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Citation for published version (APA):
Prokert, G. (1996). *Existence results for Hele-Shaw flow driven by surface tension*. (RANA : reports on applied and numerical analysis; Vol. 9619). Technische Universiteit Eindhoven.

Document status and date:
Published: 01/01/1996

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Download date: 23. Aug. 2022

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 96–19
October 1996

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by

G. Prokert

Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507

Existence results for Hele-Shaw flow driven by surface tension

G. Prokert

Technical University Eindhoven, Faculty of Mathematics and Computer Science, P.O. Box 513, 5600 MB Eindhoven, The Netherlands; e-mail prokert@win.tue.nl

Abstract: This paper addresses short-time existence and uniqueness of a solution to the N-dimensional Hele-Shaw flow problem with surface tension as driving mechanism. Global existence in time and exponential decay of the solution near equilibrium are also proved. The results are obtained in Sobolev spaces H^s with sufficiently large s. The main tools are perturbations of a fixed reference domain, linearization with respect to these perturbations, a quasilinearization argument based on a geometric invariance property, and a priori energy estimates.

1 Introduction

This article is concerned with the Hele-Shaw moving boundary flow problem where the motion is driven by surface tension only. It occurs in the modeling of flow in porous media or in a narrow confinement between two parallel plates. Although this problem is originally two-dimensional, it is of interest in higher dimensions, too, both for its own sake and because of its applications to physical problems: The problem (1.1), (1.2) below is also obtained in the description of the motion of phase boundaries by capillarity and volume diffusion in metallurgy [15]. We will deal here with the simplest version of the one-phase problem which can be described as follows:

Let $\Omega(t) \subset \mathbb{R}^N$ be the domain occupied by the fluid at time t and let v and p be the velocity and pressure fields, respectively, satisfying

$$\left.\begin{array}{rcl} v & = & -\nabla p \\ \operatorname{div} v & = & 0 \end{array}\right\} \qquad \text{in } \Omega(t),$$

$$p = -\kappa(t) \qquad \text{on } \Gamma(t) = \partial\Omega(t).$$

We denote by $\kappa(t)$ the $((N-1)$-fold) mean curvature with the sign chosen such that $\kappa(t)$ is negative if $\Omega(t)$ is convex. Note that

$$\kappa(t)n(t) = \Delta_{\Gamma(t)}x$$

where $n(t)$ and $\Delta_{\Gamma(t)}$ are the outer normal vector and the Laplace-Beltrami operator on Γ_0, respectively. x is the $\mathbb{R}^N$-valued function that assigns to each point of $\Gamma(t)$ its coordinate vector, and $\Delta_{\Gamma(t)}$ is to be applied componentwise.

As usual in free boundary flow problems, the motion of the boundary of the liquid domain is determined by prescribing its normal velocity V_n as

$$V_n = v \cdot n(t)$$

which is an equivalent formulation for the demand that the set of the "particles" at the surface does not change in time.

Setting $u = -p$ we have to consider the moving boundary problem

$$\left.\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega(t) \\ u & = & \kappa(t) & \text{on } \Gamma(t), \end{array}\right\} \tag{1.1}$$

$$V_n = \frac{\partial u}{\partial n(t)} \quad \text{on } \Gamma(t). \tag{1.2}$$

For $N = 2$, this problem as well as various related ones have been studied by methods from complex function theory (see [12] and the references therein). In [5, 6] an evolution equation for a conformal mapping onto $\Omega(t)$ is derived and its global solvability in time and exponential decay are shown in suitable spaces of analytic functions under the assumption that $\Omega(t)$ is the complement of a bounded domain which is near a circle. Moreover, a short-time existence proof for a weak solution is given in [8] where $\Omega(t)$ is the domain below the graph of a function mapping $\mathbb{R}$ to $\mathbb{R}$. Recently, also for $N = 2$, short-time existence of a solution as well as stability of equilibria for a related two-phase problem have been proved [4].

The approach we use here is essentially of geometric nature. We consider small perturbations of a fixed smooth reference domain Ω_0 near the initial domain and investigate the dependence of the solution of (1.1) on these perturbations. The linearization of the corresponding operator is shown to be a coercive pseudodifferential operator. (This technique is also used in [1, 2] for stationary free boundary problems for the Navier-Stokes equations.) Assuming that Ω_0 is strictly star-shaped, a quasilinearization argument based on the invariance of (1.1) with respect to rotations enables one to control the nonlinearity, and a short-time existence and uniqueness proof can be given employing energy estimates in Sobolev spaces of sufficiently high order. Finally, global existence of the solution in time and exponential decay of this solution near equilibrium are shown.

The problem of Stokes flow driven by surface tension can be treated in completely analogous manner [11].

We will start the investigation by listing some useful elementary properties of the evolution of $\Omega(t)$ by (1.1), (1.2):

Lemma 1 *(Properties of Hele-Shaw flow driven by surface tension)*

(i) Let $\{\Omega(t)\}$ *be a (sufficiently smooth) family of domains evolving according to* (1.1), (1.2). *Then the quantities*

$$V = \int_{\Omega(t)} dx, \qquad M = \int_{\Omega(t)} x \, dx$$

representing the volume and the center of gravity of $\Omega(t)$, respectively, are independent of t, and the surface area

$$A(t) = \int\limits_{\Gamma(t)} d\Gamma(t)$$

is nonincreasing in t.

(ii) A domain Ω yields a stationary solution of (1.1), (1.2) *iff its boundary has constant mean curvature.*

Proof: (i) can be shown directly by calculating, using integration by parts,

$$\begin{aligned}
\frac{dV}{dt} &= \int\limits_{\Omega(t)} \operatorname{div} v \, dx = 0, \\
\frac{dM}{dt} &= \int\limits_{\Omega(t)} v \, dx + \int\limits_{\Omega(t)} x \operatorname{div} v \, dx = -\int\limits_{\Omega(t)} \nabla p \, dx \\
&= \int\limits_{\Gamma(t)} \kappa n(t) \, d\Gamma(t) = \int\limits_{\Gamma(t)} \Delta_{\Gamma(t)} x \, d\Gamma(t) = 0, \\
\frac{dA(t)}{dt} &= -\int\limits_{\Gamma(t)} \kappa(t) n(t) \cdot v \, d\Gamma(t) = -\int\limits_{\Gamma(t)} u \frac{\partial u}{\partial n(t)} \, d\Gamma(t) \\
&= -\int\limits_{\Omega(t)} |\nabla u|^2 \, dx \leq 0. \qquad (1.3)
\end{aligned}$$

(ii) The "if"-part is obvious. Let Ω yield a stationary solution, i.e. suppose that the normal component of v vanishes on $\Gamma = \partial\Omega$. Hence

$$\int_\Gamma \kappa n \cdot v \, d\Gamma = 0$$

and this implies by (1.3) that u and hence κ are constant. ■

2 Perturbations of the domain and analytic expansions

Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded, simply-connected C^∞-domain with boundary Γ_0 and outer unit normal n. The (inverse) trace theorem ensures that for all real $s > \frac{1}{2}$ the trace operator T is continuous from $H^s(\Omega_0)$ to $H^{s-\frac{1}{2}}(\Gamma_0)$ and has a continuous right inverse which will be denoted by T^{-1}. Furthermore, let $\zeta \in (C^\infty(\Gamma_0))^N$ be a fixed vector-valued function on Γ_0 such that

$$\gamma(\xi) = \zeta(\xi) \cdot n(\xi) > 0 \qquad \forall \xi \in \Gamma_0. \qquad (2.1)$$

Lemma 2 *(Description of perturbed domains)*

Let $s > 2 + \frac{N}{2}$. There is a $\delta_0 > 0$ depending only on Ω_0, ζ, and s such that for all $r \in B_0(\delta_0, H^s(\Gamma_0))$ the following holds:

(i) The set

$$\Gamma_r = \{\xi + \zeta(\xi)r(\xi) \mid \xi \in \Gamma_0\}$$

is the boundary of a simply connected domain Ω_r.

(ii) There is a global C^2-diffeomorphism $z = z(r)$ mapping Ω_0 onto Ω_r such that $z \in \left(H^{s+\frac{1}{2}}(\Omega_0)\right)^N$ and

$$\|z - \mathrm{id}\|_{(C^2(\Omega_0))^N} \leq C\|r\|_s^{\Gamma_0}$$

with C independent of r.

Proof: (i) From [10], Appendix, Lemma 1 follows that there is a $\delta_1 > 0$ and a C^∞-diffeomorphism $\phi = (\phi_1, \phi_2)$ mapping the domain

$$M_{\delta_1} = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x, \Gamma_0) < \delta_1\}$$

onto $\Gamma_0 \times (-\delta_1, \delta_1)$ with

$$\begin{aligned} \phi_2(x) &= \pm\,\mathrm{dist}(x, \Gamma_0), \\ x &= \phi_1(x) + \phi_2(x)n(\phi_1(x)) \end{aligned}$$

for all $x \in M_{\delta_1}$, where the sign is negative for $x \in \Omega$ and positive for $x \neq \overline{\Omega}$. Consider now for $\xi \in \Gamma_0$ and real t with $|t|$ sufficiently small the mapping

$$\Phi(\xi, t) = \phi(\xi + t\zeta(\xi)).$$

From $\Phi_1(\xi, 0) = \xi$, $\Phi_2(\xi, 0) = 0$, and $D_2\Phi_2(\xi, 0) = \gamma(\xi) \neq 0$ follows by the local diffeomorphism theorem and a compactness argument that there is a $\delta_2 > 0$ such that Φ maps $\Gamma_0 \times (-\delta_2, \delta_2)$ into $\Gamma_0 \times (-\delta_1, \delta_1)$ as a C^∞-diffeomorphism.

Consequently, the mapping $\phi^{-1} \circ \Phi$ from $\Gamma_0 \times (-\delta_2, \delta_2)$ into M_{δ_1} is a C^∞-diffeomorphism, and for any fixed $r \in B_0(\delta_2, C(\Gamma_0))$ the set Γ_r is homeomorph to Γ_0. The assertion follows now from the continuity of the embedding $H^s(\Gamma_0) \hookrightarrow C(\Gamma_0)$.

(ii) Using the above parametrizations of the neighborhood of Γ_0 and compactness arguments it is not hard to show that there are positive constants δ_3 and C such that for all Frechet-differentiable mappings g of Ω_0 into an arbitrary normed space E and for all $x_0, x_1 \in \Omega_0$ $\|x_1 - x_0\| < \delta_1$ the estimate

$$\|g(x_1) - g(x_0)\|_E \leq C \sup_{x \in \Omega_0 \cap B_{x_0}(C\|x_1 - x_0\|, E)} \|g'(x)\|\,\|x_1 - x_0\| \tag{2.2}$$

holds, which in particular implies that g is Lipschitz continuous.

We construct z by setting $z(r) = T^{-1}(r\zeta) + \mathrm{id}$. The estimate for $\|z(r) - \mathrm{id}\|_{(C^2(\Omega_0))^N}$ is a consequence of the continuity of the embedding $H^{s+\frac{1}{2}}(\Omega_0) \hookrightarrow$

$C^2(\Omega)$. This estimate ensures for sufficiently small δ_0 that $z(r)$ is a local diffeomorphism, it remains to show that it is also global, i.e. that $z(x_1) = z(x_0)$ implies $x_1 = x_0$ for all $x_0, x_1 \in \Omega_0$. For this purpose, the equation $z(x) = z(x_0)$ may be rewritten equivalently as

$$x = S(x) := x - z'(x_0)^{-1}(z(x) - z(x_0)) \tag{2.3}$$

The mapping S is differentiable in all $x \in \Omega_0$ and has the derivative

$$S'(x) = I - z'(x_0)^{-1}z'(x) = z'(x_0)^{-1}(z'(x_0) - z'(x)).$$

According to the above remark, $z \in \left(C^2(\Omega)\right)^N$ implies Lipschitz-continuity of z', hence

$$\|S'(x)\| \leq C\|z'(x_0) - z'(x)\| \leq C\|x - x_0\|.$$

Consequently,

$$\begin{aligned} \sup_{x\in\Omega_0\cap B_{x_0}(C\|x_1-x_0\|,\mathbb{R}^N)} \|S'(x)\| &\leq C \sup_{x\in\Omega_0\cap B_{x_0}(C\|x_1-x_0\|,\mathbb{R}^N)} \|x - x_0\| \\ &\leq C\|x_1 - x_0\|. \end{aligned}$$

Moreover, assuming $z(x_1) = z(x_0)$,

$$\|x_1 - x_0\| \leq \|x_1 - z(x_1)\| + \|z(x_0) - x_0\| \leq C\delta_0, \tag{2.4}$$

hence for sufficiently small δ_0 (2.2) may be applied to S and this yields

$$\|x_1 - x_0\| = \|S(x_1) - S(x_0)\| \leq C\|x_1 - x_0\|^2,$$

i.e. $x_0 = x_1$ or $\|x_1 - x_0\| \geq C^{-1}$, but the latter of these two possibilities is in contradiction with (2.4) for small δ_0. ∎

We denote by $\tilde{\kappa}(r)$ and $\tilde{\nu}(r)$ the curvature and the outer normal vector of Γ_r as a scalar and a vector-valued function defined on it, respectively. If $s > 2 + \frac{N}{2}$ and $H^s(\Gamma_0)r$ is sufficiently small, then the Dirichlet problem

$$\begin{aligned} \Delta\tilde{u} &= 0 \quad \text{in}\Omega_r \\ \tilde{u}|_{\Gamma_r} &= \tilde{\kappa}(r) \end{aligned}$$

can be transformed to the domain Ω_0, using the C^2-diffeomorphism $z(r)$. Writing $u(r) = \tilde{u} \circ z(r)$, $\kappa(r) = \tilde{\kappa}(r) \circ z(r)$, $\nu(r) = \tilde{\nu}(r) \circ z(r)$ we get

$$\left.\begin{aligned} \Delta_r u(r) &= 0 \quad \text{in } \Omega_0 \\ u(r)|_{\Gamma_0} &= \kappa(r) \end{aligned}\right\} \tag{2.5}$$

with

$$\begin{aligned} \Delta_r &= a^{ji}\frac{\partial}{\partial x_j}\left(a^{ki}\frac{\partial}{\partial x_k}\right), \\ \kappa(r) &= \Delta_{\Gamma_r}(\xi + \zeta r) \cdot \nu(r), \\ a^{ij} &= a^{ij}(r) = \left[\left(\frac{\partial z(r)}{\partial x}\right)^{-1}\right]_{ij}. \end{aligned}$$

Here and in the following, summation has to be carried out over all indices that occur twice in the same term.

Let X and Y be Banach spaces. We will call a mapping F defined on $B_{x_0}(\varepsilon, X)$ with positive ε and values in Y values *analytic near* x_0 iff F has a series representation

$$F(x) = \sum_{k=0}^{\infty} F(x - x_0, \ldots, x - x_0) \qquad \forall x \in B_{x_0}(\varepsilon, X) \tag{2.6}$$

where the F_k are bounded k-linear symmetric operators from X^k to Y for which the majorant series

$$\sum_{k=0}^{\infty} ||F_k|| \varepsilon^k$$

converges, which implies the absolute convergence of the series in (2.6) and the boundedness of F. As usual, we will call F analytic in the open set $U \subset X$ iff F is analytic near any point of U.

The rules for calculations with convergent power series generalize to our situation. More precisely, we will use the following results ([16] Ch. 8.2., Corollary 4.23, [7] Ch. IX.2.):

- F is analytic in $B_{x_0}(\varepsilon, X)$.
- For any $k \in \mathbb{N}$, the k-th order Frechet derivative of F exists and is an analytic function near x_0 with values in $\mathcal{L}(X^k, Y)$ (and the same ε).
- If $F^{(1)}$ and $F^{(2)}$ are analytic near x_0 then $F^{(1)} \equiv F^{(2)}$ near x_0 if and only if $F^{(1)}_k = F^{(2)}_k$ for all $k \in \mathbb{N}$ ("comparison of coefficients").
- If G is analytic near $F(x_0)$ from Y to the Banach space Z then the composition $G \circ F$ is analytic near x_0 from X to Z.
- If Y is a Banach algebra and H is analytic near x_0 from X to Y then the pointwise product FH is analytic near x_0 from X to Y.
- If additionally to the usual assumptions of the Implicit Function theorem, $\tilde{F}$ is analytic near (x_0, y_0) then the function $y(\cdot)$ given by $\tilde{F}(x, y(x))$ is analytic near x_0. This implies in particular for any Banach algebra X that the mapping $u \mapsto \sqrt{u}$ is analytic near any u_0 for which $\sqrt{u_0}$ exists and is invertible and the mapping $u \mapsto u^{-1}$ is analytic near any invertible u_0.

In the following, let $s > \frac{N+1}{2}$.

Lemma 3 *(Analytic dependence of the data)*
(i)The mapping $r \mapsto \Delta_r$ is analytic near 0 from $H^{s+1}(\Gamma_0)$ to $\mathcal{L}\left(H^{s+\frac{3}{2}}(\Omega_0), H^{s-\frac{1}{2}}(\Omega_0)\right)$.
(ii) The mapping $r \mapsto \kappa(r)$ is analytic near 0 from $H^{s+3}(\Gamma_0)$ to $H^{s+1}(\Gamma_0)$.

Proof: (i) The Jacobian matrix

$$A(r) = \frac{\partial z(r)}{\partial x} = \frac{\partial T^{-1}(r\zeta)}{\partial x} + I$$

is analytic near 0 as a function from $H^{s+1}(\Gamma_0)$ into the Banach algebra $\left(H^{s+\frac{1}{2}}(\Omega_0)\right)^{N\times N}$. $A(0) = I$ is invertible in this algebra, hence the inverse $A(r)^{-1}$ and all its elements depend analytically near 0 on $r \in H^{s+1}(\Gamma_0)$ in the spaces $\left(H^{s+\frac{1}{2}}(\Omega_0)\right)^{N\times N}$ and $H^{s+\frac{1}{2}}(\Omega_0)$, respectively. The assertion follows now from the Banach algebra property of $H^{s-\frac{1}{2}}(\Omega_0)$.

(ii) Let $\Gamma_0 = \bigcup_m \Gamma_0^{(m)}$ be a finite covering of Γ_0 by coordinate patches $\Gamma_0^{(m)}$ and $\{\chi_m\}$ a smooth partition of unity subordinate to it. Let $\xi^{(m)} = \xi^{(m)}(w)$, $w \in W_m$, be smooth regular parametrizations of $\Gamma_0^{(m)}$. Without loss of generality one can assume that the W_m are bounded and have smooth boundary. In view of the definitions of Sobolev space on manifolds it is enough to show that $(\chi_m \kappa(r)) \circ \xi^{(m)}$ depends analytically near 0 on $r \in H^{s+3}(\Gamma_0)$ with values in $H^{s+1}(W_m)$ for all m. Suppressing for the sake of brevity the pull-back by $\xi^{(m)}$ in the notation, on W_m we have

$$\chi_m \kappa(r) = \frac{\chi_m}{\sqrt{g(\tilde{\chi}_m r)}} \frac{\partial}{\partial w_i} \left(\sqrt{g(\tilde{\chi}_m r)} g^{ij}(\tilde{\chi}_m r) \frac{\partial(\xi + \tilde{\chi}_m r\zeta)}{\partial w_j} \right) \cdot \nu(\tilde{\chi}_m r)$$

where $\tilde{\chi}_m \in C^\infty(\Gamma_0)$, $\operatorname{supp} \tilde{\chi}_m \subset\subset \Gamma_0^{(m)}$, $\tilde{\chi}_m \equiv 1$ in $\operatorname{supp} \chi_m$,

$$\begin{aligned} g(r) &= \det G(r) \\ g^{ij}(r) &= [G(r)^{-1}]_{ij} \\ G(r) &= \left(\frac{\partial(\xi + r\zeta)}{\partial w}\right)^T \left(\frac{\partial(\xi + r\zeta)}{\partial w}\right). \end{aligned}$$

It is clear that the mapping $r \mapsto \tilde{\chi}_m r$ is analytic from $H^{s+3}(\Gamma_0)$ to $H^{s+3}(W_m)$ and the mappings $r \mapsto g(r)$ and $r \mapsto G(r)$ are analytic near 0 from $H^{s+3}(W_m)$ to the Banach algebras $H^{s+2}(W_m)$ and $\left(H^{s+2}(W_m)\right)^{N\times N}$. Consider, furthermore, the equation

$$F(r, \tilde{\nu}) = M(r)\tilde{\nu} - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = 0$$

with

$$M(r) = \begin{pmatrix} \left(\frac{\partial(\xi + r\zeta)}{\partial w}\right)^T \\ n^T \end{pmatrix}$$

where F maps analytically near $(0, \tilde{\nu})$ from $H^{s+3}(W_m) \times \left(H^{s+2}(W_m)\right)^N$ to $\left(H^{s+2}(W_m)\right)^N$ for any $\tilde{\nu} \in \left(H^{s+2}(W_m)\right)^N$, and

$$D_2 F(0, \tilde{\nu})[h] = M(0)h \qquad \forall h \in \left(H^{s+2}(W_m)\right)^N.$$

Due to the regularity of the parametrization $\xi^{(m)}$, we have that $M(0)$ is a C^∞-function on W_m whose values are invertible matrices, hence $M(0)$ is invertible in $\left(H^{s+2}(W_m)\right)^{N\times N}$. Thus, the Implicit Function theorem yields that the mapping $r \mapsto \tilde{\nu}(r)$ defined by $F(r,\nu(r)) = 0$ is analytic near 0 from $H^{s+3}(W_m)$ to $\left(H^{s+2}(W_m)\right)^N$. Geometrically, $\tilde{\nu}(r)$ corresponds to a field of outward-directed vectors orthogonal to Γ_r, hence

$$\nu(r) = \frac{\tilde{\nu}(r)}{|\tilde{\nu}(r)|} = \frac{\tilde{\nu}(r)}{\sqrt{\tilde{\nu}(r)^T\tilde{\nu}(r)}}$$

and taking into account that $\tilde{\nu}(0) = n$ we get, by the above remarks, that also $r \mapsto \nu(r)$ is analytic near 0 from $H^{s+3}(W_m)$ to $\left(H^{s+2}(W_m)\right)^N$.

The assertion follows now easily from the Banach algebra properties of the spaces $H^{s+2}(W_m)$ and $H^{s+1}(W_m)$ together with the above remarks on the inverse and square root from the fact that $g(0)$ is a smooth positive function, hence both $g(0)$ and its square root are invertible in these Banach algebras. ■

We rewrite (2.5) now as an operator equation

$$F(r,u(r)) = L(r)u(r) - \begin{pmatrix} 0 \\ \kappa(r) \end{pmatrix} = 0$$

in $H^{s-\frac{1}{2}}(\Omega_0) \times H^{s+1}(\Gamma_0)$ with $L(r)$ defined for small $r \in H^{s+3}(\Gamma_0)$ by

$$L(r)u = \begin{pmatrix} \Delta_r u \\ u|_{\Gamma_0} \end{pmatrix}.$$

We have $D_2F(0,u) = L(0)$ and the invertibility of this operator in the spaces given above is a standard result from the regularity theory of elliptic boundary value problems (see e.g. [14]). Hence, by the Implicit Function theorem, $u(r)$ is well-defined for small $r \in H^{s+3}(\Gamma_0)$, and the mapping $r \mapsto u(r)$ is analytic near 0 from $H^{s+3}(\Gamma_0)$ to $H^{s+\frac{3}{2}}(\Omega_0)$.

If we use now the local parametrization of $\mathbb{R}^N$ near Γ_0 to describe the moving boundary by

$$\Gamma(t) = \Gamma_{r(t)}$$

the kinematic boundary condition (1.2) takes the form

$$\frac{\partial r}{\partial t} = \frac{(\nabla_r u(r))|_{\Gamma_0} \cdot \nu(r)}{\zeta \cdot \nu(r)} = \rho(r) \tag{2.7}$$

with

$$(\nabla_r u)_i = a^{ji}(r)\frac{\partial u}{\partial x_j},$$

i.e. we have reformulated the moving boundary problem as as nonlinear nonlocal evolution equation for r.

Using the analytic dependence of u and ν on $r \in H^{s+3}(\Gamma_0)$ near 0, the fact that $H^s(\Gamma_0)$ and $H^{s+\frac{1}{2}}(\Omega_0)$ are Banach algebras one obtains by the same arguments as in the proof of lemma 3

Lemma 4 *(Analyticity of ρ)*
The operator ρ on the right side of (2.7) *is analytic near* 0 *from $H^{s+3}(\Gamma_0)$ to $H^s(\Gamma_0)$.*

This implies, in particular, that there is a constant $M_s > 0$ such that

$$\|\rho_k(h_1, \ldots, h_k)\|_s^{\Gamma_0} \leq M_s^k \|h_1\|_{s+3}^{\Gamma_0} \cdots \|h_k\|_{s+3}^{\Gamma_0} \tag{2.8}$$

for all $k \in \mathbb{N}$, $h_1, \ldots, h_k \in H^{s+3}(\Gamma_0)$.

3 A chain rule and continuity of ρ

The evolution equation (2.7) is obviously fully nonlinear in the sense that both the term ρ_1 representing the linearization of ρ around 0 and the nonlinear remainder are "third order operators" in the Hilbert scale $\{H^s(\Gamma_0)\}$.

In the further analysis a technique will be applied which is essentially a "quasilinearization" of the problem, based on a chain rule which expresses the invariance of our moving boundary problem with respect to rotations around a fixed point in space. Consequently, the differential operators for which this chain rule holds will be generated by these rotations.

We will start by giving an abstract version of it in the following general framework: Let G be a Lie group, $\mathcal{G}$ its Lie algebra, $a_1, \ldots, a_d$ a basis of $\mathcal{G}$ and for $i = 1, \ldots, d$ let $t \mapsto e^{-ta_i}$ be the one-parameter subgroup of G generated by a_i. Let X an Y be Banach spaces and let

$$\begin{aligned} U: \quad & G \longrightarrow \mathcal{L}(X) \\ V: \quad & G \longrightarrow \mathcal{L}(Y) \end{aligned}$$

be strongly continuous representations of G on X and Y, respectively. We denote by $D_i^{(X)}$ and $D_i^{(Y)}$ the generators of the strongly continuous semigroups of operators $t \mapsto U(e^{-ta_i})$ and $t \mapsto V(e^{-ta_i})$ on X and Y, respectively. For the sake of brevity we will suppress the indication of the spaces X and Y in the notation for the generators.

For any multiindex $\alpha = (\alpha_1 \ldots \alpha_d) \in \mathbb{N}^d$ we define $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$. Note that due to the structure equations of $\mathcal{G}$ we have

$$D_i D_j - D_j D_i = \sum_{k=1}^{d} c_{ij}^k D_k \qquad i, j = 1, \ldots, d$$

and this implies

$$D^\alpha D^\beta = D^{\alpha+\beta} + \sum_{|\gamma| < |\alpha+\beta|} C_{\alpha\beta\gamma} D^\gamma \tag{3.1}$$

for arbitrary multiindices α, β.

By the Hille-Yosida theorem, the operators D_i are closed, hence for all $n \in \mathbb{N}$ the spaces

$$X^{(n)} = \bigcap_{|\alpha| \leq n} \mathcal{D}(D^\alpha)$$

normed by

$$\|u\|_{X^{(n)}} = \sum_{|\alpha|\le n} \|D^\alpha u\|_X$$

are Banach spaces, and Banach spaces $Y^{(n)}$ are defined analogously. From (3.1) it follows that if $|\alpha| \le n$ then D^α maps $X^{(n)}$ continuously into $X^{(n-|\alpha|)}$, corresponding results hold for the $Y^{(n)}$. It is a routine task to check that the spaces $X^{(n)}$, $Y^{(n)}$ are, up to equivalence of norms, independent of the basis choice in $\mathcal{G}$.

We will consider a situation in which X and Y are spaces of real-valued functions on a manifold on which the Lie group G is acting as a group of diffeomorphisms. In this case, the spaces $X^{(n)}$ and $Y^{(n)}$ can be seen as subspaces of X and Y containing the functions which are "n times differentiable" with respect to the differential operators that constitute $\mathcal{G}$. The following lemma will make this idea more precise in the case where we will need it.

Lemma 5 *(A characterization for $H^\sigma(S^{N-1})$)*

Let the Lie group $G = SO(N)$ be represented by the rotations of $\mathbb{R}^N$ around the origin. For arbitrary $\sigma \in \mathbb{R}$ set $X = H^\sigma(S^{N-1})$, $U(g)u = u \circ g$. Then $X^{(n)} = H^{\sigma+n}(S^{N-1})$ with equivalent norms.

This is a special case of theorem 3.17 in [9].

Lemma 6 *(Regularity and a chain rule from equivariance)*

(i) Let $\mathcal{U} \subset X$ be open, $F : \mathcal{U} \longrightarrow Y$ K times Frechet-differentiable, $n \le K$. If the equivariance relation

$$V(g)F(u) = F(U(g)u) \tag{3.2}$$

holds for all $u \in \mathcal{U}$ and all g near the unit element in G then the restriction of F to $\mathcal{U} \cap X^{(n)}$ maps $\mathcal{U} \cap X^{(n)}$ $K-n$ times Frechet-differentiable into $Y^{(n)}$, and for all $\alpha \in \mathbb{N}^d$ and $u \in \mathcal{U} \cap X^{(|\alpha|)}$ one has the chain rule

$$D^\alpha F(u) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1\ldots\beta_k} F^{(k)}(u)[D^{\beta_1}u, \ldots, D^{\beta_k}u] \tag{3.3}$$

where only $\beta_l \ne 0$ occur and $C_\alpha = 1$.

(ii) If, in particular, F is analytic near $u_0 \in \mathcal{U}$ then there is a $\varepsilon > 0$ such that the restriction of F to $B_{u_0}(\varepsilon, X) \cap X^{(n)}$ is analytic and bounded into $Y^{(n)}$ for all $n \in \mathbb{N}$.

Proof: (i) The proof of (i) will be given by induction over $|\alpha|$. Suppose $u \in \mathcal{U} \cap X^{(1)}$. By assumption, we have for sufficiently small $|t|$ and all $i = 1, \ldots, d$

$$V(e^{-ta_i})F(u) = F(U(e^{-ta_i})u).$$

The right side is differentiable with respect to t at $t = 0$, hence the same holds for the expression on the left, therefore $F(u) \in Y^{(1)}$. Carrying out the differentiation yields

$$D_i F(u) = F'(u)[D_i u], \qquad i = 1, \ldots, d. \tag{3.4}$$

The expression on the right is a $K-1$ times differentiable function from $\mathcal{U}\cap X^{(1)}$ into Y, hence all assumptions are proved for $|\alpha|=1$. In particular, if $k\leq K-1$ and $h_1,\ldots,h_k\in X^{(1)}$ then $F^{(k)}(u)[h_1,\ldots,h_k]\in Y^{(1)}$, and calculating the k-th order Frechet derivative on both sides of (3.4) yields

$$\begin{aligned}
&(D_jF)^{(k)}(u)[h_1,\ldots,h_k]=D_j(F^{(k)}(u)[h_1,\ldots,h_k])\\
=\;&\sum_{l=1}^{k}F^{(k)}(u)[h_1,\ldots,h_{l-1},D_jh_l,h_{l+1},\ldots,h_k]\\
&+F^{(k+1)}(u)[D_ju,h_1,\ldots,h_k]
\end{aligned}\tag{3.5}$$

for all $h_1,\ldots,h_k\in X^{(1)}$ which is easily proved by induction.

Suppose the assertions hold for all α' with $|\alpha'|\leq m\leq K-1$, consider α with $|\alpha|=m+1$, $u\in\mathcal{U}\cap X^{(m+1)}$. We can write $D^\alpha=D_jD^{\alpha'}$ and apply the induction assumption to $D^{\alpha'}F(u)$. Due to $D^\beta u\in X^{(1)}$ for all β with $|\beta|\leq|\alpha'|$ we have $D^{\alpha'}F(u)\in Y^{(1)}$ and (3.5) may be applied. Rearranging the terms according to the order of the Frechet derivatives and noting that the expressions on the right are $K-m-1$ times Frechet-differentiable from $\mathcal{U}\cap X^{(m+1)}$ to $Y^{(m+1)}$ completes the proof of (3.3).

(ii) In view of the definition of the space $Y^{(n)}$ it is sufficient to show that the mappings $u\mapsto D^\alpha F(u)$ are analytic and bounded from $B_{u_0}(\varepsilon,X^{(n)})\cap X^{(n)}$ to Y. The analyticity follows immediately from the above remark on the analyticity of the Frechet derivatives. The boundedness of the $F^{(k)}$ implies

$$\|F^{(k)}(u)\|_{\mathcal{L}(X^k,Y)}\leq C_k\qquad\forall u\in B_{u_0}(\varepsilon,X)$$

and if we demand $\|u\|_{X^{(n)}}\leq M$ then

$$\|D^\beta u\|_X\leq C_{\beta,n}M\qquad\forall\beta:\;|\beta|\leq n$$

and hence by (3.3)

$$\|D^\alpha F(u)\|_Y\leq C_k(C_nM)^k\leq C_nM^n.$$

■

It has to be pointed out that, in contrast to the usual chain rule, (which is, of course, contained as a special case in the given one) the mapping F may be nonlocal, which will be the case in our application.

In order to apply lemma 6 to our problem, however, a restriction has to be made: As we deal with rotational invariance, it is natural to choose $G=SO(N)$ and to represent it in Banach spaces of real-valued functions on the sphere S^{N-1}. Our problem can be fit into this framework only if its geometry is such that it can be formulated in terms of such functions in a way that (3.2) is preserved.

More precisely, we demand additionally that Ω_0 is strictly star-shaped, i.e. there is a smooth positive real-valued function R_0 defined on the unit sphere S^{N-1} such that (after a suitable translation)

$$\Gamma_0=\left\{\theta R_0(\theta)\,|\,\theta\in S^{N-1}\right\}.$$

The mapping $\Phi : S^{N-1} \longrightarrow \Gamma_0$ defined by $\Phi(\theta) = \zeta R_0(\theta)$ is a C^∞- diffeomorphism between S^{N-1} and Γ_0, hence the direct image map Φ^* defined by $(\Phi^*\varphi)(\theta) = \varphi(\Phi(\theta))$ is an isomorphism from $C^\infty(\Gamma_0)$ to $C^\infty(S^{N-1})$ and from $H^\sigma(\Gamma_0)$ to $H^\sigma(S^{N-1})$ for all $\sigma \in \mathbb{R}$. We choose $\zeta(\xi) = \frac{\xi}{|\xi|}$ which obviously meets the demands on ζ that were made in section 2. With this choice, we have for any sufficiently small $r \in H^s(\Gamma_0)$

$$\Gamma_r = \{\theta R(\theta) \mid \theta \in S^{N-1}\} = \tilde{\Gamma}_R$$

where $R = \Phi^* r + R_0$.

On a small ball around R_0 in $H^{s+3}(S^{N-1})$ we define the operators $\tilde{\rho}$ and $\tilde{\nu}$ by

$$\begin{aligned} \tilde{\rho}(R) &= \Phi^*\rho(r) = \Phi^*\rho \circ \Phi^{*^{-1}}(R - R_0) \\ \tilde{\nu}(r) &= \Phi^*\nu(r) = \Phi^*\nu \circ \Phi^{*^{-1}}(R - R_0) \end{aligned} \tag{3.6}$$

These operators are obviously analytic near R_0 from $H^{s+3}(S^{N-1})$ to $H^s(S^{N-1})$ and $\left(H^{s+2}(S^{N-1})\right)^N$, respectively. Taking the k-th Frechet derivative of (3.6) yields

$$\tilde{\rho}^{(k)}(R)[h_1, \ldots, h_k] = \Phi^*\rho^{(k)}(r)[\Phi^{*^{-1}}h_1, \ldots, \Phi^{*^{-1}}h_k] \tag{3.7}$$

for all $h_1, \ldots, h_k \in H^{s+3}(S^{N-1})$.

Let $Q : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be an arbitrary rotation around the origin with $\|Q - I\|$ sufficiently small. Then clearly $\|R \circ Q - R_0\|_{s+3}^{S^{N-1}}$ is small, $\tilde{\Gamma}_{R\circ Q} = Q^{-1}[\tilde{\Gamma}_R]$, and thus

$$\tilde{\nu}(R \circ Q) = Q^{-1}\tilde{\nu}(R) \circ Q.$$

Moreover, writing

$$r_Q = \Phi^{*^{-1}}(R \circ Q - R_0)$$

we find from the rotational invariance of the Laplacian

$$\Phi^*\left((\nabla_{r_Q} u(r_Q))|_{\Gamma_0}\right) = Q^{-1}\Phi^*\left((\nabla_r u(r))|_{\Gamma_0}\right) \circ Q.$$

Using this, we get

$$\begin{aligned} \tilde{\rho}(R \circ Q) &= \Phi^*\rho(r_Q) = \frac{\Phi^*((\nabla_{r_Q} u(r_Q))|_{\Gamma_0}) \cdot \Phi^*\nu(r_Q)}{\Phi^*\zeta \cdot \Phi^*\nu(r_Q)} \\ &= \frac{Q^{-1}\Phi^*((\nabla_r u(r))|_{\Gamma_0}) \circ Q \cdot \tilde{\nu}(R \circ Q)}{\Phi^*\zeta \cdot \tilde{\nu}(R \circ Q)} \\ &= \frac{Q^{-1}\Phi^*((\nabla_r u(r))|_{\Gamma_0}) \circ Q \cdot Q^{-1}\tilde{\nu}(R) \circ Q}{Q^{-1}\Phi^*\zeta \circ Q \cdot Q^{-1}\tilde{\nu}(R) \circ Q} \\ &= \Phi^*\left(\frac{(\nabla_r u(r))|_{\Gamma_0} \cdot \nu(r)}{\zeta \cdot \nu(r)}\right) \circ Q \\ &= \Phi^*\rho(\Phi^{*^{-1}}(R - R_0)) \circ Q = \tilde{\rho}(R) \circ Q. \end{aligned}$$

Hence, if we set $G = SO(N)$, $X = H^{s+3}(S^{N-1})$, $Y = H^s(S^{N-1})$, $U(g)u = V(g)u = u \circ g$, $F = \tilde{\rho}$ then, in view of lemma 4, all assumptions of lemma 6 are satisfied.

Defining, moreover, differential operators on Γ_0 by

$$\tilde{D}_j = \Phi^{*-1} D_j \Phi^*$$

and $\tilde{D}^\alpha$ analogously to D^α and writing $\mathcal{R}_0 = \Phi^{*-1} R_0$ we find the following results:

Lemma 7 *(Properties of ρ)*

There is an $\varepsilon > 0$ such that

(i) ρ is analytic and bounded from

$$B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$$

to $H^{s+n}(\Gamma_0)$ for all $n \in \mathbb{N}$.

(ii) ρ is weakly sequentially continuous from $B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$ to $H^{s+n}(\Gamma_0)$ for all integer $n \geq 1$.

(iii) For all $n \in \mathbb{N}$, $r \in B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$, and $\alpha \in \mathbb{N}^{\binom{N}{2}}$ with $|\alpha| \leq n$ we have

$$\tilde{D}^\alpha \rho(r) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1 + \ldots + \beta_k = \alpha} C_{\beta_1, \ldots, \beta_k} \rho^{(k)}(r)[\tilde{D}^{\beta_1}(r + \mathcal{R}_0), \ldots, \tilde{D}^{\beta_k}(r + \mathcal{R}_0)]$$

where only multiindices $\beta_i \neq 0$ occur and $C_\alpha = 1$.

Proof: The assertions (i) and (iii) follow easily by lemma 5, applying lemma 6 to $\tilde{\rho}$ and recalling that $\rho(r) = \Phi^{*-1}\tilde{\rho}(\Phi^* r + R_0)$ and (3.7).

In the following, we will use the notations $x_n \overset{X}{\rightarrow} x$ for norm convergence and $x_n \overset{X}{\rightharpoonup} x$ for weak convergence in the (Banach) space X.

In order to prove (ii), consider an arbitrary sequence

$$\{r_n\} \subset B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$$

such that $r_n \overset{H^{s+n+3}(\Gamma_0)}{\rightharpoonup} r^*$. Due to the compactness of the embedding

$$H^{s+n+3}(\Gamma_0) \hookrightarrow\hookrightarrow H^{s+n+2}(\Gamma_0)$$

this implies $r_n \overset{H^{s+n+2}(\Gamma_0)}{\rightarrow} r^*$ and thus by (i)

$$\rho(r_n) \overset{H^{s+n-1}(\Gamma_0)}{\rightarrow} \rho(r^*). \tag{3.8}$$

On the other hand, $\{r_n\}$ is bounded in $H^{s+n+3}(\Gamma_0)$ and thus by (i) $\{\rho(r_n)\}$ is bounded in $H^{s+n}(\Gamma_0)$. Consider now an arbitrary subsequence $\{\rho(r_{n'})\}$ of $\{\rho(r_n)\}$ such that $\rho(r_{n'}) \overset{H^{s+n}(\Gamma_0)}{\rightharpoonup} \rho^*$. This implies, by compactness of the embedding $H^{s+n}(\Gamma_0) \hookrightarrow\hookrightarrow H^{s+n-1}(\Gamma_0)$, $\rho(r_{n'}) \overset{H^{s+n-1}(\Gamma_0)}{\longrightarrow} \rho^*$ and thus by (3.8) $\rho^* = \rho(r^*)$. Hence we can conclude ([16], Proposition 10.13 (4)) $\rho(r_n) \overset{H^{s+n}(\Gamma_0)}{\longrightarrow} \rho(r^*)$. ■

As an important consequence of the above lemma we note that for all $n \in \mathbb{N}$ and all α with $|\alpha| \leq n$ the mapping $\tilde{D}^\alpha \rho$ is analytic and bounded from $B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$ to $H^s(\Gamma_0)$ and

$$\begin{aligned}
&(\tilde{D}^\alpha \rho)^{(m)}(0)[h_1, \dots, h_k] = \tilde{D}^\alpha(\rho_m(h_1, \dots, h_m)) \\
= \; & \frac{1}{m!} \sum_\pi \sum_{l=0}^{m} \sum_{k=\max\{1,m-l\}}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k} \frac{(k+l)!}{l!\,(m-l)!\,(k-m+l)!} \sum_\sigma \\
& \rho_{k+l}(h_{\pi(1)}, \dots, h_{\pi(l)}, \tilde{D}^{\beta_{\sigma(1)}} h_{\pi(l+1)}, \dots, \tilde{D}^{\beta_{\sigma(m-l)}} h_{\pi(m)}, \tilde{D}^{\beta_{\sigma(m-l+1)}} \mathcal{R}_0, \\
& \dots, \tilde{D}^{\beta_{\sigma(k)}} \mathcal{R}_0) \qquad (3.9)
\end{aligned}$$

for all $h_j \in H^{s+n+3}(\Gamma_0)$ where π and σ run over all permutations of the index sets $\{1, \dots, m\}$ and $\{1, \dots, k\}$, respectively.

In the following it will be convenient to work with equivalent scalar products on the spaces $H^\sigma(\Gamma_0)$ which are generated by the operators $\tilde{D}^\alpha$. As usual, for any $\sigma \in \mathbb{R}$ we define the standard scalar product $(\cdot, \cdot)_\sigma$ in these spaces by

$$(u, v)_\sigma = (P^\sigma u, P^\sigma v)_0 \qquad (3.10)$$

with

$$\begin{aligned}
P &= (I - \Delta_{\Gamma_0})^{\frac{1}{2}}, \\
(u, v)_0 &= \int_{\Gamma_0} uv \, d\Gamma_0.
\end{aligned}$$

Moreover, the fact that Φ^* is a diffeomorphism from $H^\sigma(\Gamma_0)$ to $H^\sigma(S^{N-1})$ for arbitrary $\sigma \in \mathbb{R}$ and lemma 5 imply that the scalar product $(\cdot, \cdot)_{\sigma,n}$ defined by

$$(u, v)_{\sigma,n} = \sum_{|\alpha| \leq n} (\tilde{D}^\alpha u, \tilde{D}^\alpha v)_\sigma$$

generates the usual topology on $H^{\sigma+n}(\Gamma_0)$ for all $\sigma \in \mathbb{R}$, $n \in \mathbb{N}$. Note that

$$(u, v)_{\sigma,n} = (S_{\sigma,n} u, v)_0$$

where

$$S_{\sigma,n} = \sum_{|\alpha| \leq n} (\tilde{D}^\alpha)^* P^{2\sigma} \tilde{D}^\alpha$$

is an elliptic self-adjoint pseudodifferential operator of order $2(n + \sigma)$ which is positive definite for $\sigma \geq 0$. The tilde on the operators D^α will be suppressed in the sequel.

4 Linearization

For the further investigation of the evolution equation (2.7) we have to analyze the linearization ρ_1 of ρ around $r = 0$. We find, for any sufficiently smooth real-valued function h defined on Γ_0,

$$\rho_1(h) = -\frac{1}{\gamma^2}\frac{\partial u_0}{\partial n}\zeta\cdot\nu_1(h) + \frac{1}{\gamma}\left(\left(a_1^{ji}(h)\frac{\partial u_0}{\partial x_j}\right)\Big|_{\Gamma_0} n_i + \frac{\partial}{\partial n}u_1(h) + \nabla u_0|_{\Gamma_0}\nu_1(h)\right).$$

Taking into account that u_0 is smooth, the components of ν_1 are first order differential operators on Γ_0 corresponding to smooth vector fields and

$$a_1^{ji}(h) = -\frac{\partial(T^{-1}(h\zeta_j))}{\partial x_i}$$

we have

$$\rho_1(h) = \frac{1}{\gamma}\frac{\partial}{\partial n}u_1(h) + \Lambda_1 h$$

where Λ_1 is a bounded linear mapping from $H^\sigma(\Gamma_0)$ to $H^{\sigma-1}(\Gamma_0)$ for all $\sigma > 1$.

It remains to investigate

$$u_1(h) = L(0)^{-1}\begin{bmatrix} 0 \\ \kappa_1(h) \end{bmatrix} - L(0)^{-1}L_1(h)L(0)^{-1}\begin{bmatrix} 0 \\ \kappa(0) \end{bmatrix}.$$

Note that the second term on the right equals

$$L(0)^{-1}\begin{bmatrix} a_1^{ji}(h)\frac{\partial}{\partial x_j}\left(a^{ki}(0)\frac{\partial u_0}{\partial x_k}\right) + a^{ji}(0)\frac{\partial}{\partial x_j}\left(a_1^{ki}(h)\frac{\partial u_0}{\partial x_k}\right) \\ 0 \end{bmatrix}.$$

An elementary but somewhat tedious calculation involving a reparametrization of a neighborhood of Γ_0 where the normal vector field n is used instead of ζ shows

$$\kappa_1(h) = \gamma\Delta_{\Gamma_0}h + \Lambda_2 h + \mu h$$

where Λ_2 is a first order differential operator on Γ_0 corresponding to a smooth vector field and μ is a smooth function on Γ_0.

Summarizing and using the regularity results on the boundary value problem represented by the operator $L(0)$ one has

$$\rho_1(h) = \frac{1}{\gamma}A\gamma\Delta_{\Gamma_0}h + \Lambda_3 h$$

where Λ_3 is a bounded linear operator from $H^\sigma(\Gamma_0)$ to $H^{\sigma-2}(\Gamma_0)$ for all $\sigma \geq \frac{3}{2}$ and the operator A defined by

$$A\varphi = \frac{\partial}{\partial n}L(0)^{-1}\begin{bmatrix} 0 \\ \varphi \end{bmatrix}$$

is the so-called Dirichlet-to-Neumann operator of the Laplacian. A is known to be an (elliptic) pseudodifferential operator of order 1. Applying the commutator properties of such operators we find

$$\rho_1 = A\Delta_{\Gamma_0} + \Lambda_4$$

where Λ_4 has the same properties which are stated above for Λ_3.

Lemma 8 *(Coercivity of $-\rho_1$)*
For all real $\sigma \geq \frac{1}{2}$ there are positive constants c_σ, C_σ such that

$$(-\rho_1(h), h)_\sigma \geq c_\sigma \|h\|_{\sigma+\frac{3}{2}}^{\Gamma_0\,2} - C_\sigma \|h\|_{\sigma+1}^{\Gamma_0\,2} \qquad \forall h \in H^{\sigma+3}(\Gamma_0). \tag{4.1}$$

Proof: As a first step, we give an elementary $H^0(\Gamma_0)$-coercivity estimate for A. For any $\varphi \in H^{\frac{1}{2}}(\Gamma_0)$, let $\tilde{u}$ be the solution of

$$\begin{aligned} \Delta\tilde{u} &= 0 \text{ in } \Omega_0, \\ \tilde{u}|_{\Gamma_0} &= \varphi \end{aligned}$$

and set

$$u = \tilde{u} - \frac{1}{|\Omega_0|}\int_{\Omega_0} \tilde{u}\, dx.$$

Then $A\varphi = \frac{\partial\tilde{u}}{\partial n}$ and by the well-known "dual estimate"

$$\|\tilde{u}\|_0^{\Omega_0} \leq C\|\varphi\|_{-\frac{1}{2}}^{\Gamma_0}$$

(see e.g. [14]) we have

$$\begin{aligned} \|\tilde{u}\|_1^{\Omega_0\,2} &\leq C\left(\|u\|_1^{\Omega_0\,2} + \|\tilde{u} - u\|_1^{\Omega_0\,2}\right) \leq C\left(\|u\|_1^{\Omega_0\,2} + \|\tilde{u}\|_0^{\Omega_0\,2}\right) \\ &\leq C\left(\|u\|_1^{\Omega_0\,2} + \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}\right). \end{aligned}$$

Hence, using the trace theorem,

$$\|u\|_1^{\Omega_0\,2} \geq c\|\tilde{u}\|_1^{\Omega_0\,2} - \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2} \geq c\|\varphi\|_{\frac{1}{2}}^{\Gamma_0\,2} - \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}$$

and by the Green formula and the Poincaré inequality

$$(A\varphi, \varphi)_0 = \|\nabla\tilde{u}\|_0^{\Omega_0\,2} = \|\nabla u\|_0^{\Omega_0\,2} \geq c\|u\|_1^{\Omega_0\,2} \geq c\|\varphi\|_{\frac{1}{2}}^{\Gamma_0\,2} - C\|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}. \tag{4.2}$$

For any linear operator Λ that maps $H^{\sigma+1}(\Gamma_0)$ continuously into $H^{\sigma-1}(\Gamma_0)$ we have

$$(\Lambda h, h)_\sigma = (P^{\sigma-1}\Lambda h, P^{\sigma+1}h)_0 \leq C_\sigma \|h\|_{n+1}^{\Gamma_0\,2}.$$

Therefore it is sufficient to show (4.1) with $-\rho_1$ replaced by AP^2. We get

$$(AP^2h, h)_\sigma = (P^\sigma AP^2 h, P^\sigma h)_0 = (AP^{\sigma+1}h, P^{\sigma+1}h)_0 + (\Lambda_5 h, P^{\sigma+1}, h)_0$$

where

$$\Lambda_5 = P^{\sigma-1}AP^2 - AP^{\sigma+1}$$

is a pseudodifferential operator of order $\sigma + 1$. Hence, using (4.2),

$$\begin{aligned}(AP^2h, h)_\sigma &\geq c\left\|P^{\sigma+1}h\right\|_{\frac{1}{2}}^{\Gamma_0\,2} - C\left\|P^{\sigma+1}h\right\|_{-\frac{1}{2}}^{\Gamma_0\,2} - C_\sigma\|h\|_{\sigma+1}^{\Gamma_0\,2} \\ &\geq c_\sigma\|h\|_{\sigma+\frac{3}{2}}^{\Gamma_0\,2} - C_\sigma\|h\|_{\sigma+1}^{\Gamma_0\,2}.\end{aligned}$$

■

It has to be pointed out that the results of this section have been obtained without the use of the equivariance property of ρ. They also hold in the general case of domains Ω_0 which are not necessarily strictly star-shaped.

5 Local Existence and Uniqueness

Using (3.9) and interpolation inequalities in the scale $\{H^\theta(\Gamma_0)\}$ one can show that the behaviour of ρ near 0 is essentially governed by its linearization. We prefer to give an estimate for the scalar product $(\cdot,\cdot)_{s,n}$. This choice is justified by the equivalence of this scalar product with $(\cdot,\cdot)_{s+n}$.

Lemma 9 *(A priori estimate for small r)*
Let $s > \frac{N+4}{2}$. There is an $\varepsilon > 0$ depending only on s such that for all positive integer n and all $r \in B_0(\varepsilon, H^{s+3}(\Gamma_0)) \cap H^{s+n+3}(\Gamma_0)$ an estimate

$$(\rho(r), r)_{s,n} \leq -c_{s,n}\|r\|_{s+n+\frac{3}{2}}^{\Gamma_0\,2} + C_{s,n}\left(\|r\|_{s+n+1}^{\Gamma_0\,2} + 1\right)$$

holds.

Proof: We decompose

$$\begin{aligned}(\rho(r), r)_{s,n} &= (\rho(0), r)_{s,n} \\ &\quad + \sum_{|\alpha|\leq n}\left((D^\alpha\rho_1(r), D^\alpha r)_s + \sum_{m=2}^{\infty}(D^\alpha\rho_m(r,\ldots,r), D^\alpha r)_s\right)\end{aligned}$$

and estimate the terms on the right separately. We will restrict our attention to the case $|\alpha| > 0$, the estimates for $\alpha = 0$ are immediate.

1. Obviously,

$$(\rho(0), r)_{s,n} \leq C_{s,n}\|r\|_{s+n}^{\Gamma_0} \leq C_{s,n}\left(\|r\|_{s+n+1}^{\Gamma_0\,2} + 1\right). \tag{5.1}$$

2. In equation (3.9) one has $C_\alpha = 1$ if $k = 1$ and $|\beta_j| < |\alpha|$ for all $j = 1,\ldots,k$ if $k \geq 2$. Using this in the case $m = 1$ and (2.8) with s replaced by $s-1$ one can easily show the commutator estimate

$$\|D^\alpha\rho_1(r) - \rho_1(D^\alpha r)\|_{s-1}^{\Gamma_0} \leq C_{s,\alpha}\|r\|_{s+1+|\alpha|}^{\Gamma_0}$$

holds for all $r \in H^{s+2+|\alpha|}(\Gamma_0)$. Using this and and lemma 8 one obtains

$$\begin{aligned}(D^\alpha \rho_1(r), D^\alpha r)_s &= (\rho_1(D^\alpha r), D^\alpha r)_s + (D^\alpha \rho_1(r) - \rho_1(D^\alpha r), D^\alpha r)_s \\ &\leq -c_s^{(1)} \|r\|_{s+\frac{3}{2}}^{\Gamma_0\,2} + C_s \|D^\alpha \rho_1(r) - \rho_1(D^\alpha r)\|_{s-1}^{\Gamma_0} \|D^\alpha r\|_{s+1}^{\Gamma_0} \\ &\leq -c_s^{(1)} \|r\|_{s+\frac{3}{2}}^{\Gamma_0\,2} + C_{s,n} \|r\|_{s+n+1}^{\Gamma_0\,2}. \end{aligned} \quad (5.2)$$

3. The remaining terms can be estimated using (3.9):

$$\begin{aligned}(D^\alpha \rho_m(r,\ldots,r), D^\alpha r)_s &\leq C_s \|D^\alpha \rho_m(r,\ldots,r)\|_{s-\frac{3}{2}}^{\Gamma_0} \|D^\alpha r\|_{s+\frac{3}{2}}^{\Gamma_0} \\ &\leq C_s \sum_{(l,k,\underline{\beta},\sigma)\in \mathcal{I}_{m,\alpha}} \mathcal{K}(l,k,m,\underline{\beta}) T_{k+l,m,\underline{\beta},\sigma} \|D^\alpha r\|_{s+\frac{3}{2}}^{\Gamma_0}\end{aligned}$$

with the shortcut notations

$$\begin{aligned}\underline{\beta} &= (\beta_1,\ldots,\beta_k), \\ \mathcal{I}_{m,\alpha} &= \{(l,k,\underline{\beta},\sigma)\,|\, 0 \leq l \leq m,\ \max\{1, m-l\} \leq k \leq |\alpha|, \\ &\qquad \beta_1 + \ldots + \beta_k = \alpha,\ \sigma \in \mathcal{S}_k\}, \\ \mathcal{K}(l,k,m,\underline{\beta}) &= C_{\underline{\beta}} \frac{(k+l)!}{l!\,(m-l)!\,(k-m+l)!}, \\ T_{k+l,m,\underline{\beta},\sigma} &= \big\|\rho_{k+l}(r,\ldots,r, D^{\beta_{\sigma(1)}}r,\ldots,D^{\beta_{\sigma(m-l)}}r, \\ &\qquad D^{\beta_{\sigma(m-l+1)}}\mathcal{R}_0, D^{\beta_{\sigma(k)}}\mathcal{R}_0)\big\|_{s-\frac{3}{2}}^{\Gamma_0}\end{aligned}$$

where $\mathcal{S}_k$ denotes the set of all permutations of $\{1,\ldots,k\}$. Note that it has $k!$ elements and that due to $k \leq |\alpha|$, $l \leq m$ we have

$$\mathcal{K}(l,k,m,\underline{\beta})\, k! \leq C_\alpha \frac{(k+l)!}{l!} \binom{k}{m-l} \leq (m+|\alpha|)^{|\alpha|} 2^{|\alpha|-1} \leq C_\alpha m^{|\alpha|} \quad (5.3)$$

for all $m \geq 2$ and all $(l,k,\underline{\beta},\sigma) \in \mathcal{I}_{m,\alpha}$. We will estimate the terms $T_{k+l,m,\underline{\beta},\sigma}$ using (2.8) with $s - \frac{3}{2}$ in place of s and the decomposition

$$\begin{aligned}\mathcal{I}_{m,\alpha} &= \mathcal{I}_{m,\alpha}^{(1)} \cup \mathcal{I}_{m,\alpha}^{(2)} \cup \mathcal{I}_{m,\alpha}^{(3)} \\ \mathcal{I}_{m,\alpha}^{(1)} &= \{(l,k,\underline{\beta},\sigma) \in \mathcal{I}_{m,\alpha} \,|\, l+1 = m,\ k = 1\} \\ \mathcal{I}_{m,\alpha}^{(2)} &= \{(l,k,\underline{\beta},\sigma) \in \mathcal{I}_{m,\alpha} \,|\, l+k = m,\ k > 1\} \\ \mathcal{I}_{m,\alpha}^{(3)} &= \{(l,k,\underline{\beta},\sigma) \in \mathcal{I}_{m,\alpha} \,|\, l+k > m\}.\end{aligned}$$

3.1. The set $\mathcal{I}_{m,\alpha}^{(1)}$ contains only the element $(m, 1, (\alpha), (1))$ corresponding to the term

$$m T_{m,m,(\alpha),(1)} \leq C_s m M_s^m \|r\|_{s+\frac{3}{2}}^{\Gamma_0\,m-1} \|D^\alpha r\|_{s+\frac{3}{2}}^{\Gamma_0}.$$

If we choose ε small enough and perform the summation over $m \geq 2$ we get

$$\sum_{m=2}^{\infty} m T_{m,m,(\alpha),(1)} \|D^\alpha r\|_{s+\frac{3}{2}}^{\Gamma_0} \leq \frac{c_s^{(1)}}{4} \|D^\alpha r\|_{s+\frac{3}{2}}^{\Gamma_0\,2}. \quad (5.4)$$

3.2. If $(l,k,\underline{\beta},\sigma) \in \mathcal{I}^{(2)}_{m,\alpha}$ then $|\beta_j| < |\alpha|$ for all $j = 1,\dots,k$ and thus the inequality

$$s + |\beta_j| + \frac{3}{2} \leq \left(1 - \frac{|\beta_j|}{|\alpha|}\right)(s+2) + \frac{|\beta_j|}{|\alpha|}(s + |\alpha| + \frac{3}{2} - \zeta)$$

holds with a sufficiently small positive ζ. Using the interpolation inequalities

$$\begin{aligned} \|D^\beta r\|^{\Gamma_0}_{s+\frac{3}{2}} &\leq C_{s,\beta}\|r\|^{\Gamma_0}_{s+|\beta|+\frac{3}{2}} \leq C_{s,\alpha}\|r\|^{\Gamma_0\;1-\frac{|\beta|}{|\alpha|}}_{s+2}\|r\|^{\Gamma_0\;\frac{|\beta|}{|\alpha|}}_{s+|\alpha|+\frac{3}{2}-\zeta} \\ \|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}-\zeta} &\leq \delta\|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}} + C_{s,\alpha,\delta}\|r\|^{\Gamma_0}_{s+\frac{3}{2}} \end{aligned}$$

holding for all β with $|\beta| < |\alpha|$ and all $\delta > 0$ we find

$$T_{m,m,\underline{\beta},\sigma} \leq C_{s,\alpha}M_s^m \|r\|^{\Gamma_0\;m-1}_{s+\frac{3}{2}}(\delta\|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}} + C_{s,\alpha,\delta}\|r\|^{\Gamma_0}_{s+\frac{3}{2}}).$$

Hence, by (5.3)

$$\begin{aligned} &\sum_{(l,k,\underline{\beta},\sigma)\in\mathcal{I}^{(2)}_{m,\alpha}} \mathcal{K}(l,k,m,\underline{\beta})T_{m,m,\underline{\beta},\sigma}\|D^\alpha r\|^{\Gamma_0}_{s+\frac{3}{2}} \\ \leq\;& C_{s,\alpha}M_s^m m^{|\alpha|}\|r\|^{\Gamma_0\;m-1}_{s+\frac{3}{2}}(\delta\|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}} + C_{s,\alpha,\delta}\|r\|^{\Gamma_0}_{s+\frac{3}{2}})\|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}}. \end{aligned}$$

If we demand $\varepsilon < \frac{1}{M_s}$ and choose δ sufficiently small then

$$\sum_{m=2}^{\infty}\sum_{(l,k,\underline{\beta},\sigma)\in\mathcal{I}^{(2)}_{m,\alpha}} \mathcal{K}(l,k,m,\underline{\beta})T_{m,m,\underline{\beta},\sigma}\|D^\alpha r\|^{\Gamma_0}_{s+\frac{3}{2}} \leq \frac{c_s^{(1)}c_{n,s}^{(2)}}{4\mathcal{N}_n}\|r\|^{\Gamma_0\;2}_{s+n+\frac{3}{2}} + C_{s,n} \tag{5.5}$$

where $\mathcal{N}_n$ is the number of elements in the set $\{\alpha \mid |\alpha| \leq n\}$ and $c^{(2)}_{s,n}$ is a small positive constant such that

$$(r,r)_{s+\frac{3}{2},n} \geq c^{(2)}_{s,n}\|r\|^{\Gamma_0\;2}_{s+n+\frac{3}{2}} \tag{5.6}$$

for all $r \in H^{s+n+\frac{3}{2}}(\Gamma_0)$.

3.3. If $(l,k,\underline{\beta},\sigma) \in \mathcal{I}^{(2)}_{m,\alpha}$ then $\underline{\beta}$ satisfies

$$b = \frac{1}{|\alpha|}\sum_{j=1}^{m-l}|\beta_{\sigma(j)}| < 1$$

and using the interpolation inequality

$$\|D^\beta r\|^{\Gamma_0}_{s+\frac{3}{2}} \leq C_{s,\beta}\|r\|^{\Gamma_0}_{s+|\beta|+\frac{3}{2}} \leq C_{s,\alpha}\|r\|^{\Gamma_0\;1-\frac{|\beta|}{|\alpha|}}_{s+\frac{3}{2}}\|r\|^{\Gamma_0\;\frac{|\beta|}{|\alpha|}}_{s+|\alpha|+\frac{3}{2}}$$

holding for all β with $|\beta| \leq |\alpha|$ and Youngs inequality we find

$$\begin{aligned} T_{k+l,m,\underline{\beta},\sigma} &\leq C_{s,\alpha}M^m\|r\|^{\Gamma_0\;m-1}_{s+\frac{3}{2}}\|r\|^{\Gamma_0\;1-b}_{s+\frac{3}{2}}\|r\|^{\Gamma_0\;b}_{s+|\alpha|+\frac{3}{2}} \\ &\leq C_{s,\alpha}M^m\|r\|^{\Gamma_0\;m-1}_{s+\frac{3}{2}}(\delta\|r\|^{\Gamma_0}_{s+|\alpha|+\frac{3}{2}} + C_{\delta,b}\|r\|^{\Gamma_0}_{s+\frac{3}{2}}). \end{aligned}$$

In the same way as in 3.2. we find from this

$$\sum_{m=2}^{\infty} \sum_{(l,k,\underline{\beta},\sigma)\in\mathcal{I}^{(3)}_{m,\alpha}} \mathcal{K}(l,k,m,\underline{\beta})T_{m,m,\underline{\beta},\sigma}\|D^\alpha r\|^{\Gamma_0}_{s+\frac{3}{2}} \leq \frac{c^{(1)}_s c^{(2)}_{n,s}}{4\mathcal{N}_n}\|r\|^{\Gamma_0}_{s+n+\frac{3}{2}}{}^2 + C_{s,n} \tag{5.7}$$

and the proof is completed by adding the inequalities (5.1)–(5.5) and (5.7), performing the summation over α and applying (5.6). ■

On the basis of this a priori estimate it is possible to give a short-time existence proof for the solution of (2.7), essentially in the same way as in [13]. Some modifications are necessary, due to the fact that the assumptions on ρ hold only in a neighbourhood of 0 in $H^{s+3}(\Gamma_0)$. Thus we will have work with Galerkin approximations that remain small in this space. The following lemma which generalizes the idea of diagonalizing the Gram matrix provides a preparation for this.

Lemma 10 *(Orthogonal basis for a pair of Sobolev spaces)*
Let $\sigma_1, \sigma_2 \in \mathbb{R}$, n_1, n_2 integer such that $n_2 + \sigma_2 > n_1 + \sigma_1$. There is an orthonormal basis of $H^{\sigma_1,n_1}(\Gamma_0)$, consisting of smooth functions, which is an orthogonal basis for $H^{\sigma_2,n_2}(\Gamma_0)$.

Proof: The unbounded linear operator S on $H^{\sigma_1,n_1}(\Gamma_0)$, defined by

$$\begin{aligned} D(S) &= H^{2(\sigma_2+n_2)-\sigma_1-n_1}(\Gamma_0), \\ S &= S^{-1}_{\sigma_1,n_1} S_{\sigma_2,n_2} \end{aligned}$$

satisfies

$$(u,v)_{\sigma_2,n_2} = (Su,v)_{\sigma_1,n_1} = (u,Sv)_{\sigma_1,n_1} \qquad \forall u,v \in D(S)$$

and is an elliptic pseudodifferential operator of order $2(\sigma_2 + n_2 - \sigma_1 - n_1)$. Hence $R(S) = H^{\sigma_1,n_1}(\Gamma_0)$ and thus S is self-adjoint. By Rellich's theorem, the compactness of the embedding $H^{\sigma_2,n_2}(\Gamma_0) \hookrightarrow\hookrightarrow H^{\sigma_1,n_1}(\Gamma_0)$ implies that S has a purely discrete spectrum and thus a complete orthonormal system of eigenfunctions $\{e_j\}$ which is obviously an orthogonal basis of $H^{\sigma_2,n_2}(\Gamma_0)$. The smoothness of the e_j follows from elliptic regularity theory. ■

Proposition 1 *(Short-time existence and uniqueness)*
Assume that Ω_0 is strictly star-shaped, $s > \frac{N+4}{2}$.

(i) There are positive constants ε and T such that for any integer $n > 3$ and any $r_0 \in B_0(\varepsilon, H^{s,3}(\Gamma_0)) \cap H^{s+n}(\Gamma_0)$ the initial value problem

$$\left.\begin{aligned} \frac{\partial r}{\partial t} &= \rho(r), \\ r(0) &= r_0 \end{aligned}\right\} \tag{5.8}$$

has a solution r in $C_w(IT, H^{s+n}(\Gamma_0)) \cap C^1_w(IT, H^{s+n-3}(\Gamma_0))$.

(ii) There are positive constants ε and T such that for any

$$r_0 \in B_0(\varepsilon, H^{s,3}(\Gamma_0))$$

(5.8) *has at most one solution in*

$$C^1(IT, H^{s+3}(\Gamma_0)) \cap L^\infty(IT, H^{s+\frac{9}{2}}(\Gamma_0)).$$

In both assertions, ε and T depend only on Ω_0 and s.

Here and in the sequel, we use the notations IT for the interval $[0,T]$ and $C_w^k(IT, X)$ for the space of the k times weakly continuously differentiable functions into the Banach space X, i.e. $g \in C_w^k(IT, X)$ iff the mapping $t \mapsto \langle \varphi, g(t) \rangle$ is in $C^k(IT)$ for all $\varphi \in X'$.

Proof: (i) The proof can be given as a modification of the proof of theorem A in [13]. We take

$$\begin{aligned} A(\cdot, t) &= -\rho, \\ \{V, H, X\} &= \{H^{s+n+3}(\Gamma_0), H^{s,n}(\Gamma_0), H^{s+n-3}(\Gamma_0)\} \\ \langle u, v \rangle &= \left(S_{s,n}^{\frac{s+n+3}{2(s+n)}} u, S_{s,n}^{\frac{s+n-3}{2(s+n)}} v \right)_0 \end{aligned}$$

and conclude that $\{V, H, X\}$ is an admissible triplet.

Choose ε small enough that $\|r\|_{s,n}^{\Gamma_0} \leq 2\varepsilon$ ensures by lemma 9 together with two interpolation inequalities

$$(\rho(r), r)_{s,3} \leq C_s \left(1 + \|r\|_{s,3}^{\Gamma_0\,2}\right) \tag{5.9}$$

$$(\rho(r), r)_{s,n} \leq C_{s,n} \left(1 + \|r\|_{s,n}^{\Gamma_0\,2}\right) \tag{5.10}$$

and by lemma 7 (ii) the weak sequential continuity of ρ.

By lemma 10 there is an orthonormal basis of $H^{s,3}(\Gamma_0)$ which is also an orthogonal basis of $H^{s,n}(\Gamma_0)$. Let P_j be the orthogonal projection in $H^{s,3}(\Gamma_0)$ onto $M_j = \operatorname{span}\{e_1, \ldots, e_j\}$. Clearly the restriction of P_j to $H^{s,n}(\Gamma_0)$ is the orthogonal projection onto M_j in $H^{s,n}(\Gamma_0)$. For all positive $j \in \mathbb{N}$ define the Galerkin approximations r_j as usual by

$$\frac{\partial r_j}{\partial t} = P_j \rho(r_j), \qquad r_j(0) = P_j r_0.$$

We have to prove now that there is a $T > 0$ such that

$$\|r_j(t)\|_{s,n}^{\Gamma_0} \leq K \qquad \forall t \in IT\ \forall j, \tag{5.11}$$

the proposition will follow then by the arguments in [13].

Consider the unique solution m of the initial value problem

$$\frac{\partial m}{\partial t} = 2C_s^*(1+m), \qquad m(0) = \varepsilon^2$$

where C_s^* is the constant from (5.9) and choose T to be the (uniquely defined) positive number for which $m(T) = 4\varepsilon^2$. At first we will show that

$$\|r_j(t)\|_{s,3}^{\Gamma_0} \leq 2\varepsilon \qquad \forall t \in IT \; \forall j. \tag{5.12}$$

Suppose the opposite: This would imply that for some j there is a $T^* \in (0, T)$ such that

$$\|r(T^*)\|_{s,3}^{\Gamma_0} = 2\varepsilon, \quad \|r(t)\|_{s,3}^{\Gamma_0} < 2\varepsilon \; \forall t \in [0, T).$$

On IT^* we get from (5.9) the differential inequality

$$\frac{d}{dt}\left(\|r_j(t)\|_{s,3}^{\Gamma_0\,2}\right) \leq C_s^*\left(1 + \|r_j(t)\|_{s,3}^{\Gamma_0\,2}\right)$$

and integrating it and using the strict monotonicity of m we find

$$\|r_j(T^*)\|_{s,3}^{\Gamma_0\,2} \leq m(T^*) < 4\varepsilon^2$$

in contradiction to the definition of T^*. Hence (5.12) holds, and on the basis of this it is easy to prove (5.11), using (5.10) instead of (5.9).

(ii) Proceeding similarly to the proof of lemma 9 we obtain

$$\begin{aligned}(\rho(r_1) - \rho(r_2), r_1 - r_2)_{s,3} \;\leq\;& -c_s\|r_1 - r_2\|_{s+\frac{9}{2}}^{\Gamma_0\,2} + C_s\|r_1 - r_2\|_{s+4}^{\Gamma_0\,2} \\ & + C_s \max_{i=1,2}\|r_i\|_{s+\frac{9}{2}}^{\Gamma_0}\|r_1 - r_2\|_{s+\frac{9}{2}}^{\Gamma_0}\|r_1 - r_2\|_{s+4}^{\Gamma_0}\end{aligned} \tag{5.13}$$

for all $r_1, r_2 \in B_0(\tilde{\varepsilon}, H^{s,3}(\Gamma_0)) \cap H^{s+\frac{9}{2}}(\Gamma_0)$ with $\tilde{\varepsilon} > 0$ sufficiently small. We choose $\varepsilon > 0$ small enough that $\|r_1\|_{s,3}^{\Gamma_0}, \|r_2\|_{s,3}^{\Gamma_0} \leq 2\varepsilon$ implies (5.13) and $\|r\|_{s,3}^{\Gamma_0} \leq 2\varepsilon$ implies (5.9).

Suppose now $r_1(\cdot)$, $r_2(\cdot) \in C^1(IT, H^{s+3}(\Gamma_0)) \cap L^\infty(IT, H^{s+\frac{9}{2}}(\Gamma_0))$ are two solutions of (5.8). From (5.9) one concludes $\|r_1(t)\|_{s,3}^{\Gamma_0}, \|r_2(t)\|_{s,3}^{\Gamma_0} < 2\varepsilon$ for all $t \in IT$ with a certain $T > 0$ in the same way as the corresponding estimates on the r_j in the proof of (i). Thus, (5.13) yields together with an interpolation inequality and the generalized Schwarz inequality

$$\begin{aligned}\frac{d}{dt}\left(\|r_1(t) - r_2(t)\|_{s,3}^{\Gamma_0\,2}\right) &= 2(\rho(r_1(t)) - \rho(r_2(t)), r_1(t) - r_2(t))_{s,3} \\ &\leq C_{r_1,r_2,s}\|r_1(t) - r_2(t)\|_{s,3}^{\Gamma_0\,2}\end{aligned}$$

for almost all $t \in IT$ and from the Gronwall inequality follows $r(t) = v(t)$ for all $t \in IT$. ■

Assuming higher smoothness it is possible to show by similar arguments the continuous dependence of solutions for fixed t on the initial value.

6 Near equilibrium

For the sake of brevity, we will restrict our attention in this section to the case $N = 3$, the generalization to arbitrary N is straightforward. Without loss of generality, we assume (see lemma 1) $V = \frac{4\pi}{3}$, $M = 0$. From lemma 1 (ii) it follows by a well-known theorem in global differential geometry [3] that the (unique) stationary solution corresponding to this choice is given by the unit ball centered at the origin. For the study of small perturbations of the equilibrium it is therefore natural to choose

$$\Omega_0 = B_0(1, \mathbb{R}^N), \qquad \zeta = n \tag{6.1}$$

which obviously satisfies all assumptions made above.

This implies $\Gamma_0 = S^2$, $\gamma \equiv 1$, $\mathcal{R}_0 \equiv 1$ and hence by (3.9)

$$D^\alpha \rho_1 = \rho_1 D^\alpha.$$

Elementary calculations yield

$$\kappa_1(h) = \Delta_{S^2} h + 2h$$

and the linearization of ρ becomes

$$\rho_1 = A(\Delta_{S^2} + 2I)$$

where A denotes the Dirichlet-to-Neumann operator on the unit sphere. Note that for both A and Δ_{S^2} a complete system of eigenfunctions in all $H^s(\Gamma_0)$, $s \in \mathbb{R}$ (with the standard scalar products (3.10)) is given by the spherical harmonics Y_{kl}, $l = 0, 1, 2, \ldots$, $k = -l, \ldots, l$. The corresponding eigenvalues are l for A and $-l(l+1)$ for Δ_{S^2}.

Furthermore, we define on all $H^s(\Gamma_0)$ the orthogonal projection $\mathcal{P}$ by

$$\mathrm{Im}(I - \mathcal{P}) = \mathrm{span}\{Y_{kl} \mid l < 2\}$$

and write

$$[u, v]_s = (\mathcal{P}u, \mathcal{P}v)_s, \quad |r|_s = \|\mathcal{P}r\|_s^{\Gamma_0}.$$

It is easily seen that $\mathcal{P}$ commutes both with D^α and with ρ_1. Hence, expanding r with respect to spherical harmonics, we find

$$-[\rho_1 h, h]_s \geq c|h|^2_{s+\frac{3}{2}} \tag{6.2}$$

where

$$c = \inf_{l \geq 2} \frac{l(l(l+1) - 2)}{(l(l+1)+1)^{\frac{3}{2}}} > 0.$$

The idea of the following arguments is to use the fact that no "lower order terms" occur in the energy estimates if one works with the seminorms $|\cdot|_s$ instead of the full norm. On the other hand, for actual evolutions this full norm can still be controlled due to the conservation of V and M.

Using spherical coordinates it is not difficult to obtain the expressions

$$\begin{aligned} V(r) &= \frac{1}{3}\int_{\Gamma_0} (1+r)^3\, d\Gamma_0, \\ M(r) &= \frac{1}{4}\int_{\Gamma_0} (1+r)^4 n\, d\Gamma_0 \end{aligned}$$

for the volume and the center of gravity of the domain Ω_r, respectively. On $H^s(\Gamma_0)$, $s > \frac{N-1}{2}$, we define the analytic function

$$F:\ H^s(\Gamma_0) \longrightarrow \mathbb{R}\times\mathbb{R}^3$$

by

$$F(r) = \begin{bmatrix} V(r) - \frac{4}{3}\pi \\ M(r) \end{bmatrix}.$$

and note that $F(0) = 0$,

$$F'(0)[h] = \begin{bmatrix} \int_{\Gamma_0} h\, d\Gamma_0 \\ \int_{\Gamma_0} hn\, d\Gamma_0 \end{bmatrix}.$$

For all $s > \frac{N-1}{2}$ we define

$$\mathcal{M}_s = \{r \in H^s(\Gamma_0) \mid F(r) = 0\}$$

Lemma 11 *(Norms and seminorms on $H^s(\Gamma_0)$ and $\mathcal{M}_s$)*
There are positive constants C and ε depending only on s such that

$$\begin{aligned} \|r\|_s^{\Gamma_0} &\le C(|r|_s + \|F(r)\|_{\mathbb{R}\times\mathbb{R}^3}) \quad \forall r \in B_0(\varepsilon, H^s(\Gamma_0)), && (6.3)\\ \|r\|_s^{\Gamma_0} &\le (1 + C|r|_s)|r|_s \quad \forall r \in B_0(\varepsilon, H^s(\Gamma_0)) \cap \mathcal{M}_s. && (6.4) \end{aligned}$$

Proof: (6.3) is a consequence of the Local Diffeomorphism theorem applied to the mapping

$$\Phi:\ H^s(\Gamma_0) \longrightarrow \mathcal{P}[H^s(\Gamma_0)] \times (\mathbb{R}\times\mathbb{R}^3)$$

defined by

$$\Phi(r) = \begin{bmatrix} \mathcal{P}r \\ F(r) \end{bmatrix}$$

in the neighbourhood of 0.

Due to the orthogonality of $\mathcal{P}$ we have

$$\|r\|_s^{\Gamma_0\,2} = |r|_s^2 + \|\overline{r}\|_s^{\Gamma_0\,2}$$

with $\overline{r} = (I - \mathcal{P})r$.

Because of $r \in \mathcal{M}_s$, $\overline{r}$ satisfies the equation

$$\tilde{F}(\mathcal{P}r, \overline{r}) = F(\mathcal{P}(r) + \overline{r}) = 0.$$

This implies by the Implicit Function theorem that $\bar{r}$ is a function of $\mathcal{P}r$, and using that the Frechet derivative of $\tilde{F}$ with respect to the first argument at $(0,0)$ is the zero operator we find $||\bar{r}||_s^{\Gamma_0} \leq C|r|_s^2$ if $|r|_s$ is sufficiently small. (6.4) follows easily from this. ■

Remark: It is obvious that the lemma holds (with different constants) for any equivalent norm on $H^s(\Gamma_0)$.

In the following, we use the notations

$$[u,v]_{s,n} = (\mathcal{P}u, \mathcal{P}v)_{s,n}, \quad |u|_{s,n} = ||\mathcal{P}u||_{s,n}^{\Gamma_0}.$$

Lemma 12 *(A priori estimate near equilibrium)*

Suppose $s > \frac{N+4}{2}$ and (6.1). *There are positive constants ε, c, and C depending on s such that*

$$[\rho(r), r]_{s,3} \leq -c|r|_{s,3}^2 + C||F(r)||_{\mathbb{R}\times\mathbb{R}^3}^2 \quad \forall r \in B_0(\varepsilon, H^{s,3}(\Gamma_0))$$

Proof: Proceeding as in the proof of lemma 9 and using $\rho(0) = 0$, (6.2), (6.3), and the fact that D^α, $\mathcal{P}$, and ρ_1 commute we obtain

$$\begin{aligned}
[\rho(r), r]_{s,3} &\leq \sum_{|\alpha|\leq 3} \left([D^\alpha \rho_1 r, D^\alpha r]_s + \sum_{k\geq 2} [D^\alpha \rho_k(r,\ldots,r), D^\alpha r]_s \right) \\
&\leq -c \sum_{|\alpha|\leq 3} |D^\alpha r|_{s+\frac{3}{2}}^2 + C \sum_{k\geq 2} {||r||_{s+3}^{\Gamma_0}}^k {||r||_{s+\frac{9}{2}}^{\Gamma_0}}^2 \\
&\leq -c|r|_{s+\frac{9}{2}}^2 + C \sum_{k\geq 2} {||r||_{s+3}^{\Gamma_0}}^k \left(|r|_{s+\frac{9}{2}}^2 + ||F(r)||_{\mathbb{R}\times\mathbb{R}^3}^2 \right)
\end{aligned}$$

and the assertion follows by choosing ε sufficiently small. ■

Proposition 2 *(Global existence and exponential decay near equilibrium)*

Under the assumptions (6.1) *and $s > \frac{N+4}{2}$ there are positive constants ε, c, and C such that for all*

$$r_0 \in \mathcal{M}_{s+7} \cap B_0(\varepsilon, H^{s,3}(\Gamma_0))$$

the initial value problem (5.8) *has a solution*

$$r \in C_w([0,\infty), H^{s+7}(\Gamma_0)) \cap C_w^1([0,\infty), H^{s+4}(\Gamma_0))$$

that satisfies an estimate

$$||r(t)||_{s,3}^{\Gamma_0} \leq Ce^{-ct}||r_0||_{s,3}^{\Gamma_0}. \tag{6.5}$$

Proof: From theorem 1 follows that if ε is sufficiently small there is a $T > 0$ such that (5.8) has a solution

$$r \in C_w(IT, H^{s+7}(\Gamma_0)) \cap C_w^1(IT, H^{s+4}(\Gamma_0))$$

which implies $r, \mathcal{P}r \in C^1(IT, H^{s+3}(\Gamma_0))$. Moreover, due to lemma 1 we have $r(t) \in \mathcal{M}_{s+7}$ for all $t \in IT$, and thus from lemma 12 follows

$$|r(t)|_{s,3} \leq |r_0|_{s,3}\, e^{-ct} \tag{6.6}$$

for all $t \in IT$ and further, using (6.4),

$$\|r(T)\|^{\Gamma_0}_{s,3} \leq (1 + C|r(T)|_{s,3})|r(T)|_{s,3} \leq (1 + C\varepsilon)e^{-cT}|r_0|_{s,3} < \varepsilon$$

if ε is sufficiently small. Hence it is possible to extend the solution to $[T, 2T]$ and by induction to $[nT, (n+1)T]$ for all integer n. It is clear that (6.6) holds then for all $t \geq 0$, and (6.5) follows from this and (6.3). ■

Remark: The assumption $r_0 \in H^{s+7}(\Gamma_0)$ is not necessary;

$$r_0 \in \mathcal{M}_{s+3} \cap B_0(\varepsilon, H^{s,3}(\Gamma_0))$$

is sufficient for the existence of a global solution in time satisfying the estimate (6.5). The higher smoothness has been assumed here to render possible the above simple proof. A proof without it can be given as in [11], applying lemma 12 to the Galerkin approximations in order to obtain (6.6).

Acknowledgements

The author is indebted to Prof. M. Günther for introducing him to various crucial techniques used in this article and to Prof. J. de Graaf for a lot of stimulating discussions. The reference yielding the proof of lemma 5 has been given by Dr. A. F. M. ter Elst.

References

[1] ABERGEL, F. 1993 A geometric approach to the study of stationary free surface flows for viscous liquids; *Proc. Royal Soc. Edinburgh* **123A**, 209–229

[2] ABERGEL, F. & ROUY, E. 1995 Interfaces stationnaires pour les équations de Navier-Stokes. (in French) *INRIA research report No. 2651*

[3] BLASCHKE, W. & LEICHTWEISS, K. 1973 *Elementare Differentialgeometrie.* (in German) Springer Verlag

[4] CHEN, X. 1993 The Hele-Shaw Problem and Area-preserving Curve-shortening Motions. *Arch. Rat. Mech. Anal.* **123**, 117–151

[5] CONSTANTIN, P. & KADANOFF, L. 1991 Dynamics of a Complex Interface. *Physica* **47D** 3, 450–460

[6] CONSTANTIN, P. & PUGH, M. 1993 Global solution for small data to the Hele-Shaw problem *Nonlinearity* **6**, 393–415

[7] DIEUDONNÉ, J. 1970 *Foundations of modern analysis.* Academic Press

[8] DUCHON, J. & ROBERT, R. 1984 Évolution d'une interface par capillarité et diffusion de volume I. Existence locale en temps. (in French) *Ann. Inst. Henri Poincaré Analyse non linéaire* **1** 5 361–378

[9] ELST, A. F. M. TER & ROBINSON, D. W. 1994 Subcoercivity and subelliptic operators on Lie Groups I: Free nilpotent groups. *Potential analysis* **3**, 283–337

[10] GILBARG, D. & TRUDINGER, N. S. 1977 *Elliptic partial differential equations of second order.* Springer Verlag

[11] GÜNTHER, M. & PROKERT, G. Existence results for the quasistationary motion of a viscous capillary liquid drop. Submitted to *Zeitschrift für Analysis und ihre Anwendungen*

[12] HOWISON, S. D. 1992 Complex variable methods in Hele-Shaw moving boundary problems. *Euro Jnl of Applied Mathematics* **3**, 209–224

[13] KATO, T. & LAI, C. Y. 1984 Nonlinear Evolution Equations and the Euler Flow. *Journ. of Funct. Anal.* **56**, 15–28

[14] LIONS, J.-L. & MAGENES, E. 1972 *Non-homogeneous boundary value problems and applications I.* Springer Verlag

[15] MULLINS, W. W. 1960 Grain boundary grooving by volume diffusion. *Transactions of the metallurgical society of AIME* **218**. 354–361

[16] ZEIDLER, E. 1988 *Nonlinear functional analysis and its applications I.* Springer Verlag

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