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EXISTENCE RESULTS FOR NEUTRAL FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

K. Shri Akiladevi[1], K. Balachandran[2] and J. K. Kim[3]

[1]Department of Mathematics
Bharathiar University, Coimbatore - 641 046, India
e-mail: shriakiladevi@gmail.com

[2]Department of Mathematics
Bharathiar University, Coimbatore - 641 046, India
e-mail: kb.maths.bu@gmail.com

[3]Department of Mathematics Education
Kyungnam University, Changwon - 631 701, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this paper, we study the existence and uniqueness of solutions for the neutral fractional integrodifferential equations with fractional integral boundary conditions by using fixed point theorems. The fractional derivative considered here is in the Caputo sense. Examples are provided to illustrate the results.

1. Introduction

The subject of fractional calculus has been receiving a great deal of attention from many researchers and scientists during the past few decades. This is mainly due to the fact that it provides an excellent tool in the modelling of dynamical systems which arise in science and engineering. For an extensive collection of such results, one can refer [15, 20, 21, 25, 26].

In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations [12]. The most important advantage of using them is their non-local property. It is well known that the integer

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order differential operator is a local operator but the fractional order differential operator is non-local. This means that the future state of a system depends not only on its current state but also upon all its past states. This is probably the most relevant feature for making this fractional tool useful from an applied standpoint and interesting from a mathematical standpoint. They appear naturally in control theory of dynamical systems, fluid flow, chemical physics, rheology, dynamical processes in self-similar and porous structures, viscoelasticity, optics and signal processing, electrical networks, electrochemistry of corrosion and so on. For some recent contributions on fractional initial value problems, see [2, 3, 9, 10, 11, 18, 32].

In recent years, boundary value problems of fractional differential equations involving a variety of boundary conditions have been investigated by several researchers [1, 6, 14, 19, 22, 29, 30]. In particular, integral boundary conditions have various applications in applied fields such as blood flow problems, thermoelasticity, chemical engineering, underground water flow, cellular systems, heat transmission, plasma physics, population dynamics and so forth. For a detailed description of these boundary conditions, one can refer the papers [7, 13, 17, 24, 27, 28]. Also integrodifferential equations arise in many engineering and scientific disciplines. The recent results of fractional boundary value problems with integrodifferential equations can be found in [4, 5, 8, 31] and the references therein.

In the first part of this paper, we discuss the existence and uniqueness of solutions to the nonlinear neutral fractional boundary value problem

$$\left.\begin{aligned} {}^{C}D^{q}_{0+}[x(t)-g(t,x(t))] &= f(t,x(t)),\ t\in[0,1],\ 0<q\le 1, \\ x(0) &= \alpha I^{p}x(\eta), \quad 0<\eta<1, \end{aligned}\right\} \tag{1.1}$$

where ${}^{C}D^{q}_{0+}$ denotes the Caputo fractional derivative of order q. The function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Here $\alpha\in\mathbb{R}$ is such that $\alpha\neq\frac{\Gamma(p+1)}{\eta^{p}}$ and $I^{p},\ 0<p<1$, is the Riemann-Liouville fractional integral of order p. The results generalise those of [23].

In the second part, we study the existence and uniqueness of solutions to the nonlinear neutral fractional integrodifferential boundary value problem

$$\left.\begin{aligned} {}^{C}D^{q}_{0+}[x(t)-g(t,x(t))] &= f\left(t,x(t),\int_0^t k(t,s,x(s))ds\right), \\ & \qquad t\in[0,1],\ 0<q\le 1, \\ x(0) &= \alpha I^{p}x(\eta), \quad 0<\eta<1, \end{aligned}\right\} \tag{1.2}$$

where the functions $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $k:\Omega\times\mathbb{R}\to\mathbb{R}$ are continuous, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable with $\Omega=\{(t,s):0\le s<t\le 1\}$.

The paper is organized as follows: In Section 2, we give some preliminary results. In Section 3, we discuss the existence and uniqueness results for (1.1) by using Krasnoselskii's, Leray-Schauder fixed point theorems and the Banach contraction principle respectively. The nonlinear neutral fractional integrodifferential equation (1.2) is considered in Section 4 where the existence and uniqueness results are studied using the same technique as in Section 3. Examples are also provided to illustrate the main results. To the best of the authors' knowledge, no paper has considered the existence of solutions to the nonlinear neutral fractional differential equation with fractional integral boundary conditions.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [20].

Definition 2.1. The Riemann-Liouville fractional integral of a function $f \in L^1(\mathbb{R}^+)$ of order q is defined as

$$I_{0+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0, \tag{2.1}$$

provided the integral exists.

Definition 2.2. The Caputo fractional derivative of order q is defined as

$${}^C D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \quad n-1 < q \le n, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$. In particular, if $0 < q \le 1$,

$${}^C D_{0+}^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds,$$

where $f'(s) = Df(s) = \frac{df(s)}{ds}$.

Lemma 2.3. ([1]) *Let $p, q \ge 0$, $f \in L^1[a,b]$. Then $I^p I^q f(t) = I^{p+q} f(t) = I^q I^p f(t)$ and ${}^c D^q I^q f(t) = f(t)$, for all $t \in [a,b]$.*

3. Nonlinear neutral equations

Definition 3.1. A function $x(t) \in C([0,1], \mathbb{R})$ is said to be a solution of (1.1) if it satisfies the equation

$${}^C D^q [x(t) - g(t, x(t))] = f(t, x(t)), \quad t \in [0,1]$$

and the boundary condition

$$x(0) = \alpha I^p x(\eta), \quad 0 < \eta < 1.$$

To study the nonlinear problem (1.1), we first consider the linear problem and obtain its solution.

Lemma 3.2. *Let $\alpha \neq \frac{\Gamma(p+1)}{\eta^p}$. Then, for given $f \in C([0,1], \mathbb{R})$, $g \in C^1([0,1], \mathbb{R})$, the solution of the fractional differential equation*

$$^{C}D^q[x(t) - g(t)] = f(t), \quad 0 < q \leq 1, \tag{3.1}$$

subject to the boundary condition

$$x(0) = \alpha I^p x(\eta) \tag{3.2}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} g(0) + g(t) + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \\ & \times \left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} g(s)ds + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s)ds \right). \end{aligned} \tag{3.3}$$

Proof. Suppose that x is a solution of (1.1), then from [20], we have, for some constant $c_0 \in \mathbb{R}$,

$$x(t) = c_0 - g(0) + g(t) + I^q f(t). \tag{3.4}$$

Taking Riemann-Liouville fractional integral of order p on both sides of (3.4), we get

$$\begin{aligned} I^p x(t) &= \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left(c_0 - g(0) + g(s) + I^q f(s)\right) ds \\ &= \frac{c_0}{\Gamma(p+1)} t^p - \frac{g(0)}{\Gamma(p+1)} t^p + I^p g(t) + I^p I^q f(t). \end{aligned}$$

Using Lemma 2.3, we have

$$I^p x(\eta) = \frac{c_0}{\Gamma(p+1)} \eta^p - \frac{g(0)}{\Gamma(p+1)} \eta^p + I^p g(\eta) + I^{p+q} f(\eta). \tag{3.5}$$

Using (3.5) in (3.2), we have

$$c_0 = \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \left(-\frac{g(0)}{\Gamma(p+1)} \eta^p + I^p g(\eta) + I^{p+q} f(\eta) \right).$$

Substituting the value of c_0 in (3.4), we get

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} g(0) + g(t)$$
$$+ \frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} \left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} g(s)ds + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s)ds \right).$$

□

3.1. **Existence and Uniqueness Results.** Let $\mathcal{C} = C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. In view of Lemma 3.2, we transform (1.1) as

$$x = F(x), \tag{3.6}$$

where $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is given by

$$(Fx)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \frac{\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} g(0,x(0))$$
$$+ g(t,x(t)) + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} \left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} g(s,x(s))ds \right.$$
$$\left. + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s,x(s))ds \right), \quad t \in [0,1]. \tag{3.7}$$

Observe that the problem (1.1) has solutions if the operator equation (3.6) has fixed points.

Assume that the following conditions hold:

(A1) The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there exist positive constants L_1, L_2 such that, for $t \in [0,1]$, $x, y \in \mathbb{R}$,
(i) $|f(t,x) - f(t,y)| \leq L_1|x-y|$,
(ii) $|g(t,x) - g(t,y)| \leq L_2|x-y|$.

(A2) Let $\delta_1 = L_1\lambda_1 + L_2\lambda_2 < 1$,
where $\lambda_1 = \left(\frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|\Gamma(p+q+1)} \right)$ and $\lambda_2 = \left(\frac{|\Gamma(p+1)-\alpha\eta^p| + |\alpha|\eta^p}{|\Gamma(p+1)-\alpha\eta^p|} \right)$.

(A3) For each $(t,x) \in [0,1] \times \mathbb{R}$ and $\mu_1, \mu_2 \in C([0,1],\mathbb{R}^+)$, we have
(i) $|f(t,x)| \leq \mu_1(t)$,
(ii) $|g(t,x)| \leq \mu_2(t)$.

Theorem 3.3. *Assume that f, g satisfy the hypotheses (A1) and (A2). Then the boundary value problem (1.1) has a unique solution on $[0,1]$.*

Proof. Let $M_1 = \sup_{t\in[0,1]} |f(t,0)|$, $M_2 = \sup_{t\in[0,1]} |g(t,0)|$ and consider $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$, where $r \geq \frac{\delta_2}{1-\delta_1}$ with

$$\delta_2 = M_1\lambda_1 + M_2\lambda_2 + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}|g(0,x(0))|$$

and δ_1 given by the assumption $(A2)$. Now we show that $FB_r \subset B_r$, where $F : \mathcal{C} \to \mathcal{C}$ is defined by (3.7). For $x \in B_r$, we have

$$\begin{aligned}
\|(Fx)(t)\| \leq& \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}|g(0,x(0))| \right. \\
&+|g(t,x(t))| + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}|g(s,x(s))|ds\right. \\
&\left.\left.+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))|ds\right)\right\} \\
\leq& \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right) ds \right. \\
&+\frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}|g(0,x(0))| + \left(|g(t,x(t)) - g(t,0)| + |g(t,0)|\right) \\
&+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} \left(|g(s,x(s)) - g(s,0)|\right.\right. \\
&\left.+|g(s,0)|\right) ds + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} \left(|f(s,x(s)) - f(s,0)|\right. \\
&\left.\left.+|f(s,0)|\right) ds)\right\} \\
\leq& (L_1r + M_1)\lambda_1 + (L_2r + M_2)\lambda_2 + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}|g(0,x(0))| \\
\leq& \left[L_1\lambda_1 + L_2\lambda_2\right] r + \left[M_1\lambda_1 + M_2\lambda_2 + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|}|g(0,x(0))|\right] \\
\leq& \delta_1 r + \delta_2 \leq r.
\end{aligned}$$

This shows that $FB_r \subset B_r$. Next, for $x, y \in \mathcal{C}$ and $t \in [0,1]$, we obtain

$$\begin{aligned}
\|Fx - Fy\| \leq& \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\right. \\
&+|g(t,x(t)) - g(t,y(t))| + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \\
&\times\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}|g(s,x(s)) - g(s,y(s))|ds\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s))-f(s,y(s))|ds\Big)\Big\} \\
&\le |x-y|\,[L_1\lambda_1+L_2\lambda_2] \\
&\le \delta_1|x-y|.
\end{aligned}
$$

Here δ_1 depends only on the parameters involved in the problem. By assumption $(A2)$, $\delta_1<1$ and therefore F is a contraction. Hence, by the Banach contraction principle, the problem (1.1) has a unique solution on $[0,1]$. □

Now we prove the existence of solutions of (1.1) by applying Krasnoselskii's fixed point theorem.

Theorem 3.4. ([16], Krasnoselskii Theorem) *Let S be a closed, convex, non-empty subset of a Banach space X. Let $\mathcal{P},\mathcal{Q}$ be two operators such that*

(i) *$\mathcal{P}x+\mathcal{Q}y\in S$, whenever $x,y\in S$,*
(ii) *$\mathcal{P}$ is compact and continuous,*
(iii) *$\mathcal{Q}$ is a contraction mapping.*

Then there exists $z\in S$ such that $z=\mathcal{P}z+\mathcal{Q}z$.

Theorem 3.5. *Suppose that the assumptions $(A1)$ and $(A3)$ hold with*

$$
\begin{aligned}
L &= \frac{1}{|\Gamma(p+1)-\alpha\eta^p|}\left\{L_2\left(|\Gamma(p+1)-\alpha\eta^p|+|\alpha|\eta^p\right)+\frac{L_1|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)}\right\} \\
&< 1. \qquad (3.8)
\end{aligned}
$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Let $\sup_{t\in[0,1]}|\mu_i(t)|=\|\mu_i\|,\ i=1,2,$ and $B_r=\{x\in\mathcal{C}:\|x\|\le r\}$. Now we decompose F as F_1+F_2 on B_r, where

$$
\begin{aligned}
(F_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds,\ t\in[0,1], \\
(F_2x)(t) &= -\frac{\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}g(0,x(0))+g(t,x(t))+\frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} \\
&\quad\times\left(\int_0^\eta\frac{(\eta-s)^{p-1}}{\Gamma(p)}g(s,x(s))ds+\int_0^\eta\frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds\right),
\end{aligned}
$$

for $t\in[0,1]$. Choose

$$
\begin{aligned}
r \ge \|\mu\|\Bigg[&\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|\Gamma(p+q+1)} \\
&+\frac{|\Gamma(p+1)-\alpha\eta^p|+|\alpha|\eta^p}{|\Gamma(p+1)-\alpha\eta^p|}+\frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\Bigg].
\end{aligned}
$$

For $x, y \in B_r$, we find that

$$
\begin{aligned}
&\|F_1x + F_2y\| \\
&\leq \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} |g(0,y(0))| \right. \\
&\qquad + |g(t,y(t))| + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} |g(s,y(s))| ds \right. \\
&\qquad \left.\left. + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} |f(s,y(s))| ds \right) \right\} \\
&\leq \frac{\|\mu_1\|}{\Gamma(q+1)} + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} |g(0,y(0))| + \|\mu_2\| \\
&\qquad + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \left(\frac{\eta^p}{\Gamma(p+1)} \|\mu_2\| + \frac{\eta^{p+q}}{\Gamma(p+q+1)} \|\mu_1\| \right).
\end{aligned}
$$

Let $\mu = \max\{\mu_1, g(0,y(0)), \mu_2\}$. Then, by simplification, we have

$$
\begin{aligned}
\|F_1x + F_2y\| &\leq \|\mu\| \left[\frac{1}{\Gamma(q+1)} + \frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|\Gamma(p+q+1)} \right. \\
&\qquad \left. + \frac{|\Gamma(p+1) - \alpha\eta^p| + |\alpha|\eta^p}{|\Gamma(p+1) - \alpha\eta^p|} + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \right] \\
&\leq r.
\end{aligned}
$$

Thus $F_1x + F_2y \in B_r$. Next we prove that F_2 is a contraction.

$$
\begin{aligned}
\|F_2x - F_2y\| &\leq \sup_{t\in[0,1]} \{ |g(t,x(t)) - g(t,y(t))| \\
&\quad + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} \left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} |g(s,x(s)) - g(s,y(s))| ds \right. \\
&\quad \left. \left. + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s)) - f(s,y(s))| ds \right) \right\} \\
&\leq |x-y| \left\{ \frac{L_2 \left(|\Gamma(p+1) - \alpha\eta^p| + |\alpha|\eta^p \right)}{|\Gamma(p+1) - \alpha\eta^p|} \right. \\
&\quad \left. + \frac{L_1|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|\Gamma(p+q+1)} \right\} \\
&\leq L|x-y|.
\end{aligned}
$$

Hence F_2 is a contraction. Continuity of f implies that the operator F_1 is continuous. Also F_1 is uniformly bounded on B_r as

$$\|(F_1x)(t)\| \leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\right\} \leq \frac{\|\mu_1\|}{\Gamma(q+1)}.$$

To prove that the operator F_1 is compact, it remains to show that F_1 is equicontinuous. For that, let $\bar{f} = \sup_{(t,x)\in[0,1]\times B_r}|f(t,x)|$. Now, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$, we have

$$\begin{aligned}\|(F_1x)(t_2)-(F_1x)(t_1)\| \leq{}& \sup_{(t,x)\in[0,1]\times B_r}\left\{\int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\right.\\ &\left.|f(s,x(s))|ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\right\}\\ \leq{}& \frac{\bar{f}}{\Gamma(q+1)}[t_2^q - t_1^q],\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus F_1 is equicontinuous. By Arzela-Ascoli Theorem, F_1 is compact. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $x \in \mathcal{C}$ such that $Fx = x$ which is a solution to the boundary value problem (1.1). □

The next result is based on Leray-Schauder nonlinear alternative.

Theorem 3.6. ([16], Leray-Schauder nonlinear alternative) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \to C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\bar{U}$ or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.7. *Assume that the following hypotheses hold:*

(A4) *There exist continuous nondecreasing functions $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ and $\phi_1, \phi_2 \in L^1([0,1],\mathbb{R}^+)$ such that, for each $(t,x) \in [0,1]\times\mathbb{R}$,*
   (i) $|f(t,x)| \leq \phi_1(t)\psi_1(\|x\|)$,
   (ii) $|g(t,x)| \leq \phi_2(t)\psi_2(\|x\|)$.

(A5) *There exists a constant $M > 0$ such that $\frac{M}{\Lambda} \geq 1$, where*

$$\Lambda = \psi(M)\left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)I^p(I^q\phi_1(\eta)+\phi_2(\eta))}{|\Gamma(p+1)-\alpha\eta^p|} + \phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\right].$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Observe that the operator $F : \mathcal{C} \to \mathcal{C}$ defined by (3.7) is continuous. Next we show that F maps bounded sets into bounded sets in $\mathcal{C}$.

For a positive number k, let $B_k = \{x \in \mathcal{C} : \|x\| \le k\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then we have

$$
\begin{aligned}
&\|(Fx)(t)\| \\
&\le \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} |g(0,x(0))| \\
&\quad + |g(t,x(t))| + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} \Bigg(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} |g(s,x(s))|ds \\
&\quad + \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} |f(s,x(s))|ds \Bigg) \Bigg\} \\
&\le \psi_1(\|x\|) \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \phi_1(s)ds + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} |g(0,x(0))| \\
&\quad + \phi_2(1)\psi_2(\|x\|) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} \Bigg(\psi_2(\|x\|) \int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} \\
&\quad \phi_2(s)ds + \psi_1(\|x\|) \int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} \phi_1(s)ds \Bigg) \\
&\le \psi_1(k) \left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1) I^{p+q}\phi_1(\eta)}{|\Gamma(p+1)-\alpha\eta^p|} \right] + \frac{\Gamma(p+1)|g(0,x(0))|}{|\Gamma(p+1)-\alpha\eta^p|} \\
&\quad + \psi_2(k) \left[\phi_2(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} I^p\phi_2(\eta) \right].
\end{aligned}
$$

Choosing $\psi(k) = \max\{\psi_1(k), \psi_2(k), g(0,x(0))\}$, we have

$$
\begin{aligned}
\|(Fx)(t)\| \;\le\; & \psi(k) \Bigg[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} I^p(I^q\phi_1(\eta) + \phi_2(\eta)) \\
& + \phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|} \Bigg].
\end{aligned}
$$

Now we show that F maps bounded sets into equicontinuous sets in B_k. For that, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then, for $x \in B_k$,

$$
\begin{aligned}
\|(Fx)(t_2) - (Fx)(t_1)\| \le & \int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds + |g(t_2,x(t_2))| \\
& - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} |f(s,x(s))|ds - |g(t_1,x(t_1))|
\end{aligned}
$$

$$\begin{aligned}
&\leq \psi_1(k)\int_0^{t_1}\left[\frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{\Gamma(q)}\right]\phi_1(s)ds\\
&\quad+|g(t_2,x(t_2))-g(t_1,x(t_1))|\\
&\quad+\psi_1(k)\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\phi_1(s)ds.
\end{aligned}$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independently of $x \in B_k$. Thus F maps bounded sets into equicontinuous sets in B_k. By Arzela-Ascoli's Theorem, F is completely continuous.

Now let $x = \lambda Fx$ where $\lambda \in (0,1)$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
x(t) &= \lambda\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds - \frac{\lambda\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}g(0,x(0))\\
&\quad+\lambda g(t,x(t)) + \frac{\lambda\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}g(s,x(s))ds\right.\\
&\quad\left.+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s))ds\right).
\end{aligned}$$

Then, using the computations of the first step, we have

$$\begin{aligned}
|x(t)| \;\leq\; \psi(\|x\|)&\left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^p(I^q\phi_1(\eta)+\phi_2(\eta))\right.\\
&\left.+\phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\right].
\end{aligned}$$

Consequently

$$\begin{aligned}
\|x\| \;\leq\; \psi(\|x\|)&\left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^p(I^q\phi_1(\eta)+\phi_2(\eta))\right.\\
&\left.+\phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\right].
\end{aligned}$$

In view of $(A5)$, there exists M such that $\|x\| \neq M$. Let us set

$$U = \{\, x \in \mathcal{C} : \|x\| < M\}\,.$$

Note that the operator $F : \bar{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder theorem, we deduce that F has a fixed point $x \in \bar{U}$ which is a solution to the problem (1.1). □

3.2. **Examples.**

Example 3.8. Consider the following fractional boundary value problem

$$\left.\begin{aligned} {}^{C}D^{1/2}\left[x(t) - \tfrac{1+e^{-t}}{38+e^{t}}\tfrac{x(t)}{1+x(t)}\right] &= \tfrac{t}{4} + \tfrac{\sin t}{25}|x(t)|, \qquad t \in [0,1], \\ x(0) &= \sqrt{5} I^{1/2} x\left(\tfrac{1}{5}\right). \end{aligned}\right\} \tag{3.9}$$

Here $q = \frac{1}{2}$, $\alpha = \sqrt{5}$, $p = \frac{1}{2}$, $\eta = \frac{1}{5}$, $f(t,x) = \frac{t}{4} + \frac{\sin t}{25}|x(t)|$, $g(t,x) = \frac{1+e^{-t}}{38+e^{t}}\frac{x(t)}{1+x(t)}$. Also $\alpha = \sqrt{5} \neq \frac{\Gamma(p+1)}{\eta^{p}} = \Gamma(3/2)/(1/5)^{1/2}$. Now

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left|\frac{t}{4} + \frac{\sin t}{25}|x(t)| - \frac{t}{4} - \frac{\sin t}{25}|y(t)|\right| \\ &\leq \frac{\sin t}{25}|x-y| \\ &\leq \frac{1}{25}|x-y|, \\ |g(t,x) - g(t,y)| &= \left|\frac{1+e^{-t}}{38+e^{t}}\frac{x(t)}{1+x(t)} - \frac{1+e^{-t}}{38+e^{t}}\frac{y(t)}{1+y(t)}\right| \\ &\leq \frac{1+e^{-t}}{38+e^{t}}\frac{|x-y|}{(1+|x|)(1+|y|)} \\ &\leq \frac{1+e^{-t}}{38+e^{t}}|x-y| \\ &\leq \frac{2}{39}|x-y|. \end{aligned}$$

Condition $(A1)$ holds with $L_1 = \frac{1}{25}$, $L_2 = \frac{2}{39}$. Further, for the above values of $L_1, L_2, p, q, \alpha, \eta$, we get the value of $\delta_1 = 0.6865 < 1$. Thus all the conditions of the Theorem 3.3 are satisfied. Hence, by Theorem 3.3, the boundary value problem (3.9) has a unique solution on $[0,1]$.

Example 3.9. Consider the following fractional boundary value problem

$$\left.\begin{aligned} {}^{C}D^{1/4}\left[x(t) - \tfrac{e^{-t}}{1+16e^{t}}\tfrac{x(t)}{1+x(t)}\right] &= \tfrac{1}{(t+7)^2}\tfrac{|x(t)|}{1+|x(t)|}, \qquad t \in [0,1], \\ x(0) &= \sqrt{2} I^{1/4} x\left(\tfrac{1}{2}\right). \end{aligned}\right\} \tag{3.10}$$

Here $q = \frac{1}{4}$, $\alpha = \sqrt{2}$, $p = \frac{1}{4}$, $\eta = \frac{1}{2}$, $f(t,x) = \frac{1}{(t+7)^2}\frac{|x(t)|}{1+|x(t)|}$, $g(t,x) = \frac{e^{-t}}{1+16e^{t}}\frac{x(t)}{1+x(t)}$. Also $\alpha = \sqrt{2} \neq \frac{\Gamma(p+1)}{\eta^{p}} = \Gamma(5/4)/(1/2)^{1/4}$. Now

$$
\begin{aligned}
|f(t,x)-f(t,y)| &= \left|\frac{1}{(t+7)^2}\frac{|x(t)|}{1+|x(t)|}-\frac{1}{(t+7)^2}\frac{|y(t)|}{1+|y(t)|}\right| \\
&\leq \frac{1}{(t+7)^2}\frac{|x-y|}{(1+|x|)(1+|y|)} \\
&\leq \frac{1}{(t+7)^2}|x-y| \leq \frac{1}{49}|x-y|, \\
|g(t,x)-g(t,y)| &= \left|\frac{e^{-t}}{(1+16e^t)}\frac{x(t)}{1+x(t)}-\frac{e^{-t}}{(1+16e^t)}\frac{y(t)}{1+y(t)}\right| \\
&\leq \frac{e^{-t}}{(1+16e^t)}\frac{|x-y|}{(1+|x|)(1+|y|)} \\
&\leq \frac{e^{-t}}{(1+16e^t)}|x-y| \leq \frac{1}{17}|x-y|.
\end{aligned}
$$

Also $|f(t,x)| = \left|\frac{1}{(t+7)^2}\frac{|x(t)|}{1+|x(t)|}\right| \leq \frac{1}{49}$ and $|g(t,x)| = \left|\frac{e^{-t}}{1+16e^t}\frac{x(t)}{1+x(t)}\right| \leq \frac{1}{17}$. Hence the conditions $(A1)$ and $(A3)$ holds with $L_1 = \frac{1}{49},\ L_2 = \frac{1}{17},\ \mu_1(t) = \frac{1}{49},\ \mu_2(t) = \frac{1}{17}$. For the above values of $L_1,\ L_2,\ p,\ q,\ \alpha,\ \eta$, we get the value of $L = 0.37997 < 1$. Thus all the conditions of the Theorem 3.5 are satisfied. Hence, by Theorem 3.5, the problem (3.10) has at least one solution on $[0,1]$.

4. Nonlinear neutral integrodifferential equations

Definition 4.1. A function $x(t) \in C([0,1],\mathbb{R})$ is said to be a solution of (1.2) if it satisfies the equation

$$
{}^{C}D^{q}[x(t)-g(t,x(t))] = f\left(t, x(t), \int_0^t k(t,s,x(s))ds\right), \qquad t \in [0,1],
$$

and the boundary condition

$$
x(0) = \alpha I^p x(\eta), \qquad 0 < \eta < 1.
$$

Taking $Kx(t) = \displaystyle\int_0^t k(t,s,x(s))ds$, equation (1.2) is equivalent to the following integral equation

$$
\begin{aligned}
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s),Kx(s))ds - \frac{\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p} g(0,x(0)) \\
+g(t,x(t)) + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)} g(s,x(s))ds\right. \\
\left.+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)} f(s,x(s),Kx(s))ds\right).
\end{aligned}
$$

Define the mapping $F : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(Fx)(t) \; &= \; \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), Kx(s))ds - \frac{\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} g(0, x(0)) \\ &\quad + g(t, x(t)) + \frac{\alpha\Gamma(p+1)}{\Gamma(p+1) - \alpha\eta^p} \left(\int_0^\eta \frac{(\eta - s)^{p-1}}{\Gamma(p)} g(s, x(s))ds \right. \\ &\quad \left. + \int_0^\eta \frac{(\eta - s)^{p+q-1}}{\Gamma(p+q)} f(s, x(s), Kx(s))ds \right), \qquad (4.1)\end{aligned}$$

for $t \in [0, 1]$ and we have to show that F has a fixed point. This fixed point is then a solution to the boundary value problem (1.2).

4.1. **Existence and Uniqueness Results.** Assume that the following conditions hold:

- (B1) The function $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there exist positive constants L_1, L_2 such that
  - (i) $|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L_1 \{|x_1 - x_2| + |y_1 - y_2|\}$, $t \in [0, 1]$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$,
  - (ii) $|g(t, x_1) - g(t, x_2)| \le L_2|x_1 - x_2|, \quad t \in [0, 1],\ x_1, x_2 \in \mathbb{R}$.
- (B2) The function $k : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists constant $L_3 > 0$, such that $|k(t, s, x_1) - k(t, s, x_2)| \le L_3|x_1 - x_2|, \quad \forall t, s \in [0, 1],\ x_1, x_2 \in \mathbb{R}$.
- (B3) Let $\rho_1 = L_1(1 + L_3)\lambda_1 + L_2\lambda_2 < 1$.
- (B4) For $\mu_1, \mu_2 \in C([0, 1], \mathbb{R}^+)$, we have
  - (i) $|f(t, x, y)| \le \mu_1(t), \quad (t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$,
  - (ii) $|g(t, x)| \le \mu_2(t), \quad (t, x) \in [0, 1] \times \mathbb{R}$.

Theorem 4.2. *Assume that f, g satisfy the hypotheses $(B1) - (B3)$. Then the boundary value problem (1.2) has a unique solution on $[0, 1]$.*

Proof. Let $M_1 = \sup\limits_{t\in[0,1]} |f(t, 0, 0)|$, $M_2 = \sup\limits_{t\in[0,1]} |g(t, 0)|$, $M_3 = \sup\limits_{t,s\in[0,1]} |k(t, s, 0)|$ and consider $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$, where $r \ge \frac{\rho_2}{1-\rho_1}$ with

$$\rho_2 = \left[(L_1M_3 + M_1)\lambda_1 + M_2\lambda_2 + \frac{\Gamma(p+1)}{|\Gamma(p+1) - \alpha\eta^p|} |g(0, x(0))| \right]$$

and ρ_1 given by the assumption $(B3)$. Now we show that $FB_r \subset B_r$, where $F : \mathcal{C} \to \mathcal{C}$ is defined by (4.1). For $x \in B_r$, we have

$$
\begin{aligned}
&\|(Fx)(t)\| \\
&\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s))|ds+\frac{\Gamma(p+1)|g(0,x(0))|}{|\Gamma(p+1)-\alpha\eta^p|}\right.\\
&\qquad +|g(t,x(t))|+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}|g(s,x(s))|ds\right.\\
&\qquad \left.\left.+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s),Kx(s))|ds\right)\right\}\\
&\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(|f(s,x(s),Kx(s))-f(s,0,0)|+|f(s,0,0)|\right)ds\right.\\
&\qquad +\frac{\Gamma(p+1)|g(0,x(0))|}{|\Gamma(p+1)-\alpha\eta^p|}+\left(|g(t,x(t))-g(t,0)|+|g(t,0)|\right)\\
&\qquad +\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}\left(|g(s,x(s))-g(s,0)|\right.\right.\\
&\qquad \left.+|g(s,0)|\right)ds+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}\left(|f(s,x(s),Kx(s))-f(s,0,0)|\right.\\
&\qquad \left.+|f(s,0,0)|\right)ds)\}\\
&\leq \frac{1}{\Gamma(q+1)}\left[L_1\left\{(1+L_3)r+M_3\right\}+M_1\right]+\frac{\Gamma(p+1)|g(0,x(0))|}{|\Gamma(p+1)-\alpha\eta^p|}\\
&\qquad +(L_2r+M_2)+(L_2r+M_2)\frac{|\Gamma(p+1)-\alpha\eta^p|+|\alpha|\eta^p}{|\Gamma(p+1)-\alpha\eta^p|}\\
&\qquad +\frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|\Gamma(p+q+1)}\left[L_1\left\{(1+L_3)r+M_3\right\}+M_1\right]\\
&\leq \left[L_1(1+L_3)\lambda_1+L_2\lambda_2\right]r+\left[(L_1M_3+M_1)\lambda_1+M_2\lambda_2\right.\\
&\qquad \left.+\frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}|g(0,x(0))|\right]\\
&\leq \quad \rho_1 r+\rho_2\leq r.
\end{aligned}
$$

This shows that $FB_r \subset B_r$. Next, for $x, y \in \mathcal{C}$ and $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\|\\
&\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s),Kx(s))-f(s,y(s),Ky(s))|ds\right.\\
&\qquad +|g(t,x(t))-g(t,y(t))|+\frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\\
&\qquad \times\left(\int_0^\eta \frac{(\eta-s)^{p-1}}{\Gamma(p)}|g(s,x(s))-g(s,y(s))|ds\right.
\end{aligned}
$$

$$+\int_0^\eta \frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}|f(s,x(s),Kx(s))-f(s,y(s),Ky(s))|ds\Big)\Big\}$$
$$\leq |x-y|\left[L_1(1+L_3)\left\{\frac{1}{\Gamma(q+1)}+\frac{|\alpha|\eta^{p+q}\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|\Gamma(p+q+1)}\right\}\right.$$
$$\left.+L_2\frac{(|\Gamma(p+1)-\alpha\eta^p|+|\alpha|\eta^p)}{|\Gamma(p+1)-\alpha\eta^p|}\right]$$
$$\leq \rho_1|x-y|.$$

Here ρ_1 depends only on the parameters involved in the problem. By $(B3)$, $\rho_1<1$ and therefore F is a contraction. Hence, by the Banach contraction principle, the problem (1.2) has a unique solution on $[0,1]$. □

Now we prove the existence result based on Krasnoselskii's fixed point theorem.

Theorem 4.3. *Suppose that the assumptions* $(B1),(B2)$ *and* $(B4)$ *hold with*

$$L = \frac{1}{|\Gamma(p+1)-\alpha\eta^p|}\left\{L_2\left(|\Gamma(p+1)-\alpha\eta^p|+|\alpha|\eta^p\right)\right.$$
$$\left.+\frac{L_1(1+L_3)|\alpha|\eta^{p+q}\Gamma(p+1)}{\Gamma(p+q+1)}\right\}<1. \tag{4.2}$$

Then the boundary value problem (1.2) *has at least one solution on* $[0,1]$.

Proof. Let $\sup_{t\in[0,1]}|\mu_i(t)|=\|\mu_i\|,\ i=1,2$ and $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$. Now we decompose F as F_1+F_2 on B_r where

$$(F_1x)(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s),Kx(s))ds,\ t\in[0,1],$$
$$(F_2x)(t)=-\frac{\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}g(0,x(0))+g(t,x(t))$$
$$+\frac{\alpha\Gamma(p+1)}{\Gamma(p+1)-\alpha\eta^p}\left(\int_0^\eta\frac{(\eta-s)^{p-1}}{\Gamma(p)}g(s,x(s))ds\right.$$
$$\left.+\int_0^\eta\frac{(\eta-s)^{p+q-1}}{\Gamma(p+q)}f(s,x(s),Kx(s))ds\right),\quad t\in[0,1].$$

As in Theorem 3.5, we can show that $F_1x+F_2y\in B_r$, F_2 is a contraction with L given by (4.2) and F_1 is compact and continuous. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $x\in\mathcal{C}$ such that $Fx=x$ which is a solution to the boundary value problem (1.2). □

Next we apply Leray-Schauder nonlinear alternative to prove the existence results for (1.2).

Theorem 4.4. *Assume that the following hypotheses hold:*

(B5) *There exist continuous nondecreasing functions* $\psi_1, \psi_2 : [0,\infty) \to (0,\infty)$ *and* $\phi_1, \phi_2 \in L^1([0,1],\mathbb{R}^+)$ *such that*
  (i) $|f(t,x,y)| \le \phi_1(t)\psi_1(\|x\|), \quad (t,x,y) \in [0,1]\times\mathbb{R}\times\mathbb{R},$
  (ii) $|g(t,x)| \le \phi_2(t)\psi_2(\|x\|), \quad (t,x) \in [0,1]\times\mathbb{R}.$

(B6) *There exists a constant* $N > 0$ *such that* $\frac{N}{\Lambda} \ge 1$, *where*

$$\Lambda = \psi(N)\left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^p(I^q\phi_1(\eta)+\phi_2(\eta)) \right.$$
$$\left. +\phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\right].$$

Then the boundary value problem (1.2) *has at least one solution on* $[0,1]$.

Proof. For a positive number k, let $B_k = \{x \in \mathcal{C} : \|x\| \le k\}$ be a bounded ball in $C([0,1],\mathbb{R})$.
By a similar argument as in Theorem 3.7, it is easy to prove that F is continuous, compact and

$$\|x\| \le \psi(\|x\|)\left[I^q\phi_1(1) + \frac{|\alpha|\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}I^p(I^q\phi_1(\eta)+\phi_2(\eta)) \right.$$
$$\left. +\phi_2(1) + \frac{\Gamma(p+1)}{|\Gamma(p+1)-\alpha\eta^p|}\right].$$

In view of $(B6)$, there exists N such that $\|x\| \ne N$. Let us set

$$U = \{\, x \in \mathcal{C} : \|x\| < N\}\,.$$

Note that the operator $F : \bar{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda F x$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder theorem, we deduce that F has a fixed point $x \in \bar{U}$ which is a solution to the boundary value problem (1.2). □

4.2. **Examples.**

Example 4.5. Consider the following fractional boundary value problem

$$\left.\begin{aligned} {}^C D^{1/2}\left[x(t) - \tfrac{e^{-t}}{26+e^t}\tfrac{x(t)}{1+x(t)}\right] &= \tfrac{1}{(t+6)^2}\tfrac{|x(t)|}{1+|x(t)|} + \tfrac{1}{36}\int_0^t e^{\frac{-1}{5}x(s)}ds,\ t \in [0,1],\\ x(0) &= \sqrt{2}I^{1/2}x\left(\tfrac{1}{2}\right). \end{aligned}\right\} \quad (4.3)$$

Here $q = \frac{1}{2},\ \alpha = \sqrt{2},\ p = \frac{1}{2},\ \eta = \frac{1}{2},\ f(t,x,Kx) = \frac{1}{(t+6)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{36}Kx(t),$

where $Kx(t) = \int_0^t e^{\frac{-1}{5}x(s)}ds,\ g(t,x) = \frac{e^{-t}}{26+e^t}\frac{x(t)}{1+x(t)}$. Also $\alpha = \sqrt{2} \neq \frac{\Gamma(p+1)}{\eta^p} = \Gamma(3/2)/(1/2)^{1/2}$. Now

$$|k(t,s,x(s)) - k(t,s,y(s))| = \left|e^{\frac{-1}{5}x} - e^{\frac{-1}{5}y}\right| \leq \frac{1}{5}|x-y|,$$

$$\begin{aligned}|f(t,x,Kx) - f(t,y,Ky)| &\leq \frac{1}{(t+6)^2}\frac{|x-y|}{(1+|x|)(1+|y|)} + \frac{1}{36}|Kx(t) - Ky(t)| \\ &\leq \frac{1}{36}\left[|x-y| + |Kx - Ky|\right],\end{aligned}$$

and

$$|g(t,x) - g(t,y)| = \left|\frac{e^{-t}}{26+e^t}\frac{x(t)}{1+x(t)} - \frac{e^{-t}}{26+e^t}\frac{y(t)}{1+y(t)}\right| \leq \frac{1}{27}|x-y|.$$

Hence the conditions $(B1), (B2)$ hold with $L_1 = \frac{1}{36},\ L_2 = \frac{1}{27},\ L_3 = \frac{1}{5}$. Further, for the above values of $L_1,\ L_2,\ L_3,\ p,\ q,\ \alpha,\ \eta$, we get the value of $\rho_1 = 0.58378 < 1$. All the conditions of the Theorem 4.2 are satisfied. Hence (4.3) has a unique solution on $[0,1]$.

Example 4.6. Consider the following fractional boundary value problem

$$\left.\begin{aligned} {}^CD^{1/2}\left[x(t) - \frac{e^{-t}}{1+36e^t}\frac{x(t)}{1+x(t)}\right] &= \frac{1}{(t+4)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{16}\int_0^t \frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds, \\ &\quad t \in [0,1], \\ x(0) &= \sqrt{1}1 I^{1/2}x\left(\tfrac{1}{11}\right). \end{aligned}\right\} \quad (4.4)$$

Here $q = \frac{1}{2},\ \alpha = \sqrt{1}1,\ p = \frac{1}{2},\ \eta = \frac{1}{11},\ f(t,x,Kx) = \frac{1}{(t+4)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{16}Kx(t)$, where $Kx(t) = \int_0^t \frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds,\ g(t,x) = \frac{e^{-t}}{1+36e^t}\frac{x(t)}{1+x(t)}$. Also $\alpha = \sqrt{1}1 \neq \frac{\Gamma(p+1)}{\eta^p} = \Gamma(3/2)/(1/11)^{1/2}$. Now

$$\begin{aligned}|k(t,s,x(s)) - k(t,s,y(s))| &= \left|\frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|} - \frac{e^{-s}}{9}\frac{|y(t)|}{1+|y(t)|}\right| \\ &\leq \frac{1}{9}|x-y|,\end{aligned}$$

$$\begin{aligned}|f(t,x,Kx) - f(t,y,Ky)| &\leq \frac{1}{(t+4)^2}\frac{|x-y|}{(1+|x|)(1+|y|)} + \frac{1}{16}|Kx(t) - Ky(t)| \\ &\leq \frac{1}{(t+4)^2}|x-y| + \frac{1}{16}|Kx - Ky| \\ &\leq \frac{1}{16}\left[|x-y| + |Kx - Ky|\right],\end{aligned}$$

and

$$\begin{aligned}|g(t,x)-g(t,y)| &= \left|\frac{e^{-t}}{(1+36e^{t})}\frac{x(t)}{1+x(t)}-\frac{e^{-t}}{(1+36e^{t})}\frac{y(t)}{1+y(t)}\right| \\ &\leq \frac{e^{-t}}{(1+36e^{t})}\frac{|x-y|}{(1+|x|)(1+|y|)} \\ &\leq \frac{e^{-t}}{(1+36e^{t})}|x-y| \leq \frac{1}{37}|x-y|.\end{aligned}$$

Also $|f(t,x,y)| = \left|\frac{1}{(t+4)^2}\frac{|x(t)|}{1+|x(t)|}+\frac{1}{16}\int_0^t \frac{e^{-s}}{9}\frac{|x(t)|}{1+|x(t)|}ds\right| \leq \frac{5}{72}$ and $|g(t,x)| = \left|\frac{e^{-t}}{1+36e^{t}}\frac{x(t)}{1+x(t)}\right| \leq \frac{1}{37}$. Here $L_1=\frac{1}{16},\ L_2=\frac{1}{37},\ L_3=\frac{1}{9},\ \mu_1(t)=\frac{5}{72},\ \mu_2(t)=\frac{1}{37}$. For the above values of $L_1,\ L_2,\ L_3,\ p,\ q,\ \alpha,\ \eta$, we get the value of $L = 0.42769 < 1$. All the conditions of the Theorem 4.3 are satisfied. Hence (4.4) has at least one solution on $[0,1]$.

References

[1] R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math., **109** (2010), 973–1033.

[2] R.P. Agarwal, V. Lakshmikantham and J.J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72** (2010), 2859–2862.

[3] R.P. Agarwal, Y. Zhou and Y. He, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl., **59** (2010), 1095–1100.

[4] B. Ahmad, A. Alsaedi and S. Alsulami, *A nonlocal integral boundary value problem of nonlinear integrodifferential equations of fractional order*, Int. J. Phys. Sci., **7** (2012), 5648–5653.

[5] B. Ahmad and J.J. Nieto, *Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions*, Bound. Value Probl., (2009), Article ID 708576.

[6] B. Ahmad and J.J. Nieto, *Anti-periodic fractional boundary value problems*, Comput. Math. Appl., **62** (2011), 1150–1156.

[7] B. Ahmad and S. Sivasundaram, *Existence of solutions for impulsive integral boundary value problems of fractional order*, Nonlinear Anal. Hybrid Syst., **4** (2010), 134–141.

[8] B. Ahmad and S. Sivasundaram, *On four-point nonlocal boundary value problems of nonlinear integrodifferential equations of fractional order*, Appl. Math. Comput., **217** (2010), 480–487.

[9] K. Balachandran and S. Kiruthika, *Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators*, Comput. Math. Appl., **62** (2011), 1350–1358.

[10] K. Balachandran, S. Kiruthika and J.J. Trujillo, *Existence of solutions of nonlinear fractional pantograph equations*, Acta. Math. Sci., **33** (2013), 712–720.

[11] K. Balachandran and J.J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal., **72** (2010), 4587–4593.

[12] B. Bonilla, M. Rivero, L. Rodriguez-Germa and J.J. Trujillo, *Fractional differential equations as alternative models to nonlinear differential equations*, Appl. Math. Comput., **187** (2007), 79–88.

[13] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, Nonlinear Anal., **70** (2009), 364–371.

[14] F. Chen, J.J. Nieto and Y. Zhou, *Global attractivity for nonlinear fractional differential equations*, Nonlinear Anal. Real World Appl., **13** (2012), 287–298.

[15] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin(2010).

[16] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York(2003).

[17] A. Guezane-Lakoud and R. Khaldi, *Solvability of a fractional boundary value problem with fractional integral condition*, Nonlinear Anal., **75** (2012), 2692–2700.

[18] E. Hernández, D.O' Regan and K. Balachandran, *Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators*, Indag. Math., **24** (2013), 68–82.

[19] X. Liu and Y. Liu, *Fractional differential equations with fractional non-seperated boundary conditions*, Electron. J. Differ. Equ., **2013(25)** (2013), 1–13.

[20] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam(2006).

[21] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York(1993).

[22] S.A. Murad and S.B. Hadid, *Existence and uniqueness theorem for fractional differential equation with integral boundary condition*, J. Frac. Calc. Appl., **3** (2012), 1–9.

[23] S.K. Ntouyas, *Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions*, J. Frac. Calc. Appl., **3** (2012), 1–14.

[24] S.K. Ntouyas, *Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions*, Opuscula Math., **33** (2013), 117–138.

[25] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, London(1974).

[26] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York(1999).

[27] I. Podlubny, *Geometric and physical interpretation of fractional differential equation with integral boundary condition*, Fract. Calc. Appl. Anal., **5** (2002), 367–386.

[28] G. Wang, W. Liu and C. Ren, *Existence of solutions for multi-point nonlinear differential equations of fractional orders with integral boundary conditions*, Electron. J. Differ. Equ., **2012(54)** (2012), 1–10.

[29] J. Wang, Y. Zhou and M. Fečkan, *On recent developments in the theory of boundary value problems for impulsive fractional differential equations*, Comput. Math. Appl., **64** (2012), 3008–3020.

[30] J. Zhao, P. Lu and Y. Liu, *Existence and numerical simulation of solutions for fractional equations involving two fractional orders with nonlocal boundary conditions*, J. Appl. Math., (2013), Article ID 268347.

[31] W. Zhou, Y. Chu, *Existence of solutions for fractional differential equations with multi-point boundary conditions*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 1142–1148.

[32] Y. Zhou and F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. Real World Appl., **11** (2010), 4465–4475.