_v \rho\theta - p$ follows from a result of the first law of thermodynamics law $\rho^2 \frac{\partial e}{\partial \rho} = p - \alpha K_\theta \theta$ and the relation $\alpha \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_p = \frac{\gamma c_v \rho}{K_\theta}$, where α is the thermal expansion coefficient and K_θ is the isothermal bulk modulus. The viscous coefficients satisfy the Stokes relation $\mu + \lambda = a\mu$ for some positive number a. The application for this model is possible because combining the regularity results of Lamé system (see Section 5 of [5]) with the Stokes like relation above, one can verify the regularity results like Lemma 2.8 below in which some estimates independent of the size of domain are shown. In the case of the bounded domain we can remove the Stokes relation. We will however not present here the details of results on the above problem. Without Stokes like relation, it is much difficult to handle more general compressible fluids in an unbounded domain because it is hard to derive an elliptic estimate independent of the size of domain. This will be very interesting problem to try.

2. Preliminary results

Let Ω be a bounded domain, an exterior domain or the whole space, and let us define

$$R_0 = R_0(\Omega) = \begin{cases} \sup_{x \in \partial\Omega} |x| + 2 & \text{if } \partial\Omega \text{ is nonempty}, \\ 3 & \text{if } \partial\Omega \text{ is empty} \end{cases} \tag{11}$$

and

$$\Omega_R = \{x \in \Omega : |x| < R\} \qquad \text{for} \quad 2R_0 < R \leq \infty.$$

Note that $\Omega_R = \Omega$ if Ω is bounded or $R = \infty$, and Ω_R is the intersection of an unbounded domain Ω and the open ball $B_R = B_R(0)$ otherwise.

In this section, we derive some estimates in Ω_R which are independent of the radius R. These results will be crucial technical tools in proving our main theorem for unbounded domains via the method of domain expansions. Throughout the paper, we adapt the following notations for the sake of conciseness. For (semi)-normed spaces X and Y, we define

$$|\cdot|_{X \cap Y} = |\cdot|_X + |\cdot|_Y.$$

Moreover, for a mapping $\mathcal{F}$ from X into a space Y' with $Y \subset Y'$, we mean by the inequality

$$|\mathcal{F}x|_Y \le C|x|_X \tag{12}$$

that if $x \in X$, then $\mathcal{F}x \in Y$ and (12) holds for some C independent of x.

2.1. Sobolev embedding results

First we prove fundamental embedding inequalities for Sobolev spaces, which are referred in this paper to *Sobolev inequalities.*

Lemma 2.1. *Assume that* $2R_0 < R \le \infty$ *and* $3 < q < \infty$. *Then*

$$|u|_{L^6(\Omega_R)} \le C|u|_{D_0^1(\Omega_R)}, \tag{13}$$

$$|u|_{L^6(\Omega_R)} \le C|u|_{H^1(\Omega_R)}, \quad |u|_{C^{0,1-\frac{3}{q}}(\overline{\Omega}_R)} \le C|u|_{W^{1,q}(\Omega_R)} \tag{14}$$

and

$$|u|_{C^{0,1-\frac{3}{q}}(\overline{\Omega}_R)} \le C|u|_{D_0^1(\Omega_R)\cap D^{1,q}(\Omega_R)}. \tag{15}$$

Here C *denotes a positive constant depending only on* q *and* Ω *but independent of* R *and for* $0 < \alpha < 1$, $C^{0,\alpha}(\overline{\Omega}_R)$ *is the Hölder space consisting of all scalar (or vector) fields* u *in* Ω_R *such that*

$$|u|_{C^{0,\alpha}(\overline{\Omega}_R)} \equiv \sup_{x\in\Omega_R} |u(x)| + \sup_{x,y\in\Omega_R,\, x\ne y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

Proof. The first inequality (13) is an immediate consequence of the corresponding Sobolev embedding inequality in the whole space because $C_c^\infty(\Omega_R)$ is dense in $D_0^1(\Omega_R)$; see Theorem 6.1 in [10, Chapter II]. Moreover, since Ω_R satisfies the cone condition uniformly on R, it follows from classical Sobolev embedding results (see [1, Chapter 5] for instance) that

$$|u|_{L^6(\Omega_R)} \le C|u|_{H^1(\Omega_R)} \quad \text{and} \quad |u|_{C(\overline{\Omega}_R)} + |u|_{C^{0,1-\frac{3}{q}}(\overline{\Omega}_{2R_0})} \le C|u|_{W^{1,q}(\Omega_R)}.$$

Hence in order to prove (14), we have only to show that if $u \in W^{1,q}(\Omega_R)$, then

$$|u(x_1) - u(x_2)| \le C|\nabla u|_{L^q(\Omega_R)}|x_1 - x_2|^{1-\frac{3}{q}} \tag{16}$$

for $x_1, x_2 \in B_R \setminus B_{\frac{3}{2}R_0}$ with $|x_1 - x_2| \le 1$. Let us denote

$$r = \frac{|x_1 - x_2|}{2} \quad \text{and} \quad \overline{x} = \frac{x_1 + x_2}{2}.$$

Then since the set $B_r(\overline{x}) \cap B_R$ is convex,

$$\operatorname{diam}(B_r(\overline{x}) \cap B_R) = 2r \quad \text{and} \quad |B_r(\overline{x}) \cap B_R| > \frac{1}{2}|B_r(\overline{x})| = \frac{1}{2}\pi r^3,$$

it follows from Lemma 7.16 in [12] that

$$\left|u(x) - u_{B_r(\overline{x})\cap B_R}\right| \le \frac{16}{3\pi}\int_{B_r(\overline{x})\cap B_R} |\nabla u(y)||x-y|^{-2}\,dy \tag{17}$$

for all $x \in B_r(\overline{x}) \cap B_R$, where

$$u_{B_r(\overline{x})\cap B_R} = \frac{1}{|B_r(\overline{x}) \cap B_R|} \int_{B_r(\overline{x})\cap B_R} u(y)\, dy.$$

By virtue of Hölder inequality, we deduce that if $x \in B_r(\overline{x}) \cap B_R$, then

$$\begin{aligned}\int_{B_r(\overline{x})\cap B_R} |\nabla u(y)||x-y|^{-2}\, dy &\le |\nabla u|_{L^q(\Omega_R)} \left(\int_{B_{2r}(x)} |x-y|^{-\frac{2q}{q-1}}\, dy \right)^{\frac{q-1}{q}} \\ &\le C|\nabla u|_{L^q(\Omega_R)} r^{1-\frac{3}{q}}.\end{aligned}$$

Combining this and (17), we have

$$|u(x_1) - u(x_2)| \le |u(x_1) - \overline{u}| + |u(x_2) - \overline{u}| \le C|\nabla u|_{L^q(\Omega_R)} r^{1-\frac{3}{q}},$$

which implies (16) and thus the second inequality (14).

It remains to prove the last inequality (15). Let $u \in D_0^1(\Omega_R) \cap D^{1,q}(\Omega_R)$. Then defining $\overline{u}$ by

$$\overline{u} = u \quad \text{in} \quad \Omega_R \quad \text{and} \quad \overline{u} = 0 \quad \text{outside} \quad \Omega_R,$$

we easily show that $\overline{u} \in D^1(\mathbf{R}^3) \cap D^{1,q}(\mathbf{R}^3)$ and $\nabla\overline{u} = 0$ a.e. in $\mathbf{R}^3 \setminus \Omega_R$. Hence adapting the proof of (16), we deduce the classical result due to Morrey that $\overline{u}$ is Hölder continuous in $\mathbf{R}^3$ with exponent $\alpha = 1 - \frac{3}{q}$ and

$$|\overline{u}(x) - \overline{u}(y)| \le C|\nabla\overline{u}|_{L^q(\mathbf{R}^3)}|x-y|^\alpha \quad \text{for} \quad x, y \in \mathbf{R}^3. \tag{18}$$

Moreover, it follows from (17) with $r = 1$ and $R = \infty$ that

$$|\overline{u}(x)| \le C\int_{B_1(x)} |\overline{u}(y)|\, dy + C\int_{B_1(x)} |\nabla\overline{u}(y)||x-y|^{-2}\, dy$$

for all $x \in \mathbf{R}^3$. Hence in view of Hölder inequality again, we have

$$|\overline{u}(x)| \le C\left(|\overline{u}|_{L^6(B_1(x))} + |\nabla\overline{u}|_{L^q(B_1(x))}\right) \quad \text{for} \quad x \in \mathbf{R}^3. \tag{19}$$

Now the inequality (15) follows immediately from (18), (19), and (13). This completes the proof of Lemma 2.1. □

2.2. A linear transport equation

Let us consider the following linear hyperbolic problem

$$\rho_t + \operatorname{div}(\rho v) = 0 \quad \text{in} \quad (0,T) \times \Omega_R \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{in} \quad \Omega_R, \tag{20}$$

where v is a known vector field in $(0,T) \times \Omega_R$ such that

$$v \in C([0,T]; D_0^1(\Omega_R) \cap D^2(\Omega_R)) \cap L^2(0,T; D^{2,q}(\Omega_R))$$

for some $3 < q \le 6$. Here it should be noted that v need not be divergence-free in $(0,T) \times \Omega_R$. Using exactly the same arguments as in [7], we can prove the following existence and regularity results.

Lemma 2.2. *Assume that*

$$\rho_0 \in L^{\frac{3}{2}}(\Omega_R) \cap H^1(\Omega_R) \cap W^{1,q}(\Omega_R) \quad \textit{and} \quad \rho_0 \geq 0 \quad \textit{in} \quad \Omega_R.$$

Then

(i) *there exists a unique solution* ρ *to the problem* (20) *such that*

$$\rho \in C([0,T]; L^{\frac{3}{2}}(\Omega_R) \cap H^1(\Omega_R) \cap W^{1,q}(\Omega_R)),$$

(ii) *the solution* ρ *satisfies the following estimate*

$$|\rho(t)|_{L^{\frac{3}{2}}(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)} \leq |\rho_0|_{L^{\frac{3}{2}}(\Omega_R)\cap H^1(\Omega_R)\cap W^{1,q}(\Omega_R)} \times \exp\left(C\int_0^t |\nabla v(s)|_{H^1(\Omega_R)\cap D^{1,q}(\Omega_R)}\, ds\right)$$

for $0 \leq t \leq T$ *and finally,*

(iii) *the solution* ρ *is represented by the formula*

$$\rho(t,x) = \rho_0(\,U(0,t,x)\,)\exp\left[-\int_0^t \operatorname{div} v(s, U(s,t,x)\,)\, ds\right], \tag{21}$$

where $U \in C([0,T]\times[0,T]\times\overline{\Omega}_R)$ *is the solution to the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t}U(t,s,x) = v(t, U(t,s,x)\,), & 0 \leq t \leq T, \\ U(s,s,x) = x, \quad 0 \leq s \leq T, & x \in \overline{\Omega}_R. \end{cases} \tag{22}$$

Remark 2.3. It follows immediately from the equation in (20) that

$$\rho_t \in C([0,T]; L^{\frac{3}{2}}(\Omega_R) \cap L^q(\Omega_R)).$$

Proof. The lemma except the result (ii) has been well-known in case when Ω_R is a bounded domain: see [26] for instance. A very related estimate to that in (ii) was obtained in [5, 7] by means of a standard energy method. Moreover the case of unbounded domains can be reduced to that of bounded domains using a regularization technique and a cut-off technique. We omit its detailed proof of the lemma and refer the readers to our previous papers [5, 7]. □

2.3. The nonhomogeneous Stokes equations

Next, we consider the boundary value problem for the nonhomogeneous Stokes equations

$$-\operatorname{div}(2\mu\, du) + \nabla p = F, \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega_R, \tag{23}$$

$$u = 0 \text{ on } \partial\Omega_R \quad \text{and} \quad u(x) \to 0 \text{ as } |x| \to \infty, \;\; x \in \Omega_R, \tag{24}$$

where F and μ are known vector and scalar fields in Ω_R such that

$$F \in D^{-1}(\Omega_R), \quad \mu, \mu^{-1} \in L^\infty(\Omega_R) \quad \text{and} \quad \underline{\mu} \leq \mu \leq \overline{\mu} \quad \text{in} \quad \Omega_R \tag{25}$$

for some constants $\underline{\mu}$ and $\overline{\mu}$ with $0 < \underline{\mu} \leq 1 \leq \overline{\mu}$.

By weak solutions to the problem (23) and (24), we mean either

Definition 2.4. A vector field $u \in D^1_{0,\sigma}(\Omega_R)$ is a weak solution to the boundary value problem (23) and (24) provided that

$$\int 2\mu \mathrm{d}u : \mathrm{d}v \, dx = \langle F, v \rangle \quad \text{for all } v \in D^1_{0,\sigma}(\Omega_R) \tag{26}$$

or

Definition 2.5. A pair $(u,p) \in D^1_{0,\sigma}(\Omega_R) \times L^2_0(\Omega_R)$ of vector and scalar fields in Ω_R is a weak solution to the boundary value problem (23) and (24) provided that

$$\int 2\mu \mathrm{d}u : \mathrm{d}v \, dx - \int p \operatorname{div} v \, dx = \langle F, v \rangle \quad \text{for all } v \in D^1_0(\Omega_R). \tag{27}$$

Here the space $L^2_0(\Omega_R)$ is defined by

$$L^2_0(\Omega_R) = \begin{cases} \left\{ p \in L^2(\Omega_R) : \int_{\Omega_R} p \, dx = 0 \right\} & \text{if } \Omega_R \text{ is bounded,} \\ L^2(\Omega_R) & \text{otherwise.} \end{cases}$$

These two definitions are actually equivalent. If $(u,p) \in D^1_{0,\sigma}(\Omega_R) \times L^2_0(\Omega_R)$ is a weak solution to (23) and (24) in the sense of Definition 2.5, then it is obvious that u is a weak solution in the sense of Definition 2.4. The converse is an immediate consequence of the following result.

Lemma 2.6. *Let $u \in D^1_{0,\sigma}(\Omega_R)$ be a weak solution to the problem* (23) *and* (24) *in the sense of Definition* 2.4. *Then there exists a unique scalar field $p \in L^2_0(\Omega_R)$ such that (u,p) is a weak solution to* (23) *and* (24) *in the sense of Definition* 2.5. *This field p is called the pressure associated with u.*

Proof. Let us define a bounded linear functional $\mathcal{F}$ on $D^1_0(\Omega_R)$ by

$$\mathcal{F}(v) = \int 2\mu \mathrm{d}u : \mathrm{d}v \, dx - \langle F, v \rangle \quad \text{for all } v \in D^1_0(\Omega_R). \tag{28}$$

Then since $\mathcal{F}$ vanishes identically on $D^1_{0,\sigma}(\Omega_R)$, it follows from Theorem 5.2 and Corollary 5.1 in [10, Chapter III] that there exists a unique scalar field $p \in L^2_0(\Omega_R)$ such that

$$\mathcal{F}(v) = \int p \operatorname{div} v \, dx \quad \text{for all } v \in D^1_0(\Omega_R). \tag{29}$$

Note that the statement (29) holds if and only if (u,p) satisfies (27). This completes the proof of Lemma 2.6. □

The existence of a unique weak solution to the problem (23) and (24) can be easily proved using the Riesz representation theorem and Lemma 2.6. In fact, the left hand side of (26) defines an inner product on the Hilbert space $D^1_{0,\sigma}(\Omega_R)$ because $|\mathrm{d}v|_{L^2(\Omega_R)} = |\nabla v|_{L^2(\Omega_R)} = |v|_{D^1_{0,\sigma}(\Omega_R)}$ for all $v \in D^1_{0,\sigma}(\Omega_R)$. Hence by virtue of the Riesz representation theorem, we deduce the existence of a unique weak solution $u \in D^1_{0,\sigma}(\Omega_R)$ in the sense of Definition 2.4. Then the existence of a unique pressure $p \in L^2_0(\Omega_R)$ follows from Lemma 2.6. We have proved the existence part of the following result.

Lemma 2.7. *For each $F \in D^{-1}(\Omega_R)$, there exists a unique weak solution $(u,p) \in D^1_{0,\sigma}(\Omega_R) \times L^2_0(\Omega_R)$ to the boundary value problem (23) and (24). Moreover we have the the following estimate*

$$|u|_{D^1_0(\Omega_R)} + |p|_{L^2(\Omega_R)} \le C\,\overline{\mu}\,\underline{\mu}^{-1}\,|F|_{D^{-1}(\Omega_R)}. \tag{30}$$

Proof. It remains to derive the estimate (30). Taking $v = u \in D^1_{0,\sigma}(\Omega_R)$ in (26), we first have

$$\int 2\mu|\mathrm{d}u|^2\,dx \;=\; \langle F, u\rangle \;\le\; |F|_{D^{-1}(\Omega_R)}|u|_{D^1_0(\Omega_R)}.$$

But since $\mu \ge \underline{\mu} > 0$ in Ω_R and $|\mathrm{d}u|_{L^2(\Omega_R)} = |\nabla u|_{L^2(\Omega_R)} = |u|_{D^1_0(\Omega_R)}$, we readily deduce that

$$|u|_{D^1_0(\Omega_R)} \le (2\underline{\mu})^{-1}|F|_{D^{-1}(\Omega_R)}. \tag{31}$$

To derive the estimate for p, we need to show that there is a vector field v in $D^1_0(\Omega_R)$ such that

$$\operatorname{div} v = p \quad \text{in} \quad \Omega_R \quad \text{and} \quad |v|_{D^1_0(\Omega_R)} \le C|p|_{L^2(\Omega_R)}. \tag{32}$$

To show this, we adapt the proof of Theorem 3.4 in [10, Chapter III]. See also [17]. Let us extend p to B_R by zero outside Ω_R.[1] Then since $p \in L^2_0(B_R)$, it follows from a classical result due to Bogovskiĭ [3] (see also Theorem 3.1 in [10, Chapter III]) that there exists a vector field $v_1 \in D^1_0(B_R)$ such that

$$\operatorname{div} v_1 = p \quad \text{in} \quad B_R \quad \text{and} \quad |v_1|_{D^1_0(B_R)} \le C|p|_{L^2(\Omega_R)}.$$

Moreover, we observe that

$$\int_{\partial\Omega} v_1 \cdot \nu\, d\sigma = -\int_{B_{R_0}\setminus\Omega} \operatorname{div} v_1\, dx = -\int_{B_{R_0}\setminus\Omega} p\, dx = 0.$$

Hence by virtue of a corollary (Exercise 3.4) of Theorem 3.1 in [10, Chapter III], there exists a vector field v_2 such that

$$\begin{aligned} &\operatorname{div} v_2 = 0 \quad \text{in} \quad \Omega_{2R_0}, \quad v_2 = 0 \quad \text{on} \quad \partial B_{2R_0}, \quad v_2 = -v_1 \quad \text{on} \quad \partial\Omega \\ &\qquad v_2 \in D^1(\Omega_{2R_0}) \quad \text{and} \quad |v_2|_{D^1(\Omega_{2R_0})} \le C|\operatorname{div}(\varphi v_1)|_{L^2(\Omega_{2R_0})}, \end{aligned} \tag{33}$$

where $\varphi \in C_c^\infty(B_{2R_0})$ is a cut-off function with $\varphi = 1$ in B_{R_0}. We extend v_2 to Ω_R by zero outside Ω_{2R_0}. Then from (33), we easily deduce that

$$\begin{aligned} &\operatorname{div} v_2 = 0 \quad \text{in} \quad \Omega_R, \quad v_2 = 0 \quad \text{on} \quad \partial B_R, \quad v_2 = -v_1 \quad \text{on} \quad \partial\Omega \\ &\qquad v_2 \in D^1(\Omega_R) \quad \text{and} \quad |v_2|_{D^1(\Omega_R)} \le C|v_1|_{D^1_0(B_R)}. \end{aligned}$$

[1] $B_\infty = \mathbf{R}^3$ by convention.

It is now easy to show that if we define v by $v = v_1 + v_2$, then v satisfies (32). Hence from (28), (29) and (31), we deduce that

$$\begin{aligned} \int p^2\,dx &= \mathcal{F}(v) = \int 2\mu \mathrm{d}u : \mathrm{d}v\,dx - \langle F, v\rangle \\ &\le C\left(\bar{\mu}|u|_{D_0^1(\Omega_R)} + |F|_{D^{-1}(\Omega_R)}\right)|v|_{D_0^1(\Omega_R)} \\ &\le C\,\bar{\mu}\,\underline{\mu}^{-1}\,|F|_{D^{-1}(\Omega_R)}|p|_{L^2(\Omega_R)}. \end{aligned}$$

This completes the proof of Lemma 2.7. □

One of the main purposes of this section is to prove the following higher regularity estimates for weak solutions to the problem (23) and (24).

Proposition 2.8. *Let $(u,p) \in D^1_{0,\sigma}(\Omega_R) \times L^2_0(\Omega_R)$ be a weak solution to the boundary value problem* (23) *and* (24). *Assume in addition to* (25) *that*

$$\nabla\mu \in L^3(\Omega_R) \cap L^q(\Omega_R) \qquad \text{for some} \quad q \in (3,6].$$

Then we have the following regularity estimates:

$$|\nabla u|_{H^1(\Omega_R)} + |p|_{H^1(\Omega_R)} \le CM(\mu)^N |F|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)} \tag{34}$$

and

$$|\nabla u|_{W^{1,q}(\Omega_R)} + |p|_{W^{1,q}(\Omega_R)} \le CM(\mu)^N |F|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)\cap L^q(\Omega_R)}, \tag{35}$$

where

$$M(\mu) = \bar{\mu}\,\underline{\mu}^{-1}\left(1 + |\nabla\mu|_{L^2(\Omega_R)\cap L^q(\Omega_R)}\right) \quad \text{and} \quad N = N(q) > 1.$$

Proof. Our proof consists of three steps.

Step 1: We first consider the case that Ω_R is bounded and $\mu = 1$ identically in Ω_R. In this case, (23) reduces to the classical Stokes equations

$$-\Delta u + \nabla p = F \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in} \quad \Omega_R. \tag{36}$$

We will show that if $(u,p) \in H^1_0(\Omega_R) \cap L^2_0(\Omega_R)$ is a weak solution of (36), then

$$|\nabla u|_{W^{1,r}(\Omega_R)} + |p|_{W^{1,r}(\Omega_R)} \le C|F|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)\cap L^r(\Omega_R)} \tag{37}$$

for $2 \le r \le 6$. To show this, let $F \in L^r(\Omega_R)$ for some $r \in [2,6]$. Then it follows from a standard regularity result that $u \in W^{2,r}(\Omega_R)$, $p \in W^{1,r}(\Omega_R)$ and $|u|_{W^{2,r}(\Omega_R)} + |p|_{W^{1,r}(\Omega_R)} \le C_R|F|_{L^r(\Omega_R)}$ for some constant C_R but depending on R. An R-independent estimate can be derived using a cut-off technique and Poincaré inequality in Ω_{2R_0}. That is, adapting the arguments in [5, Section 5] and [16, Section 3], we can show that

$$|\nabla u|_{W^{1,r}(\Omega_R)} + |p|_{W^{1,r}(\Omega_R)} \le C\left(|F|_{L^r(\Omega_R)} + |\nabla u|_{L^r(\Omega_R)} + |p|_{L^r(\Omega_R)}\right). \tag{38}$$

A detailed derivation of (38) is omitted. Combining (30) and (38), we immediately obtain the estimate (37) for $r = 2$. Assume that $2 < r \le 6$. Then in view of Sobolev inequality, we deduce from (38) and (37) with $r = 2$ that

$$\begin{aligned} |\nabla u|_{W^{1,r}(\Omega_R)} + |p|_{W^{1,r}(\Omega_R)} &\le C\left(|F|_{L^r(\Omega_R)} + |\nabla u|_{L^r(\Omega_R)} + |p|_{L^r(\Omega_R)}\right) \\ &\le C\left(|F|_{L^r(\Omega_R)} + |\nabla u|_{H^1(\Omega_R)} + |p|_{H^1(\Omega_R)}\right) \\ &\le C|F|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)\cap L^r(\Omega_R)}. \end{aligned}$$

This proves the regularity estimate (37).

Step 2. Next we prove the proposition for the case that Ω_R is bounded and $\mu \in C^1(\overline{\Omega}_R)$. Let $(u,p) \in H_0^1(\Omega_R) \times L_0^2(\Omega_R)$ be a weak solution to (23) and (24) with $F \in L^r(\Omega_R)$ for $r = 2$ or q. Then it follows from a regularity result in [11] that $u \in W^{2,r}(\Omega_R)$ and $p \in W^{1,r}(\Omega_R)$. Hence it remains to derive (34) and (35). From now on, we drop Ω_R and write L^r instead of $L^r(\Omega_R)$ for simplicity. The same notations are also adapted to Sobolev spaces.

We begin with rewriting (23) as

$$-\Delta u + \nabla \tilde{p} = \tilde{F} \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in} \quad \Omega, \tag{39}$$

where $\tilde{p} = \mu^{-1}p$ and $\tilde{F} = \mu^{-1}\left(F + 2\nabla\mu \cdot du - \tilde{p}\,\nabla\mu\right)$. Using the estimate (30), Hölder inequality and Sobolev inequalities, we have

$$\begin{aligned} |\tilde{F}|_{D^{-1}} &\le \widehat{C}\underline{\mu}^{-2}\left(1 + |\nabla\mu|_{L^3}\right)\left(|F|_{D^{-1}} + |\nabla u|_{L^2} + |p|_{L^2}\right) \\ &\le CM(\mu)^3|F|_{D^{-1}} \end{aligned}$$

and

$$\begin{aligned} |\tilde{F}|_{L^2} &\le C\,\underline{\mu}^{-1}\left(|F|_{L^2} + |\,|\nabla\mu||\nabla u|\,|_{L^2} + |\tilde{p}\,\nabla\mu|_{L^2}\right) \\ &\le C\,\underline{\mu}^{-1}|F|_{L^2} + C\,\underline{\mu}^{-2}|\nabla\mu|_{L^q}\left(|\nabla u|_{L^{\frac{2q}{q-2}}} + |p|_{L^{\frac{2q}{q-2}}}\right) \\ &\le C\,\underline{\mu}^{-1}|F|_{L^2} + C\,\underline{\mu}^{-2}|\nabla\mu|_{L^q}\left(|\nabla u|_{L^2}^{1-\frac{3}{q}}|\nabla u|_{H^1}^{\frac{3}{q}} + |p|_{L^2}^{1-\frac{3}{q}}|p|_{H^1}^{\frac{3}{q}}\right) \\ &\le C\,\underline{\mu}^{-1}|F|_{L^2} + \eta^{-1}CM(\mu)^{\frac{2q}{q-3}}(|\nabla u|_{L^2} + |p|_{L^2}) + \eta(|\nabla u|_{H^1} + |p|_{H^1}) \\ &\le C\eta^{-1}M(\mu)^{\frac{2q}{q-3}}|F|_{D^{-1}\cap L^2} + \eta(|\nabla u|_{H^1} + |p|_{H^1}) \end{aligned}$$

for any small constant $\eta \in (0,1)$. Hence it follows from the estimate (37) that

$$\begin{aligned} |\nabla u|_{H^1} + |\nabla\tilde{p}|_{L^2} &\le C|\tilde{F}|_{D^{-1}\cap L^2} \\ &\le C\eta^{-1}M(\mu)^{\frac{3q}{q-3}}|F|_{D^{-1}\cap L^2} + \eta(|\nabla u|_{H^1} + |p|_{H^1}). \end{aligned} \tag{40}$$

On the other hand, we observe that

$$\begin{aligned} |p|_{H^1} &\le |\tilde{p}\nabla\mu|_{L^2} + |\mu\nabla\tilde{p}|_{L^2} + |p|_{L^2} \\ &\le C\underline{\mu}^{-1}|\nabla\mu|_{L^q}|p|_{L^2}^{1-\frac{3}{q}}|p|_{H^1}^{\frac{3}{q}} + C\overline{\mu}|\nabla\tilde{p}|_{L^2} + |p|_{L^2} \\ &\le CM(\mu)^{\frac{q}{q-3}}|F|_{D^{-1}} + C\overline{\mu}|\nabla\tilde{p}|_{L^2} + \frac{1}{2}|p|_{H^1} \end{aligned}$$

and so

$$|p|_{H^1} \le CM(\mu)^{\frac{q}{q-3}}|F|_{D^{-1}} + C\overline{\mu}|\nabla\tilde{p}|_{L^2}.$$

Substituting this into (40) and choosing $\eta = (2C\overline{\mu})^{-1}$, we deduce that

$$|\nabla u|_{H^1} + |p|_{H^1} \le CM(\mu)^{\frac{5q}{q-3}}|F|_{D^{-1}\cap L^2}. \tag{41}$$

This proves the first estimate (34).

To derive the second one, we use the estimate (37) again to deduce that

$$|\nabla u|_{W^{1,q}} + |\nabla \tilde{p}|_{L^q} \le C|\tilde{F}|_{D^{-1}\cap L^2\cap L^q}. \tag{42}$$

By virtue of (41), Hölder inequality and Sobolev inequalities, we obtain

$$|\tilde{F}|_{D^{-1}\cap L^2} \le CM(\mu)^{\frac{5q}{q-3}}|F|_{D^{-1}\cap L^2}$$

and

$$\begin{aligned}|\tilde{F}|_{L^q} &\le C\underline{\mu}^{-1}\left(|F|_{L^q} + \big||\nabla\mu||\nabla u|\big|_{L^q} + |\tilde{p}\nabla\mu|_{L^q}\right)\\ &\le C\underline{\mu}^{-1}|F|_{L^q} + C\underline{\mu}^{-2}|\nabla\mu|_{L^q}\left(|\nabla u|_{L^\infty} + |p|_{L^\infty}\right)\\ &\le C\underline{\mu}^{-1}|F|_{L^q} + C\underline{\mu}^{-2}|\nabla\mu|_{L^q}\left(|\nabla u|_{H^1}^{1-\gamma}|\nabla u|_{W^{1,q}}^{\gamma} + |p|_{H^1}^{1-\gamma}|p|_{W^{1,q}}^{\gamma}\right)\\ &\le C\eta^{-1}M(\mu)^N|F|_{D^{-1}\cap L^2\cap L^q} + \eta\left(|\nabla u|_{W^{1,q}} + |p|_{W^{1,q}}\right)\end{aligned}$$

for some $\gamma = \gamma(q) \in (0,1)$ and $N = N(q) > 1$. Similarly, we have

$$\begin{aligned}|p|_{W^{1,q}} &\le |\tilde{p}\nabla\mu|_{L^q} + |\mu\nabla\tilde{p}|_{L^q} + |p|_{L^q}\\ &\le C\underline{\mu}^{-1}|\nabla\mu|_{L^q}|p|_{H^1}^{1-\gamma}|p|_{W^{1,q}}^{\gamma} + C\overline{\mu}|\nabla\tilde{p}|_{L^q} + |p|_{H^q}\\ &\le CM(\mu)^N|F|_{D^{-1}\cap L^2} + C\overline{\mu}|\nabla\tilde{p}|_{L^q} + \frac{1}{2}|p|_{W^{1,q}}\end{aligned}$$

and so

$$|p|_{W^{1,q}} \le CM(\mu)^N|F|_{D^{-1}\cap L^2} + C\overline{\mu}|\nabla\tilde{p}|_{L^q}.$$

Substituting these estimates into (42) and choosing η appropriately, we easily derive the second estimate (35).

Step 3. Finally we prove the propositioin without any restriction. In Step 2, we proved the proposition under the additional assumption that Ω_R is bounded and $\mu \in C^1(\overline{\Omega}_R)$. But since the estimates (34) and (35) are independent of the C^1-regularity of μ, a simple regularization argument allows us to prove the lemma for the case that Ω_R is bounded. Hence to complete the proof, it remains to prove the proposition for the case that Ω is unbounded and $R = \infty$. This can be deduced from the uniform estimates (34) and (35) on $R < \infty$ using the method of domain expansions. Let $(u,p) \in D^1_{0,\sigma}(\Omega)\cap L^2_0(\Omega)$ be a weak solution to the problem (23) and (24) with $R = \infty$ and $F \in D^{-1}(\Omega)\cap L^r(\Omega)$ for $r = 2$ or q. Then since

$$M(\mu|_{\Omega_R}) \le M(\mu)$$

and

$$|F|_{\Omega_R}|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)\cap L^r(\Omega_R)} \le |F|_{D^{-1}(\Omega)\cap L^2(\Omega_R)\cap L^r(\Omega)}$$

for each $R < \infty$, it follows from Lemma 2.7 and the validity of the present proposition in case of bounded domains that for each R with $2R_0 < R < \infty$, there exists a unique weak solution $(u^R, p^R) \in H^1_{0,\sigma}(\Omega_R) \times L^2_0(\Omega_R)$ to the problem (23) and (24), which satisfies the uniform estimate

$$\begin{aligned} &|u^R|_{D^1_0(\Omega_R)} + |\nabla u^R|_{W^{1,r}(\Omega_R)} + |p^R|_{W^{1,r}(\Omega_R)}r \\ &\le CM(\mu)^N |F|_{D^{-1}(\Omega)\cap L^2(\Omega)\cap L^r(\Omega)}. \end{aligned} \tag{43}$$

Extending (u^R, p^R) to Ω by zero outside Ω_R, we find that $(u^R, p^R) \in D^1_{0,\sigma}(\Omega) \times L^2_0(\Omega)$. Then adapting the proof of Proposition 6 in [8] (see also the proof of Lemma 4.1 below), we easily show that

$$u^R \to u \quad \text{in} \quad D^1_{0,\sigma}(\Omega) \quad \text{as} \quad R \to \infty,$$

which implies immediately that

$$\nabla p^R \to \nabla p \quad \text{in} \quad D^{-1}_{loc}(\Omega) \quad \text{as} \quad R \to \infty.$$

Hence from (43), we deduce that

$$|\nabla u|_{W^{1,r}(\Omega)} + |\nabla p|_{W^{1,r}(\Omega)} \le CM(\mu)^N |F|_{D^{-1}(\Omega)\cap L^2(\Omega)\cap L^r(\Omega)}.$$

Combining this estimate and (30), we complete the proof of Lemma 2.8. □

Remark 2.9. In our previous paper [6], we derived similar estimates to (34) and (35). But they depend on the radius R because Poincaré inequality in Ω_R was used in an essential way.

2.4. The stationary heat conduction equation

Adapting the previous arguments, we can also prove the similar regularity results on the boundary value problem for the stationary heat conduction equation

$$\begin{aligned} &-\operatorname{div}(\kappa \nabla \theta) = G \quad \text{in} \quad \Omega_R \\ &\theta = 0 \text{ on } \partial\Omega_R \quad \text{and} \quad \theta(x) \to 0 \text{ as } |x| \to \infty, \quad x \in \Omega_R, \end{aligned} \tag{44}$$

where G and κ are known scalar fields in Ω_R such that

$$G \in D^{-1}(\Omega_R), \quad \kappa^{-1} \in L^\infty(\Omega_R) \quad \text{and} \quad \underline{\kappa} \le \kappa \quad \text{in} \quad \Omega_R \tag{45}$$

for some constant $\underline{\kappa}$ with $0 < \underline{\kappa} \le 1$.

Proposition 2.10. *For each $G \in D^{-1}(\Omega_R)$, there exists a unique weak solution $\theta \in D^1_0(\Omega_R)$ to the boundary value problem (44), which satisfies the estimate*

$$|\theta|_{D^1_0(\Omega_R)} \le C\underline{\kappa}^{-1}|G|_{D^{-1}(\Omega_R)}.$$

Moreover, if κ satisfies the additional regularity

$$\nabla\kappa \in L^3(\Omega_R) \cap L^q(\Omega_R) \qquad \textit{for some} \quad q \in (3, 6],$$

then we have the following regularity results:

$$|\nabla\theta|_{H^1(\Omega_R)} \le CM(\kappa)^N |G|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)}$$

and

$$|\nabla\theta|_{W^{1,q}(\Omega_R)} \le CM(\kappa)^N |G|_{D^{-1}(\Omega_R)\cap L^2(\Omega_R)\cap L^q(\Omega_R)},$$

where

$$M(\kappa) = \underline{\kappa}^{-1}\left(1 + |\nabla\kappa|_{L^3(\Omega_R)\cap L^q(\Omega_R)}\right) \quad \text{and} \quad N = N(q) > 1.$$

3. A priori estimates for a linearized problem

To prove Theorem 1.1, we consider the following linearized problem

$$\operatorname{div} u = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{46}$$

$$\rho_t + \operatorname{div}(\rho v) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{47}$$

$$\begin{aligned}(\rho u)_t + \operatorname{div}(\rho v\otimes u) - \operatorname{div}(2\mu_0 \mathrm{d}u) + \nabla p \\ = \operatorname{div}(2(\mu-\mu_0)\mathrm{d}v) + \rho f \quad \text{in} \quad (0,T)\times\Omega,\end{aligned} \tag{48}$$

$$\begin{aligned}c_v\left((\rho\theta)_t + \operatorname{div}(\rho\theta v)\right) - \operatorname{div}(\kappa_0\nabla\theta) \\ = \operatorname{div}((\kappa-\kappa_0)\nabla\eta) + 2\mu|\mathrm{d}v|^2 + \rho h \quad \text{in} \quad (0,T)\times\Omega,\end{aligned} \tag{49}$$

$$(\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0) \text{ in } \Omega,\ (u,\theta) = (0,0) \text{ on } (0,T)\times\partial\Omega, \tag{50}$$

$$(\rho,u,\theta) \to (0,0,0) \text{ in } (0,T)\times\Omega \text{ as } |x|\to\infty, \tag{51}$$

where we write

$$\mu = \mu(\rho,\rho\eta), \quad c_v = c_v(\rho,\rho\eta), \quad \kappa = \kappa(\rho,\rho\eta),$$
$$\mu_0 = \mu(\rho(0),\rho(0)\eta(0)) \quad \text{and} \quad \kappa_0 = \kappa(\rho(0),\rho(0)\eta(0))$$

for simplicity. Throughout this section, we assume that the data ρ_0, u_0, θ_0, f, h satisfy

$$\begin{aligned}&\rho_0 \ge 0,\ \rho_0 \in L^{\frac{3}{2}}\cap H^1\cap W^{1,q},\ u_0 \in D^1_{0,\sigma}\cap D^2,\ \theta_0 \in D^1_0\cap D^2,\\ &(h,f)\in C([0,T];L^2)\cap L^2([0,T];L^q),\ (h_t,f_t)\in L^2(0,T;H^{-1}),\\ &-\operatorname{div}(2\mu_0 \mathrm{d}u_0) + \nabla p_0 = \rho_0^{\frac{1}{2}} g_1,\ -\operatorname{div}(\kappa_0\nabla\theta_0) - 2\mu_0|\mathrm{d}v(0)|^2 = \rho_0^{\frac{1}{2}} g_2\end{aligned} \tag{52}$$

for some $q\in(3,6]$, $p_0\in H^1$ and $(g_1,g_2)\in L^2$. We assume further that the pair (v,η) of known vector and scalar fields satisfies

$$(v,\eta)\in C([0,T];D^1_0\cap D^2)\cap L^2(0,T;D^{2,q}),\ (v_t,\eta_t)\in L^2(0,T;D^1_0). \tag{53}$$

Here we emphasize again that v need not be divergence-free in $(0,T)\times\Omega$.

First, we prove an existence result for the problem (46)-(50) for the case that ρ_0 is bounded below away from zero and Ω is a bounded domain.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary. In addition to* (52) *and* (53), *we assume that $\rho_0 \ge \delta$ in Ω for some constant $\delta > 0$. Then there exists a unique solution (ρ,u,p,θ) to the linearized problem* (46)-(50) *such that*

$$\begin{aligned}&\rho\in C([0,T_*];W^{1,q}),\ (u,\theta)\in C([0,T_*];H^1_0\cap H^2)\cap L^2(0,T_*;W^{2,q}),\\ &p\in C([0,T_*];H^1)\cap L^2(0,T_*,W^{1,q}),\ \rho_t\in C([0,T_*];L^q),\\ &(u_t,\theta_t)\in C([0,T];L^1)\cap L^2(0,T_*;H^1_0) \text{ and } \rho\ge\underline{\delta} \text{ in } (0,T)\times\Omega\end{aligned} \tag{54}$$

for some constant $\underline{\delta} > 0$.

Proof. The existence and regularity of a unique solution ρ to the linear hyperbolic problem (47) and (50) were proved in Lemma 2.2. Then since $\rho \geq \underline{\delta} > 0$ in $(0,T) \times \Omega$, we can rewrite (48) as a nonhomogeneous Stokes equations. Hence the existence and regularity of a unique solution (u,p) to the linear problem (48) and (50) can be proved by standard methods like a semi-discrete Galerkin method in [6]. We omit its details and refer to [6]. Similarly, we can solve the linear parabolic problem (49) and (50). This completes the proof of Lemma 3.1. □

Assume that ρ_0, u_0, θ_0, f, h, v, η and Ω satisfy the hypotheses of Lemma 3.1. Then it follows from Lemma 3.1 that there exists a unique strong solution (ρ, u, p, θ) to the linear problem (46)-(50) satisfying the regularity (54). The purpose of this section is to derive some local (in time) a priori estimates for (ρ, u, p, θ) which are independent of the lower bound δ of ρ_0 and the size of Ω. For this purpose, we choose a fixed constant $c_0 > 1$ so that

$$c_0 \geq 1 + |\rho_0|_{L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}} + |(u_0, \theta_0)|_{D_0^1} + |(g_1, g_2)|_{L^2}$$

and assume that

$$\begin{aligned} &|(v(0), \eta(0))|_{D_0^1} \leq 2c_0, \\ \text{(55)} \quad &\sup_{0 \leq t \leq T_*} |v(t)|_{D_0^1} + \int_0^{T_*} \left(|v_t(t)|_{D_0^1}^2 + |v(t)|_{D^{2,q}}^2 \right) dt \leq 2c_1, \\ &\sup_{0 \leq t \leq T_*} \left(|v(t)|_{D^2} + |\eta(t)|_{D_0^1 \cap D^2} \right) + \int_0^{T_*} \left(|\eta_t(t)|_{D_0^1}^2 + |\eta(t)|_{D^{2,q}}^2 \right) dt \leq 2c_2, \\ &\sup_{0 \leq t \leq T_*} |\eta(t) - \eta(0)|_{D_0^1 \cap D^{1,4}} \leq 2c_2^{-1} \quad \text{and} \quad \int_0^{T_*} |\rho_0 \eta_t(t)|_{L^2}^2 \, dt \leq 2c_2^{-6} \end{aligned}$$

for some constants c_1, c_2 and T_* with $1 < c_0 \leq c_1 \leq c_2$ and $0 < T_* \leq T$, which will be determined later and depend only on c_0 and the parameters of C. Throughout this and next sectoins, we denote by C a generic positive constant depending only on the fixed constants q, T, $|\mu|_{C^1(\mathbf{R}^2)}$, $|\kappa|_{C^1(\mathbf{R}^2)}$, $|c_v|_{C^1(\mathbf{R}^2)}$ and the norms of f and h. Moreover, $M = M(\cdot)$ denotes a generic increasing continuous function from $[1, \infty)$ to $[1, \infty)$ which depends only on the parameters of C. We also adopt the simplified notation $\mu(t) = \mu(\rho(t), \rho(t)\eta(t))$, etc.

3.1. Estimates for the density ρ

We first derive estimates for the density ρ, which is the solution to the hyperbolic problem (47) and (50). By virtue of Lemma 2.2, we have

$$\text{(56)} \qquad |\rho(t)|_{L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}} \leq |\rho_0|_{L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}} \exp\left(C \int_0^t |\nabla v|_{H^1 \cap D^{1,q}} \, ds \right)$$

for $0 \le t \le T$. Then since

$$\int_0^t |\nabla v|_{H^1\cap D^{1,q}}\,ds \le t^{\frac{1}{2}}\left(\int_0^t |\nabla v|^2_{H^1\cap D^{1,q}}\,ds\right)^{\frac{1}{2}} \le Cc_2t + C(c_2)t)^{\frac{1}{2}}, \tag{57}$$

it follows from (56) and (47) that

$$|\rho(t)|_{L^{\frac{3}{2}}\cap H^1\cap W^{1,q}} \le Cc_0 \quad \text{and} \quad |\rho_t(t)|_{L^{\frac{3}{2}}\cap L^q} \le Cc_2^2 \tag{58}$$

for $0 \le t \le \min(T_*, T_1)$, where $T_1 = c_2^{-1}$. On the other hand, from Lemma 2.2, we also deduce that

$$\rho(t,x) = \rho_0(U(0,t,x))\exp\left[-\int_0^t \operatorname{div} v(s, U(s,t,x))\,ds\right], \tag{59}$$

where $U \in C([0,T]\times[0,T]\times\overline{\Omega})$ is the solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}U(t,s,x) = v(t, U(t,s,x)), & 0 \le t \le T \\ U(s,s,x) = x, \quad 0 \le s \le T, & x \in \overline{\Omega}. \end{cases}$$

In view of the classical embedding result $W^{1,q} \hookrightarrow C^{0,1-\frac{3}{q}}$, we have

$$|\rho_0(U(0,t,x)) - \rho_0(x)| \le C|\rho_0|_{W^{1,q}}|U(0,t,x) - x|^{1-\frac{3}{q}},$$

while

$$|U(0,t,x) - x| \le \int_0^t \left|\frac{\partial}{\partial s}U(s,t,x)\right| ds \le \int_0^t |v(s)|_{L^\infty}\,ds \le c_2t.$$

Hence using the estimate (57), we obtain

$$\begin{aligned} &|\rho(t,x) - \rho_0(x)| \\ &\le \left|(\rho_0(U(0,t,x)) - \rho_0(x))\exp\left(-\int_0^t \operatorname{div} v(s,U(s,t,x))\,ds\right)\right| \\ &\quad + \rho_0(x)\left|1 - \exp\left(-\int_0^t \operatorname{div} v(s,U(s,t,x))\,ds\right)\right| \\ &\le C|\rho_0|_{W^{1,q}}\left(|U(0,t,x) - x|^{1-\frac{3}{q}} + \int_0^t |\nabla v(s)|_{L^\infty}\,ds\right) \\ &\quad \times \exp\left(C\int_0^t |\nabla v(s)|_{L^\infty}\,ds\right) \\ &\le Cc_0\left((c_2t)^{1-\frac{3}{q}} + c_2t + (c_2t)^{\frac{1}{2}}\right) \end{aligned} \tag{60}$$

for $0 \le t \le \min(T_*, T_1)$ and $x \in \Omega$. Therefore, taking

$$T_2 = \min(T_1, c_2^{-\alpha_1}) \quad \text{with} \quad \alpha_1 = \max\left(\frac{42q-3}{q-3}, 100\right),$$

we conclude that

$$|\rho(t) - \rho_0|_{L^\infty} \le Cc_2^{-40} \quad \text{for} \quad 0 \le t \le \min(T_*, T_2). \tag{61}$$

Using this and (58), we also conclude that

$$|\rho(t)-\rho_0|_{L^3\cap L^6} \le Cc_2^{-20} \quad \text{for} \quad 0 \le t \le \min(T_*, T_2). \tag{62}$$

3.2. Estimates for the coefficients μ, κ and c_v

Note that μ, κ and c_v are positive C^1-functions of $(\rho, \rho\eta)$. But by virtue of Lemma 2.1, (55) and (61), we have

$$|\eta(t)-\eta(0)|_{L^\infty} \le C|\eta(t)-\eta(0)|_{D_0^1\cap D^{1,4}} \le Cc_2^{-1}$$

and

$$|\rho(t)\eta(t)-\rho(0)\eta(0)|_{L^\infty} \le |(\rho(t)-\rho_0)\eta(t)|_{L^\infty} + |\rho_0(\eta(t)-\eta(0))|_{L^\infty} \le Cc_2^{-1}$$

for $0 \le t \le \min(T_*, T_2)$. Using this observation together with (58) and (61), we easily show that

$$\begin{aligned} M(c_0)^{-1} &\le \mu(t), \kappa(t), c_v(t) \le M(c_0), \\ |(\mu(t)-\mu_0, \kappa(t)-\kappa_0)|_{L^\infty} &\le M(c_0)c_2^{-1}, \\ |(\mu(t), \kappa(t), c_v(t))|_{D^1\cap D^{1,q_1}} &\le M(c_0), \\ |(\mu(t), \kappa(t), c_v(t))|_{D^{1,q}} &\le M(c_0)c_2 \end{aligned} \tag{63}$$

for $0 \le t \le \min(T_*, T_2)$, where $q_1 = \min(q, 4)$. Moreover, in view of (58) and (62), we have

$$\begin{aligned} &\int_0^t |(\mu_t(s), \kappa_t(s), (c_v)_t(s))|_{L^3}^2\, ds \\ &\le M(c_0)\int_0^t \left(|\rho_t|_{L^3}^2 + |\rho_t|_{L^3}^2|\eta|_{L^\infty}^2 + |\rho\eta_t|_{L^3}^2\right) ds \\ &\le M(c_0)\left(c_2^4 t + \int_0^t \left(|\rho-\rho_0|_{L^6}^2|\nabla\eta_t|_{L^2}^2 + |\rho_0|_{L^\infty}|\rho_0\eta_t|_{L^2}|\nabla\eta_t|_{L^2}\right) ds\right) \\ &\le M(c_0)\left(c_2^{-2} + \int_0^t \left(c_2^{-3}|\nabla\eta_t|_{L^2}^2 + |\rho_0\eta_t|_{L^2}|\nabla\eta_t|_{L^2}\right) ds\right) \end{aligned}$$

for $0 \le t \le \min(T_*, T_2)$. Hence it follows from (55) that

$$\int_0^t |(\mu_t(s), \kappa_t(s), (c_v)_t(s))|_{L^3}^2\, ds \le M(c_0)c_2^{-2} \tag{64}$$

for $0 \le t \le \min(T_*, T_2)$.

Remark 3.2. If μ, κ and c_v are C^1-functions of ρ and η, then

$$\begin{aligned} \int_0^t |(\mu_t(s), \kappa_t(s), (c_v)_t(s))|_{L^3}^2\, ds &\le M(c_0)\int_0^t \left(|\rho_t(s)|_{L^3}^2 + |\eta_t(s)|_{L^3}^2\right) ds \\ &\le M(c_0)\left(c_2^{-2} + \tilde{C}\int_0^t |\eta_t(s)|_{D_0^1}^2\, ds\right) \\ &\le M(c_0)\tilde{C}c_2^2 \end{aligned}$$

for $0 \leq t \leq \min(T_*, T_2)$, where $\tilde{C}$ is a constant depending also on the size of Ω. However, we have failed to derive an analogue of (64), which is necessary in particular to estimate the third term of the right hand side of (66).

3.3. Estimates for the velocity u

In view of (47), we can rewrite (48) as

$$\rho u_t + \rho v \cdot \nabla u - \operatorname{div}(2\mu_0 \mathrm{d}u) + \nabla p = \operatorname{div}\left(2(\mu - \mu_0)\mathrm{d}v\right) + \rho f. \tag{65}$$

Differentiating this with respect to t, we have

$$\begin{aligned} &\rho u_{tt} + \rho v \cdot \nabla u_t - \operatorname{div}(2\mu_0 \mathrm{d}u_t) + \nabla p_t \\ = &-\rho_t u_t - (\rho v)_t \cdot \nabla u + \operatorname{div}\left(2(\mu - \mu_0)\mathrm{d}v\right)_t + (\rho f)_t. \end{aligned}$$

We multiplying this by u_t and integrating over Ω. Then by virtue of (47), we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho |u_t|^2\,dx + \int 2\mu_0 |\mathrm{d}u_t|^2\,dx \\ = &-\int \rho_t |u_t|^2\,dx - \int ((\rho v)_t \cdot \nabla u) \cdot u_t\,dx \\ &-\int 2\left((\mu - \mu_0)\mathrm{d}v\right)_t : \nabla u_t\,dx + \langle (\rho f)_t, u_t \rangle. \end{aligned} \tag{66}$$

Using Hölder and Sobolev inequalities together with (58), (63), and (64), we can estimate each term of the right hand side of (66) as follows.

$$\begin{aligned} -\int \rho_t |u_t|^2\,dx = &\int \operatorname{div}(\rho v)|u_t|^2\,dx \\ = &-2\int (\rho v \cdot \nabla u_t) \cdot u_t\,dx \\ \leq &|\rho|_{L^\infty}^{\frac{1}{2}} |v|_{L^\infty} |\nabla u_t|_{L^2} |\sqrt{\rho} u_t|_{L^2} \\ \leq &C c_0 c_2^2 |\sqrt{\rho} u_t|_{L^2}^2 + \frac{1}{8}\underline{\mu} |u_t|_{D_0^1}^2, \end{aligned}$$

$$\begin{aligned} &-\int ((\rho v)_t \cdot \nabla u) \cdot u_t\,dx \\ = &-\int (\rho_t v \cdot \nabla u + \rho v_t \cdot \nabla u) \cdot u_t\,dx \\ \leq &C|\rho_t|_{L^3} |v|_{L^\infty} |u|_{D_0^1} |u_t|_{D_0^1} + C|\rho|_{L^\infty}^{\frac{1}{2}} |v_t|_{D_0^1} |\nabla u|_{L^3} |\sqrt{\rho} u_t|_{L^2} \\ \leq &C c_2^6 |u|_{D_0^1}^2 + C c_0 \varepsilon |v_t|_{D_0^1}^2 |\sqrt{\rho} u_t|_{L^2}^2 + \varepsilon^{-1} |\nabla u|_{L^3}^2 + \frac{1}{8}\underline{\mu} |u_t|_{D_0^1}^2 \\ \leq &C c_0 \left(c_2^6 + \varepsilon |v_t|_{D_0^1}^2 + \varepsilon^{-2}\right)\left(|\sqrt{\rho} u_t|_{L^2}^2 + |u|_{D_0^1}^2\right) + |\nabla u|_{H^1}^2 + \frac{1}{8}\underline{\mu} |u_t|_{D_0^1}^2, \end{aligned}$$

$$
\begin{aligned}
&-\int 2\left((\mu-\mu_0)dv\right)_t : \nabla u_t\,dx \\
&\le C\left(|\mu_t|_{L^3}|\nabla v|_{H^1} + |\mu-\mu_0|_{L^\infty}|\nabla v_t|_{L^2}\right)|\nabla u_t|_{L^2} \\
&\le C\left(c_2^2|\mu_t|_{L^3}^2 + |\mu-\mu_0|_{L^\infty}^2|v_t|_{D_0^1}^2\right) + \frac{1}{8}\underline{\mu}|u_t|_{D_0^1}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\langle(\rho f)_t, u_t\rangle &= \langle\rho_t f, u_t\rangle + \langle f_t, \rho u_t\rangle \\
&\le C|\rho_t|_{L^3}|f|_{L^2}|u_t|_{D_0^1} + |f_t|_{H^{-1}}|\rho u_t|_{H_0^1} \\
&\le C\left(c_2^4|f|_{L^2}^2 + c_0^2|f_t|_{H^{-1}}^2\right) + |\sqrt{\rho}u_t|_{L^2}^2 + \frac{1}{8}\underline{\mu}|u_t|_{D_0^1}^2.
\end{aligned}
$$

Here $\varepsilon > 0$ is a small constant and $\underline{\mu} = \inf_\Omega \mu_0$. Substituting all the estimates into (66), taking $\varepsilon = c_2^{-1} < 1$ and observing that

$$
\underline{\mu} \ge M(c_0)^{-1}, \quad |\mu-\mu_0|_{L^\infty} \le M(c_0)c_2^{-1} \quad \text{and} \quad |\mathrm{d}u_t|_{L^2} = |\nabla u_t|_{L^2} = |u_t|_{D_0^1},
$$

we deduce that

$$
\begin{aligned}
&\frac{d}{dt}|\sqrt{\rho}u_t|_{L^2}^2 + M(c_0)^{-1}|u_t|_{D_0^1}^2 \\
&\le M(c_0)\left(c_2^6 + c_2^{-1}|v_t|_{D_0^1}^2\right)\left(1 + |\sqrt{\rho}u_t|_{L^2}^2 + |u|_{D_0^1}^2\right) \\
&\quad + C\left(c_2^2|\mu_t|_{L^3}^2 + c_0^2|f_t|_{H^{-1}}^2 + |\nabla u|_{H^1}^2\right).
\end{aligned}
$$

Hence integrating this over (τ, t) and using (64), we have

$$
\begin{aligned}
&|\sqrt{\rho}u_t(t)|_{L^2}^2 + M(c_0)^{-1}\int_\tau^t |u_t|_{D_0^1}^2\,ds \\
&\le M(c_0) + |\sqrt{\rho}u_t(\tau)|_{L^2}^2 + C\int_\tau^t |\nabla u|_{H^1}^2\,ds \qquad (67)\\
&\quad + M(c_0)\int_\tau^t \left(c_2^6 + c_2^{-1}|v_t|_{D_0^1}^2\right)\left(1 + |\sqrt{\rho}u_t|_{L^2}^2 + |u|_{D_0^1}^2\right)ds
\end{aligned}
$$

for $0 < \tau \le t \le \min(T_*, T_2)$. On the other hand, since u_t is divergence-free in $(0,T)\times\Omega$, it follows from (65) that

$$
\begin{aligned}
&\int \rho|u_t|^2\,dx \\
&= \int \left(-\rho v\cdot\nabla u + \mathrm{div}\,(2\mu_0\mathrm{d}u) - \nabla p + \mathrm{div}\,(2(\mu-\mu_0)\mathrm{d}v) + \rho f\right)\cdot u_t\,dx \\
&= \int \left(-\rho v\cdot\nabla u + \mathrm{div}\,(2\mu_0\mathrm{d}u) - \nabla p_0 + \mathrm{div}\,(2(\mu-\mu_0)\mathrm{d}v) + \rho f\right)\cdot u_t\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
\int \rho|u_t|^2\,dx &\le C\int\left(\rho|v|^2|\nabla u|^2 + \rho|f|^2\right)dx + \int \rho^{-1}|\mathrm{div}\,(2\mu_0\mathrm{d}u) - \nabla p_0|^2\,dx \\
&\quad + C\int \rho^{-1}|\mathrm{div}\,(2(\mu-\mu_0)\mathrm{d}v)|^2\,dx.
\end{aligned}
$$

Noting that $\rho \geq \underline{\delta} > 0$, $\rho, \mu \in C([0,T_*]; W^{1,q})$, $u \in C([0,T_*]; H^2)$ and

$$\operatorname{div}(2\mu_0 \mathrm{d}u(t)) \to \operatorname{div}(2\mu_0 \mathrm{d}u_0) = \nabla p_0 + \rho_0^{\frac{1}{2}} g_1 \quad \text{in} \quad L^2 \quad \text{as} \quad t \to 0,$$

we thus have

$$\begin{aligned}&\limsup_{\tau \to 0} |\sqrt{\rho} u_t(\tau)|_{L^2}^2 \\ \leq\ & C|\rho_0|_{L^\infty} \left(|\nabla v(0)|_{H^1}^2 |u_0|_{D_0^1}^2 + |f(0)|_{L^2} \right) + |g_1|_{L^2}^2 \leq C c_0^5.\end{aligned}$$

Letting $\tau \to 0$ in (67) and using the fact that

$$|\nabla u(t)|_{L^2}^2 \leq C|\nabla u_0|^2 + C \int_0^t |\nabla u_t|_{L^2}^2 \, ds \quad \text{for} \quad 0 \leq t \leq T_*,$$

we obtain

$$\begin{aligned}&\left(1 + |\sqrt{\rho} u_t(t)|_{L^2}^2 + |u(t)|_{D_0^1}^2\right) + \int_0^t |u_t|_{D_0^1}^2 \, ds \\ &\leq M(c_0) \left(1 + \int_0^t |\nabla u|_{H^1}^2 \, ds + \int_0^t \left(c_2^6 + c_2^{-1} |v_t|_{D_0^1}^2 \right) \left(1 + |\sqrt{\rho} u_t|_{L^2}^2 + |u|_{D_0^1}^2 \right) ds \right)\end{aligned}$$

for $0 \leq t \leq \min(T_*, T_2)$. Hence, in view of Gronwall's inequality, we deduce that

$$(68) \quad \left(|\sqrt{\rho} u_t(t)|_{L^2}^2 + |u(t)|_{D_0^1}^2 \right) + \int_0^t |u_t|_{D_0^1}^2 \, ds \leq M(c_0) \left(1 + \int_0^t |\nabla u|_{H^1}^2 \, ds \right)$$

for $0 \leq t \leq \min(T_*, T_2)$. To estimate $|\nabla u|_{H^1}$, we observe that for each $t \in [0, T_*]$, $u = u(t) \in D_0^1 \cap D^2$ is a solution of the Stokes equations

$$-\operatorname{div}(2\mu_0 \mathrm{d}u) + \nabla p = F \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega,$$

where

$$F = \rho(f - u_t - v \cdot \nabla u) + \operatorname{div}(2(\mu - \mu_0)\mathrm{d}v)$$

satisfies

$$\begin{aligned}|F|_{D^{-1} \cap L^2} \leq C|\rho|_{L^{\frac{3}{2}} \cap L^\infty}^{\frac{1}{2}} |\sqrt{\rho} u_t|_{L^2} + C|\rho|_{L^3 \cap L^\infty} \left(|v|_{D_0^1} |\nabla u|_{L^3} + |f|_{L^2} \right) \\ + C|\mu - \mu_0|_{L^\infty} |\nabla v|_{H^1} + C|\nabla(\mu - \mu_0) \cdot \mathrm{d}v|_{L^2}\end{aligned}$$

and

$$\begin{aligned}|F|_{L^q} \leq\ & C|\rho|_{L^\infty}^{\frac{1}{2}} |\sqrt{\rho} u_t|_{L^2} + C|\rho|_{L^\infty} |u_t|_{D_0^1} + C|\rho|_{L^\infty} |\nabla v|_{H^1} |\nabla u|_{H^1} \\ &+ C|\rho|_{L^\infty} |f|_{H^1} + C|\mu - \mu_0|_{L^\infty} |\nabla^2 v|_{L^q} + C|\nabla(\mu - \mu_0)|_{L^q} |\nabla v|_{L^\infty}.\end{aligned}$$

Hence by virtue of Proposition 2.8 and (63), we have

$$
\begin{aligned}
|(\nabla u,p)|_{H^1} &\le M(c_0)\,|F|_{D^{-1}\cap L^2}\\
&\le M(c_0)\left(1+|\sqrt{\rho}u_t|_{L^2}+|v|_{D_0^1}|\nabla u|_{L^3}\right)\\
&\quad + C\left(|\mu-\mu_0|_{L^\infty}|\nabla v|_{H^1}+C|\nabla(\mu-\mu_0)|_{L^{q_1}}|\nabla v|_{L^{\frac{2q_1}{q_1-2}}}\right)\\
&\le M(c_0)c_1\left(1+|\sqrt{\rho}u_t|_{L^2}+|\nabla u|_{L^2}^{\frac12}|\nabla u|_{H^1}^{\frac12}+|\nabla v|_{L^2}^{1-\frac{3}{q_1}}|\nabla v|_{H^1}^{\frac{3}{q_1}}\right)
\end{aligned}
$$

and thus

$$
|(\nabla u,p)|_{H^1}\le M(c_0)c_1^2\left(c_2^{\frac{3}{q_1}}+|\sqrt{\rho}u_t|_{L^2}+|u|_{D_0^1}\right).
$$

Therefore, substituting this into (68) and applying Gronwall's inequality again, we conclude that

$$
(69)\qquad
\begin{aligned}
|\sqrt{\rho}u_t(t)|_{L^2}^2+|u(t)|_{D_0^1}^2+\int_0^t|u_t(s)|_{D_0^1}^2\,ds &\le M(c_0),\\
|(\nabla u(t),p(t))|_{H^1} &\le M(c_0)c_1^2c_2^{\frac{3}{q_1}}
\end{aligned}
$$

for $0\le t\le\min(T_*,T_2)$. Moreover, it follows also from Proposition 2.8, (63) and (69) that

$$
\begin{aligned}
&|(\nabla u,p)|_{W^{1,q}}\\
&\le M(c_0)\,|F|_{D^{-1}\cap L^2\cap L^q}\\
&\le M(c_0)\left(c_1^2c_2^{\frac{3}{q_1}}+|\sqrt{\rho}u_t|_{L^2}+|u_t|_{D_0^1}+|\nabla v|_{H^1}|\nabla u|_{H^1}+|f|_{H^1}\right)\\
&\quad + M(c_0)\left(|\mu-\mu_0|_{L^\infty}|v|_{D^{2,q}}+|\nabla(\mu-\mu_0)|_{L^q}|\nabla v|_{L^\infty}\right)\\
&\le M(c_0)\left(c_2^4+|f|_{H^1}+|u_t|_{D_0^1}+c_2^{-1}|v|_{D^{2,q}}+c_2^2|\nabla v|_{H^1}^{1-\gamma}|\nabla v|_{W^{1,q}}^{\gamma}\right)
\end{aligned}
$$

for some $\gamma=\gamma(q)\in(0,1)$. Hence taking

$$
T_3=c_2^{\alpha_2}\qquad\text{with}\quad \alpha_2=\min\left(\alpha_1,\frac{6-2\gamma}{1-\gamma}\right),
$$

we deduce that

$$
(70)\qquad \int_0^t\left(|u(s)|_{D^{2,q}}^2+|p(s)|_{D^{1,q}}^2\right)ds\le M(c_0)(1+c_2^{6-2\gamma}t^{1-\gamma})\le M(c_0)
$$

for $0\le t\le\min(T_*,T_3)$.

3.4. Estimates for the temperature θ

Differentiating (49) in time, we obtain

$$
\begin{aligned}
&c_v\rho\theta_{tt}-\operatorname{div}(\kappa_0\nabla\theta_t)\\
&=\ -(c_v\rho)_t\theta_t-(c_v\rho v\cdot\nabla\theta)_t+\operatorname{div}\left((\kappa-\kappa_0)\nabla\eta\right)_t+2\left(\mu|\mathrm{d}v|^2+\rho h\right)_t.
\end{aligned}
$$

Then multiplying this by θ_t and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int c_v\rho|\theta_t|^2\,dx + \int \kappa_0|\nabla\theta_t|^2\,dx \\
(71)\qquad &= \int\left(-\frac{1}{2}(c_v)_t\rho|\theta_t|^2 + \left(2\mu|\mathrm{d}v|^2\right)_t\theta_t\right)dx - \int\frac{1}{2}c_v\rho_t|\theta_t|^2\,dx \\
&\quad - \int(c_v\rho v\cdot\nabla\theta)_t\theta_t\,dx - \int((\kappa-\kappa_0)\nabla\eta)_t\cdot\nabla\theta_t\,dx + \langle(\rho h)_t,\theta_t\rangle.
\end{aligned}$$

Using (58), (63), (64) and (69), we can estimate each term of the right hand side in (71) as follows.

$$\begin{aligned}
&\int\left(-\frac{1}{2}(c_v)_t\rho|\theta_t|^2 + \left(2\mu|\mathrm{d}v|^2\right)_t\theta_t\right)dx \\
\leq\ & M(c_0)\left(|(c_v)_t|_{L^3}|\sqrt{\rho}\theta_t|_{L^2} + |\mu_t|_{L^3}|\nabla v|^2_{L^4} + |\nabla v|_{L^3}|\nabla v_t|_{L^2}\right)|\theta_t|_{D_0^1} \\
\leq\ & M(c_0)\left(|(c_v)_t|^2_{L^3}|\sqrt{\rho}\theta_t|^2_{L^2} + c_1c_2^3|\mu_t|^2_{L^3} + c_1c_2|\nabla v_t|^2_{L^2}\right) + \frac{1}{8}\underline{\kappa}|\theta_t|^2_{D_0^1},
\end{aligned}$$

$$\begin{aligned}
&-\int\frac{1}{2}c_v\rho_t|\theta_t|^2\,dx \\
&= \frac{1}{2}\int \mathrm{div}\,(\rho v)c_v|\theta_t|^2\,dx \\
&\leq \int\rho|v||\nabla c_v||\theta_t|^2\,dx + \int\rho|v|c_v|\theta_t||\nabla\theta_t|\,dx \\
&\leq C|\rho|^{\frac{1}{2}}_{L^\infty}|v|_{D_0^1}|\nabla c_v|_{L^q}|\sqrt{\rho}\theta_t|_{L^{\frac{3q}{2q-3}}}|\nabla\theta_t|_{L^2} \\
&\quad + C|\rho|^{\frac{1}{2}}_{L^\infty}|v|_{L^\infty}|\sqrt{\rho}\theta_t|_{L^2}|\nabla\theta_t|_{L^2} \\
&\leq M(c_0)c_1c_2^2|\sqrt{\rho}\theta_t|^{\frac{3}{2}-\frac{3}{q}}_{L^2}|\nabla\theta_t|^{\frac{1}{2}+\frac{3}{q}}_{L^2} + M(c_0)c_2|\sqrt{\rho}\theta_t|_{L^2}|\nabla\theta_t|_{L^2} \\
&\leq M(c_0)(c_1c_2^2)^{\frac{4q}{3q-6}}|\sqrt{\rho}\theta_t|^2_{L^2} + \frac{1}{8}\underline{\kappa}|\theta_t|^2_{D_0^1},
\end{aligned}$$

$$\begin{aligned}
&-\int(c_v\rho v\cdot\nabla\theta)_t\theta_t\,dx \\
\leq\ & M(c_0)\left(c_2^2|(c_v)_t|^2_{L^3} + c_2^6 + \varepsilon|\nabla v_t|^2_{L^2}\right)\left(|\nabla\theta|^2_{L^2} + |\sqrt{\rho}\theta_t|^2_{L^2}\right) \\
&+\varepsilon^{-1}|\nabla\theta|^2_{L^3} + \frac{1}{8}\underline{\kappa}|\theta_t|^2_{D_0^1};
\end{aligned}$$

$$\int((\kappa-\kappa_0)\nabla\eta)_t\cdot\nabla\theta_t\,dx \leq C(c_2^2|\kappa_t|^2_{L^3} + c_2^{-2}|\eta_t|^2_{D_0^1}) + \frac{1}{8}\underline{\kappa}|\theta_t|^2_{D_0^1}$$

and

$$\langle(\rho h)_t,\theta_t\rangle \leq C\left(c_2^4 + c_0^2|h_t|^2_{H^{-1}}\right) + |\sqrt{\rho}\theta_t|^2_{L^2} + \frac{1}{8}\underline{\kappa}|\theta_t|^2_{D_0^1}.$$

Here $\varepsilon > 0$ is a small number and $\underline{\kappa} = \inf_\Omega \kappa_0$. Substituting all the estimates into (71) and taking $\varepsilon = c_1^{-1}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int c_v\rho|\theta_t|^2\,dx + \underline{\kappa}\int|\nabla\theta_t|^2\,dx \\
&\le M(c_0)\Big(\Big(c_2^2|(c_v)_t|_{L^3}^2 + c_2^{\frac{4q}{q-2}} + c_1^{-2}|\nabla v_t|_{L^2}^2 + c_2^6\Big)\Big(|\sqrt{\rho}\theta_t|_{L^2}^2 + |\theta|_{D_0^1}^2\Big) \\
&\quad + |\nabla\theta|_{H^1}^2 + c_1c_2^3|\mu_t|_{L^3}^2 + c_1c_2|\nabla v_t|_{L^2}^2 + c_2^2|\kappa_t|_{L^3}^2 + c_2^4 + |h_t|_{H^{-1}}^2 + c_2^{-2}|\eta_t|_{D_0^1}^2\Big).
\end{aligned}$$

Hence integrating over (τ, t), we derive an analogue of (67):

$$\begin{aligned}
&|\sqrt{\rho}\theta_t(t)|_{L^2}^2 + \int_\tau^t |\theta_t|_{D_0^1}^2\,ds \\
\le\ & M(c_0)\left(c_1^3c_2 + \int_\tau^t|\nabla\theta|_{H^1}^2\,ds\right) + |\sqrt{\rho}\theta_t(\tau)|_{L^2}^2 \\
&+ M(c_0)\int_\tau^t\left(c_2^2|(c_v)_t|_{L^3}^2 + c_2^{\frac{4q}{q-2}} + c_1^{-2}|\nabla v_t|_{L^2}^2\right)\left(|\sqrt{\rho}\theta_t|_{L^2}^2 + |\theta|_{D_0^1}^2\right)ds
\end{aligned}$$

for $0 < \tau \le t \le \min(T_*, T_3)$. Then observing that

$$-\mathrm{div}(\kappa_0\nabla\theta(t)) \to \mu_0|\mathrm{d}v(0)|^2 + \rho_0^{\frac{1}{2}}g_2 \quad \text{in} \quad L^2 \quad \text{as} \quad t\to 0$$

and

$$|\nabla\theta(t)|_{L^2} \le C|\nabla\theta_0|_{L^2} + C\int_0^t|\nabla\theta_t|_{L^2}^2\,ds \quad \text{for} \quad 0\le t\le T_*,$$

we easily deduce that

$$\begin{aligned}
&|\sqrt{\rho}\theta_t(t)|_{L^2}^2 + |\theta(t)|_{D_0^1}^2 + \int_0^t|\theta_t|_{D_0^1}^2\,ds \le M(c_0)\left(c_1^3c_2 + \int_\tau^t|\nabla\theta|_{H^1}^2\,ds\right) \\
&+ M(c_0)\int_\tau^t\left(c_2^2|(c_v)_t|_{L^3}^2 + c_2^{\frac{4q}{q-2}} + c_1^{-2}|\nabla v_t|_{L^2}^2\right)\left(|\sqrt{\rho}\theta_t|_{L^2}^2 + |\theta|_{D_0^1}^2\right)ds
\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. Hence in view of Gronwall's inequality, we have

$$\begin{aligned}
&|\sqrt{\rho}\theta_t(t)|_{L^2}^2 + |\theta(t)|_{H_0^1}^2 + \int_0^t|\theta_t|_{H_0^1}^2\,ds \\
&\le M(c_0)\left(c_1^3c_2 + \int_\tau^t|\nabla\theta|_{H^1}^2\,ds\right) \qquad (72)
\end{aligned}$$

for $0 \le t \le \min(T_*, T_3)$. On the other hand, since each $\theta = \theta(t) \in D_0^1 \cap D^2$ is a solution of the elliptic equation

$$-\mathrm{div}(\kappa_0\nabla\theta) = G \qquad \text{in} \quad \Omega,$$

where $G = -c_v\rho(\theta_t + v\cdot\nabla\theta) + \mathrm{div}\,((\kappa-\kappa_0)\nabla\eta) + 2\mu|\mathrm{d}v|^2 + \rho h$, it follows from Proposition 2.10 that

$$|\nabla\theta|_{W^{1,r}} \le M(c_0)\,|G|_{D^{-1}\cap L^2\cap L^r} \qquad \text{for} \quad r = 2, q.$$

Combining this result and (72), we can easily show that

$$|\sqrt{\rho}\theta_t(t)|_{L^2}^2 + |\nabla\theta(t)|_{H^1}^2 + \int_0^t \left(|\theta_t(s)|_{D_0^1}^2 + |\theta(s)|_{D^{2,q}}^2\right) ds$$

$$\leq \quad M(c_0)c_1^4 c_2^{\frac{3}{q_1}} \tag{73}$$

for $0 \leq t \leq \min(T_*, T_3)$. As a consequence, we have

$$|\theta(t) - \theta_0|_{D_0^1} \leq \left(t\int_0^t |\theta_t|_{D_0^1}^2\, ds\right)^{\frac{1}{2}} \leq M(c_0)(c_2^5 t)^{\frac{1}{2}} \leq M(c_0)c_2^{-8} \tag{74}$$

and thus

$$|\theta(t) - \theta_0|_{D^{1,4}} \leq C|\theta(t) - \theta_0|_{D_0^1}^{\frac{1}{4}} |\theta(t) - \theta_0|_{D^2}^{\frac{3}{4}} \leq M(c_0)c_2^{-2} \tag{75}$$

for $0 \leq t \leq \min(T_*, T_3)$. Moreover, since

$$\int_0^t |\rho_0\theta_t|_{L^2}^2\, ds \leq 2\int (|(\rho - \rho_0)\theta_t|_{L^2}^2 + |\rho\theta_t|_{L^2}^2)\, ds$$

$$\leq C \sup_{0\leq s\leq t} |\rho(s) - \rho_0|_{L^3} \int_0^t |\theta_t|_{D_0^1}^2\, ds + Cc_0^{\frac{1}{2}} t \sup_{0\leq s\leq t} |\sqrt{\rho}\theta_t(s)|_{L^2}^2,$$

it follows from (62) and (73) that

$$\int_0^t |\rho_0\theta_t(s)|_{L^2}^2\, ds \leq M(c_0)c_2^{-7} \qquad \text{for} \quad 0 \leq t \leq \min(T_*, T_3). \tag{76}$$

3.5. Conclusion

From (58), (69), (73), (74), (75), and (76), it follows that if $0 \leq t \leq \min(T_*, T_3)$, then there exists a constant $M(c_0) \geq 1$ depending increasingly on c_0 such that

$$|u(t)|_{D_0^1} + \int_0^t \left(|u_t|_{D_0^1}^2 + |u|_{D^{2,q}}^2 + |\nabla p|_{L^q}^2\right) ds \leq M(c_0),$$

$$|u(t)|_{D^2} + |\theta(t)|_{D_0^1\cap D^2} + \int_0^t \left(|\theta_t|_{D_0^1}^2 + |\theta|_{D^{2,q}}^2\right) ds \leq M(c_0)c_1^4 c_2^{\frac{3}{q_1}},$$

and

$$|\theta(t) - \theta_0|_{D_0^1\cap D^{1,4}} \leq M(c_0)c_2^{-2}, \quad \int_0^t |\rho_0\theta_t(s)|_{L^2}^2\, ds \leq M(c_0)c_2^{-7},$$

$$|\rho(t)|_{L^{\frac{3}{2}}\cap H^1\cap W^{1,q}} + |\rho_t(t)|_{L^{\frac{3}{2}}\cap L^q} + |p(t)|_{H^1}$$

$$+|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t)(t)|_{L^2} + \int_0^t |p(s)|_{W^{1,q}}^2\, ds \leq M(c_0)c_1^4 c_2^{\frac{3}{q_1}}.$$

Therefore, choosing c_1, c_2 and T_* so that

$$c_1 = M(c_0), \quad c_2 = M(c_0)c_1^4 c_2^{\frac{3}{q_1}} \quad \text{and} \quad 0 < T_* \leq T_{**} \equiv \min(T_*, T_4), \tag{77}$$

we conclude that

$$\sup_{0\le t\le T_*}|u(t)|_{D_0^1}+\int_0^{T_*}(|u_t(t)|_{D_0^1}+|u(t)|^2_{D^{2,q}})\,dt \le c_1,$$

$$\sup_{0\le t\le T_*}(|u(t)|_{D^2}+|\theta(t)|_{D_0^1\cap D^2})+\int_0^{T_*}(|\theta_t(t)|^2_{D_0^1}+|\theta(t)|^2_{D^{2,q}})\,dt \le c_2,$$

$$\sup_{0\le t\le T_*}|\theta(t)-\theta(0)|_{D_0^1\cap D^{1,4}}\le c_2^{-1},\qquad \int_0^{T_*}|\rho_0\theta_t(t)|^2_{L^2}\,dt \le c_2^{-6}, \tag{78}$$

$$\sup_{0\le t\le T_*}(|\rho(t)|_{L^{\frac{3}{2}}\cap H^1\cap W^{1,q}}+|\rho_t(t)|_{L^{\frac{3}{2}}\cap L^q}+|p(t)|_{H^1})$$
$$+\sup_{0\le t\le T_*}|(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)(t)|_{L^2}+\int_0^{T_*}|p(t)|^2_{W^{1,q}}\,dt \le c_2.$$

Remark 3.3. It should be pointed out that the constant C doesn't depend on the lower bound δ of the initial density ρ_0 and further, the radius R in case when Ω is the intersection of an unbounded domain and a large open ball with radius R.

4. Proof of Theorem 1.1

In this section, we provide a complete proof of Theorem 1.1. Let (ρ_0,u_0,θ_0) be a given initial data satisfying the hypotheses of Theorem 1.1. To prove the theorem, we construct a sequence $\{(\rho^k,u^k,p^k,\theta^k)\}_{k\ge1}$ of approximate solutions solving the linearized problem (47)-(51) successively.

4.1. The construction of $\{(\rho^k,u^k,p^k,\theta^k)\}_{k\ge1}$

First, let u^0 be the solution in $C([0,\infty);D_0^1\cap D^2)\cap L^2(0,\infty;D^3)$ of the Stokes equations

$$u_t^0-\Delta u^0+\nabla p^0=0\quad\text{and}\quad \mathrm{div}u^0=0$$

in $(0,\infty)\times\Omega$ with the initial data $u^0(0)=u_0$. We also denote by θ^0 the solution in $C([0,\infty);D_0^1\cap D^2)\cap L^2(0,\infty;D^3)$ of the linear parabolic equation $\theta_t^0-\Delta\theta^0=0$ in $(0,\infty)\times\Omega$ with the initial data $\theta^0(0)=\theta_0$. It is easy to show that

$$\sup_{0\le t\le 1}|u^0(t)|_{D_0^1\cap D^2}+\int_0^1\left(|u_t^0(t)|^2_{D_0^1}+|u^0(t)|^2_{D^3}\right)dt$$
$$\le\ C(1+|u_0|^2_{D_0^1\cap D^2}) \tag{79}$$

and

$$\sup_{0\le t\le 1}|\theta^0(t)|_{D_0^1\cap D^2}+\int_0^1\left(|\theta_t^0(t)|^2_{D_0^1}+|\theta^0(t)|^2_{D^3}\right)dt$$
$$\le\ C(1+|\theta_0|^2_{D_0^1\cap D^2}). \tag{80}$$

Next, let us define c_0 by

$$c_0 = 2 + |\rho_0|_{L^{\frac{3}{2}}\cap H^1\cap W^{1,q}} + |(u_0, \theta_0)|_{D_0^1} + |(g_1, g_2)|_{L^2},$$

and we choose the positive constants c_1, c_2 and T_{**} as in (77), which depend only on c_0 and the parameters of C. Then since $u_0 \in D_0^1 \cap D^2$ is a solution of the stationary Stokes equations

$$-\mathrm{div}\,(2\mu_0 du_0) + \nabla p_0 = \rho_0^{\frac{1}{2}} g_1 \quad \text{and} \quad \mathrm{div} u_0 = 0$$

in Ω, it follows immediately from Proposition 2.8 that

$$|u_0|_{D_0^1\cap D^2} \leq M(c_0) \tag{81}$$

for some increasing function $M = M(\cdot)$. Similarly, using Proposition 2.10, we easily deduce that

$$|\theta_0|_{D_0^1\cap D^2} \leq M(c_0). \tag{82}$$

By virtue of (77), (79), (80), (81), and (82), we may assume without loss of generality that

$$\begin{aligned} &\sup_{0\leq t\leq 1} |(u^0, \theta^0)(t)|_{D_0^1\cap D^2} + \int_0^1 (|(u_t^0, \theta_t^0)(t)|_{D_0^1}^2 + |(u^0, \theta^0)(t)|_{D^{2,q}}^2)\,dt \\ (83)\quad \leq\ & c_1. \end{aligned}$$

Moreover, since $\theta^0 \in C([0,\infty); D_0^1 \cap D^2)$, $\theta_t^0 \in L^2(0,\infty; D_0^1)$ and $\rho_0 \in L^3$, there is a small time $T_* \in (0, T_{**})$ such that

$$\sup_{0\leq t\leq T_*} |\theta^0(t) - \theta_0|_{D_0^1\cap D^{1,4}} \leq c_2^{-1} \quad \text{and} \quad \int_0^{T_*} |\rho_0\theta_t^0(t)|_{L^2}^2\,dt \leq c_2^{-6}. \tag{84}$$

Our construction of the sequence $\{(\rho^k, u^k, p^k, \theta^k)\}_{k\geq 1}$ is based on the following key lemma to the proof of Theorem 1.1.

Lemma 4.1. *Let (v, η) be a pair of vector and scalar fields satisfying the regularity (53) with T replaced by T_*. Assume further that (v, η) satisfies*

$$\begin{aligned} &(v(0),\, \eta(0)) = (u_0,\, \theta_0), \\ &\sup_{0\leq t\leq T_*} |v(t)|_{D_0^1} + \int_0^{T_*} \left(|v_t(t)|_{D_0^1} + |v(t)|_{D^{2,q}}^2\right) dt \leq c_1, \\ (85)\quad &\sup_{0\leq t\leq T_*} \left(|v(t)|_{D^2} + |\eta(t)|_{D_0^1\cap D^2}\right) + \int_0^{T_*} \left(|\eta_t(t)|_{D_0^1}^2 + |\eta(t)|_{D^{2,q}}^2\right) dt \leq c_2, \\ &\sup_{0\leq t\leq T_*} |\eta(t) - \eta(0)|_{D_0^1\cap D^{1,4}} \leq c_2^{-1} \quad \text{and} \quad \int_0^{T_*} |\rho_0\eta_t(t)|_{L^2}^2\,dt \leq c_2^{-6}. \end{aligned}$$

Then there exists a unique strong solution (ρ, u, p, θ) to the linearized problem (47)-(51) *in* $[0, T_*]$ *satisfying the estimate* (78) *as well as the regularity*

$$\begin{aligned} &\rho \in C([0,T_*]; L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}), \quad \rho_t \in C([0,T_*]; L^{\frac{3}{2}} \cap L^q), \\ &(u,\theta) \in C([0,T_*]; D_0^1 \cap D^2) \cap L^2(0,T_*; D^{2,q}), \\ &p \in C([0,T_*]; H^1) \cap L^2(0,T_*, W^{1,q}), \\ &(u_t, \theta_t) \in L^2(0,T_*; D_0^1) \quad \text{and} \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T_*; L^2). \end{aligned} \tag{86}$$

Proof. Let $\varphi \in C_c^\infty(B_1)$ be a smooth cut-off function such that $\varphi = 1$ in $B_{1/2}$, and we define

$$\varphi^R(x) = \varphi(x/R), \quad v^R(t,x) = \varphi^R(x) v(t,x) \quad \text{and} \quad \eta^R(t,x) = \varphi^R(x)\eta(t,x)$$

for $(t,x) \in [0,T_*] \times \Omega$ and $2R_0 < R < \infty$, where $R_0 \geq 3$ is the constant defined in (11). Note that if $\Omega \subset \mathbf{R}^3$, then $v^R = v$ and $\eta^R = \eta$ for each $R > 2R_0$ and otherwise, they are supported in $[0,T_*] \times \Omega_R$, where $\Omega_R = \Omega \cap B_R$. Moreover it is easy to show that $\{(v^R, \eta^R)\}$ and $\{(v_t^R, \eta_t^R)\}$ converge to (v,η) and (v_t, η_t) in $C([0,T]; D_0^1 \cap D^2) \cap L^2(0,T; D^{2,q})$ and $L^2(0,T; D_0^1)$, respectively.

For each $R > 2R_0$, let $(u_0^R, p_0^R) \in \left(D_0^1 \cap D^2\right)(\Omega_R) \times H^1(\Omega_R)$ be the solution to the Stokes equations

$$-\operatorname{div}(2\mu_0^R \mathrm{d}u_0^R) + \nabla p_0^R = (\rho_0^R)^{\frac{1}{2}} g_1^R \quad \text{and} \quad \operatorname{div} u_0^R = 0 \qquad \text{in} \quad \Omega_R, \tag{87}$$

where

$$\rho_0^R = \rho_0 + R^{-3}, \quad \mu_0^R = \mu(\rho_0^R, \rho_0^R \eta^R(0)) \quad \text{and} \quad (\rho_0^R)^{\frac{1}{2}} g_1^R = \rho_0^{\frac{1}{2}} g_1.$$

Then extending (u_0^R, p_0^R) to Ω by zero outside Ω_R, we will show that

$$u_0^R \to u_0 \quad \text{in} \quad D_{0,\sigma}^1(\Omega) \quad \text{as} \quad R \to \infty. \tag{88}$$

To show this, we first observe from (8) and (87) that

$$\int_\Omega 2\mu_0^R |\mathrm{d}u_0^R|^2 \, dx = \int_\Omega \rho_0^{\frac{1}{2}} g_1 \cdot u_0^R \, dx = \int_\Omega 2\mu_0 \mathrm{d}u_0 : \mathrm{d}u_0^R \, dx. \tag{89}$$

But since $|\rho_0^R - \rho_0|_{L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}} \leq CR^{-1}$ and

$$|\mu_0^R - \mu_0|_{L^\infty} \leq M(c_2) \left(R^{-3}(1 + |\eta(0)|_{L^\infty}) + |\eta^R(0) - \eta(0)|_{L^\infty}\right), \tag{90}$$

it follows immediately from (89) that $\{u_0^R\}$ is bounded in $D_{0,\sigma}^1(\Omega)$. Hence there exists a sequence $\{R_j\}$, $R_j \to \infty$, such that $\{u_0^{R_j}\}$ converges weakly in $D_{0,\sigma}^1(\Omega)$ to a limit u_0^∞. Moreover in view of (90), we deduce from (88) that $u_0^\infty \in D_{0,\sigma}^1(\Omega)$ is a weak solution of the Stokes equations

$$-\operatorname{div}(2\mu_0 \mathrm{d}u_0^\infty) + \nabla p_0^\infty = \rho_0^{\frac{1}{2}} g_1 \quad \text{and} \quad \operatorname{div} u_0^\infty = 0 \quad \text{in} \quad \Omega$$

with some pressure p_0^∞. Since $u_0 \in D_{0,\sigma}^1(\Omega)$ is also a weak solution of the same equations, it follows from Lemma 2.6 that $u_0^\infty = u_0$ in Ω. Then by virtue of (88) and (90), we easily show that $\{u_0^{R_j}\}$ converges strongly to u_0 in $D_{0,\sigma}^1(\Omega)$. Since the above argument also shows that every subsequence of $\{u_0^R\}$ has a subsequence converging in $D^1_{0,\sigma}(\Omega)$ to the same limit u_0, we conclude that the whole sequence $\{u_0^R\}$ converges to u_0 in $D^1_{0,\sigma}(\Omega)$ as $R \to \infty$, which proves (88).

Similarly, if we denote by $\theta_0^R \in (D_0^1 \cap D^2)(\Omega_R)$ the solution of the equation

$$-\text{div}(\kappa_0^R \nabla \theta_0^R) - 2\mu_0^\delta |dv^R(0)|^2 = (\rho_0^R)^{\frac{1}{2}} g_2^R \qquad \text{in} \quad \Omega_R,$$

where $\kappa_0^R = \mu(\rho_0^R, \rho_0^R \eta^R(0))$ and $(\rho_0^R)^{\frac{1}{2}} g_2^R = \rho_0^{\frac{1}{2}} g_2$, and if we extend θ_0^R to Ω by zero outside Ω_R, then

$$\theta_0^R \to \theta_0 \quad \text{in} \quad D_0^1(\Omega) \quad \text{as} \quad R \to \infty. \tag{91}$$

Combining all the above arguments, we deduce that there exists a large number $R_1 > 2R_0$ such that if $R > R_1$, then the initial data $(\rho_0^R, u_0^R, \theta_0^R)$ satisfies

$$1 + |\rho_0^R|_{L^{\frac{3}{2}} \cap H^1 \cap W^{1,q}} + |(u_0^R, \theta_0^R)|_{D_0^1} + |(g_1^R, g_2^R)|_{L^2} < c_0$$

and the pair (v^R, η^R) satisfies the estimate (55). Hence it follows from the results in Section 3 that there exists a unique solution $(\rho^R, u^R, p^R, \theta^R)$ to the initial boundary value problem (46)-(50) with $(\rho_0, u_0, \theta_0, v, \eta, T, \Omega)$ replaced by $(\rho_0^R, u_0^R, \theta_0^R, v^R, \eta^R, T_*, \Omega_R)$ and the solution $(\rho^R, u^R, p^R, \theta^R)$ satisfies the uniform estimate (78) on R.

Let us now extend $(\rho^R, u^R, p^R, \theta^R)$ to Ω by zero outside Ω_R. Then by virtue of standard weak compactness results in Sobolev spaces, we can choose a sequence $(R_j), R_j \to \infty$, such that $(\rho^{R_j}, u^{R_j}, p^{R_j}, \theta^{R_j})$ converges to a limit (ρ, u, p, θ) in the following weak or weak-$*$ sense:

$$(u^{R_j}, \theta^{R_j}) \overset{*}{\rightharpoonup} (u, \theta) \quad \text{in} \quad L^\infty(0, T_*; D_0^1(\Omega) \cap D^2_{loc}(\Omega)),$$
$$(\rho^{R_j}, p^{R_j}) \overset{*}{\rightharpoonup} (\rho, p) \quad \text{in} \quad L^\infty(0, T_*; W^{1,q}_{loc}(\Omega) \times H^1_{loc}(\Omega))$$

and

$$u_t^{R_j}, \theta_t^{R_j}) \rightharpoonup (u_t, \theta_t) \quad \text{in} \quad L^2(0, T_*; D_0^1(\Omega)).$$

Moreover, it follows from a compactness result in [24] that for each sufficiently large $R > 2R_0$, some subsequences of $\{(u^{R_j}, \theta^{R_j})\}$ and $\{\rho^{R_j}\}$ converge to (u, θ) and ρ in $C([0, T_*]; D_0^1(\Omega_R))$ and $C([0. T_*]; L^q(\Omega_R))$, respectively. Using these convergence results together with (78), (88) and (91), we easily conclude that (ρ, u, p, θ) is a strong solution to the original problem (47)-(51) satisfying the estimate (78). Then the uniqueness and time continuity of the solution can be proved (in an easier way) following the arguments in Section 4.3 below for the corresponding results on the nonlinear problem. We may omit its details. We have completed the proof of Lemma 4.1. □

Lemma 4.1 enables us to construct the sequence $\{(\rho^k, u^k, p^k, \theta^k)\}_{k \geq 1}$ of approximate solutions. From (83) and (84), it follows that the pair $(v, \eta) = (u^0, \theta^0)$ satisfies the hypotheses of Lemma 4.1. Hence there exists a unique strong solution $(\rho, u, p, \theta) = (\rho^1, u^1, p^1, \theta^1)$ to the linearized problem(47)-(51) in $(0, T_*)$ with $(v, \eta) = (u^0, \theta^0)$, which satisfies the estimates (78). Note that

$(v,\eta) = (u^1, \theta^1)$ also satisfies the hypotheses of Lemma 4.1. Hence there exists a unique strong solution $(\rho, u, p, \theta) = (\rho^2, u^2, p^2, \theta^2)$ to the linearized problem(47)-(51) in $(0, T_*)$ with $(v,\eta) = (u^1, \theta^1)$, which satisfies the estimates (78). By an obvious inductive argument, we can define a sequence $\{(\rho^k, u^k, p^k, \theta^k)\}_{k\geq 1}$ such that for each $k \geq 1$, $(\rho, u, p, \theta) = (\rho^k, u^k, p^k, \theta^k)$ is the unique solution to the problem (47)-(51) in $(0, T_*)$ with $(v,\eta) = (u^{k-1}, \theta^{k-1})$ and satisfies the uniform bound

$$
\begin{aligned}
&\sup_{0\leq t\leq T_*} \left(|\rho^k(t)|_{L^{\frac{3}{2}}\cap H^1\cap W^{1,q}} + |\rho_t^k(t)|_{L^{\frac{3}{2}}\cap L^2\cap L^q}\right) \\
&+ \sup_{0\leq t\leq T_*} \left(|(u^k, \theta^k)(t)|_{D_0^1\cap D^2} + |p^k(t)|_{H^1}\right) \\
&+ \operatorname{ess\,sup}_{0\leq t\leq T_*} |(\sqrt{\rho^k} u^k, \sqrt{\rho^k}\theta^k)(t)|_{L^2} \\
&+ \int_0^{T_*} \left(|(u_t^k, \theta_t^k)(t)|_{D_0^1}^2 + |(u^k, \theta^k)(t)|_{D^{2,q}}^2 + |p^k(t)|_{W^{1,q}}^2\right) dt \leq \tilde{C}.
\end{aligned} \tag{92}
$$

Throughout the proof, we denote by $\tilde{C}$ a generic positive constant depending only on c_0 and the parameters of C, but independent of k.

4.2. Proof of the convergence

From now on, we show that the full sequence $\{(\rho^k, u^k, p^k, \theta^k)\}_{k\geq 1}$ converges to a solution to the original nonlinear problem (2)-(4) in a strong sense. To show this, let us define

$$
(\overline{\rho}^{k+1}, \overline{u}^{k+1}, \overline{p}^{k+1}, \overline{\theta}^{k+1}) = (\rho^{k+1} - \rho^k, u^{k+1} - u^k, p^{k+1} - p^k, \theta^{k+1} - \theta^k).
$$

Then from (47)-(49) with $(v, \eta, \rho, u, p, \theta) = (u^k, \theta^k, \rho^{k+1}, u^{k+1}, p^{k+1}, \theta^{k+1})$ and $(v, \eta, \rho, u, p, \theta) = (u^{k-1}, \theta^{k-1}, \rho^k, u^k, p^k, \theta^k)$, we derive the difference equations

$$
\overline{\rho}_t^{k+1} + u^k \cdot \nabla\overline{\rho}^{k+1} + \overline{u}^k \cdot \nabla\rho^k = 0, \tag{93}
$$

$$
\begin{aligned}
&\rho^{k+1}\overline{u}_t^{k+1} + \rho^{k+1}u^k \cdot \nabla\overline{u}^{k+1} - \operatorname{div}\left(2\mu_0 \mathrm{d}\overline{u}^{k+1}\right) + \nabla\overline{p}^{k+1} \\
= \ &\overline{\rho}^{k+1}(f - u_t^k - u^{k-1}\cdot\nabla u^k) - \rho^{k+1}\overline{u}^k \cdot \nabla u^k \\
&+\operatorname{div}\left(2(\mu^{k+1} - \mu^k)\mathrm{d}u^k\right) + \operatorname{div}\left(2(\mu^k - \mu_0)\mathrm{d}\overline{u}^k\right),
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
&c_v^{k+1}\rho^{k+1}\left(\overline{\theta}_t^{k+1} + u^k \cdot \nabla\overline{\theta}^{k+1}\right) - \operatorname{div}\left(\kappa_0 \nabla\overline{\theta}^{k+1}\right) \\
= \ &\overline{\rho}^{k+1}\left(h - c_v^k(\theta_t^k + u^{k-1}\cdot\nabla\theta^k)\right) - c_v^k\rho^{k+1}\overline{u}^k \cdot \nabla\theta^k \\
&-(c_v^{k+1} - c_v^k)\rho^{k+1}(\theta_t^k + u^k\cdot\nabla\theta^k) \\
&+\operatorname{div}\left((\kappa^{k+1} - \kappa^k)\nabla\theta^k\right) + \operatorname{div}\left((\kappa^k - \kappa_0)\nabla\overline{\theta}^k\right) \\
&+2(\mu^{k+1} - \mu^k)|\mathrm{d}u^k|^2 + 2\mu^k(|\mathrm{d}u^k|^2 - |\mathrm{d}u^{k-1}|^2),
\end{aligned} \tag{95}
$$

where $\mu^{k+1} = \mu(\rho^{k+1}, \rho^{k+1}\theta^k)$, etc.

First, multiplying (93) by $\operatorname{sgn}(\overline{\rho}^{k+1})|\overline{\rho}^{k+1}|^{\frac{1}{2}}$ and integrating over Ω, we obtain

$$\frac{d}{dt}\int |\overline{\rho}^{k+1}|^{\frac{3}{2}}\,dx \le C\int |\nabla u^k||\overline{\rho}^{k+1}|^{\frac{3}{2}} + |\nabla\rho^k||\overline{u}^k||\overline{\rho}^{k+1}|^{\frac{1}{2}}\,dx$$
$$\le \; C|\nabla u^k|_{W^{1,q}}|\overline{\rho}^{k+1}|^{\frac{3}{2}}_{L^{\frac{3}{2}}} + C|\rho^k|_{H^1}|\nabla\overline{u}^k|_{L^2}|\overline{\rho}^{k+1}|^{\frac{1}{2}}_{L^{\frac{3}{2}}}.$$

Hence multiplying this by $|\overline{\rho}^{k+1}|^{\frac{1}{2}}_{L^{\frac{3}{2}}}$, we have

$$\frac{d}{dt}|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}} \le A^k_\varepsilon(t)|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}} + \varepsilon|\nabla\overline{u}^k|^2_{L^2}, \tag{96}$$

where $A^k_\varepsilon(t) = C|\nabla u^k(t)|_{W^{1,q}} + \varepsilon^{-1}C|\rho^k(t)|^2_{H^1}$. Notice from the uniform bound (92) that $\int_0^t A^k_\varepsilon(s)\,ds \le \tilde{C}\left(1+\varepsilon^{-1}t\right)$ for all $k\ge 1$ and $t\in[0,T_*]$. In a similar manner, we can also show that

$$\frac{d}{dt}|\overline{\rho}^{k+1}|^2_{L^2} \le B^k_\varepsilon(t)|\overline{\rho}^{k+1}|^2_{L^2} + \varepsilon|\nabla\overline{u}^k|^2_{L^2} \tag{97}$$

for some $B^k_\varepsilon(t)\in L^1(0,T_*)$ such that $\int_0^t B^k_\varepsilon(s)\,ds \le \tilde{C}\left(1+\varepsilon^{-1}t\right)$ for $k\ge 1$ and $0\le t\le T_*$.

Next, multiplying (94) by $\overline{u}^{k+1}$, integrating over Ω and recalling that

$$\rho^{k+1}_t + \operatorname{div}(\rho^{k+1}u^k) = 0 \quad \text{in} \quad \Omega\times(0,T_*),$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^{k+1}|\overline{u}^{k+1}|^2\,dx + \int 2\mu_0|d\overline{u}^{k+1}|^2\,dx\\
\le\;& \int |\overline{\rho}^{k+1}||f-u^k_t-u^{k-1}\cdot\nabla u^k||\overline{u}^{k+1}| + |\rho^{k+1}||\overline{u}^k||\nabla u^k||\overline{u}^{k+1}|\\
&+2|\mu^{k+1}-\mu^k||\nabla u^k||\nabla\overline{u}^{k+1}| + 2|\mu^k-\mu_0||\nabla\overline{u}^k||\nabla\overline{u}^{k+1}|\,dx.
\end{aligned} \tag{98}$$

In view of Hölder and Sobolev inequalities, we easily show that the first two terms in the right hand side of (98) are bounded above by

$$C^k|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}\cap L^2} + \tilde{C}\varepsilon^{-1}|\sqrt{\rho^{k+1}}\overline{u}^{k+1}|^2_{L^2} + \varepsilon|\nabla\overline{u}^k|^2_{L^2} + \frac{1}{4}\underline{\mu}|\nabla\overline{u}^{k+1}|^2_{L^2}$$

for some $C^k(t)\in L^1(0,T_*)$ such that $\int_0^{T_*} C^k(t)\,dt \le \tilde{C}$ for all $k\ge 1$, where $\underline{\mu} = \inf_\Omega \mu_0$. The third term is bounded above by

$$\tilde{C}\int \left(|\overline{\rho}^{k+1}| + |\overline{\rho}^{k+1}||\theta^k| + \rho^k|\overline{\theta}^k|\right)|\nabla u^k||\nabla\overline{u}^{k+1}|\,dx$$
$$\le\; D^k|\overline{\rho}^{k+1}|^2_{L^2} + \tilde{C}\delta^{-1}|\sqrt{\rho^k}\overline{\theta}^k|^2_{L^2} + \delta|\nabla\overline{\theta}^k|^2_{L^2} + \frac{1}{4}\underline{\mu}|\nabla\overline{u}^{k+1}|^2_{L^2}$$

for some $D^k(t)\in L^1(0,T_*)$ such that $\int_0^{T_*} D^k(t)\,dt \le \tilde{C}$ for $k\ge 1$. The last term is bounded above by

$$\tilde{C}|\mu^k-\mu_0|^2_{L^\infty}|\nabla\overline{u}^k|^2_{L^2} + \frac{1}{4}\underline{\mu}|\nabla\overline{u}^{k+1}|^2_{L^2}.$$

Substituting all the above estimates into (98), we have

$$
\begin{aligned}
&\frac{d}{dt}\int \rho^{k+1}|\overline{u}^{k+1}|^2\,dx + \underline{\mu}|\nabla\overline{u}^{k+1}|_{L^2}^2 \\
(99)\qquad \leq\ & E_\varepsilon^k\left(|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}}^2 + |\overline{\rho}^{k+1}|_{L^2}^2 + |\sqrt{\rho^{k+1}}\overline{u}^{k+1}|_{L^2}^2\right) \\
&+\tilde{C}\delta^{-1}|\sqrt{\rho^k}\overline{\theta}^k|_{L^2}^2 + 2\delta|\nabla\overline{\theta}^k|_{L^2}^2 + (\tilde{C}|\mu^k-\mu_0|_{L^\infty}^2+\varepsilon)|\nabla\overline{u}^k|_{L^2}^2
\end{aligned}
$$

for some $E_\varepsilon^k(t)\in L^1(0,T_*)$ such that $\int_0^t E_\varepsilon^k(s)\,ds \leq \tilde{C}\left(1+\varepsilon^{-1}t\right)$ for $k\geq 1$ and $0\leq t\leq T_*$.

Finally, multiplying (95) by $\overline{\theta}^{k+1}$ and integrating over Ω, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int c_v^{k+1}\rho^{k+1}|\overline{\theta}^{k+1}|^2\,dx + \int \kappa_0|\nabla\overline{\theta}^{k+1}|^2\,dx \\
\leq\ & \int |(c_v)_t^{k+1}|\rho^{k+1}|\overline{\theta}^{k+1}|^2 + |\nabla c_v^{k+1}|\rho^{k+1}|u^k||\overline{\theta}^{k+1}|^2 \\
&+|\overline{\rho}^{k+1}||h-c_v^k(\theta_t^k+u^{k-1}\cdot\nabla\theta^k)||\overline{\theta}^{k+1}| \\
&+|c_v^k|\rho^{k+1}|\overline{u}^k||\nabla\theta^k||\overline{\theta}^{k+1}| \\
(100)\qquad &+|c_v^{k+1}-c_v^k|\rho^{k+1}(|\theta_t^k|+|u^k||\nabla\theta^k|)|\overline{\theta}^{k+1}| \\
&+|\kappa^{k+1}-\kappa^k||\nabla\theta^k||\nabla\overline{\theta}^{k+1}| \\
&+2|\mu^{k+1}-\mu^k||\nabla u^k|^2|\overline{\theta}^{k+1}| \\
&+|\kappa^k-\kappa_0||\nabla\overline{\theta}^k||\nabla\overline{\theta}^{k+1}| \\
&+2\mu^k(|\nabla u^k|+|\nabla u^{k-1}|)|\nabla\overline{u}^k||\overline{\theta}^{k+1}|\,dx.
\end{aligned}
$$

Recall from (63) and (64) that

$$
c_v^{k+1}\geq \tilde{C}^{-1} \quad\text{and}\quad \int_0^{T_*}\left(|(c_v)_t^{k+1}(t)|_{L^3}^2 + |\nabla c_v^{k+1}(t)|_{L^3}^2\right)dt \leq \tilde{C}
$$

for all $k\geq 1$. Hence arguing as before, we can show that the right hand side of (100) is bounded above by

$$
\begin{aligned}
&F^k\left(|\overline{\rho}^{k+1}|_{L^{\frac{3}{2}}\cap L^2}^2 + |\sqrt{c_v^{k+1}\rho^{k+1}}\overline{\theta}^{k+1}|_{L^2}^2\right) + \tilde{C}\delta^{-1}|\sqrt{c_v^k\rho^k}\theta^k|_{L^2}^2 \\
&+\left(\tilde{C}|\kappa^k-\kappa_0|_{L^\infty}^2+\delta\right)|\nabla\overline{\theta}^k|_{L^2}^2 + \tilde{C}|\nabla\overline{u}^k|_{L^2}^2 + \frac{1}{2}\underline{\kappa}|\nabla\overline{\theta}^{k+1}|_{L^2}^2
\end{aligned}
$$

for some $F^k(t)\in L^1(0,T_*)$ such that $\int_0^{T_*}F^k(s)\,ds\leq\tilde{C}$ for all $k\geq 1$, where $\underline{\kappa}=\inf_\Omega\kappa_0$. Substituting this estimate into (100), we obtain

$$
(101)\qquad \frac{d}{dt}\int c_v^{k+1}\rho^{k+1}|\overline{\theta}^{k+1}|^2\,dx + \underline{\kappa}\int|\nabla\overline{\theta}^{k+1}|^2\,dx
$$

$$\leq \quad F^k \left(|\overline{\rho}^{k+1}|^2_{L^{\frac{3}{2}}} + |\overline{\rho}^{k+1}|^2_{L^2} + |\sqrt{c_v^{k+1}\rho^{k+1}}\overline{\theta}^{k+1}|^2_{L^2} \right)$$
$$+\tilde{C}\delta^{-1}|\sqrt{c_v^k\rho^k}\theta^k|^2_{L^2} + \left(\tilde{C}|\kappa^k - \kappa_0|^2_{L^\infty} + 2\delta\right)|\nabla\overline{\theta}^k|^2_{L^2} + \tilde{C}|\nabla\overline{u}^k|^2_{L^2}.$$

Now for a small fixed $\varepsilon > 0$, let us define φ^k and ψ^k by

$$\varphi^k(t) = |\overline{\rho}^k(t)|^2_{L^{\frac{3}{2}}} + |\overline{\rho}^k(t)|^2_{L^2} + |\sqrt{\rho^k}\overline{u}^k(t)|^2_{L^2} + \varepsilon\tilde{C}^{-1}|\sqrt{c_v^k\rho^k}\overline{\theta}^k(t)|^2_{L^2}$$

and

$$\psi^k(t) = \underline{\mu}|\nabla\overline{u}^k(t)|^2_{L^2} + \varepsilon\tilde{C}^{-1}\underline{\kappa}|\nabla\overline{\theta}^k(t)|^2_{L^2}.$$

Then combining (96), (97), (99) and (101), we have

$$\begin{aligned} &\frac{d}{dt}\varphi^{k+1}(t) + \psi^{k+1}(t) \\ (102) \qquad \leq\ & G_\varepsilon^k(t)\varphi^{k+1}(t) + \tilde{C}\delta^{-1}\varepsilon^{-1}\varphi^k(t) \\ &+\tilde{C}\varepsilon^{-1}\left(|\mu^k(t) - \mu_0|^2_{L^\infty} + |\kappa^k(t) - \kappa_0|^2_{L^\infty} + \delta\right)\psi^k(t) \end{aligned}$$

for some $G_\varepsilon^k(t) \in L^1(0,T_*)$ such that $\int_0^t G_\varepsilon^k(s)\,ds \leq \tilde{C}\left(1 + \varepsilon^{-1}t\right)$ for $k \geq 1$ and $0 \leq t \leq T_*$. But by virtue of (60), (74), (75) and Lemma 2.1, there is a small number $\gamma = \gamma(\tilde{C}) \in (0,1)$ such that

$$|\mu^k(t) - \mu_0|^2_{L^\infty} + |\kappa^k(t) - \kappa_0|^2_{L^\infty} \leq \tilde{C}t^\gamma \quad \text{for} \quad t \in [0,T_*].$$

Hence taking $\delta = \varepsilon^2$ in (102), we obtain

$$\frac{d}{dt}\varphi^{k+1}(t) + \psi^{k+1}(t) \leq G_\varepsilon^k(t)\varphi^{k+1}(t) + \tilde{C}\varepsilon^{-3}\varphi^k(t) + \tilde{C}\left(\varepsilon^{-1}t^\gamma + \varepsilon\right)\psi^k(t),$$

which implies in view of Gronwall's inequality that

$$\begin{aligned} &\varphi^{k+1}(t) + \int_0^t \psi^{k+1}(s)\,ds \\ \leq\ & \tilde{C}\left(\varepsilon^{-3}\int_0^t \varphi^k(s)\,ds + \left(\varepsilon^{-1}t^\gamma + \varepsilon\right)\int_0^t \psi^k(s)\,ds\right)\exp\left(\varepsilon^{-1}\tilde{C}t\right) \end{aligned}$$

for $0 \leq t \leq T_*$. Therefore, if we choose ε and T_1 so small that

$$\tilde{C}\left(\varepsilon^{-1}T_1^\gamma + \varepsilon\right) \leq \frac{1}{4}, \quad 0 < T_1 < T_* \quad \text{and} \quad \exp\left(\varepsilon^{-1}\tilde{C}T_1\right) \leq 2,$$

then we deduce that

$$\varphi^{k+1}(t) + \int_0^t \psi^{k+1}(s)\,ds \leq \tilde{C}\int_0^t \varphi^k(s)\,ds + \frac{1}{2}\int_0^t \psi^k(s)\,ds$$

for $k \geq 1$ and $0 \leq t \leq T_1$. Fixing a large $K > 1$, we sum this over $1 \leq k \leq K$ to yield

$$\sum_{k=1}^K \left(\varphi^{k+1}(t) + \int_0^t \psi^{k+1}(s)\,ds\right) \leq \tilde{C} + \tilde{C}\int_0^t \left(\sum_{k=1}^K \psi^{k+1}(s)\right)ds.$$

Using Gronwall's inequality again, we finally have

$$\sum_{k=1}^{\infty}\left(\sup_{0\le t\le T_1}\varphi^{k+1}(t)+\int_0^{T_1}\psi^{k+1}(t)\,dt\right)\le \tilde{C},$$

which implies obviously that

$$\rho^k\to\rho \text{ in } L^\infty(0,T_1;L^{\frac{3}{2}}\cap L^2) \quad\text{and}\quad (u^k,\theta^k)\to(u,\theta) \text{ in } L^2(0,T_1;D_0^1)$$

as $k\to\infty$ for some limits ρ, u and θ. By virtue of this strong convergence, one easily verify that (ρ,u,θ) is a weak solution to the original nonlinear problem (1)-(5) for some pressure p. Moreover, it follows from the uniform bound (92) that

$$\begin{gathered}\rho\in L^\infty(0,T_1;L^{\frac{3}{2}}\cap H^1\cap W^{1,q}),\quad \rho_t\in L^\infty(0,T_1;L^{\frac{3}{2}}\cap L^q),\\ (u,\theta)\in L^\infty(0,T_1;D_0^1\cap D^2)\cap L^2(0,T_*;D^{2,q}),\\ p\in L^\infty(0,T_1;H^1)\cap L^2(0,T_1,W^{1,q}),\\ (u_t,\theta_t)\in L^2(0,T_1;D_0^1)\quad\text{and}\quad(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty(0,T_1;L^2).\end{gathered}\tag{103}$$

4.3. Continuity and uniqueness

The proof of the continuity of ρ is standard and omitted; see the papers [5, 6] and [7]. To prove the continuity of (u,p,θ), we first observe that

$$(u,\theta)\in C([0,T_1];D_0^1)\cap C([0,T_1];D^2-\text{weak}).$$

Then by virtue of Sobolev inequality, we deduce that

$$(u,\theta)\in C([0,T_1];D_0^1\cap D^{1,q_1})\hookrightarrow C([0,T_1]\times\overline{\Omega}),$$

which implies in particular that

$$(\mu,\kappa,c_v)\in C([0,T_1]\times\overline{\Omega})\quad\text{and}\quad(\nabla\mu,\nabla\kappa)\in C([0,T_1];L^{q_1}).$$

Here we denote $q_1=\min(q,4)>3$.

For the continuity of (u,p), we first observe that $(\rho u_t)_t\in L^2(0,T_*;H_\sigma^{-1})$. In fact, from (3) we have

$$\begin{aligned}\frac{d}{dt}\langle\rho u_t,w\rangle_{H_\sigma}&=\langle\rho(f_t-u_t\cdot\nabla u-u\cdot\nabla u_t)+\rho_t(f-u\cdot\nabla u),w\rangle_{H_\sigma}\\&\quad+\langle 2\mu_t du+2\mu du_t,\nabla w\rangle_{L^2}\le H(t)|w|_{H_0^1}\end{aligned}$$

for some positive function $H(t)\in L^2(0,T)$ and for all $w\in H^1_{0,\sigma}$. Hence it follows from the well-known result (see Chapter 3 in [27]) that $(\rho u_t)_t\in L^2(0,T_*;H_\sigma^{-1})$ and $\frac{d}{dt}\langle\rho u_t,w\rangle_{H_\sigma}=\langle(\rho u_t)_t,w\rangle_{H_\sigma}$. Then since

$$\rho u_t\in L^\infty(0,T_*;L^2)\quad\text{and}\quad L^2\hookrightarrow H_\sigma^{-1},$$

it follows from the standard embedding results that $\rho u_t\in C([0,T_*],H_\sigma^{-1})\cap C([0,T_*];L^2-\text{weak})$. Moreover, from the identity (66) with u instead of v, we deduce that the function $t\mapsto|\sqrt{\rho}u_t(t)|_{L^2}^2$ is continuous on $[0,T_*]$. Therefore, recalling that $\rho\in C([0,T_*];W^{1,q})$, we conclude that $\rho u_t\in C([0,T_*],L^2)$.

Now observe that for each $t \in [0, T_*]$, $u = u(t) \in D^1_{0,\sigma} \cap D^2$ is a solution of the stationary Stokes equations

$$-\operatorname{div}(2\mu \mathrm{d}u) + \nabla p = F \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega,$$

where $F = \rho f - \rho u_t - \rho u \cdot \nabla u \in C([0, T_*], L^2)$. The continuity of F in L^2 follows from the already known continuity properties of ρ and u. Then following the argument for the proof of Lemma 2.8, we easily deduce the continuity of (u, p). For details see Section 3.2 of [6].

Now we prove the uniqueness. Suppose that $\rho_i, u_i, p_i, \theta_i (i = 1, 2)$ are the solutions of the equations (2)-(5). Then letting $\overline{\rho} = \rho_1 - \rho_2$, $\overline{u} = u_1 - u_2$, $\overline{p} = p_1 - p_2$ and $\overline{\theta} = \theta_1 - \theta_2$, the differences of solutions satisfy the following equations:

$$\overline{\rho}_t = \nabla \overline{\rho} \cdot u_1 + \nabla \rho_2 \cdot \overline{u}, \tag{104}$$

$$\begin{aligned} &\rho_1 \overline{u}_t + \rho_1 u_1 \cdot \nabla \overline{u} + \rho_1 \overline{u} \cdot \nabla u_2 - \operatorname{div}(2\mu_1 \mathrm{d}\overline{u}) + \nabla \overline{p} \\ &= \overline{\rho} f - \overline{\rho}(u_2)_t - \overline{\rho} u_2 \cdot \nabla u_2 + \operatorname{div}(2\overline{\mu} \mathrm{d}u_2), \end{aligned} \tag{105}$$

$$\begin{aligned} &c_{v,1}(\rho_1 \overline{\theta}_t + \rho_1 u_t \cdot \nabla \overline{\theta}) - \operatorname{div}(\kappa_1 \nabla \overline{\theta}) \\ &= 2\overline{\mu} |\mathrm{d}u_1|^2 + 2\mu_1 (|\mathrm{d}u_1|^2 - |\mathrm{d}u_2|^2) + \overline{\rho} h \\ &\quad - c_{v,1}(\overline{\rho}(\theta_2)_t + \overline{\rho} u_1 \cdot \nabla \theta_2 + \rho_2 \overline{u} \cdot \nabla \theta_2) \\ &\quad - \overline{c_v}(\rho_2(\theta_2)_t + \rho_2 u_2 \cdot \nabla \theta_2) + \operatorname{div}(\overline{\kappa} \nabla \theta_2), \end{aligned} \tag{106}$$

where

$$\begin{aligned} \overline{\mu} = \mu_1 - \mu_2, \quad \overline{\kappa} = \kappa_1 - \kappa_2, \quad \overline{c_v} = c_{v,1} - c_{v,2} \\ \mu_i = \mu(\rho_i, \rho_i \theta_i) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \text{etc.} \end{aligned}$$

By the similar method to the proof of convergence in Section 4.2, we can easily derive that

$$\begin{aligned} &\frac{d}{dt}(|\overline{\rho}|_{L^{\frac{3}{2}}} + |\overline{\rho}|^2_{L^2}) \le \tilde{C}(|\overline{\rho}|_{L^{\frac{3}{2}}} + |\overline{\rho}|^2_{L^2}) + (4\tilde{C})^{-1} |\nabla \overline{u}|^2_{L^2}, \\ &\frac{d}{dt} |\sqrt{\rho_1} \overline{u}|^2_{L^2} + \tilde{C}^{-1} |\nabla \overline{u}|^2_{L^2} \\ \le\ &E(t)(|\overline{\rho}|^2_{L^{\frac{3}{2}}} + |\overline{\rho}|^2_{L^2} + |\sqrt{\rho_1} |\overline{u}|^2_{L^2} + |\sqrt{\rho_1} \overline{\theta}|^2_{L^2}), \\ &\frac{d}{dt} |\sqrt{c_{v,1} \rho_1} \overline{\theta}|^2_{L^2} + \tilde{C}^{-1} |\nabla \overline{\theta}|^2_{L^2} \\ \le\ &F(t)(|\overline{\rho}|^2_{L^{\frac{3}{2}}} + |\overline{\rho}|^2_{L^2} + |\sqrt{\rho_1} \overline{\theta}|^2_{L^2}) + \tilde{C} |\nabla \overline{u}|^2_{L^2}, \end{aligned} \tag{107}$$

where E and F are some functions such that $\int_0^{T_*} (E(t) + F(t))\, dt \le \tilde{C}$. Letting

$$\Psi(t) = |\overline{\rho}|_{L^{\frac{3}{2}}} + |\overline{\rho}|^2_{L^2} + |\sqrt{\rho_1} \overline{u}|^2_{L^2} + \frac{c_{v*}}{4\tilde{C}^2} |\sqrt{\rho_1} \overline{\theta}|^2_{L^2}$$

and combining (107), we obtain

$$\Psi(t) \le \tilde{C} \int_0^t \Psi(s)\, ds$$

for all $t \in [0, T_*]$. Thus Gronwall's inequality yields that

$$\overline{\rho} = 0, \quad \rho_1 \overline{u} = 0, \quad \rho_1 \overline{\theta} = 0.$$

Therefore the uniqueness follows from (107).

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