Annali di Matematica 182, 53–79 (2003)

Digital Object Identifier (DOI) 10.1007/s10231-002-0056-y

Angelo Alvino · Lucio Boccardo · Vincenzo Ferone · Luigi Orsina · Guido Trombetti

Existence results for nonlinear elliptic equations with degenerate coercivity

Received: January 18, 2002
Published online: March 12, 2003 – © Springer-Verlag 2003

Mathematics Subject Classification (2000). 35J70, 35J65, 35B45

1. Introduction

In this paper we are interested in the existence of solutions for some nonlinear elliptic equations with principal part having degenerate coercivity. The model case is:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\,\nabla u}{(1+|u|)^{\theta\,(p-1)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with Ω a bounded open subset of $\mathbf{R}^N$, $N \geq 2$, $p > 1$, $\theta \geq 0$, and f a measurable function on whose summability we will make different assumptions. It is clear from the above example that the differential operator is defined on $W_0^{1,p}(\Omega)$, but that it may not be coercive on the same space as u becomes large. Due to this lack of coercivity, standard existence theorems for solutions of nonlinear elliptic equations cannot be applied. In this paper, we will prove several existence and regularity results (depending on the summability of the datum f) for the solutions of (1.1).

Let us give the precise assumptions on the problems that we will study.

Let Ω be a bounded open subset of $\mathbf{R}^N$, $N \geq 2$.

Let $1 < p < N$, and let $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function (that is, $a(\cdot, t, \xi)$ is measurable on Ω for every (t, ξ) in $\mathbf{R} \times \mathbf{R}^N$, and $a(x, \cdot, \cdot)$ is continuous on $\mathbf{R}\times\mathbf{R}^N$ for almost every x in Ω), such that the following assumptions hold:

$$a(x, t, \xi) \cdot \xi \geq b(|t|)\, |\xi|^p\,, \tag{1.2}$$

A. Alvino, V. Ferone, G. Trombetti: Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Univ. Monte S. Angelo, Via Cintia, 8126 Napoli, Italy

L. Boccardo, L. Orsina: Dipartimento di Matematica "Guido Castelnuovo", Università degli Studi di Roma "La Sapienza", Piazzale A. Moro 5, 00185 Roma, Italy

for almost every x in Ω, for every (t, ξ) in $\mathbf{R} \times \mathbf{R}^N$, where

$$b(t) = \frac{\alpha}{(1+t)^{\theta(p-1)}}\,, \tag{1.3}$$

for some $\theta \geq 0$ and some $\alpha > 0$;

$$|a(x, t, \xi)| \leq \beta\,[\,j(x) + |t|^{p-1} + |\xi|^{p-1}]\,, \tag{1.4}$$

for almost every x in Ω, for every (t, ξ) in $\mathbf{R} \times \mathbf{R}^N$, where j is a non-negative function in $L^{p'}(\Omega)$, and $\beta > 0$;

$$[a(x, t, \xi) - a(x, t, \xi')] \cdot [\xi - \xi'] > 0\,, \tag{1.5}$$

for almost every x in Ω, for every t in $\mathbf{R}$, for every ξ, ξ' in $\mathbf{R}^N$, with $\xi \neq \xi'$. Remark that assumption (1.2) implies that

$$a(x, t, 0) = 0\,, \tag{1.6}$$

for almost every x in Ω and for every t in $\mathbf{R}$. We will then define, for u in $W_0^{1,p}(\Omega)$, the nonlinear elliptic operator

$$A(u) = -\operatorname{div}(a(x, u, \nabla u))\,,$$

which, thanks to (1.4), maps $W_0^{1,p}(\Omega)$ into its dual space $W^{-1,p'}(\Omega)$.

In this paper, we are interested in proving the existence results for the nonlinear elliptic problem:

$$\begin{cases} A(u) = -\operatorname{div}(a(x, u, \nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

under various assumptions on the function f and on θ (appearing in (1.3)). As stated before, due to assumption (1.3), A may not be coercive on $W_0^{1,p}(\Omega)$, so that the standard Leray–Lions surjectivity theorem cannot also be applied even in the case in which f belongs to $W^{-1,p'}(\Omega)$. To overcome this problem, we will reason by approximation, "cutting" by means of truncatures the nonlinearity $a(x, t, \xi)$ in order to get a pseudomonotone and coercive differential operator on $W_0^{1,p}(\Omega)$, obtaining some *a priori* estimates on approximate solutions, a technical result of almost everywhere convergence for the gradients of the approximate solutions, and then passing to the limit.

Our results are a generalization, in the direction of a nonlinear operator with respect to the gradient, of the results contained in [11] and [1], where the case $p = 2$ is studied. For related results, see also [10], where a part of the results of [11] are proved with an easier technique; see also [21], for the uniqueness of solutions and [16] for regularity results.

Our first result is the following:

Theorem 1.1. *Let* $0 \le \theta \le 1$, *and let* f *belong to* $L^m(\Omega)$, *with* $m > N/p$. *Then there exists at least a solution* u *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of problem* (1.7), *in the sense that*

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx = \int_\Omega f \, v \, dx \,, \qquad \forall v \in W_0^{1,p}(\Omega) \,. \tag{1.8}$$

Remark 1.2. Observe that the assumption on f given in the preceding theorem is the one which yields $L^\infty(\Omega)$ solutions for nonlinear coercive elliptic equations ([12]). The result (which is independent of θ) is not surprising, since if one looks for bounded solutions then the lack of coercivity of the operator A (which is created by unbounded functions) "disappears".

If we decrease the summability of f, the situation is no longer the same: there is now an "interaction" between f and θ in order to still have $W_0^{1,p}(\Omega)$ solutions. Before giving the statement of the theorem, let us define

$$\widetilde{m} = \frac{Np}{(N-p)\,(1-\theta)\,(p-1) + p^2} \,. \tag{1.9}$$

Theorem 1.3. *Let* $0 \le \theta < 1$, *and let* f *be in* $L^m(\Omega)$, *with*

$$\widetilde{m} \le m < \frac{N}{p} \,. \tag{1.10}$$

Then there exists at least a solution u *of* (1.7) *in the sense of* (1.8). *Moreover,* u *belongs to* $W_0^{1,p}(\Omega) \cap L^s(\Omega)$, *with*

$$s = \frac{Nm\,(p-1)\,(1-\theta)}{N - mp} \,. \tag{1.11}$$

Remark 1.4. If $\theta = 1$, assumption (1.10) becomes empty, being $\widetilde{m} = N/p$ in this case. Observe that, for every θ in $[0, 1)$, we have

$$\widetilde{m} \ge \frac{Np}{Np - N + p} = (p^*)' \,,$$

so that the datum f always belongs to $W^{-1,p'}(\Omega)$. In this sense, it is natural to expect a $W_0^{1,p}(\Omega)$ solution; anyway, the solution is no longer bounded, so that the operator A is actually not coercive.

If we continue to decrease the summability of f, we no longer obtain solutions in the "energy space" $W_0^{1,p}(\Omega)$. Furthermore, we will have a solution whose gradient will have a regularity depending on both θ and the summability assumptions on f. Since we assumed a different growth from above and from below on a (see (1.2) and (1.4)), it is not possible to deduce *a priori* from the regularity of u that $|a(x, u, \nabla u)|$ belongs to $L^1(\Omega)$. Thus, it is not possible to use the notion of solution in the sense of distributions. In order to overcome this problem, we will need to introduce a different definition of solution, which also involves a different definition of gradient for a measurable function u.

We begin by defining, for $k > 0$, and t in $\mathbf{R}$, the truncation function

$$T_k(t) = \max\{-k, \min\{k, t\}\}\,, \tag{1.12}$$

and we recall the following result (see [8], Lemma 2.1):

Theorem 1.5. *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists a unique measurable function $v : \Omega \to \mathbf{R}^N$ such that*

$$v\chi_{\{|u|<k\}} = \nabla T_k(u), \qquad \textit{for almost every } x \in \Omega,\ \forall k > 0, \tag{1.13}$$

where χ_E is the characteristic function of a measurable set E. If, moreover, u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

In view of the above result, for every measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, we *define* ∇u, the weak gradient of u, as the unique function v which satisfies (1.13). The definition of the weak gradient allows us to give the following definition of an entropy solution for problem (1.7) (see [8]):

Definition 1.6. Let f be in $L^m(\Omega)$, $m \geq 1$. A measurable function u is an entropy solution of (1.7) if $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, and if

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi)\, dx \leq \int_\Omega f\, T_k(u - \varphi)\, dx\,, \tag{1.14}$$

for every $k > 0$ and for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Note that the left-hand side is well defined since the integral is only on the set $|u - \varphi| \leq k$, and on this set $|u| \leq k + \|\varphi\|_{L^\infty(\Omega)} = M$, so that we have

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - \varphi)\, dx = \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot \nabla T_k(u - \varphi)\, dx\,,$$

which is finite by the growth assumptions on a.

If u is an entropy solution of (1.7) and is such that $|a(x, u, \nabla u)|$ belongs to $L^1(\Omega)$, then u is also a solution of (1.7) in the sense of distributions (see [8]).

We are now ready to state the existence results for solutions not in the energy space $W_0^{1,p}(\Omega)$. As before, we define

$$\overline{m} = \frac{N}{[N(1-\theta) + \theta](p-1) + 1}\,. \tag{1.15}$$

Theorem 1.7. *Let* $0 \le \theta < 1$*, and let* f *be in* $L^m(\Omega)$*, with*

$$\max\{1, \overline{m}\} < m < \widetilde{m}\,. \tag{1.16}$$

Then there exists at least an entropy solution u *in* $W_0^{1,q}(\Omega) \cap L^s(\Omega)$ *of* (1.7)*, with* s *as in* (1.11)*, that is*

$$s = \frac{Nm\,(p-1)\,(1-\theta)}{N-mp}\,,$$

and

$$q = \frac{Nm\,(p-1)(1-\theta)}{N-m\,[1+\theta\,(p-1)]}\,. \tag{1.17}$$

Remark 1.8. Once again, the case $\theta = 1$ cannot be considered, since assumption (1.16) becomes empty (both $\widetilde{m}$ and $\overline{m}$ become equal to N/p). The number q given by (1.17) is always smaller than p, so that the solution u does not necessarily belong to $W_0^{1,p}(\Omega)$. Observe, however, that it is possible to have m satisfying both (1.16) and $m \ge (p^*)'$, so that the datum f belongs to $W^{-1,p'}(\Omega)$. Furthermore, note that we always have $q > 1$, so that the solution belongs to some Sobolev space; hence, as stated before, the weak gradient of u coincides with its standard distributional gradient.

Starting from (1.4), and using the fact that u belongs to $W_0^{1,q}(\Omega)$ with q as in the statement, we have that $|a(x,u,\nabla u)|^{q/(p-1)}$ belongs to $L^1(\Omega)$. Thus, we have that, *a priori*, $|a(x,u,\nabla u)|$ belongs to $L^1(\Omega)$ if and only if $q \ge p-1$, a fact that is not necessarily true under our assumptions on m and θ; indeed, it is satisfied if and only if m is such that

$$m \ge \underline{m} = \frac{N}{N(1-\theta)+1+\theta(p-1)}\,,$$

and, if $p > 2$, one has that $\underline{m} > \overline{m}$, so that it is possible to have entropy solutions which may not be solutions in the sense of distributions.

The last possible choice of summability for f is now f in $L^m(\Omega)$, with $1 \le m \le \max\{1, \overline{m}\}$. In this case, the solutions we will obtain no longer belong to Sobolev spaces (and, in some cases, not even to $L^1(\Omega)$). We will prove the following existence theorem for entropy solutions of (1.7):

Theorem 1.9. *Let* $0 \le \theta < 1$*, and let* f *in* $L^m(\Omega)$*, with*

$$1 \le m \le \max\{1, \overline{m}\}\,. \tag{1.18}$$

Then there exists at least an entropy solution u *of problem* (1.7)*. Moreover,* u *belongs to the Marcinkiewicz space* $M^s(\Omega)$*, with* s *given by* (1.11)*, that is*

$$s = \frac{Nm\,(p-1)\,(1-\theta)}{N-mp}\,,$$

and ∇u, the weak gradient of u, belongs to the Marcinkiewicz $(M^q(\Omega))^N$, with q given by (1.17)*, that is*

$$q = \frac{Nm\,(p-1)(1-\theta)}{N - m\,[1+\theta\,(p-1)]}$$

(see Section 2 for the definition of Marcinkiewicz spaces).

The plan of the paper is as follows: in Section 2 we will prove some *a priori* estimates in Lebesgue spaces on the solutions u_n of some approximating problems, and this will be done using rearrangement techniques. Section 3 will be devoted to the estimates on the gradients of u_n, while in Section 4 a theorem of almost everywhere convergence for the gradients of u_n is proved. Section 5 contains the proof of the existence theorems, obtained by putting together the results of Sections 2 and 3. The final section of the paper studies the case $\theta > 1$, giving a detailed picture of both existence and non-existence results for solutions of (1.7).

2. A priori estimates in Lebesgue spaces

Let $n \in \mathbf{N}$, and define, for u in $W_0^{1,p}(\Omega)$, the differential operator

$$A_n(u) = -\operatorname{div}(a(x, T_n(u), \nabla u)),$$

which turns out to be pseudomonotone from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Moreover, by (1.2), we have

$$\langle A_n(u), u\rangle = \int_\Omega a(x, T_n(u), \nabla u)\cdot\nabla u\,dx \geq \frac{1}{(1+n)^{\theta(p-1)}}\int_\Omega |\nabla u|^p\,dx.$$

Hence, A_n is also coercive on $W_0^{1,p}(\Omega)$. Thus, if f_n belongs to $L^\infty(\Omega)$, there exists at least one solution u_n in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ of

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n), \nabla u_n)) = f_n & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

solution in the sense that

$$\int_\Omega a(x, T_n(u_n), \nabla T_n(u_n))\cdot\nabla v\,dx = \int_\Omega f_n\,v\,dx, \qquad \forall v \in W_0^{1,p}(\Omega).$$

Before studying problem (2.1), we recall the definition of decreasing rearrangement of a measurable function $w : \Omega \to \mathbf{R}$. If one denotes by $|E|$ the Lebesgue measure of a set E, one can define the distribution function $\mu_w(t)$ of w as:

$$\mu_w(t) = |\{x\in\Omega : |w(x)| > t\}|, \qquad t \geq 0.$$

The decreasing rearrangement w^* of w is then defined as the generalized inverse function of μ_w:

$$w^*(\sigma) = \inf\{t\in\mathbf{R} : \mu_w(t) \leq \sigma\}, \qquad \sigma\in(0, |\Omega|).$$

We recall that w and w^* are equimeasurable, i.e.,

$$\mu_w(t) = \mu_{w^*}(t)\,, \qquad t \ge 0\,.$$

This implies, for example, that, for any monotone function ψ, it holds that:

$$\int_\Omega \psi\big(|w(x)|\big)\,dx = \int_0^{|\Omega|} \psi\big(w^*(\sigma)\big)\,d\sigma\,,$$

and, in particular,

$$\|w^*\|_{L^p(0,|\Omega|)} = \|w\|_{L^p(\Omega)}\,, \qquad 1 \le p \le \infty\,. \tag{2.2}$$

The theory of rearrangements is well known and exhaustive treatments of it can be found for example in [7], [17], [20], [4].

Using the notation just introduced above, we say that a measurable function $w : \Omega \to \mathbf{R}$ belongs to the Marcinkiewicz space $M^r(\Omega)$, $r > 0$, if there exists a constant c such that

$$\mu_w(t) \le \frac{c}{t^r}, \qquad \forall t > 0.$$

We observe that the above condition is equivalent to say that $w^*(\sigma) \le c\sigma^{-1/r}$, $\forall \sigma \in (0, |\Omega|)$, for some positive constant c, and we put $\|w\|_{M^r(\Omega)} = \sup_{\sigma\in(0,\Omega|)} w^*(\sigma)\sigma^{1/r}$. We also recall that if $w \in L^r(\Omega)$ then $w \in M^r(\Omega)$. Indeed, we have

$$w^*(\sigma) \le \frac{1}{\sigma}\int_0^\sigma w^*(\sigma)\,d\sigma \le \frac{1}{\sigma^{1/r}}\|w\|_{L^r(\Omega)}.$$

For solutions of (2.1) one can prove the following differential inequality:

Theorem 2.1. *Suppose u_n is a solution of* (2.1)*, and define:*

$$B(t) = \int_0^t b(\eta)^{\frac{1}{p-1}}\,d\eta = \int_0^t \frac{\alpha^{\frac{1}{p-1}}}{(1+\eta)^\theta}\,d\eta\,, \qquad t \ge 0\,. \tag{2.3}$$

The following inequality holds, for a.e. $s \in (0, |\Omega|)$:

$$\frac{-d}{d\sigma}B\left(u_n^*(\sigma)\right) \le \frac{1}{\left(NC_N^{1/N}\sigma^{1-1/N}\right)^{p'}}\left(\int_0^\sigma f_n^*(\tau)\,d\tau\right)^{p'/p}\,, \tag{2.4}$$

where C_N is the measure of the unit ball in $\mathbf{R}^N$.

Proof. For $t > 0$ and $k > 0$, we use in the formulation of solution (1.8) the test function $v = T_k(u_n - T_t(u_n))$, obtaining:

$$\int_{t<|u_n|\le t+k} a(x, T_n(u_n), \nabla u_n)\cdot\nabla u_n\,dx \le k\int_{|u_n|>t}|f_n|\,dx\,.$$

Dividing both sides by k and using (1.2) we get:

$$\begin{aligned}&\frac{\alpha}{k}\int_{t<|u_n|\le t+k}\frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}\,dx\\&\quad\le\frac{\alpha}{k}\int_{t<|u_n|\le t+k}\frac{|\nabla u_n|^p}{(1+|T_n(u_n)|)^{\theta(p-1)}}\,dx\le\int_{|u_n|>t}|f_n|\,dx\,.\end{aligned}\tag{2.5}$$

The above inequality and Hölder's inequality imply:

$$\begin{aligned}&\left(\frac{\alpha}{k}\int_{t<|u_n|\le t+k}\frac{|\nabla u_n|}{(1+|u_n|)^{\theta(p-1)}}\,dx\right)^p\\&\quad\le\left(\frac{\alpha}{k}\int_{t<|u_n|\le t+k}\frac{1}{(1+|u_n|)^{\theta(p-1)}}\,dx\right)^{p-1}\left(\int_{|u_n|>t}|f_n|\,dx\right).\end{aligned}\tag{2.6}$$

We can pass to the limit as k goes to 0 in (2.6) to get, after simplification,

$$\frac{\alpha}{(1+t)^{\theta(p-1)}}\left(\frac{d}{dt}\int_{|u_n|\le t}|\nabla u_n|\,dx\right)^p\le\left[-\mu'_{u_n}(t)\right]^{p-1}\int_0^{\mu_{u_n}(t)}f_n^*(\tau)\,d\tau\,.\tag{2.7}$$

It is well known that a consequence of the Fleming–Rishel formula (see [15]) and the isoperimetric inequality (see [14], [3]), is the following inequality:

$$N\,C_N^{1/N}\mu_{u_n}(t)^{1-1/N}\le\frac{d}{dt}\int_{|u_n|\le t}|\nabla u_n|\,dx\,,\tag{2.8}$$

where C_N denotes the measure of the unit ball in $\mathbf{R}^N$. Then (2.7) and (2.8) give:

$$\frac{\alpha^{\frac{1}{p-1}}}{(1+t)^{\theta}}\le\frac{\left[-\mu'_{u_n}(t)\right]}{\left(N\,C_N^{1/N}\mu_{u_n}(t)^{1-1/N}\right)^{p'}}\left(\int_0^{\mu_{u_n}(t)}f_n^*(\tau)\,d\tau\right)^{p'/p}.$$

Using the properties of rearrangements one easily obtains (2.4). □

Remark 2.2. Looking at the above proof, it appears clear that the result of Theorem 2.1 does not depend on the choice of the function $b(t)$ made in (1.3). We explicitly remark that if $b(t)$ is any continuous bounded positive function, then (2.4) still holds true, being $B(t)$ defined as in (2.3).

Corollary 2.3. *Suppose that u_n is a solution of* (2.1)*, and assume:*

$$0\le\theta\le1\,.\tag{2.9}$$

We have:

(i) if $m>N/p\ge1$, then:

$$\|u_n\|_{L^\infty(\Omega)}\le B^{-1}\left(\frac{|\Omega|^{\frac{p'}{N}-\frac{p'}{pm}}}{\left(N\,C_N^{1/N}\right)^{p'}}\frac{Nm(p-1)}{pm-N}\|f_n\|_{L^m(\Omega)}^{\frac{p'}{p}}\right),\tag{2.10}$$

where B^{-1} denotes the inverse function of B, defined in (2.3)*;*

(ii) *if* $1 < m < N/p$, *then:*

$$\|B(|u_n|)^r\|_{L^1(\Omega)} \leq A\|f_n\|_{L^m(\Omega)}^{\frac{rp'}{p}}, \tag{2.11}$$

where $r = Nm(p-1)/(N-mp)$ *and* A *is a constant which depends on* N, p, m;

(iii) *if* $m = 1$, *then*

$$\|B(|u_n|)\|_{M^{\frac{N(p-1)}{N-p}}(\Omega)} \leq \frac{N'}{(p'-N')\left(N\,C_N^{1/N}\right)^{p'}}\|f_n\|_{L^1(\Omega)}^{\frac{p'}{p}}. \tag{2.12}$$

The above corollary can be proven using the following technical result which can be found, for example, in [2].

Lemma 2.4. *Let* $\phi : (0,+\infty) \to (0,+\infty)$ *be a decreasing function. For* $\varepsilon \geq 0$ *and* $\lambda \neq 1$, *let* $F_\lambda(\sigma)$ *be defined as follows:*

$$F_\lambda(\sigma) = \begin{cases} \displaystyle\int_0^\sigma \tau^\varepsilon \phi(\tau)\,d\tau & \text{if } \lambda < 1 \\ \displaystyle\int_\sigma^\infty \tau^\varepsilon \phi(\tau)\,d\tau & \text{if } \lambda > 1. \end{cases}$$

If $r > 0$, *then:*

$$\int_0^{+\infty} \left(\frac{F_\lambda(\sigma)}{\sigma}\right)^r \sigma^{r\lambda}\,\frac{d\sigma}{\sigma} \leq c\int_0^{+\infty} \big(\phi(\sigma)\big)^r \sigma^{r(\varepsilon+\lambda)}\,\frac{d\sigma}{\sigma},$$

where c *is a constant which depends only on* ε, r *and* λ.

Proof of Corollary 2.3. The corollary is a consequence of the fact that, if we integrate (2.4) between s and $|\Omega|$, we have:

$$B(u_n^*(\sigma)) \leq \frac{1}{\left(N\,C_N^{1/N}\right)^{p'}}\int_\sigma^{|\Omega|}\left(\int_0^\rho f_n^*(\tau)\,d\tau\right)^{p'/p}\frac{d\rho}{\rho^{p'/N'}}. \tag{2.13}$$

Immediately we get part (i) by evaluating $B(u_n^*(0))$. Also part (iii) is immediate. Indeed, taking into account the fact that B is a positive increasing function we can say that the decreasing rearrangement of $B(|u_n(x)|)$ coincides with $B(u_n^*(\sigma))$. On the other hand (2.13) implies:

$$B(u_n^*(\sigma)) \leq \frac{N'\|f_n\|_{L^1(\Omega)}^{p'/p}}{(p'-N')\left(N\,C_N^{1/N}\right)^{p'}\;\sigma^{p'/N'-1}},$$

which gives (2.12).

As regards part (ii), we observe that (2.13) gives:

$$\|B(|u_n|)^r\|_{L^1(\Omega)} \le \frac{1}{\left(N\,C_N^{1/N}\right)^{p'}}\left(\int_0^{|\Omega|}\left(\int_\sigma^{|\Omega|}\left(\int_0^\rho f_n^*(\tau)\,d\tau\right)^{p'/p}\frac{d\rho}{\rho^{p'/N'}}\right)^r d\sigma\right).$$

Recalling that the function $\bar{f}_n(\rho) = \dfrac{1}{\rho}\displaystyle\int_0^\rho f_n^*(\tau)\,d\tau$ is decreasing, we can use two times Lemma 2.4 to obtain:

$$\|B(|u_n|)^r\|_{L^1(\Omega)} \le c\left(\int_0^{|\Omega|}(f^*(\tau))^{rp'/p}\tau^{rp'/N}\,d\tau\right),$$

where c is a suitable constant. Recalling that, in particular, $f_n \in M^m(\Omega)$ (with m as in (ii)), the above inequality implies (2.11). □

Remark 2.5. Looking at (2.13) in the proof of Corollary 2.3 one realizes that (2.10) can be proven under the weaker assumption:

$$\int_0^{|\Omega|}\left(\int_0^\sigma f_n^*(\tau)\,d\tau\right)^{p'/p}\frac{d\sigma}{\sigma^{p'/N'}} < +\infty.$$

The above condition is equivalent to saying that f belongs to the Lorentz space $L(N/p, p'/p)$. Clearly statement (ii) can also be given in terms of suitable estimates in Lorentz spaces. We also remark that in the case $p = N$ in statement (ii) (which we have not studied here), similar results can be given. More precisely, one can prove that u belongs to a suitable Orlicz space (see [25] when $N = p = 2$).

Remark 2.6. Using the explicit form of the function B, Corollary 2.3 thus implies the following:

a) if $m > \frac{N}{p} \ge 1$, then for every θ in $[0, 1]$ the norm of u_n in $L^\infty(\Omega)$ is bounded by a constant depending on the norm of f_n in $L^m(\Omega)$;
b) if $1 < m < \frac{N}{p}$, then for every θ in $[0, 1)$ the norm of $u_n^{(1-\theta)\frac{Nm(p-1)}{N-mp}}$ is bounded in $L^1(\Omega)$ by a constant depending on the norm of f_n in $L^m(\Omega)$;
c) if $m = 1$, then for every θ in $[0, 1)$ the norm of u_n in $M^{\frac{N(p-1)(1-\theta)}{N-p}}(\Omega)$ is bounded by a constant depending on the norm of f_n in $L^1(\Omega)$.

We explicitly observe that in b) the case $\theta = 1$ is a limit case where it is not possible, in general, to prove that $|u_n|^q$ is bounded in $L^1(\Omega)$ for some $q > 0$. As a consequence of Corollary 2.3 one can only say that $\log(1 + |u_n|)$ is bounded in $L^{\frac{Nm(p-1)}{N-mp}}(\Omega)$. A similar observation holds true in c).

It is possible to get information about u_n also when assumption (2.9) is not satisfied. For example, we have the following result:

Corollary 2.7. *Suppose that u_n is a solution of* (2.1)*, and assume*

$$\theta > 1 . \tag{2.14}$$

If f_n is such that

$$\frac{1}{\left(N C_N^{1/N}\right)^{p'}} \int_0^{|\Omega|} \left(\int_0^{\sigma} f_n^*(\tau)\, d\tau\right)^{p'/p} \frac{d\sigma}{\sigma^{p'/N'}} < \frac{\alpha^{\frac{1}{p-1}}}{\theta - 1}, \tag{2.15}$$

then,

$$\|u_n\|_{L^\infty(\Omega)} \le B^{-1}\left(\frac{1}{\left(N C_N^{1/N}\right)^{p'}} \int_0^{|\Omega|} \left(\int_0^{\sigma} f_n^*(\tau)\, d\tau\right)^{p'/p} \frac{d\sigma}{\sigma^{p'/N'}}\right), \tag{2.16}$$

where B^{-1} denotes the inverse function of B.

Proof. The proof follows the arguments of Corollary 2.3. Inequality (2.16) is an immediate consequence of (2.13), (2.14) and (2.15). □

Remark 2.8. The results stated in Theorem 2.1, in Corollaries 2.3, 2.7 and in Remarks 2.5, 2.6, are sharp in the sense that the estimates for u_n^* become equalities when the problem has a spherical symmetry. More precisely, consider the model problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla v|^{p-2}\nabla v}{(1+|v|)^{\theta(p-1)}}\right) = f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}, \end{cases} \tag{2.17}$$

where $\Omega^{\#}$ is the ball centered at the origin such that $|\Omega^{\#}| = |\Omega|$ and $f^{\#}(x) = f^*(C_N|x|^N)$, $x \in \Omega^{\#}$. It is easy to show that inequality (2.4) can be written as

$$-\frac{d}{d\sigma} B(u_n^*(\sigma)) \le -\frac{d}{d\sigma} B(v^*(\sigma)).$$

This means that all the estimates which are direct consequences of (2.4) cannot be improved. Using this kind of argument the optimality of Corollary 2.7 will be further analysed in Section 6.

3. Gradient estimates

In this section we will give some *a priori* estimates on the gradients of the solutions of (2.1), depending on the various regularity assumptions on f_n as stated in Theorems 1.1, 1.3, 1.7 and 1.9.

We start with the simplest result concerning the assumptions on f_n which yield bounded solutions. We observe that if u_n is bounded in $L^\infty(\Omega)$, then it is also natural to expect a boundedness in $W_0^{1,p}(\Omega)$, since in this case the lack of coerciveness of the operator A “disappears”.

Theorem 3.1. *Let $0 \le \theta \le 1$, let $m > N/p$, and let u_n be a solution of* (2.1). *Then there exists a constant c, continuously depending on the norm of f_n in $L^m(\Omega)$, and independent of n, such that*

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le c\,. \tag{3.1}$$

Proof. We choose u_n as a test function in (2.1). We obtain, using (1.2) and (1.3),

$$\alpha \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}\,dx \le \int_\Omega |f_n|\,|u_n|\,dx\,,$$

which implies

$$\int_\Omega |\nabla u_n|^p\,dx \le c\,(1+\|u_n\|_{L^\infty(\Omega)})^{1+\theta(p-1)}\,\|f_n\|_{L^m(\Omega)}\,,$$

for some positive constant c. Since, by Remark 2.6, a), the norm of u_n in $L^\infty(\Omega)$ is bounded by a constant depending on the norm of f_n in $L^m(\Omega)$, (3.1) is proved. □

The next result deals with the cases studied in Theorems 1.3 and 1.7.

Theorem 3.2. *Let u_n be a solution of* (2.1) *under the assumptions* (1.2) *and* (1.3), *with $0 \le \theta < 1$, and let $\widetilde{m}$, $\overline{m}$, s and q be as in* (1.9), (1.15), (1.11) *and* (1.17), *respectively. We have:*

(a) if $\widetilde{m} \le m < N/p$, then the norm of u_n in $W_0^{1,p}(\Omega)$ and in $L^s(\Omega)$ is bounded by a constant continuously depending on the norm of f_n in $L^m(\Omega)$;
(b) if $\max\{1,\overline{m}\} < m < \widetilde{m}$, then the norm of u_n in $W_0^{1,q}(\Omega)$ and in $L^s(\Omega)$ is bounded by a constant continuously depending on the norm of f_n in $L^m(\Omega)$.

Proof. We already know, from Remark 2.6, that u_n is bounded in $L^s(\Omega)$ by a constant depending on the norm of f_n in $L^m(\Omega)$. We explicitly observe that a direct consequence of the definition of $\overline{m}$ is that when $m > \overline{m}$ then $s > N' > 1$.

In order to prove the gradient estimate, we use the same arguments of Theorem 2.1 to get (2.5). Using the explicit form of $b(t)$ we have

$$\frac{d}{dt}\int_{|u_n|\le t} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)+1-s/m'}}\,dx \le c\,(1+t)^{-1+s/m'}\int_0^{\mu_{u_n}(t)} f_n^*(\tau)\,d\tau. \tag{3.2}$$

Integrating between 0 and $+\infty$ we get

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)+1-s/m'}}\,dx &\le c\int_0^{+\infty}(1+t)^{-1+s/m'}\int_0^{\mu_{u_n}(t)} f_n^*(\tau)\,d\tau\,dt\\ &= \frac{cm'}{s}\int_0^{|\Omega|} f_n^*(\tau)\left[(1+u_n^*(\tau))^{s/m'}-1\right]d\tau\,.\end{aligned} \tag{3.3}$$

Under the assumptions of part (a) we have $\theta(p-1)+1-s/m' \le 0$, so that (3.3), together with the fact that u_n is bounded in $L^s(\Omega)$, immediately implies that u_n is bounded in $W_0^{1,p}(\Omega)$ (and the norm of u_n is controlled by a constant depending on the norm of f_n in $L^m(\Omega)$). Under the assumptions of part (b) we have $\theta(p-1)+1-s/m' > 0$, so, for q as in (1.17), we get

$$\int_\Omega |\nabla u_n|^q\,dx \leq \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)+1-s/m'}}\,dx\right)^{\frac{q}{p}} \left(\int_\Omega (1+|u_n|)^s\,dx\right)^{1-\frac{q}{p}},$$

and the proof is then complete, since every term can be estimated with constants depending on the norm of u_n in $L^s(\Omega)$, hence on the norm of f_n in $L^m(\Omega)$. □

Remark 3.3. A different way of proving the previous theorem is to use $v = (1+|u_n|)^{s/m'} - 1$ as a test function in (2.1), which is admissible since u_n belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.4. We observe that in the case (a) the arguments used in Theorem 3.2 allow us to prove that not only the norm of u_n is bounded in $W_0^{1,p}(\Omega)$, but also that $\dfrac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)+1-s/m'}}$ is bounded in $L^1(\Omega)$, a slightly stronger result (recall that in this case $\theta(p-1)+1-s/m' \leq 0$).

If $1 \leq m \leq \max\{1, \overline{m}\}$ we have to turn to Marcinkiewicz spaces in order to obtain *a priori* estimates on the gradients of u_n.

Theorem 3.5. *Let u_n be a solution of* (2.1) *under the assumptions* (1.2) *and* (1.3), *with* $0 \leq \theta < 1$, *and let $\overline{m}$, s and q be as in* (1.15), (1.11) *and* (1.17), *respectively. If we have*

$$1 \leq m \leq \max\{1, \overline{m}\}, \tag{3.4}$$

then the norm of u_n in $M^s(\Omega)$ and the norm of $|\nabla u_n|$ in $M^q(\Omega)$ are bounded by constants continuously depending on the norm of f_n in $L^m(\Omega)$. Furthermore, if $\overline{m} > 1$ and $1 < m \leq \overline{m}$, then the norm of $|u_n|^s$ is bounded in $L^1(\Omega)$ (once again, by a constant continuously depending on the norm of f_n in $L^m(\Omega)$).

Before proving the above theorem we state a technical lemma which gives a sufficient condition for a function to be in a Marcinkiewicz space.

Lemma 3.6. *Let v be a measurable function belonging to $M^r(\Omega)$ for some $r > 0$, such that, for every $k \geq 0$, $T_k(v)$ belongs to $W_0^{1,p}(\Omega)$, $p > 1$. Suppose that*

$$\int_{|v|\leq k} |\nabla v|^p\,dx \leq ck^\lambda, \qquad \forall k > k_0, \tag{3.5}$$

for some non-negative λ, c and k_0. Then the weak gradient of v is such that $|\nabla v|$ belongs to $M^q(\Omega)$, with $q = \dfrac{rp}{r+\lambda}$.

Proof. The proof of the lemma is essentially the same as the proof of Lemma 3.2 in [11] (see also [8]), but, for the sake of completeness, we sketch it. If $k > 0$ is fixed, for every $t > 0$, we can write

$$|\{|\nabla v| > k\}| \le |\{|\nabla v| > k,\ |v| \le t\}| + |\{|v| > t\}|\,.$$

Using (3.5) and the fact that $v \in M^r(\Omega)$, we have, for $t > k_0$,

$$\begin{aligned} |\{|\nabla v| > k\}| &\le \frac{1}{k^p}\int_\Omega |\nabla T_t(v)|^p\,dx + |\{|v| > t\}| \\ &\le c\left(\frac{t^\lambda}{k^p} + \frac{1}{t^r}\right). \end{aligned} \tag{3.6}$$

For k sufficiently large ($k > (\lambda k_0^{r+\lambda}/r)^{1/p}$), a minimization of the right-hand side of (3.6) gives

$$|\{|\nabla v| > k\}| \le \frac{c}{t^{rp/(r+\lambda)}}.$$

Observing that, for any value of k, $|\{|\nabla v| > k\}| \le |\Omega|$, we obtain the assertion. □

Proof of Theorem 3.5. The boundedness of u_n in both Lebesgue and Marcinkiewicz spaces holds true by Remark 2.6. We explicitly observe that, in this case, s can be smaller than 1 (a fact that happens if either p or θ are close to 1).

As regards the gradient estimates, we can argue as in the proof of Theorem 3.2 in order to obtain (3.2). A simple consequence of it is the following inequality:

$$\frac{d}{dt}\int_{|u_n|\le t} |\nabla u_n|^p\,dx \le c(1+t)^{\theta(p-1)}\int_0^{\mu_{u_n}(t)} f_n^*(\tau)\,d\tau.$$

Integrating between 0 and k, $k > 0$, we get

$$\int_{|u_n|\le k} |\nabla u_n|^p\,dx \le c\int_0^k (1+t)^{\theta(p-1)}\int_0^{\mu_{u_n}(t)} f_n^*(\tau)\,d\tau\,dt. \tag{3.7}$$

If $m = 1$, from (3.7) we have

$$\int_{|u_n|\le k} |\nabla u_n|^p\,dx \le c((1+k)^{\theta(p-1)+1} - 1)\,,$$

where c is a constant which depends only on the data. This means that we can apply Lemma 3.6, obtaining that $|\nabla u_n|$ is bounded in $M^q(\Omega)$, where q is as in (1.17); that is the assertion.

If $1 < m \le \max\{1, \overline{m}\}$ (this means $\overline{m} > 1$), we have to use the fact that u_n is bounded in $M^s(\Omega)$. From (3.7) we get

$$\int_{|u_n|\le k} |\nabla u_n|^p\,dx \le c\int_0^k \frac{(1+t)^{\theta(p-1)}}{t^{s/m'}}\,dt,$$

where c is a constant which depends only on the data. The above inequality implies that, for k large, inequality (3.5) holds true with $\lambda = \theta(p-1) + 1 - s/m'$. An application of Lemma 3.6 then gives the assertion. □

4. Almost everywhere convergence of gradients

This section will be devoted to the proof of the almost everywhere convergence of the gradients of the approximate solutions u_n, a technical result which, together with the *a priori* estimates proved in the preceding sections, will allow us to pass to the limit in the approximate equations (2.1).

Recall that if u is a measurable function such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ then it is possible to define its weak gradient ∇u.

Our result is the following:

Theorem 4.1. *Let u_n be a sequence of solutions of the problems*

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n), \nabla u_n)) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with f_n strongly convergent to some f in $L^1(\Omega)$. Suppose that:

(i) u_n is such that $T_k(u_n)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$;
(ii) u_n converges almost everywhere in Ω to some measurable function u which is finite almost everywhere, and such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ (note that (i) and (ii) imply that $T_k(u_n)$ weakly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$);
(iii) u_n is bounded in $M^{r_1}(\Omega)$ for some $r_1 > 0$, and u belongs to the same $M^{r_1}(\Omega)$;
(iv) there exists $\gamma > 0$ such that $|\nabla u_n|^\gamma$ is bounded in $L^{r_2}(\Omega)$, for some $r_2 > 1$, and $|\nabla u|^\gamma$ belongs to the same $L^{r_2}(\Omega)$.

Then, up to a subsequence, ∇u_n converges almost everywhere in Ω to ∇u, the weak gradient of u.

Proof. We follow the proof which can be found in [9]. Let λ be a real number between 0 and 1, which will be chosen later. Define $a_n(x, t, \xi) = a(x, T_n(t), \xi)$ and (for the sake of simplicity, we omit the dependence of a_n on x)

$$I(n) = \int_\Omega \{[a_n(u_n, \nabla u_n) - a_n(u_n, \nabla u)] \cdot \nabla(u_n - u)\}^\lambda \, dx\,.$$

Note that $I(n)$ is well defined since the integrand function is non-negative thanks to (1.5). We fix $k > 0$ and split the integral in $I(n)$ on the sets $\{|u| > k\}$ and $\{|u| \le k\}$, obtaining

$$I_1(n, k) = \int_{\{|u|>k\}} \{[a_n(u_n, \nabla u_n) - a_n(u_n, \nabla u)] \cdot \nabla(u_n - u)\}^\lambda \, dx\,,$$

and

$$I_2(n, k) = \int_{\{|u|\le k\}} \{[a_n(u_n, \nabla u_n) - a_n(u_n, \nabla u)] \cdot \nabla(u_n - u)\}^\lambda \, dx\,.$$

We remark that, by the growth conditions on a (see (1.4)), one has

$$I_1(n, k) \le c \int_{\{|u|>k\}} \left(1 + |\nabla u_n|^{\lambda p} + |\nabla u|^{\lambda p}\right) dx\,.$$

We now choose $\lambda < 1$ such that $\lambda\, p = \gamma$. Using the Hölder inequality and (iv) we obtain

$$
\begin{aligned}
I_1(n,k) &\le c\left(\int_\Omega \left(1+|\nabla u_n|^{\gamma\, r_2}+|\nabla u|^{\gamma\, r_2}\right)dx\right)^{\frac{1}{r_2}}\ |\{|u|>k\}|^{1-\frac{1}{r_2}}\\
&\le c\,|\{|u|>k\}|^{1-\frac{1}{r_2}}\,.
\end{aligned}
$$

By (iii), and by the choice of λ, we thus have

$$
\lim_{k\to+\infty}\ \limsup_{n\to+\infty}\ I_1(n,k)=0\,. \tag{4.2}
$$

Since where $|u|\le k$ one has $u=T_k(u)$, and since the integrand function is non-negative, for $I_2(n,k)$ one has

$$
\begin{aligned}
&I_2(n,k)\le I_3(n,k)\\
&\qquad=\int_\Omega \{[a_n(u_n,\nabla u_n)-a_n(u_n,\nabla T_k(u))]\cdot\nabla(u_n-T_k(u))\}^{\lambda}\ dx\,.
\end{aligned}
$$

We fix $h>0$ and split the integral in $I_3(n,k)$ on the sets $\{|u_n-T_k(u)|>h\}$ and $\{|u_n-T_k(u)|\le h\}$, obtaining

$$
\begin{aligned}
&I_4(n,k,h)\\
&\quad=\int_{\{|u_n-T_k(u)|>h\}}\{[a_n(u_n,\nabla u_n)-a_n(u_n,\nabla T_k(u))]\cdot\nabla(u_n-T_k(u))\}^{\lambda}\ dx\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&I_5(n,k,h)\\
&\quad=\int_{\{|u_n-T_k(u)|\le h\}}\{[a_n(u_n,\nabla u_n)-a_n(u_n,\nabla T_k(u))]\cdot\nabla(u_n-T_k(u))\}^{\lambda}\ dx\,.
\end{aligned}
$$

For $I_4(n,k,h)$ one can reason as for $I_1(n,k)$, since (thanks to (iii)), the measure of the set $\{|u_n-T_k(u)|>h\}$ tends to zero as h tends to $+\infty$ uniformly in n and k. Thus (with the same choice of λ), one has

$$
\lim_{h\to+\infty}\ \limsup_{k\to+\infty}\ \limsup_{h\to+\infty}\ I_4(n,k,h)=0\,. \tag{4.3}
$$

Since $\nabla(u_n-T_k(u))=\nabla T_h(u_n-T_k(u))$ on the set $\{|u_n-T_k(u)|\le h\}$, we have

$$
I_5(n,k,h)=\int_\Omega\{[a_n(u_n,\nabla u_n)-a_n(u_n,\nabla T_k(u))]\cdot\nabla T_h(u_n-T_k(u))\}^{\lambda}\ dx\,.
$$

By the Hölder inequality (with exponents $1/\lambda$ and $1/(1-\lambda)$), we have

$$
\begin{aligned}
&I_5(n,k,h)\\
&\quad\le|\Omega|^{1-\lambda}\left\{\int_\Omega [a_n(u_n,\nabla u_n)-a_n(u_n,\nabla T_k(u))]\cdot\nabla T_h(u_n-T_k(u))\,dx\right\}^{\lambda}\,.
\end{aligned}
$$

Define

$$I_6(n,k,h) = \int_\Omega [a_n(u_n,\nabla u_n) - a_n(u_n,\nabla T_k(u))]\cdot \nabla T_h(u_n - T_k(u))\,dx\,,$$

which we split as the difference $I_7(n,k,h) - I_8(n,k,h)$, where

$$I_7(n,k,h) = \int_\Omega a_n(u_n,\nabla u_n)\cdot \nabla T_h(u_n - T_k(u))\,dx\,,$$

and

$$I_8(n,k,h) = \int_\Omega a_n(u_n,\nabla T_k(u))\cdot \nabla T_h(u_n - T_k(u))\,dx\,.$$

The integral in $I_8(n,k,h)$ is on the set where $|u_n - T_k(u)| \le h$, that is a subset of the set where $|u_n| \le k+h$; thus, if $n \ge h+k$, one has (recalling the definition of a_n), $a_n(u_n,\nabla T_k(u)) = a(u_n,\nabla T_k(u))$. Using the almost everywhere convergence of u_n to u, and the growth assumptions on a, one has that

$$a(u_n,\nabla T_k(u)) \to a(u,\nabla T_k(u)) \qquad \text{strongly in } (L^{p'}(\Omega))^N\,,$$

so that, using the weak convergence of $\nabla T_h(u_n - T_k(u))$ to $\nabla T_h(u - T_k(u))$ in $(L^p(\Omega))^N$ (a consequence of (i) and (ii)), one has

$$\lim_{n\to+\infty} I_8(n,k,h) = \int_\Omega a(u,\nabla T_k(u))\cdot \nabla T_h(u - T_k(u))\,dx = 0\,,$$

since $a(u,\nabla T_k(u)) \neq 0$ only on the set $|u| \le k$, and on this set the gradient of $T_h(u - T_k(u))$ is zero. For $I_7(n,k,h)$ we use the equation (2.1), and we obtain

$$I_7(n,k,h) = \int_\Omega f_n\, T_h(u_n - T_k(u))\,dx\,.$$

Using the strong convergence of f_n in $L^1(\Omega)$, one then has

$$\lim_{n\to+\infty} I_7(n,k,h) = \int_\Omega f\, T_h(u - T_k(u))\,dx\,,$$

so that

$$\lim_{k\to+\infty}\lim_{n\to+\infty} I_7(n,k,h) = 0\,. \tag{4.4}$$

Putting together (4.2), (4.3) and (4.4) one thus has

$$\lim_{n\to+\infty} I(n) = 0\,.$$

Since the integrand function in $I(n)$ is non-negative, this implies that

$$\{[a_n(u_n,\nabla u_n) - a_n(u_n,\nabla u)]\cdot\nabla(u_n - u)\}^\lambda \to 0\,, \quad \text{strongly in } L^1(\Omega).$$

Thus, up to subsequences still denoted by u_n,

$$[a_n(x,u_n(x),\nabla u_n(x)) - a_n(x,u_n(x),\nabla u(x))]\cdot\nabla(u_n(x) - u(x)) \to 0\,, \tag{4.5}$$

for almost every x in Ω. We now conclude the proof using the same technique as that of [18]. Let x in Ω be such that $u_n(x)$ converges to $u(x)$, that $|u(x)| < +\infty$, and that (4.1) holds true. Due to (ii), the set of x in Ω such that at least one of the above properties does not hold has zero measure. Since $|u(x)| < +\infty$, one has $|u_n(x)| \leq |u(x)| + 1 \leq n$ for n large enough, so that (4.5) becomes

$$[a(x, u_n(x), \nabla u_n(x)) - a(x, u_n(x), \nabla u(x))] \cdot \nabla(u_n(x) - u(x)) \to 0\,. \qquad (4.6)$$

Due to the growth assumptions on a with respect to ξ, and to the fact that $|u_n(x)|$ remains bounded, one has that $\{|\nabla u_n(x)|\}$ is a bounded sequence. Let ρ be a limit point of the sequence $\nabla u_n(x)$. Thanks to the continuity of a, and to (4.6), one has

$$[a(x, u(x), \rho) - a(x, u(x), \nabla u(x))] \cdot (\rho - \nabla u(x)) = 0\,,$$

and this implies, by (1.5), that $\rho = \nabla u(x)$. Since the limit is independent of the subsequence, $\nabla u_n(x)$ converges to $\nabla u(x)$, and this result holds for almost every x in Ω. □

5. Proof of the results

In this section we are going to combine the results of Sections 2, 3 and 4, in order to prove Theorems 1.1, 1.3, 1.7 and 1.9.

Let f be in $L^m(\Omega)$, with $m \geq 1$, and let f_n be a sequence of $L^\infty(\Omega)$ functions strongly convergent to f in $L^m(\Omega)$. Then let u_n be a sequence of solutions of (2.1), which exist by the result of [18]. Let $k > 0$ be fixed, and choose $T_k(u_n)$ as a test function in (2.1) to obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n)\, dx = \int_\Omega f_n\, T_k(u_n)\, dx\,.$$

Using (1.2), we have, for $n > k$,

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p\, dx \leq (1+k)^{\theta(p-1)}\, k\, \|f_n\|_{L^1(\Omega)}\,,$$

so that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ independently of n. This implies (see [8]) that there exists a subsequence of u_n (still denoted by u_n) which is almost everywhere convergent in Ω to a measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. Since f_n is at least bounded in $L^1(\Omega)$, then u_n is bounded in $M^s(\Omega)$ (with s as in (1.11)), and $|\nabla u_n|$ is bounded in $M^q(\Omega)$ (with q as in (1.17), written for $m = 1$). It is then true that u_n satisfies the assumptions i)–iv) of Theorem 4.1 (with $r_1 = s$ and $\gamma = q/2$, $r_2 = 2$), and so ∇u_n almost everywhere converges to ∇u.

Thus,

$$a(x, T_n(u_n), \nabla u_n) \quad \text{almost everywhere converges to} \quad a(x, u, \nabla u)\,. \qquad (5.1)$$

Suppose now that the assumptions of Theorems 1.1 or 1.3 hold. Then u_n is bounded in $W_0^{1,p}(\Omega)$ so that (as a consequence of (1.4)), $|a(x, T_n(u_n), \nabla u_n)|$ is bounded in

$L^{p'}(\Omega)$. Thus, by (5.1), $a(x, T_n(u_n), \nabla u_n)$ is weakly convergent to $a(x, u, \nabla u)$ in $(L^{p'}(\Omega))^N$. If v belong to $W_0^{1,p}(\Omega)$ is it then possible to pass to the limit as n tends to infinity in the identities

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v \, dx = \int_\Omega f_n \, v \, dx \,,$$

to obtain that u is a solution of (1.7) in the sense (1.8).

If we are under the assumptions of Theorems 1.7 or 1.9, we fix $k > 0$, φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and choose $T_k(u_n - \varphi)$ as a test function in (2.1). We have

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \varphi) \, dx = \int_\Omega f_n \, T_k(u_n - \varphi) \, dx \,.$$

The right-hand side converges, as n tends to infinity, to

$$\int_\Omega f \, T_k(u - \varphi) \, dx \,,$$

since f_n is strongly covergent in (at least) $L^1(\Omega)$, while $T_k(u_n - \varphi)$ converges both weakly$*$ in $L^\infty(\Omega)$ and almost everywhere to $T_k(u - \varphi)$. As the left-hand side is concerned, we split it as the sum

$$\int_{\{|u_n - \varphi| \le k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx - \int_{\{|u_n - \varphi| \le k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla \varphi \, dx \,.$$

Taking into account the fact that the second integral is on a subset of the set where $|u_n| \le k + \|\varphi\|_{L^\infty(\Omega)} = M$, we can rewrite it (taking $n > M$) as

$$\int_{\{|u_n - \varphi| \le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla \varphi \, dx \,.$$

Since $|a(x, T_M(u_n), \nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega)$, a consequence of (5.1) is that $a(x, T_M(u_n), \nabla T_M(u_n))$ is weakly convergent to $a(x, T_M(u), \nabla T_M(u))$ in $(L^{p'}(\Omega))^N$; hence, the second integral converges, as n tends to infinity, to

$$\int_{\{|u - \varphi| \le k\}} a(x, T_M(u), \nabla T_M(u)) \cdot \nabla \varphi \, dx = \int_{\{|u - \varphi| \le k\}} a(x, u, \nabla u) \cdot \nabla \varphi \, dx \,.$$

The integrand function of the first is non-negative (by (1.2)), and it is almost everywhere convergent (by (5.1)); by Fatou's lemma, we have

$$\int_{\{|u - \varphi| \le k\}} a(x, u, \nabla u) \cdot \nabla u \, dx \le \liminf_{n \to +\infty} \int_{\{|u_n - \varphi| \le k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \,.$$

Putting all the terms together, we obtain

$$\int_{\{|u - \varphi| \le k\}} a(x, u, \nabla u) \cdot \nabla (u - \varphi) \, dx = \int_\Omega f \, T_k(u - \varphi) \, dx \,,$$

which is (1.14).

As a final remark, we observe that the solution u we obtained possesses the regularity stated in the statements of the theorems, due to the results of Sections 2 and 3. □

5.1. Data in divergence form

If the datum of (1.7) is in divergence form, it is possible to give an existence result similar to those of the previous theorems. For the sake of simplicity, we consider the model problem (1.1) with data in divergence form, that is

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = -\operatorname{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

with $|F|$ in $L^{p'}(\Omega)$.

Theorem 5.1. *Let $0 \le \theta < 1$, and let $|F|$ be in $L^{p'}(\Omega)$. Then there exists at least an entropy solution of* (5.2)*, i.e., a function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k \ge 0$ and*

$$\int_\Omega \frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}} \cdot \nabla T_k(u-\varphi)\,dx = \int_\Omega F \cdot \nabla T_k(u-\varphi)\,dx\,, \tag{5.3}$$

for every φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, u is such that $|u|^{(1-\theta)\,p^*}$ belongs to $L^1(\Omega)$.

Proof. Let u_n be a sequence of solutions of

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u_n|^{p-2}\nabla u_n}{(1+|T_n(u_n)|)^{\theta(p-1)}}\right) = -\operatorname{div}(F) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega; \end{cases} \tag{5.4}$$

observe that u_n exists in $W_0^{1,p}(\Omega)$ by the result of [18] since $-\operatorname{div}(F)$ is in $W^{-1,p'}(\Omega)$.

Taking $T_k(u_n)$ as a test function in (5.4), we have

$$\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|u_n|)^{\theta(p-1)}}\,dx \le \int_\Omega F\,\nabla T_k(u_n)\,dx\,,$$

and so

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx \le (1+k)^{\theta p} \int_\Omega |F|^{p'}\,dx\,.$$

Hence, $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k \ge 0$.

Let us define

$$H(t) = \int_0^t \frac{ds}{(1+|s|)^\theta}\,,$$

and take $H(u_n)$ as a test function in (5.4); we have

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}\,dx &\le \int_\Omega F \cdot \frac{\nabla u_n}{(1+|u_n|)^\theta}\,dx \\ &\le \||F|\|_{L^{p'}(\Omega)} \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}\,dx\right)^{\frac{1}{p}}\,.\end{aligned}$$

Hence,

$$\int_\Omega |\nabla H(u_n)|^p \, dx \le \int_\Omega |F|^{p'} \, dx \,,$$

so that $H(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Moreover, by Sobolev embedding,

$$\int_\Omega |H(u_n)|^{p^*} \, dx \le c \,,$$

and since $H(u_n)$ behaves like $|u_n|^{1-\theta}$, we have

$$\int_\Omega |u_n|^{(1-\theta)\, p^*} \, dx \le c \,.$$

The fact that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and the estimate on $|u_n|^{(1-\theta)\, p^*}$ imply that (up to subsequences still denoted by u_n) u_n converges almost everywhere in Ω to some function u, which is such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$, $H(u)$ belongs to $W_0^{1,p}(\Omega)$, and $|u|^{(1-\theta)\, p^*}$ is in $L^1(\Omega)$.

We now choose $H(T_k(u_n)) - H(T_k(u))$ as a test function in (5.4), and observe that, since $H(T_k(u_n))$ weakly converges to $H(T_k(u))$ in $W_0^{1,p}(\Omega)$, then

$$\lim_{n\to+\infty} \int_\Omega F \cdot (\nabla H(T_k(u_n)) - \nabla H(T_k(u)))\, dx = 0 \,.$$

Hence

$$\lim_{n\to+\infty} \int_\Omega \frac{|\nabla u_n|^{p-2}\, \nabla u_n}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} \cdot (\nabla H(T_k(u_n)) - \nabla H(T_k(u)))\, dx = 0 \,.$$

We now write

$$\frac{|\nabla u_n|^{p-2}\, \nabla u_n}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} = \frac{|\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n)}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} + \frac{|\nabla G_k(u_n)|^{p-2}\, \nabla G_k(u_n)}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} \,,$$

and deal with the two integrals separately. We have, for $n > k$,

$$\begin{aligned} \frac{|\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n)}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} &= \frac{|\nabla T_k(u_n)|^{p-2}\, \nabla T_k(u_n)}{(1+|T_k(u_n)|)^{\theta\,(p-1)}} \\ &= |\nabla H(T_k(u_n))|^{p-2}\, \nabla H(T_k(u_n)) \,, \end{aligned}$$

so that

$$\begin{aligned} &\int_\Omega \frac{|\nabla u_n|^{p-2}\, \nabla u_n}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} \cdot (\nabla H(T_k(u_n)) - \nabla H(T_k(u)))\, dx \\ &= \int_\Omega \Big[(|\nabla H(T_k(u_n))|^{p-2}\, \nabla H(T_k(u_n)) - |\nabla H(T_k(u))|^{p-2}\, \nabla H(T_k(u))) \\ &\qquad \cdot (\nabla H(T_k(u_n)) - \nabla H(T_k(u))) \Big]\, dx \\ &\quad + \int_\Omega |\nabla H(T_k(u))|^{p-2}\, \nabla H(T_k(u)) \cdot (\nabla H(T_k(u_n)) - \nabla H(T_k(u)))\, dx \\ &\quad - \int_\Omega \frac{|\nabla G_k(u_n)|^{p-2}\, \nabla G_k(u_n)}{(1+|T_n(u_n)|)^{\theta\,(p-1)}} \cdot \nabla H(T_k(u))\, dx \,. \end{aligned}$$

Remark that

$$\lim_{n\to+\infty}\int_\Omega |\nabla H(T_k(u))|^{p-2}\,\nabla H(T_k(u))\cdot(\nabla H(T_k(u_n))-\nabla H(T_k(u)))\,dx=0\,,$$

since $\nabla H(T_k(u_n)) - \nabla H(T_k(u))$ converges weakly to zero in $(L^{p'}(\Omega))^N$ and that

$$\lim_{n\to+\infty}\int_\Omega \frac{|\nabla G_k(u_n)|^{p-2}\,\nabla G_k(u_n)}{(1+|T_n(u_n)|)^{\theta(p-1)}}\cdot\nabla H(T_k(u))\,dx=0\,,$$

since (again up to subsequences) $\frac{|\nabla G_k(u_n)|^{p-2}\,\nabla G_k(u_n)}{(1+|T_n(u_n)|)^{\theta(p-1)}}$ converges weakly in $(L^{p'}(\Omega))^N$ to $\Psi(x)\chi_{\{|u(x)|\geq k\}}$, for some $\Psi\in(L^{p'}(\Omega))^N$, and $\nabla H(T_k(u))=0$ on the set $\{|u(x)|\geq k\}$. Thus,

$$\lim_{n\to+\infty}\int_\Omega\Big[(|\nabla H(T_k(u_n))|^{p-2}\,\nabla H(T_k(u_n))-|\nabla H(T_k(u))|^{p-2}\,\nabla H(T_k(u)))\cdot(\nabla H(T_k(u_n))-\nabla H(T_k(u)))\Big]dx=0\,,$$

which implies that

$$H(T_k(u_n))\to H(T_k(u))\qquad\text{strongly in } W_0^{1,p}(\Omega).$$

Choosing $T_k(u_n-\varphi)$ as a test function in (5.4), we obtain

$$\int_\Omega |\nabla H(u_n)|^{p-2}\,\nabla H(u_n)\cdot\nabla T_k(u_n-\varphi)\,dx=\int_\Omega F\cdot\nabla T_k(u_n-\varphi)\,dx\,.$$

Observing that $\nabla T_k(u_n-\varphi)\neq 0$ only on the set $\{|u_n-\varphi|\leq k\}$, and that on this set $|u_n|\leq M=k+\|\varphi\|_{L^\infty(\Omega)}$, we have

$$\int_\Omega |\nabla H(T_M(u_n))|^{p-2}\nabla H(T_M(u_n))\cdot\nabla T_k(u_n-\varphi)dx=\int_\Omega F\cdot\nabla T_k(u_n-\varphi)dx.$$

Hence, it is possible to pass to the limit as n tends to infinity, obtaining (5.3). □

6. The case $\theta > 1$

In this section we are going to consider the case $\theta > 1$. Our first result is an existence one.

Theorem 6.1. *Let $\theta>1$, and let f be in $L^m(\Omega)$, with $m>\frac{N}{p}$. Then there exists $M>0$ such that if $\|f\|_{L^m(\Omega)}\leq M$, then there exists a solution u of* (1.7) *with u in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.*

Proof. If $\theta>1$ then the function B defined in (2.3) is bounded by $\frac{\alpha^{\frac{1}{p-1}}}{\theta-1}$. It is then enough to apply Corollary 2.7 and Theorem 3.1. □

In order to prove the next result, which "characterizes" the problem of existence and non-existence for $\theta > 1$, from now on we will make stronger growth assumptions on a with respect to both t and ξ. More precisely, we will consider $p = 2$, and functions $a(x, t, \xi)$ of the form

$$a(x, t, \xi) = a(x, t)\, \xi \,,$$

with

$$\frac{\alpha}{(1 + |t|)^{\theta}} \leq a(x, t) \leq \frac{\beta}{(1 + |t|)^{\theta}}\,, \tag{6.1}$$

for some positive real numbers α, β, and $\theta > 1$, for almost every x in Ω, and for every t in $\mathbf{R}$. Moreover, we suppose that there exists $L > 0$ such that

$$|a(x, t_1) - a(x, t_2)| \leq L\, |t_1 - t_2|\,, \tag{6.2}$$

for almost every x in Ω, and for every t_1, t_2 in $\mathbf{R}$. Our result is the following:

Theorem 6.2. *Let $\theta > 1$, let f be in $L^m(\Omega)$, with $m > \frac{N}{2}$, with $f \geq 0$. Let λ be a positive real number, and consider the problem*

$$\begin{cases} -\operatorname{div}\big(a(x, u_\lambda)\, \nabla u_\lambda\big) = \lambda\, f & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Then there exists $\lambda^* > 0$, $\lambda^* \in \mathbf{R}$, such that:

(i) for every λ in $[0, \lambda^*)$ there exists a solution u_λ of (6.3), with u_λ in $H_0^1(\Omega) \cap L^\infty(\Omega)$;
(ii) there exists no solution u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of (6.3) for $\lambda > \lambda^*$;
(iii) if $0 \leq \lambda < \mu < \lambda^*$, then $u_\lambda \leq u_\mu$;
(iv) for $\lambda = \lambda^*$ there exists an entropy solution of (6.3).

Proof. Let

$$\lambda^* = \sup\left\{\lambda > 0 : \exists u \geq 0 \text{ solution of (6.3), with } u \text{ in } H_0^1(\Omega) \cap L^\infty(\Omega)\right\}\,.$$

By Theorem 6.1, for f fixed, if λ is small enough there exists a solution u of (6.3), with u in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Since f is positive, so is u. Thus, $\lambda^* > 0$.

To prove that $\lambda^* < +\infty$ (which will yield (ii)), suppose by contradiction that for every $\lambda > 0$ there exists a solution u_λ of (6.3), with u_λ in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Then u_λ is a solution of

$$-\operatorname{div}\left(a(x, u_\lambda)\,(1 + u_\lambda)^{\theta}\, \frac{\nabla u_\lambda}{(1 + u_\lambda)^{\theta}}\right) = \lambda\, f\,.$$

Define $a_\lambda(x) = a(x, u_\lambda(x))\,(1 + u_\lambda(x))^{\theta}$, so that, by (6.1), $\alpha \leq a_\lambda(x) \leq \beta$. Thus u_λ is a solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of

$$-\operatorname{div}\left(a_\lambda(x)\, \frac{\nabla u_\lambda}{(1 + u_\lambda)^{\theta}}\right) = \lambda\, f\,.$$

Now let

$$v_\lambda = \frac{1 - (1 + u_\lambda)^{1-\theta}}{\theta - 1};$$

then

$$\nabla v_\lambda = \frac{\nabla u_\lambda}{(1 + u_\lambda)^\theta},$$

so that v_λ solves

$$-\operatorname{div}(a_\lambda(x)\,\nabla v_\lambda) = \lambda\, f\,,$$

and it is easy to see that v_λ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. By definition, and by the fact that $\theta > 1$, one also has

$$\|v_\lambda\|_{L^\infty(\Omega)} \le \frac{1}{\theta - 1}\,. \tag{6.4}$$

If we denote by $G_\lambda(x, y)$ the Green function for the operator $-\operatorname{div}(a_\lambda(x)\,\nabla u)$ in Ω with Dirichlet boundary conditions, then one has (see [22])

$$v_\lambda(x) = \lambda \int_\Omega G_\lambda(x, y)\, f(y)\, dy\,.$$

Since $\alpha \le a_\lambda(x) \le \beta$, by a result of [19] one has that for every set $K \subset\subset \Omega$ there exists a constant $c(\alpha, \beta, K)$ such that

$$G_\lambda(x, y) \ge c(\alpha, \beta, K)\, \Gamma(x, y)\,, \quad \text{for every } (x, y) \in K \times K,$$

where Γ is the Green function of the Laplacian in Ω with Dirichlet boundary conditions. Thus

$$v_\lambda(x) \ge c(\alpha, \beta, K)\, \lambda \int_\Omega \Gamma(x, y)\, f(y)\, dy = c(\alpha, \beta, K)\, \lambda\, w(x)\,,$$

where w is the (strictly positive, by the maximum principle) solution of $-\Delta w = f$. Thus, as λ tends to infinity, one has

$$v_\lambda(x) \to +\infty\,, \quad \text{for every } x \text{ in } \Omega,$$

and this contradicts (6.4).

We are now going to prove that for every λ in $[0, \lambda^*)$ there exists a solution u_λ of (6.3). Let λ in $[0, \lambda^*)$. By definition of λ^*, there exists $\bar\lambda$ in (λ, λ^*) such that (6.3) has a solution $\bar u = u_{\bar\lambda}$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $f \ge 0$, $\bar u$ is a supersolution of problem (6.3). Furthermore, $\underline{u} \equiv 0$ is a subsolution of (6.3), and one has $\underline{u} \le \bar u$. We then define $u_0 = \bar u$, and, by recurrence, let u_n be the solution of

$$-\operatorname{div}(a(x, u_{n-1})\,\nabla u_n) = \lambda\, f\,.$$

Working as in [6], and using (6.2), one proves that the sequence $\{u_n\}$ is decreasing, so that it converges towards a function u_λ, which is easily seen to be a solution of (6.3); hence, (i) is proved. Furthermore, by construction, $0 \le u_\lambda \le \bar u$. With the same kind of techniques one also proves (iii).

Finally, if $\lambda \to \lambda^*$ from below, then the corresponding solutions u_λ increase towards a function u_{λ^*}, hence u_λ converges almost everywhere in Ω to u_{λ^*}. Furthermore, choosing $T_k(u_\lambda)$ as a test function (which can be done, since u_λ, hence $T_k(u_\lambda)$, is in $H_0^1(\Omega) \cap L^\infty(\Omega)$), one has

$$\int_\Omega a(x, u_\lambda)\, |\nabla T_k(u_\lambda)|^2\, dx = \lambda \int_\Omega f\, T_k(u_\lambda)\, dx\,,$$

and so, by (6.1),

$$\int_\Omega |\nabla T_k(u_\lambda)|^2\, dx \le c\, \lambda^*\, (1+k)^{\theta+1}\,.$$

Thus, $T_k(u_\lambda)$ is bounded in $H_0^1(\Omega)$ independently of λ; this fact implies, reasoning as in the proof of Theorems 1.7 or 1.9 (see Section 5), without the need of using the almost everywhere convergence of gradients since the operator is now linear with respect to the gradient, that u_{λ^*} is an entropy solution of (6.3). □

Remark 6.3. Observe that the assumption on the Lipschitz continuity of a has not been used to prove (ii), i.e., (6.3) has no solution for λ large enough only under assumption (6.1). Note also that the bound from above on a given in (6.1) is necessary in order to have such a non-existence result, since, for example, the Laplacian operator satisfies assumption (1.2) with $b \equiv 1$ (and the equation $-\Delta u_\lambda = \lambda\, f$ has a solution for every λ).

Remark 6.4. The result of Theorem 6.2 (iv) states that problem (6.3) has an entropy solution for $\lambda = \lambda^*$, without saying whether such a solution is in $H_0^1(\Omega) \cap L^\infty(\Omega)$, in $H_0^1(\Omega)$, or if it is less regular. As a matter of fact, such a solution is found as the limit of an increasing sequence of functions in $H_0^1(\Omega) \cap L^\infty(\Omega)$, so that it does not automatically inherit the properties of the approximating sequence. We are going to show, with an example, that several possibilities may happen for $\lambda = \lambda^*$. Let $N \ge 2$, let $\Omega = B(0,1)$, the unit ball of $\mathbf{R}^N$, and let $f \equiv 1$. We are going to consider the model example, that is the solutions u_λ of the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u_\lambda}{(1+u_\lambda)^\theta}\right) = \lambda & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Defining, as in the proof of Theorem 6.2,

$$v_\lambda = \frac{1 - (1+u_\lambda)^{1-\theta}}{\theta - 1}\,,$$

one has that v_λ is a solution of

$$\begin{cases} -\Delta v_\lambda = \lambda & \text{in } \Omega, \\ v_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the Laplacian is linear, then $v_\lambda = \lambda\, v_1$, where v_1 is the unique solution of the problem

$$\begin{cases} -\Delta v_1 = 1 & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since v_1 is radially symmetric, it can be explicitly calculated, and one has

$$v_1(\rho) = \frac{1}{2N}(1-\rho^2), \qquad (\rho = |x|)$$

so that $v_\lambda(\rho) = \lambda\, v_1(\rho) = \frac{\lambda}{2N}(1-\rho^2)$. Recall that, by definition, $v_\lambda \le \frac{1}{\theta-1}$, so that one can recover an "actual" solution u_λ starting from v_λ if and only if the maximum of v_λ is strictly smaller than $\frac{1}{\theta-1}$. Since

$$\max_{B(0,1)} v_\lambda(\rho) = v_\lambda(0) = \frac{\lambda}{2N},$$

this can be done if and only if $\lambda < \frac{2N}{\theta-1}$. Thus $\lambda^* = \frac{2N}{\theta-1}$. For $\lambda = \lambda^*$ one has

$$v_{\lambda^*}(\rho) = \frac{1}{\theta-1}\,(1-\rho^2),$$

which implies

$$u_{\lambda^*}(\rho) = \frac{1}{\rho^{\frac{2}{\theta-1}}} - 1 .$$

Note that u_{λ^*} is not in $L^\infty(\Omega)$, and that it belongs to $H_0^1(\Omega)$ if and only if $\theta > \frac{N+2}{N-2}$. Moreover, a rather "bizarre" fact happens: the regularity of u_{λ^*} increases as θ increases, and this is contradiction with the properties of the solutions in the case $\theta < 1$.

Observe also that if we consider as solutions of (6.5) the solutions given starting from v_λ also in the case $\lambda > \lambda^*$, one has

$$u_\lambda(\rho) = \begin{cases} \left(1 - \frac{\lambda(\theta-1)}{2N}(1-\rho^2)\right)^{\frac{1}{1-\theta}} - 1 & \text{if } \rho_\lambda < \rho \le 1, \\ +\infty & \text{if } 0 \le \rho \le \rho_\lambda, \end{cases}$$

where

$$\rho_\lambda = \sqrt{\frac{\lambda(\theta-1) - 2N}{\lambda(\theta-1)}},$$

so that u_λ is equal to $+\infty$ on a set of positive Lebesgue measure.

References

1. Alvino, A., Ferone, V., Trombetti, G.: A priori estimates for a class of non uniformly elliptic equations. Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., 381–391 (1998)
2. Alvino, A., Ferone, V., Trombetti, G.: Estimates for the gradient of nonlinear elliptic equations with L^1 data. Ann. Mat. Pura Appl., IV. Ser. **178**, 129–142 (2000)
3. Alvino, A., Ferone, V., Trombetti, G.: Nonlinear elliptic equations with lower-order terms. Differ. Integral Equ. **14**, 1169–1180 (2001)
4. Alvino, A., Lions, P.L., Trombetti, G.: On optimization problems with prescribed rearrangements. Nonlinear Anal., Theory Methods Appl. **13**, 185–220 (1989)
5. Alvino, A., Lions, P.L., Trombetti, G.: Comparison results for elliptic and parabolic equations via Schwarz symmetrization. Ann. Inst. Henri Poincaré **7**, 37–65 (1990)

6. Artola, M.: Sur une classe de problèmes paraboliques quasi-linéaires. Boll. Unione Mat. Ital. **5**, 51–70 (1986)
7. Bandle, C.: Isoperimetric inequalities and applications. Monographs and Studies in Math., No. 7. London: Pitman 1980
8. Benilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J.L.: An L^1 theory of existence and uniqueness of nonlinear elliptic equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **22**, 240–273 (1995)
9. Boccardo, L.: Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form. "Progress in elliptic and parabolic partial differential equations (Capri, 1994)", ed. by A. Alvino et al., 43–57, Pitman Res. Notes Math. Ser. 350. Harlow: Longman 1996
10. Boccardo, L., Brezis, H.: Elliptic equations with degenerate coercivity. Preprint
11. Boccardo, L., Dall'Aglio, A., Orsina, L.: Existence and regularity results for some degenerate elliptic equations. Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., 51–81 (1998)
12. Boccardo, L., Giachetti, D.: Existence results via regularity for some nonlinear elliptic problems. Commun. Partial Differ. Equations **14**, 663–680 (1989)
13. Boccardo, L., Orsina, L.: Existence and regularity of minima for integral functionals noncoercive in the energy space. Dedicated to Ennio De Giorgi. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **25**, 95–130 (1997)
14. De Giorgi, E.: Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. Ann. Mat. Pura Appl., IV. Ser. **36**, 191–213 (1954)
15. Fleming, W., Rishel, R.: An integral formula for total gradient variation. Arch. Math. **11**, 218–222 (1960)
16. Giachetti, D., Porzio, M.M.: Existence results for some nonuniformly elliptic equations with irregular data. J. Math. Anal. Appl. **257**, 100–130 (2001)
17. Kawohl, B.: Rearrangements and convexity of level sets in P.D.E., Lecture Notes in Math., No. 1150. Berlin, New York: Springer 1985
18. Leray, J., Lions, J.-L.: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. Fr. **93**, 97–107 (1965)
19. Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **17**, 43–77 (1963)
20. Mossino, J.: Inégalités isopérimétriques et applications en physique. Collection Travaux en Cours. Paris: Hermann 1984
21. Porretta, A.: Uniqueness and homogenization for a class of noncoercive operators in divergence form. Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., 915–936 (1998)
22. Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier **15**, 189–258 (1965)
23. Talenti, G.: Elliptic equations and rearrangements. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **3**, 697–718 (1976)
24. Talenti, G.: Nonlinear elliptic equations, Rearrangements of functions and Orlicz spaces. Ann. Mat. Pura Appl., IV. Ser. **120**, 159–184 (1979)
25. Trombetti, C.: Existence and regularity for a class of non-uniformly elliptic equations in two dimensions. Differ. Integral Equ. **13**, 687–706 (2000)