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Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions

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Abstract
In this paper, we are concerned with the existence and uniqueness of solutions for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary condition. Our results are based on the Banach contraction mapping principle and the Krasnoselskii fixed point theorem. Some examples are also given to illustrate our results.

Keywords: antiperiodic boundary value problems; impulsive; fractional integro-differential equations; existence results

1 Introduction

Fractional differential equations appear naturally in a number of fields such as physics, chemistry, electromagnetic, engineering, control, and other branches; see [1–16] and the references therein. Fractional differential equations have recently gained much importance and attention. The study of fractional differential equations ranges from the theoretical aspects of the existence of solutions to the analytic and numerical methods for finding solutions.

Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden discontinuous jumps occur. Such processes are naturally seen in physics, engineering, biology, and so on. Due to their significance, it is important to study the solvability of impulsive differential equations. Impulsive differential equations of fractional order have not been much studied, and many aspects of these equations are yet to be explored. The recent results on impulsive fractional differential equations can be found in [17–32] and the references therein.

Recently, the boundary value problem of impulsive fractional differential equations with antiperiodic boundary conditions have been studied in the literature; see [33–39]. The authors of [36–38] investigated the following antiperiodic boundary value problem for impulsive differential equations of fractional order:

$$\begin{cases} {}^{c}D^{q}u(t) = f(t,u(t)), & t \in [0,T] \setminus \{t_1,t_2,\ldots,t_m\}, 1 < q \le 2, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad \Delta u'|_{t=t_k} = J_k(u(t_k)), & k = 1,2,\ldots,m, \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{cases}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order q, $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is continuous, $I_k, J_k : \mathbf{R} \to \mathbf{R}$, $\mathbf{R} = (-\infty,+\infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. By applying the Banach contraction mapping principle, Krasnoselskii fixed point theorem, Schaefer fixed point theorem, and a nonlinear alternative of the Leray-Schauder-type theorem, some existence results of solutions are obtained.

However, the existence and uniqueness of solutions to impulsive fractional differential equations for antiperiodic boundary value problems with constant coefficients seem to be rarely involved. It should be pointed out that Kilbas et al. (see (3.1.32)-(3.1.34) in [1]) obtained that the solution u of the linear fractional differential equation with constant coefficients

$$\begin{cases} {}^{c}D^{q}u(t) + \lambda u(t) = h(t), & t \in [0,1], 0 < q < 1, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

is given by

$$u(t) = E_q\bigl(-t^q\lambda\bigr)u_0 + \int_0^t (t-s)^{q-1}E_{q,q}\bigl(-(t-s)^q\lambda\bigr)h(s)\,ds, \quad t \in [0,1],$$

where E_q and $E_{q,q}$ are the so-called classical and generalized Mittag-Leffler functions.

More recently, Wang and Lin [40] studied antiperiodic boundary value problems for impulsive fractional differential equations with constant coefficients

$$\begin{cases} {}^{c}D^{q}u(t) + \lambda u(t) = f(t,u(t)), & t \in J' = J \setminus \{t_1,t_2,\ldots,t_m\}, 0 < q < 1, \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = y_k, & k = 1,2,\ldots,m, \\ u(0) = -u(1), \end{cases} \tag{1.2}$$

where $\lambda > 0$, $y_k \in \mathbf{R}$, ${}^{c}D^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$, $f : J \times \mathbf{R} \to \mathbf{R}$, $J = [0,1]$, and the fixed impulsive times t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. By means of fixed point theorems, some sufficient conditions on the existence and uniqueness of solutions for problem (1.2) are established under Lipschitz and nonlinear growth conditions.

Motivated by the works mentioned and many known results, in this paper, we are concerned with the existence and uniqueness of solutions for impulsive fractional integro-differential equation of mixed type with constant coefficients and antiperiodic boundary condition

$$\begin{cases} {}^{c}D^{q}u(t) + \lambda u(t) = f(t,u(t),Tu(t),Su(t)), & t \in J' = J \setminus \{t_1,t_2,\ldots,t_m\}, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1,2,\ldots,m, \\ u(0) = -u(1), \end{cases} \tag{1.3}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $q \in (0,1)$, $\lambda > 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $f \in C(J \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$, $J = [0,1]$, $\mathbf{R}$ is the set of real numbers, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, that is, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, T and S are the linear operators defined by

$$(Tu)(t) = \int_0^t k(t,s)u(s)\,ds \quad \text{and} \quad (Su)(t) = \int_0^1 h(t,s)u(s)\,ds, \quad t \in J,$$

where $k \in C(D, \mathbf{R})$, $D = \{(t,s) \in J \times J : t \geq s\}$, and $h \in C(J \times J, \mathbf{R})$.

At present, the concept of solutions for impulsive fractional differential equations has been argued extensively. There are some ways to consider the notion of solution for impulsive fractional differential equations; for example, see [29–32]. In this paper, we adopt the formula of the solution in Lemma 2.4, which comes from [40].

This paper is arranged as follows. In Section 2, we present some definitions and preliminary lemmas. In Section 3, we establish the existence and uniqueness of solutions for the boundary value problem (1.3) by using the Banach contraction mapping principle and Krasnoselskii fixed point theorem. Some illustrated examples are presented in Section 4.

2 Preliminaries and lemmas

Let $J_0 = [0,t_1], J_1 = (t_1,t_2], \ldots, J_{m-1} = (t_{m-1},t_m], J_m = (t_m,1]$, and

$$\begin{aligned} PC(J,\mathbf{R}) = \bigl\{ &u : J \to \mathbf{R} : u \in C(J_k,\mathbf{R}), k = 0,1,2,\ldots,m, \\ &u\bigl(t_k^+\bigr) \text{ and } u\bigl(t_k^-\bigr) \text{ exist}, k = 1,\ldots,m, \text{and } u\bigl(t_k^-\bigr) = u(t_k)\bigr\}. \end{aligned}$$

Then $PC(J,\mathbf{R})$ is a Banach space with the norm $\|u\|_{PC} = \sup\{|u(t)| : t \in J\}$. For a measurable function $\mu : J \to \mathbf{R}$, define the norm

$$\|\mu\|_{L^p(J)} = \begin{cases} (\int_J |\mu(t)|^p\,dt)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf_{\text{mes}(\bar{J})=0}\{\sup_{t \in J \setminus \bar{J}} |u(t)|\}, & p = \infty. \end{cases}$$

Then $L^p(J,\mathbf{R})$ is the Banach space of Lebesgue-measurable functions $\mu : J \to \mathbf{R}$ with $\|\mu\|_{L^p(J)} < \infty$.

Definition 2.1 ([1]) The fractional integral of order α with lower limit zero for a function $f : [0,\infty) \to \mathbf{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $[0,+\infty)$.

Definition 2.2 ([1]) The Caputo derivative of order α for a function $f : [0,\infty) \to \mathbf{R}$ can be written as

$${}^{c}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s) - \sum_{k=0}^{n-1} \frac{s^k}{k!} f^{(k)}(0)}{(t-s)^{\alpha+1-n}}\,ds, \quad t > 0, n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of α.

Remark 2.1 ([30]) If $f \in C^n[0,+\infty)$, then

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds = I^{n-\alpha}f^{(n)}(t), \quad t>0, n=[\alpha]+1,$$

that is, Definition 2.2 is just the usual Caputo fractional derivative. In this paper, we consider an impulsive problem, so Definition 2.2 is appropriate.

Definition 2.3 A function $u \in PC(J,\mathbf{R})$ is said to be a solution of problem (1.3) if it satisfies the equation $^{c}D^{q}u(t) + \lambda u(t) = f(t,u(t),Su(t),Tu(t))$ a.e. on J' and the conditions $\Delta u|_{t=t_k} = I_k(u(t_k))$, $k=1,\dots,m$, and $u(0) = -u(1)$.

Lemma 2.1 ([41]) *The nonnegative functions E_q and $E_{q,q}$ given by*

$$E_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(qk+1)}, \qquad E_{q,q}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(qk+q)},$$

have the following properties:

(1) *For any $\lambda > 0$ and $t \in J$,*

$$E_q(-t^q\lambda) \le 1, \qquad E_{q,q}(-t^q\lambda) \le \frac{1}{\Gamma(q)}.$$

Moreover,

$$E_q(0) = 1, \qquad E_{q,q}(0) = \frac{1}{\Gamma(q)}.$$

(2) *For any $\lambda > 0$ and $t_1, t_2 \in J$,*

$$E_q(-t_2{}^q\lambda) \to E_q(-t_1{}^q\lambda) \quad \text{as } t_2 \to t_1,$$
$$E_{q,q}(-t_2{}^q\lambda) \to E_{q,q}(-t_1{}^q\lambda) \quad \text{as } t_2 \to t_1.$$

(3) *For any $\lambda > 0$ and $t_1, t_2 \in J$ such that $t_1 \le t_2$,*

$$E_q(-t_2{}^q\lambda) \le E_q(-t_1{}^q\lambda), \qquad E_{q,q}(-t_2{}^q\lambda) \le E_{q,q}(-t_1{}^q\lambda).$$

Lemma 2.2 ([42]) *Let M be a closed, convex, and nonempty subset of a Banach space X, and let A, B be operators such that*:

(1) *$Ax + By \in M$ whenever $x, y \in M$.*
(2) *A is compact and continuous.*
(3) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.3 ([43]) *Let X be a Banach space, and let $J = [0,T]$. Suppose that $W \subset PC(J,X)$ satisfies the following conditions*:

(1) *W is a uniformly bounded subset of $PC(J,X)$.*
(2) *W is equicontinuous in (t_k, t_{k+1}), $k = 0,1,\dots,m$, where $t_0 = 0$, $t_{m+1} = T$.*

(3) *Its t-sections* $W(t) = \{u(t) : u \in W, t \in J \setminus \{t_1, \ldots, t_m\}\}$, $W(t_k^+) = \{u(t_k^+) : u \in W\}$, *and* $W(t_k^-) = \{u(t_k^-) : u \in W\}$ *are relatively compact subsets of* X.

Then W *is a relatively compact subset of* $PC(J, X)$.

Lemma 2.4 ([40]) *Let* $h : J \to \mathbf{R}$ *be a continuous function. The function* u *given by*

$$u(t) = \begin{cases} \frac{-E_q(-\lambda)E_q(-t^q\lambda)}{1+E_q(-\lambda)} \sum_{i=1}^m \frac{y_i}{E_q(-t_i^q\lambda)} + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)h(s)\,ds \\ \quad - \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)h(s)\,ds, \quad t \in J_0, \\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \{\sum_{i=1}^m \frac{y_i}{E_q(-t_i^q\lambda)} - \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)h(s)\,ds\} \\ \quad - E_q(-t^q\lambda) \sum_{j=k+1}^m \frac{y_j}{E_q(-t_j^q\lambda)} \\ \quad + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)h(s)\,ds, \quad t \in J_k, k = 1,2,\ldots,m-1, \\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \{\sum_{i=1}^m \frac{y_i}{E_q(-t_i^q\lambda)} - \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)h(s)\,ds\} \\ \quad + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)h(s)\,ds, \quad t \in J_m, \end{cases}$$

is a unique solution of the impulsive problem

$$\begin{cases} {}^cD^q u(t) + \lambda u(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1,2,\ldots,m, \\ u(0) = -u(1). \end{cases}$$

It follows from Lemma 2.4 that the solution of (1.3) can be expressed by

$$u(t) = \begin{cases} \frac{-E_q(-\lambda)E_q(-t^q\lambda)}{1+E_q(-\lambda)} \sum_{i=1}^m \frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)} + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds \\ \quad - \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds, \quad t \in J_0, \\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \{\sum_{i=1}^m \frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)} - \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds\} \\ \quad - E_q(-t^q\lambda) \sum_{j=k+1}^m \frac{I_j(u(t_j))}{E_q(-t_j^q\lambda)} \\ \quad + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds, \\ \quad t \in J_k, k = 1,2,\ldots,m-1, \\ \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)} \{\sum_{i=1}^m \frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)} - \int_0^1 (1-s)^{q-1} E_{q,q}(-(1-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds\} \\ \quad + \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)f(s,u(s),Tu(s),Su(s))\,ds, \quad t \in J_m. \end{cases}$$

3 Main results

Theorem 3.1 *Assume that conditions* (H_1)-(H_3) *hold*:

(H_1) *There exist* $L_i(t) \in C(J, (0, +\infty))$ $(i = 1,2,3)$ *such that*

$$|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)| \le L_1(t)|u_1 - u_2| + L_2(t)|v_1 - v_2| + L_3(t)|w_1 - w_2|$$

for all $t \in J$ *and* $u_j, v_j, w_j \in \mathbf{R}$, $j = 1,2$.

(H_2) *There exists a constant* $L_4 > 0$ *such that*

$$|I_k(u) - I_k(v)| \le L_4|u - v|, \quad u, v \in \mathbf{R}, k = 1,2,\ldots,m.$$

(H_3)

$$\chi = \frac{3}{|1+E_q(-\lambda)|}\left(\sum_{i=1}^{m}\frac{L_4}{|E_q(-t_i^q\lambda)|}+\frac{(\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0)}{\Gamma(q+1)}\right)<1,$$

where $\overline{L_j}=\max\{L_j(t):t\in J\}$, $j=1,2,3$, $k_0=\max\{|k(t,s)|:(t,s)\in D\}$, *and* $h_0=\max\{|h(t,s)|:(t,s)\in J\times J\}$.

Then the boundary value problem (1.3) *has a unique solution*.

Proof Let $M=\sup\{|f(t,0,0,0)|:t\in J\}$, $M'=\max\{|I_i(0)|:i=1,2,\ldots,m\}$, and $B_r=\{u\in PC(J,\mathbf{R}):\|u\|_{PC}\le r\}$, where

$$r\ge\frac{\sum_{i=1}^{m}\frac{M'}{|E_q(-t_i^q\lambda)|}+\frac{M}{\Gamma(q+1)}}{\frac{|1+E_q(-\lambda)|}{3}-[\sum_{i=1}^{m}\frac{L_4}{|E_q(-t_i^q\lambda)|}+\frac{\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0}{\Gamma(q+1)}]}.$$

Define the operator $F:B_r\to PC(J,\mathbf{R})$ by

$$\begin{aligned}Fu(t)={}&\frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\left\{\sum_{i=1}^{m}\frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)}\right.\\&\left.-\int_0^1(1-s)^{q-1}E_{q,q}\bigl(-(1-s)^q\lambda\bigr)f\bigl(s,u(s),Tu(s),Su(s)\bigr)\,ds\right\}\\&-E_q\bigl(-t^q\lambda\bigr)\sum_{j=k+1}^{m}\frac{I_j(u(t_j))}{E_q(-t_j^q\lambda)}\\&+\int_0^t(t-s)^{q-1}E_{q,q}\bigl(-(t-s)^q\lambda\bigr)f\bigl(s,u(s),Tu(s),Su(s)\bigr)\,ds,\\&t\in J_k,k=0,1,2,\ldots,m.\end{aligned}$$

First, we show that $F(B_r)\subset B_r$. For any $u\in B_r$ and $t\in J$, by Lemma 2.1 we have

$$\begin{aligned}&\bigl|(Fu)(t)\bigr|\\&\le\bigl|E_q\bigl(-t^q\lambda\bigr)\bigr|\left|\frac{1}{1+E_q(-\lambda)}\left\{\sum_{i=1}^{m}\frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)}\right.\right.\\&\quad\left.\left.-\int_0^1(1-s)^{q-1}E_{q,q}\bigl(-(1-s)^q\lambda\bigr)f\bigl(s,u(s),Tu(s),Su(s)\bigr)\,ds\right\}-\sum_{j=k+1}^{m}\frac{I_j(u(t_j))}{E_q(-t_j^q\lambda)}\right|\\&\quad+\left|\int_0^t(t-s)^{q-1}E_{q,q}\bigl(-(t-s)^q\lambda\bigr)f\bigl(s,u(s),Tu(s),Su(s)\bigr)\,ds\right|\\&\le\frac{1}{|1+E_q(-\lambda)|}\left\{\sum_{i=1}^{m}\frac{|I_i(u(t_i))|}{|E_q(-t_i^q\lambda)|}+\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}\bigl|f\bigl(s,u(s),Tu(s),Su(s)\bigr)\bigr|\,ds\right\}\\&\quad+\sum_{i=1}^{m}\frac{|I_i(u(t_i))|}{|E_q(-t_i^q\lambda)|}+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\bigl|f\bigl(s,u(s),Tu(s),Su(s)\bigr)\bigr|\,ds\\&\le\frac{1+|1+E_q(-\lambda)|}{|1+E_q(-\lambda)|}\left\{\sum_{i=1}^{m}\frac{|I_i(u(t_i))-I_i(0)|+M'}{|E_q(-t_i^q\lambda)|}\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)|1+E_q(-\lambda)|}\int_0^1(1-s)^{q-1}\big|f\big(s,u(s),Tu(s),Su(s)\big)-f(s,0,0,0)\big|\,ds\\
&+\frac{1}{\Gamma(q)|1+E_q(-\lambda)|}\int_0^1(1-s)^{q-1}\big|f(s,0,0,0)\big|\,ds\\
&+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\big|f\big(s,u(s),Tu(s),Su(s)\big)-f(s,0,0,0)\big|\,ds\\
&+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\big|f(s,0,0,0)\big|\,ds\\
\le{}&\frac{3}{|1+E_q(-\lambda)|}\sum_{i=1}^m\frac{L_4r+M'}{|E_q(-t_i^q\lambda)|}+\frac{M}{\Gamma(q+1)|1+E_q(-\lambda)|}+\frac{M}{\Gamma(q+1)}\\
&+\frac{1}{\Gamma(q)|1+E_q(-\lambda)|}\int_0^1(1-s)^{q-1}\big[L_1(s)\big|u(s)\big|+L_2(s)\big|Tu(s)\big|+L_3(s)\big|Su(s)\big|\big]\,ds\\
&+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\big[L_1(s)\big|u(s)\big|++L_2(s)\big|Tu(s)\big|+L_3(s)\big|Su(s)\big|\big]\,ds\\
\le{}&\frac{3}{|1+E_q(-\lambda)|}\left\{\sum_{i=1}^m\frac{L_4r+M'}{|E_q(-t_i^q\lambda)|}+\frac{M}{\Gamma(q+1)}\right\}\\
&+\frac{1}{\Gamma(q)|1+E_q(-\lambda)|}\int_0^1(1-s)^{q-1}(\overline{L_1}r+\overline{L_2}k_0r+\overline{L_3}h_0r)\,ds\\
&+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(\overline{L_1}r+\overline{L_2}k_0r+\overline{L_3}h_0r)\,ds\\
\le{}&\frac{3}{|1+E_q(-\lambda)|}\left\{\sum_{i=1}^m\frac{M'}{|E_q(-t_i^q\lambda)|}+\frac{M}{\Gamma(q+1)}\right.\\
&\left.+\left[\sum_{i=1}^m\frac{L_4}{|E_q(-t_i^q\lambda)|}+\frac{\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0}{\Gamma(q+1)}\right]r\right\}\\
\le{}&r.
\end{aligned}
$$

Hence $F(B_r)\subset B_r$.

Next, we show that the operator F is a contraction mapping. For any $t\in J$ and $u,v\in B_r$, we obtain

$$
\begin{aligned}
&\big|(Fu)(t)-(Fv)(t)\big|\\
&=\left|\frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\left\{\sum_{i=1}^m\frac{I_i(u(t_i))-I_i(v(t_i))}{E_q(-t_i^q\lambda)}\right.\right.\\
&\quad\left.-\int_0^1(1-s)^{q-1}E_{q,q}\big(-(1-s)^q\lambda\big)\big(f\big(s,u(s),Tu(s),Su(s)\big)-f\big(s,v(s),Tv(s),Sv(s)\big)\big)\,ds\right\}\\
&\quad-E_q\big(-t^q\lambda\big)\sum_{j=k+1}^m\frac{I_j(u(t_j))-I_j(v(t_j))}{E_q(-t_j^q\lambda)}\\
&\quad\left.+\int_0^t(t-s)^{q-1}E_{q,q}\big(-(t-s)^q\lambda\big)\big(f\big(s,u(s),Tu(s),Su(s)\big)-f\big(s,v(s),Tv(s),Sv(s)\big)\big)\,ds\right|\\
&\le\left(\frac{1}{1+|E_q(-\lambda)|}+1\right)\sum_{i=1}^m\frac{L_4|u(t_i)-v(t_i)|}{|E_q(-t_i^q\lambda)|}+\frac{1}{\Gamma(q)|1+E_q(-\lambda)|}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \int_0^1 (1-s)^{q-1}\Big\{L_1(s)\big|u(s)-v(s)\big| + L_2(s)\int_0^s \big|k(s,\tau)\big|\big|u(\tau)-v(\tau)\big|\,d\tau \\
&+ L_3(s)\int_0^1 \big|h(s,\tau)\big|\big|u(\tau)-v(\tau)\big|\,d\tau\Big\}\,ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big\{L_1(s)\big|u(s)-v(s)\big| \\
&+ L_2(s)\int_0^s \big|k(s,\tau)\big|\big|u(\tau)-v(\tau)\big|\,d\tau + L_3(s)\int_0^1 \big|h(s,\tau)\big|\big|u(\tau)-v(\tau)\big|\,d\tau\Big\}\,ds \\
\le{}& \frac{3}{|1+E_q(-\lambda)|}\sum_{i=1}^m \frac{L_4\|u-v\|_{PC}}{|E_q(-t_i^q\lambda)|} + \frac{1}{\Gamma(q)|1+E_q(-\lambda)|} \\
&\cdot \int_0^1 (1-s)^{q-1}\big(\overline{L_1}\|u-v\|_{PC} + \overline{L_2}k_0\|u-v\|_{PC} + \overline{L_3}h_0\|u-v\|_{PC}\big)\,ds \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big(\overline{L_1}\|u-v\|_{PC} + \overline{L_2}k_0\|u-v\|_{PC} + \overline{L_3}h_0\|u-v\|_{PC}\big)\,ds \\
\le{}& \frac{3}{|1+E_q(-\lambda)|}\left(\sum_{i=1}^m \frac{L_4}{|E_q(-t_i^q\lambda)|} + \frac{(\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0)}{\Gamma(q+1)}\right)\|u-v\|_{PC} \\
={}& \chi\|u-v\|_{PC}.
\end{aligned}
$$

Thus $\|Fu-Fv\|_{PC} \le \chi\|u-v\|_{PC}$. Then from the Banach contraction mapping principle it follows that problem (1.3) has a unique solution. This completes the proof. □

Theorem 3.2 *Assume that condition* (H_2) *and the following conditions* (H_4)-(H_5) *hold*:

(H_4) *There exist a function* $\mu \in L^{\frac{1}{\sigma}}(J,(0,+\infty))$ $(0<\sigma<q<1)$ *and a nondecreasing function* $\overline{\omega} \in C([0,\infty),(0,+\infty))$ *such that*

$$\big|f\big(t,u(t),Tu(t),Su(t)\big)\big| \le \mu(t)\overline{\omega}\big(\|u\|_{PC}\big), \quad u \in PC(J,\mathbf{R}), t \in J.$$

(H_5)

$$\frac{3}{|1+E_q(-\lambda)|}\left(\frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}}\liminf_{r\to+\infty}\frac{\overline{\omega}(r)}{r} + \sum_{i=1}^m \frac{L_4}{|E_q(-t_i^q\lambda)|}\right) < 1.$$

Then the boundary value problem (1.3) *has at least one solution*.

Proof For $r>0$, the set $B_r = \{u \in PC(J,\mathbf{R}) : \|u\|_{PC} \le r\}$ is a bounded closed convex set in $PC(J,\mathbf{R})$. Define the operators P and Q on B_r as

$$
\begin{aligned}
(Pu)(t) ={}& \int_0^t (t-s)^{q-1}E_{q,q}\big(-(t-s)^q\lambda\big)f\big(s,u(s),Tu(s),Su(s)\big)\,ds \\
&- \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\int_0^1 (1-s)^{q-1}E_{q,q}\big(-(1-s)^q\lambda\big)f\big(s,u(s),Tu(s),Su(s)\big)\,ds, \\
(Qu)(t) ={}& \frac{E_q(-t^q\lambda)}{1+E_q(-\lambda)}\sum_{i=1}^m \frac{I_i(u(t_i))}{E_q(-t_i^q\lambda)} - E_q\big(-t^q\lambda\big)\sum_{j=k+1}^m \frac{I_j(u(t_j))}{E_q(-t_j^q\lambda)}.
\end{aligned}
$$

By (H_4) and the Hölder inequality, for any $u \in B_r$, we have

$$\begin{aligned}
&\int_0^t \left|(t-s)^{q-1} f\left(s, u(s), Tu(s), Su(s)\right)\right| ds \\
&\quad \leq \int_0^t \left|(t-s)^{q-1} \mu(s) \overline{\omega}(r)\right| ds \\
&\quad \leq \left(\int_0^t (t-s)^{\frac{q-1}{1-\sigma}} ds\right)^{1-\sigma} \left(\int_0^t \left(\overline{\omega}(r)\mu(s)\right)^{\frac{1}{\sigma}} ds\right)^{\sigma} \\
&\quad \leq \frac{t^{q-\sigma}}{(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r) \|\mu\|_{L^{\frac{1}{\sigma}}(J)} \\
&\quad \leq \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r).
\end{aligned}$$

Similarly, we have

$$\int_0^1 \left|(1-s)^{q-1} f\left(s, u(s), Tu(s), Su(s)\right)\right| ds \leq \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r).$$

Next, we show that there exists $r_0 > 0$ with $Pu + Qv \in B_{r_0}$ for $u, v \in B_{r_0}$. If this were not true, then, for each $r > 0$, there would exist $u_r, v_r \in B_r$ and $t_r \in J$ such that $|(Pu_r)(t_r) + (Qv_r)(t_r)| > r$. Assumption ($H_2$) implies $|I_i(u(t_i))| \leq |I_i(u(t_i)) - I_i(0)| + |I_i(0)| \leq L_4 r + M'$. Hence

$$\begin{aligned}
r &< \left|(Pu_r)(t_r) + (Qv_r)(t_r)\right| \\
&\leq \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)|1 + E_q(-\lambda)|(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r) \\
&\quad + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r) + \frac{1}{|1 + E_q(-\lambda)|} \sum_{i=1}^m \frac{L_4 r + M'}{|E_q(-t_i{}^q \lambda)|} + \sum_{i=1}^m \frac{L_4 r + M'}{|E_q(-t_i{}^q \lambda)|} \\
&= \left(\frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r) + \sum_{i=1}^m \frac{L_4 r + M'}{|E_q(-t_i{}^q \lambda)|}\right)\left(1 + \frac{1}{|1 + E_q(-\lambda)|}\right) \\
&\leq \frac{3}{|1 + E_q(-\lambda)|}\left(\frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \overline{\omega}(r) + \sum_{i=1}^m \frac{L_4 r + M'}{|E_q(-t_i{}^q \lambda)|}\right).
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to +\infty$, we obtain

$$1 \leq \frac{3}{|1 + E_q(-\lambda)|}\left(\frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} \liminf_{r\to\infty} \frac{\overline{\omega}(r)}{r} + \sum_{i=1}^m \frac{L_4}{|E_q(-t_i^q \lambda)|}\right),$$

which contradicts condition (H_5). Thus, there exists $r_0 > 0$ such that $Pu + Qv \in B_{r_0}$ for all $u, v \in B_{r_0}$.

For all $t \in J$ and $u, v \in B_r$, we get

$$\begin{aligned}
&\left|(Qu)(t) - (Qv)(t)\right| \\
&\quad \leq \frac{|E_q(-t^q \lambda)|}{|1 + E_q(-\lambda)|} \sum_{i=1}^m \frac{|I_i(u(t_i)) - I_i(v(t_i))|}{|E_q\left(-t_i^q \lambda\right)|}
\end{aligned}$$

$$
\begin{aligned}
&+ \left|E_q\left(-t^q\lambda\right)\right| \sum_{i=1}^{m} \frac{|I_i(u(t_i)) - I_i(v(t_i))|}{|E_q(-t_i^q\lambda)|} \\
&\le \sum_{i=1}^{m} \frac{|I_i(u(t_i)) - I_i(v(t_i))|}{|E_q(-t_i^q\lambda)|}\left(1 + \frac{1}{|1 + E_q(-\lambda)|}\right) \\
&\le \frac{3}{|1 + E_q(-\lambda)|} \sum_{i=1}^{m} \frac{L_4|u(t_i) - v(t_i)|}{|E_q(-t_i^q\lambda)|} \\
&\le \frac{3}{|1 + E_q(-\lambda)|} \sum_{i=1}^{m} \frac{L_4\|u - v\|_{PC}}{|E_q(-t_i^q\lambda)|}.
\end{aligned}
$$

Let $\chi' = \frac{3}{|1+E_q(-\lambda)|} \sum_{i=1}^{m} \frac{L_4}{|E_q(-t_i^q\lambda)|}$. From ($H_5$) we have $0 < \chi' < 1$ and $\|Qu - Qv\|_{PC} \le \chi'\|u - v\|_{PC}$, so Q is a contraction mapping.

The continuity of f implies that the operator P is continuous. We now prove that P is a compact operator. Following the procedure used in the first part of Theorem 3.1, it follows that $P(B_r)$ is uniformly bounded on $PC(J,\mathbf{R})$. We now show that $P(B_r)$ is equicontinuous on J_k ($k = 1,\dots,m$). Let $\Omega = J \times B_r \times TB_r \times SB_r$ and $\overline{f} = \sup_{(t,u,Tu,Su)\in\Omega} |f(t,u,Tu,Su)|$. Then, for any $t_k < \tau_2 < \tau_1 \le t_{k+1}$, we have

$$
\begin{aligned}
&\left|(Pu)(\tau_2) - (Pu)(\tau_1)\right| \\
&\quad\le \left|\int_0^{\tau_2} (\tau_2 - s)^{q-1} E_{q,q}\left(-(\tau_2 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \right. \\
&\qquad \left. - \int_0^{\tau_1} (\tau_1 - s)^{q-1} E_{q,q}\left(-(\tau_1 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \right| \\
&\qquad + \left|\frac{E_q(-\tau_2{}^q\lambda) - E_q(-\tau_1{}^q\lambda)}{1 + E_q(-\lambda)} \int_0^{1} (1 - s)^{q-1} E_{q,q}\left(-(1 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds\right| \\
&\quad\le \left|\int_0^{\tau_2} (\tau_2 - s)^{q-1} E_{q,q}\left(-(\tau_2 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \right. \\
&\qquad - \int_0^{\tau_2} (\tau_1 - s)^{q-1} E_{q,q}\left(-(\tau_2 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \\
&\qquad + \int_0^{\tau_2} (\tau_1 - s)^{q-1} E_{q,q}\left(-(\tau_2 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \\
&\qquad - \int_0^{\tau_2} (\tau_1 - s)^{q-1} E_{q,q}\left(-(\tau_1 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \\
&\qquad \left. - \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} E_{q,q}\left(-(\tau_1 - s)^q\lambda\right) f\left(s, u(s), Tu(s), Su(s)\right) ds \right| \\
&\qquad + \frac{|E_q(-\tau_2{}^q\lambda) - E_q(-\tau_1{}^q\lambda)|}{\Gamma(q)|1 + E_q(-\lambda)|} \int_0^{1} (1 - s)^{q-1} \left|f\left(s, u(s), Tu(s), Su(s)\right)\right| ds \\
&\quad\le \int_0^{\tau_2} \left|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\right| \left|E_{q,q}\left(-(\tau_2 - s)^q\lambda\right)\right| \overline{f}\, ds \\
&\qquad + \int_0^{\tau_2} (\tau_1 - s)^{q-1} \left|E_{q,q}\left(-(\tau_2 - s)^q\lambda\right) - E_{q,q}\left(-(\tau_1 - s)^q\lambda\right)\right| \overline{f}\, ds \\
&\qquad + \frac{\overline{f}}{\Gamma(q)} \left|\int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1}\, ds\right| + \frac{|E_q(-\tau_2{}^q\lambda) - E_q(-\tau_1{}^q\lambda)|\overline{f}}{\Gamma(q + 1)|1 + E_q(-\lambda)|} \\
&\quad\le \frac{\overline{f}}{\Gamma(q)} \left|\int_0^{\tau_2} \left((\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\right) ds\right| + \frac{(\tau_1 - \tau_2)^q \overline{f}}{\Gamma(q + 1)} + \frac{|E_q(-\tau_2{}^q\lambda) - E_q(-\tau_1{}^q\lambda)|\overline{f}}{\Gamma(q + 1)|1 + E_q(-\lambda)|}
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{f}\int_0^{\tau_2}(\tau_1-s)^{q-1}\big|E_{q,q}\big(-(\tau_2-s)^q\lambda\big)-E_{q,q}\big(-(\tau_1-s)^q\lambda\big)\big|\,ds\\
\le{}& \frac{(\tau_1-\tau_2)^q+\tau_1^q-\tau_2^q}{\Gamma(q+1)}\overline{f}+\frac{(\tau_1-\tau_2)^q\overline{f}}{\Gamma(q+1)}+\frac{|E_q(-\tau_2{}^q\lambda)-E_q(-\tau_1{}^q\lambda)|\overline{f}}{\Gamma(q+1)|1+E_q(-\lambda)|}\\
&+\overline{f}\int_0^{\tau_2}(\tau_1-s)^{q-1}\big|E_{q,q}\big(-(\tau_2-s)^q\lambda\big)-E_{q,q}\big(-(\tau_1-s)^q\lambda\big)\big|\,ds.
\end{aligned}
$$

By Lemma 2.1(2) we know that $E_{q,q}(-t^q\lambda)$ is continuous on $t\in J$, and thus $E_{q,q}(-t^q\lambda)$ is uniformly continuous on $t\in J$. Hence, for any $\varepsilon>0$, there is a sufficiently small $\delta>0$ such that, for $t_1,t_2\in J$ with $|t_1-t_2|<\delta$, we have

$$
\big|E_{q,q}\big(-t_1^q\lambda\big)-E_{q,q}\big(-t_2^q\lambda\big)\big|<\frac{\varepsilon}{\tau_2^{\frac{q}{2-q}}}.
$$

Let $\sigma_1=\frac{2-q}{2(1-q)}$ and $\sigma_2=\frac{2-q}{q}$. Then $\sigma_1>1$, $\sigma_2>1$, and $\frac{1}{\sigma_1}+\frac{1}{\sigma_2}=1$. By the Hölder inequality we have

$$
\begin{aligned}
&\int_0^{\tau_2}(\tau_1-s)^{q-1}\big|E_{q,q}\big(-(\tau_2-s)^q\lambda\big)-E_{q,q}\big(-(\tau_1-s)^q\lambda\big)\big|\,ds\\
&\le\left[\int_0^{\tau_2}(\tau_1-s)^{(q-1)\frac{2-q}{2(1-q)}}\,ds\right]^{\frac{2(1-q)}{2-q}}\\
&\quad\cdot\left[\int_0^{\tau_2}\Big(E_{q,q}\big(-(\tau_2-s)^q\lambda\big)-E_{q,q}\big(-(\tau_1-s)^q\lambda\big)\Big)^{\frac{2-q}{q}}\,ds\right]^{\frac{q}{2-q}}\\
&\le\left[\frac{\tau_1^{\frac{q}{2}}-(\tau_1-\tau_2)^{\frac{q}{2}}}{\frac{q}{2}}\right]^{\frac{2(1-q)}{2-q}}\cdot\left[\int_0^{\tau_2}\left(\frac{\varepsilon}{\tau_2^{\frac{q}{2-q}}}\right)^{\frac{2-q}{q}}ds\right]^{\frac{q}{2-q}}\\
&=\left[\frac{2\tau_1^{\frac{q}{2}}-2(\tau_1-\tau_2)^{\frac{q}{2}}}{q}\right]^{\frac{2(1-q)}{2-q}}\cdot\varepsilon,
\end{aligned}
$$

so $\int_0^{\tau_2}(\tau_1-s)^{q-1}|E_{q,q}(-(\tau_2-s)^q\lambda)-E_{q,q}(-(\tau_1-s)^q\lambda)|\,ds$ tends to zero as $\tau_2\to\tau_1$. Therefore, $|(Pu)(\tau_2)-(Pu)(\tau_1)|$ tends to zero as $\tau_2\to\tau_1$. This yields that P is equicontinuous on the interval J_k.

Combining the above arguments and the PC-type Arzelà-Ascoli theorem (Lemma 2.3 in the case $X=\mathbf{R}$), we conclude that $P:B_r\to B_r$ is compact and completely continuous. Then it follows from Lemma 2.2 that problem (1.3) has at least one solution. This completes the proof. □

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the following impulsive fractional integro-differential equation with antiperiodic boundary condition:

$$
\begin{cases}
{}^{c}D^{\frac{1}{2}}u(t)+u(t)=\frac{u(t)+1}{36(e^t+1)}+\frac{1}{t^2+15}\int_0^t\frac{u(s)}{e^{(t+3)s}}\,ds\\
\qquad\qquad+\frac{2}{\sqrt{t+49}}\int_0^1\frac{u(s)}{(8+t+s)^2}\,ds,\quad t\in[0,1]\setminus\{\frac{1}{2}\},\\
\Delta u|_{t=\frac{1}{2}}=\frac{|u(\frac{1}{2})|}{12+|u(\frac{1}{2})|},\\
u(0)=-u(1),
\end{cases}
\tag{4.1}
$$

Let

$$f(t,u,v,w)=\frac{u+1}{36(e^t+1)}+\frac{v}{t^2+15}+\frac{2w}{\sqrt{t+49}},\qquad I_k(u)=\frac{|u|}{12+|u|},$$
$$(Tu)(t)=\int_0^t e^{-(t+3)s}u(s)\,ds,\qquad (Su)(t)=\int_0^1\frac{u(s)}{(8+t+s)^2}\,ds.$$

By direct computation, $k_0=\max\{\frac{1}{e^{(t+3)s}}:0\le s\le t\le 1\}=1$ and $h_0=\max\{\frac{1}{(8+t+s)^2}:0\le s,t\le 1\}=\frac{1}{64}$. For $u_1,u_2,v_1,v_2,\omega_1,\omega_2\in\mathbf{R}$ and $t\in J$, we have

$$\begin{aligned}&\bigl|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)\bigr|\\&\quad\le\frac{1}{36(e^t+1)}|u_1-u_2|+\frac{1}{t^2+15}|v_1-v_2|+\frac{2}{\sqrt{t+49}}|w_1-w_2|,\\&\bigl|I_k(u_1)-I_k(u_2)\bigr|\le\frac{1}{12}|u_1-u_2|.\end{aligned}$$

Let

$$L_1(t)=\frac{1}{36(e^t+1)},\qquad L_2(t)=\frac{1}{t^2+15},$$
$$L_3(t)=\frac{2}{\sqrt{t+49}},\qquad L_4=\frac{1}{12}.$$

It is easy to see that $\overline{L_1}=\frac{1}{72},\overline{L_2}=\frac{1}{15},\overline{L_3}=\frac{2}{49},E_{\frac{1}{2}}(-1)=\frac{1+\frac{\pi-2}{\sqrt{\pi}}}{1+\sqrt{\pi}+(\pi-2)}\approx 0.42,E_{\frac{1}{2}}(-(\frac{1}{2})^{\frac{1}{2}})\approx 0.52$, $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}\approx 0.89$,

$$\begin{aligned}\chi&=\frac{3}{|1+E_{\frac{1}{2}}(-1)|}\left(\frac{L_4}{|E_{\frac{1}{2}}(-(\frac{1}{2})^{\frac{1}{2}})|}+\frac{(\overline{L_1}+\overline{L_2}k_0+\overline{L_3}h_0)}{\Gamma(\frac{3}{2})}\right)\\&\approx\frac{3}{1+0.42}\left(\frac{\frac{1}{12}}{0.52}+\frac{\frac{1}{72}+\frac{1}{15}+\frac{2}{49}\times\frac{1}{64}}{0.89}\right)<1.\end{aligned}$$

Then by Theorem 3.1 problem (4.1) has a unique solution.

Example 4.2 Consider the following impulsive antiperiodic problem:

$$\begin{cases}{}^cD^{\frac{1}{2}}u(t)+u(t)=(\frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}})\frac{|u(t)|}{1+|u(t)|}+\frac{\sqrt[3]{t+1}}{16e^t}\sin(\int_0^t\sin(t-s)u(s)\,ds)\\\qquad\qquad+\frac{1}{16\sqrt[3]{t+1}}\cos(\int_0^1\frac{u(s)}{1+ts}\,ds),\quad t\in[0,1]\setminus\{\frac{1}{2}\},\\\Delta u|_{t=\frac{1}{2}}=\frac{|u(\frac{1}{2})|}{12+|u(\frac{1}{2})|},\\u(0)=-u(1),\end{cases}\tag{4.2}$$

where

$$f(t,u,v,w)=\left(\frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}}\right)\frac{|u|}{1+|u|}+\frac{\sqrt[3]{t+1}}{16e^t}\sin v+\frac{1}{16\sqrt[3]{t+1}}\cos w.$$

By computation we obtain

$$\begin{aligned}\bigl|f(t,u,Tu,Su)\bigr| &\leq \frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}}+\frac{\sqrt[3]{t+1}}{16}\|u\|_{PC}+\frac{1}{16\sqrt[3]{t+1}}\|u\|_{PC}\\ &=\left(\frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}}\right)\bigl(\|u\|_{PC}+1\bigr).\end{aligned}$$

Let $\mu(t)=\frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}}$, $\sigma=\frac{1}{3}$, and $\varpi(r)=r+1$. Then $\liminf_{r\to\infty}\frac{\varpi(r)}{r}=1$ and $L_4=\frac{1}{12}$. Thus,

$$\left\{\frac{\frac{1}{12}}{E_{\frac{1}{2}}(-(\frac{1}{2})^{\frac{1}{2}})}+\frac{[\int_0^1(\frac{\sqrt[3]{t+1}}{16}+\frac{1}{16\sqrt[3]{t+1}})^3\,dt]^{\frac{1}{3}}}{\Gamma(\frac{1}{2})(\frac{\frac{1}{2}-\frac{1}{3}}{1-\frac{1}{3}})^{1-\frac{1}{3}}}\right\}\frac{3}{1+E_{\frac{1}{2}}(-1)}\approx 0.95<1.$$

By Theorem 3.2 problem (4.2) has at least one solution.

5 Conclusion

In this paper, we are concerned with the existence and uniqueness of solutions for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary condition. The paper has several new features. First, we consider the impulsive fractional integro-differential equation of mixed type, that is, the nonlinear f involves linear operators T and S. The second new feature is that we studied antiperiodic boundary value problems with constant coefficients. Our results are based on the Banach contraction mapping principle and the Krasnoselskii fixed point theorem.

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